MT2014132.

GEN501: Mathematics for IT (Probability Theory)

Discussion Test (24^{th} October 2014) Marks 50 (All questions carry equal marks)

Note: Let a be the digit obtained by the addition of last 3 digits of your roll number(see example). The problems to be solved by you are the questions given by 5 distinct numbers starting with a in the decimal expansion of $\pi = 3.14159265358979323846264338327953$. For example, if the roll number is MT2014149 then a=1+4+9=14=1+4=5. The problems to be solved are, 5,9,2,6,3. If a= 3, the problems to be solved are 3,1,4,5,9

1. a) Using the axioms of Probability theory show that i) $P(\emptyset) = 0$, ii) If A is a subset of B then $P(A) \leq P(B)$ and iii) if A and B are independent then $\bar{A}$ and B and the events $\bar{A}$ and $\bar{B}$ are also independent.
   b) In the New York state lottery six numbers are drawn from the sequence of numbers from 1 to 51. What is the probability that the six numbers drawn will have i) all one digit numbers? Ii) two one digit and four 2 digit numbers?

2. The random experiment consists of dropping a needle of length less than 1 on hard wood floor covered with parallel lines with perpendicular distance between them as 1. What is the probability that the needle crosses a parallel line on the board?

3. Give sample space for the following experiments and assign probabilities to as many subsets of the sample space in each case as possible.
   A) Tossing 2 coins,
   B) Tossing a coin till first Head appears.
   C) Roll a Die. If the outcome is 5 roll a second Die.
   D) Toss a coin. If it falls heads, throw a six faced die. If the coin falls tails, toss it again.
   E) A radioactive substance is selected at t=0 and the time of emission of a particle is observed.

4. a) Consider 3 events, A, B and C. Find expressions for the following events of A, B and C, i) only A occurs, ii) at least one of the events occur, iii) at least two of the events occur, iv) at most one of them occur and v) exactly two of them occur.
   b) A train and bus arrive at the station randomly between 9am and 10am.The train stops for 10 minutes and the bus for x minutes. Find x so that the probability that the bus and the train will meet equals .5.

5. a) Consider 3 events, A, B and C. Find simpler expressions for the following events of A,B and C, i) $(A \cup B)(A \cup \bar{B})$, ii)$(A \cup B)(\bar{A} \cup B)(A \cup \bar{B})$, iii) $(A \cup B)(B \cup C)$
   b) A closet consists of 12 pairs of shoes. If 8 shoes are randomly selected what is the probability that there will be a) no complete pair. b) Exactly one complete pair?
   Hint: The event no complete pair consists of choosing 8 pairs from 12 pairs in ${}^{12}C_8$ ways and then picking one of of the shoes from each pair which can be done in 2^8 ways.

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6. a) Given the sample space S a nonempty collection of subsets of S is called a Field F such that ,
   1. If $A \in F$ Then $\bar{A} \in F$
   2. If $A \in F$ and $B \in F$ then $A \cup B \in F$

   i) Show that $A \cap B \in F$. ii) What is the smallest Field of a sample space? iii) Find a Field of S={1,2,3,4,5,6} which contains the event {2,3,4}.
   b) Suppose that the events A,B and C have P(A)=P(B)=P(C)=1/5 and their intersections have the same probability, P(AB)=P(AC)=P(BC)=P(ABC)=p. i) If p=1/25 check if A,B,C are independent or not. ii) Repeat i) if p=1/125.

7. a) How many different 7-place license plates are possible if the first 2 places are for letters and the other 5 for numbers?
   b) Suppose box 1 contains a white balls and b black balls, and box 2 contains c white balls and d black balls. One ball of unknown colour is transferred from the first box to the second one and then a ball is drawn from the latter. What is the probability that it will be a white ball?

8. a) Repeat 8(a) under the assumption that no letter or number can be repeated in a single license plate.
   b) Let F be the set of all onto functions from A={1,2,...n} to B={x,y,z}. If a function $f \in F$ is chosen randomly what is the probability that $f^{-1}(x)$ is a singleton? A function from set A to set B is onto if for every b in B there is an a in A such that f(a)=b

9. a) Using the axioms of Probability theory show that i) $P(A) \leq 1$, where A is any subset of a sample space and ii) If A is a subset of B then $P(A) \leq P(B)$ and iii) If A,B,C are any three events of a sample space then,
   $$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(ABC)$$
   b) A box contains m white balls and n black balls. Balls are drawn at random one at a time without replacement. Find the probability of encountering a white ball by the kth draw.

"Our greatest glory is not in never failing, but in rising up every time we fail."
- Ralph Waldo Emerson

International Institute of Information Technology, Bangalore.
GEN 503 Probability and Statistics.
Mid Term Exam, 26 August 2016.

