Home Work 2

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1.

a. Let $\sigma = \frac{1}{1+e^{-x}}$

then to show: $\sigma^{'}(x) = \sigma(x)[1 - \sigma(x])$

$\sigma^{'}(x) = \frac{d}{dx}(\frac{1}{1+e^{-x}}) = \frac{d}{dx}((1+e^{-x})^{-1})$

$= e^{-x}(1+e^{-x})^{-1} = (\frac{1}{1+e^{-x}})(\frac{e^{-x}}{1+e^{-x}}) = \sigma(x)(\frac{1+e^{-x}-1}{1+e^{-x}})$

$= \sigma(x)(1 - \frac{1}{1+e^{-x}}) = \sigma(x)[1 - \sigma(x)]$

b.

negative log likelihood equation for logistic regression $= nll(\theta)$

$nll(\theta) = -\sum_i y_i log\sigma(\theta^T x_i) + (1-y_i)log(1-\sigma(\theta^T x_i))$

Taking the gradients wrt θ, we get:

$\nabla_\theta nll(\theta) = -\sum_i y_i \frac{1}{\sigma(\theta^T x_i)}\sigma^{'}(\theta^T x_i) + (1-y_i)\frac{1}{1-\sigma(\theta^T x_i)}(-\sigma^{'}(\theta^T x_i))$

this can be further simplified to yield

$\nabla_\theta nll(\theta) = \sum_i(\sigma(\theta^T x_i) - y_i)x_i$
$= \sum_i(\mu_i - y_i)x_i = X^T(\mu - y)$

where $\mu_i = (\sigma^{'}(\theta^T x_i))$ and x_i is the ith column of X^T

c.

Thus, from (b) we can find the Hessian Matrix:

So $\mathrm{H}_\theta = \nabla_\theta(\nabla_\theta nll(\theta))^T = \nabla_\theta[X^T(\mu - y)]^T = \nabla_\theta(\mu^T X - y^T X)$

$= \nabla_\theta \mu^T X = \nabla_\theta \sigma(X\theta)^T X = X^T diag(\mu(1-\mu))X = X^T S X$

where S is $\mu(1-\mu)$. Therefore we can see that H_θ is positive and semi-definite and from the equivalence above and assuming eigenvalues of S ¿0, we can also say that S is positive and semi-definite. Since S is a diagonal matrix, its eigenvalues are the entries on the diagonal, Hence:

$\mu_i(1-\mu_i) = \sigma(\theta^T x_i)(1-\sigma(\theta^T x_i)) \geq 0$ and since 0¡$\sigma(.)$¡1, we need $\sigma(.)(1-\sigma(.)) \geq 0$ and thus we can show that the hessian matrix H is positive semi definite

2.

$$\int_{\mathbb{R}} \frac{1}{Z} exp(\frac{-x^2}{2\sigma^2})dx = 1$$

$$\int_{\mathbb{R}} exp(\frac{-x^2}{2\sigma^2})dx = Z$$

$$Z^2 = \int_{\rm I\!R} exp(\frac{-x^2}{2\sigma^2})dx \int_{\rm I\!R} exp(\frac{-y^2}{2\sigma^2})dy = \iint_{\rm I\!R^2} exp(-\frac{x^2+y^2}{2\sigma^2})\, dx\, dy$$

$$= \int_0^{\infty} \int_0^{2\pi} exp(\frac{-r^2}{2\sigma^2}) r\, d\theta\, dr = 2\pi(\sigma^2) \int_0^{\infty} exp(\frac{-r^2}{2\sigma^2})(\frac{-2}{\sigma^2})dr$$

on solving, we get $= -2\pi\sigma^2(0-1) = 2\pi\sigma^2$

Hence, we get $Z^2 = 2\pi\sigma^2 \Rightarrow Z = \sqrt{}(2\pi\sigma^2) = \sqrt{}(2\pi)\sigma$

3. refer to pdf