1. How many different licence plates containing 2 letters and 3 numbers can be formed? A person, who has seen a car drive away from a crime scene, reports to the police that the licence plate of the car had KD6 as the first three symbols and is sure that the numbers in the licence plate don't repeat. How many cars can be cleared by the police that it was not used in the crime? **(6 marks)**

2. A student is to answer 7 out of 10 questions in an examination.
   (a) How many choices does the student have? **(4 marks)**
   (b) How many if the student must answer at least 3 of the first 5 questions? **(4 marks)**

3. A total of 100 people were asked the question "Do you play any musical instrument?" The answers are tabulated as shown in Table 1. Answer the following questions based on this data.
   What is the probability of a randomly selected
   (a) individual being a male who plays an instrument? **(2 marks)**
   (b) individual being a male? **(2 marks)**
   (c) individual playing an instrument? **(2 marks)**
   (d) male playing an instrument? **(3 marks)**
   (e) instrument player being male? **(3 marks)**

4. Define the following events when three fair coins are tossed:
   - A: All heads or all tails
   - B: At least two heads
   - C: At most two tails

   Of the pairs of events, (A,B), (A,C), and (B,C), which are independent and which are dependent? **(12 marks)**

5. Let A and B be independent events with $P(A) = 0.25$ and $P(A \cup B) = 2P(B) - P(A)$. Find
   - $P(B)$. **(4 marks)**
   - $P(A|B)$. **(4 marks)**
   - $P(B^C|A)$. **(4 marks)**

	Yes	No	Total
Male	19	41	60
Female	12	28	40
Total	31	69	100

Table 1: Data of people who play a musical instrument

6. a) Given the sample space S a nonempty collection of subsets of S is called a Field F such that ,
   1. If $A \in F$ Then $\bar{A} \in F$
   2. If $A \in F$ and $B \in F$ then $A \cup B \in F$

   i) Show that $A \cap B \in F$. ii) What is the smallest Field of a sample space? iii) Find a Field of S={1,2,3,4,5,6} which contains the event {2,3,4}.

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A. Borel Cantelli Lemma: Given a sequence of events $A_1, A_2, \ldots\ldots$ with probabilities $p_k = P(A_k)$, k=1,2,....

a) Suppose $\sum_{k=1}^{\infty} p_k < \infty$. Then with probability 1 only finitely many of the events $A_1, A_2, \ldots$ occur.

b) Suppose $A_1, A_2, \ldots$ are also independent events and $\sum_{k=1}^{\infty} p_k = \infty$. Then, with probability 1 infinitely many of the events.

B. Prove Total Probability result and Bayes' Theorem:

If $U = \{A_1, A_2, \ldots\ldots A_n\}$ is a partition ($A_i s$ are mutually exclusive and $A_1 U A_2 U \ldots\ldots U A_n = S$) of S and B is an arbitrary event then, $P(B) = P(B/A_1)P(A_1) + \ldots\ldots\ldots\ldots P(B/A_n)P(A_n)$ an

$$P(A_i/B) = \frac{P(B/A_i)P(A_i)}{P(B) = P(B/A_1)P(A_1) + \ldots\ldots\ldots\ldots P(B/A_n)P(A_n)}$$

"The probability that we may fail in the struggle ought not to deter us from the support of a cause we believe to be just." Abraham Lincoln

GEN501: Mathematics for IT (Probability Theory)

Discussion Assignment (20th October 2014)

Ans

1. Show that i) if $AB = \emptyset$ then $P(A) \leq P(\bar{B})$ ii) if P(A)=P(B)=P(AB), then $P(A\bar{B} \cup \bar{A}B) = 0$.
2. Prove and generalize the following identity,
   $$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(ABC)$$
3. Trains X and Y arrive at a station at random between 8am and 8.20am. Train X stops for4 minutes and train Y stops for 5 minutes. Assuming that the trains arrive independently of each other find the sample space for this experiment. Find a) the probability that train X arrives before Y and b) the probability that the trains meet at the station.
4. A train and bus arrive at the station randomly between 9am and 10am.The train stops for 10 minutes and the bus for x minutes. Find x so that the probability that the bus and the train will meet equals .5.
5. Prove that mean must always lie between the smallest and largest data values. Take mean of n data values $x(1), x(2), \ldots x(n)$ as $\bar{x} = \frac{x(1)+x(2)+\cdots x(n)}{n}$.
6. Given n particles and m > n boxes, we place at random each particle in one of the boxes. We wish to find the probability p that in n preselected boxes, one and only one particle will be found. Find p under following scenarios.
   a) Each particle is distinguishable
   b) Particles are not distinguishable
   c) Particles are not distinguishable and we assume that in each box we are allowed to place at most one particle.
7. Give sample space for the experiment of tossing a coin twice. Show that
   a) $P(HH) = a^2, P(HT) = P(TH) = ab \text{ and } P(TT) = b^2$ is valid probability assignment if a and b are positive numbers such that a+b=1.
   b) Check if the events heads at first toss (A) and heads at second toss (B) are independent or not.
8. Repeat 7) if P(HH)=.2, P(HT)=.3, P(TH)=.2 and P(TT)=.3.
9. Show that $2^n - (n+1)$ equations are needed to establish the independence of n events.
10. Show that a set S with n elements has, $\frac{n(n-1)(n-2)\ldots(n-k+1)}{1.2.3\ldots k} = \frac{n!}{k!(n-k)!}$, k element subsets.

$\frac{3^n 2^n}{3^n}$.

11. F is a set of functions from the set A = {1,2,3,…n} to B = {a,b,c}. In a random selection of functions assume that every function f ∈ F is equally likely. What is the probability that such a function has a in its range.
12. A biased coin is tossed till a head appears for the first time. What is the probability that the number of required tosses is odd?
13. We have two coins; the first coin is fair and the second two headed. We pick one of the coins at random, we toss it twice and heads shows both times. Find the probability that coin picked is fair.

GEN501: Mathematics for IT (Probability Theory)

Class Test (31st October 2014) Marks 60 Time 60 minutes

1. a) As we know mutually exclusive sets correspond to mutually exclusive events. One should guard against thinking of non-overlapping sets as corresponding to independent events. Using the axioms of Probability theory show that if A and B are independent events with non-zero probabilities, the corresponding subsets of the sample space must have at least one common point. (3 Marks)
   b) Once upon a time there was a dictator. An astrologer forecast something bad for him and the dictator awarded a death penalty to the astrologer. The latter pleaded for his life, so the dictator gave him a chance to save himself and decreed as follows: " I will allow you to put two white balls and two black balls in any manner you like in two urns without disclosing it to anybody. My executioner will choose one of the urns, dip his hand into it and take out a ball. If the ball is black, he will cut off your head. If the ball is white, your life is saved. Try save yourself if you can". What would you advise the astrologer to do, in order to give himself the maximum probability of saving his life? (7 Marks)
2. a) An experiment consists of throwing a fair die until two successive results are the same. Give the sample space and determine the probability of stopping with the nth toss, n=0,1,2,… Verify that these probabilities sum to 1. (Marks 4)
   b) Let A,B,C be three events, not necessarily disjoint, defined on a sample space. Show that,
   $P(A \cup B \cup C) \leq P(A) + P(B) + P(C), P(A \cup B \cup C) \geq P(A), P(A \cap B \cap C) \leq P(A)$.
   (Marks 6)
3. Consider the following experiment involving four urns. A ball is chosen from urn A, which contains six balls labelled B, three balls labelled C, and three balls labelled D. The letter drawn specifies the urn from which a second drawing is made. Urn B contains five red and five white balls. Urn C contains four red and six white balls. Urn D contains two red and eight white balls.
   a) Construct a sample space and probability assignment.
   b) Given that the second ball drawn is red, what is the conditional probability that the first drawing yielded B?
   c) Are the two events "first ball labelled C" and "second ball red" independent?
   (Marks 10)
4. Consider a sample space S consisting of all positive real numbers t. Show that if $P(t_0 \leq t \leq t_0 + t_1 \mid t \geq t_0) = P(t \leq t_1)$ for every t_0 and t_1, then $P(t \leq t_1) = 1 - e^{-ct_1}$, where c is a constant(10 Marks)
5. Players X and Y roll two dice alternately starting with X. The player that rolls eleven wins. Find the probability p that X wins. (10 Marks)
6. In a coin tossing experiment let probability of getting a head be, p (0 < p < 1), and let X denote the random variable which represents the number of times head appears in n independent tossings. Find mean and variance of X.(4 Marks) Show that for any $\epsilon > 0$,
   $P(|\frac{X}{n} - p| \leq \epsilon) \to 1$, as $n \to \infty$ (6 Marks) : Hint : Can we use Chebyshev inequality
   $P(|X - \mu_X| \geq a) \leq \frac{\sigma_X^2}{a^2}$, a>0?

GEN501: Mathematics for IT (Random Variables)

Discussion Assignment (29th October 2014)

Reading assignment. Read Papoulis Chapter 4, Sections 4.1(Concept of a random variable),4.2 (Distribution and Density functions) and 4.3 (Specific Random Variables)

1. Total Probability and Bayes' Theorem.
   If $U = \{A_1, A_2, A_n\}$ is a partition (A_i are mutually exclusive and $A_1 \cup A_2 \cup \ldots A_n = S$) of S and B is an arbitrary event then prove that,

   $$P(B) = P(B/A_1)P(A_1) + \ldots\ldots\ldots P(B/A_n)P(A_n) \text{ and}$$

   $$P(A_i / B) = \frac{P(B/A_i)P(A_i)}{P(B) = P(B/A_1)P(A_1) + \ldots\ldots\ldots P(B/A_n)P(A_n)}$$

2. A fair coin is tossed twice, and let the random variable X represent the number of heads. Find $F_X(x)$. Find its mean and variance.

3. A random variable X has a density function $f(x) = .75(1-x^2)$ if $-1 \le x \le 1$ and zero otherwise. Find the distribution function. Find the probabilities $P(-\frac{1}{2} \le X \le \frac{1}{2})$ and $P(\frac{1}{4} \le X \le 2)$. Find x such that $P(X \le x) = .95$

4. Let X be a random variable with density function $f_X(x)$. Find density function of Y when, Y = aX+b and a is not zero and b is a constant.

5. Determine Mean and Variance of the Gaussian random variable X with Probability density function given by,

   $$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{1}{2\sigma^2}(x-\mu)^2} \quad -\infty \le x \le \infty$$. Prove by integration that $\int_{-\infty}^{\infty} f_X(x)dx = 1$

6. If X is a nonnegative random variable show (Markov inequality) that

   $$P(X \ge a) \le \frac{E(X)}{a}$$

   Use Markov inequality to prove Chebyshev inequality given by

   $P(|X - \mu| \ge \in) \le \frac{\sigma^2}{\in^2}$ for any $\in > 0$, for any random variable X with finite mean μ and variance σ^2 . (Hint: Try this problem yourself before looking up.)

7. Moment generating function or the characteristic function of a random variable is given by,

$$\Phi_X(\omega) = E(e^{j\omega X}) = \sum_j e^{j\omega x_j} P(X = x_j)$$

Or

$$\Phi_X(\omega) = E(e^{j\omega X}) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)dx$$

Where, X is a discrete or continuous random variable. Find characteristic function of a Poisson random variable and a Gaussian random variable.

8. Find mean and variance of Cauchy distribution given by its density function,

$$f_X(x) = \frac{1}{\pi(1+x^2)}, -\infty \leq x \leq \infty$$

Find its characteristic function.

9. If X has the probability mass function $P(X = k) = \frac{a}{2^k}$, k = 0,1,2,3... Find a, mean and characteristic function of X and $P(X \geq 4)$.

10. Determine Mean and Variance of the Poisson random variable X with Probability mass function given by,

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} \quad k = 0, 1,2,3 \ldots.$$

11. Determine Mean and Variance of the Binomial random variable X with Probability mass function given by,

$$P(X = k) = {}^nC_k p^k (1-p)^{n-k} \quad k = 0,1,2....n \text{ and } p > 0.$$

It is the probability of getting k heads in n independent tossings of a coin. Probability of getting a head is p.

12. A shipment contains K good and N-K defective components. We pick at random $n \leq K$ components and test them. Let X be random variable which counts the number of good components in n. Find P(X=k) k=0,1,2..n. Find mean of X.

"The probable is what usually happens." Aristotle

IIITB, Bangalore

MTech -1, 2015 Linear Algebra

End-term Examination, December 2015

Duration: 3 Hours

Max.Marks: 80

1. Let $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 3 & 9 & 6 & 12 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$. Determine all $b \in R^3$ such that the system $AX = b$ is consistent. Is it a proper subspace of R^3? [8]

2. Let $W = \{(x, y, z, w) \in R^4 : w - z = y - x\}$. Check if it is a subspace of R^4. Does it contained in span of $(1, 0, 0, -1), (0, 1, 0, 1)$ and $(0, 0, 1, 1)$. [10]
   Find basis and dimension of W.

3. Let $M = \{(x, y, z) \in R^3 : x + y + 4z = 0\}$ and $N = \{(x, y, z) \in R^3 : x + y + z = 0\}$.Find dimension and basis of $M \cap N$. Also find basis of M and N containing basis of $M \cap N$. [12]

4. Prove or disprove : if λ_1 and λ_2 are two distinct eigen values of a matrix A and if u_1 and u_2 are corresponding eigen vectors of A then $u_1 + u_2$ can be eigen vector of A corresponding to some other eigen value μ. [8]

5. Show that if a matrix A is skew ~~symmetric~~ hermitian then it's eigen values are either zero or purely imaginary. [8]

6. Let $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$. Find A^{100} only by diagonalizing A. [8]

7. Construct a matrix A whose column space contains $(1, 1, 5)$, $(0, 3, 1)$ and whose null space contains $(1, 1, 2)$. [8]

8. Solve $AX = b$ by least square method and find $p = Ax^*$ for $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}$ [8]

9. Find an orthogonal matrix Q and upper triangular matrix R such that $A = QR$ for $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$. [10]