Lecture 1

The definition of <u>monoidal category</u> is an axiomatization of the tensor product of vector spaces. So let's review that. There are 3 equivalent ways to define it:

1) Let k be a field and V, W be k-vector spaces. $V \otimes W$ is the free abelian group generated by elements of the form $v \otimes w$ with $v \in V$, $w \in W$ modulo the relations

   1) $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$
   2) $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$
   3) If $r \in k$, $rv \otimes w = v \otimes rw$

2) The tensor classifies bilinear maps

There is a canonical bilinear map $\alpha: V \times W \to V \otimes W$ such that for all bilinear maps. So if $V \times W \xrightarrow{f} U$ is any bilinear map, $\exists!$ linear map $\hat{f}\ V \otimes W \to U$ s.t.

$$\begin{tikzcd}
V \times W \arrow[rr, "f"] \arrow[d, "\alpha"'] & & U \\
V \otimes W \arrow[urr, dashed, "\hat{f}"'] & &
\end{tikzcd}$$

3) Pick bases for V, W: $\{e_i\}_{i\in I}$ for V and $\{f_j\}_{j\in J}$ for W. Then $V\otimes W$ is the space with basis: $\{e_i\otimes f_j\}_{i\in I, j\in J}$

This last is not a good definition. Why not?

Some properties: ① If V, W are finite-dimensional $\mathbb{R}$ vector spaces with $\dim(V)=n$ & $\dim(W)=m$, then $\dim(V\otimes W)=n\cdot m$. ② If V^* is the dual space of V, then $\dim(V^*)=\dim(V)$.

③ The tensor is a functor in each variable. Let Vec be the category whose objects are vector spaces and whose arrows are linear maps. Let V be a vector space. Then

$$\left.\begin{array}{l} V\otimes(\,): \mathrm{Vec}\to\mathrm{Vec} \\ (\,)\otimes V: \mathrm{Vec}\to\mathrm{Vec}\end{array}\right\} \text{ are functors.}$$

If $f: W\to W'$ is a linear map, I need

$$V\otimes f: V\otimes W\to V\otimes W'$$

$$v\otimes w\mapsto v\otimes f(w)$$

Why is this well-defined?

Similarly, $\otimes$ is a bifunctor, i.e.

$$- \otimes - : \mathrm{Vec} \times \mathrm{Vec} \to \mathrm{Vec}$$

Also, there are "canonical" isomorphisms:

$$\left.\begin{aligned} (V \otimes W) \otimes Z &\simeq V \otimes (W \otimes Z) \\ V \otimes k &\simeq V \simeq k \otimes V \end{aligned}\right\} \text{These are not equalities}$$

Pause: What does "canonical" mean?

It should be a natural transformation, but we'll need more than that.

First, naturality: I'll write this just once:

We must have a NT $\alpha : (V \otimes W) \otimes Z \to V \otimes (W \otimes Z)$

Natural means: For all

$f : V \to V'$

$g : W \to W'$

$h : Z \to Z'$

$$\begin{tikzcd}
(V \otimes W) \otimes Z \arrow[r, "\alpha_{V,W,Z}"] \arrow[d, "(f \otimes g) \otimes h"'] & V \otimes (W \otimes Z) \arrow[d, "f \otimes (g \otimes h)"] \\
(V' \otimes W') \otimes Z' \arrow[r, "\alpha_{V',W',Z'}"'] & V' \otimes (W' \otimes Z')
\end{tikzcd}$$

But we actually need more than this.

Consider

$$(A \otimes (B \otimes C)) \otimes (D \otimes (E \otimes F))$$

$$\downarrow \cong$$

$$A \otimes (((B \otimes (C \otimes D)) \otimes E) \otimes F)$$

Using the above isomorphism α, one could construct many different paths from the top object to the bottom. We would like all these isos to be the same.

It turns out that adding one equation suffices. This is called the Mac Lane pentagon:

$$\begin{tikzcd}
 & (U \otimes V) \otimes (W \otimes Z) \arrow[dr, "\alpha"] & \\
((U \otimes V) \otimes W) \otimes Z \arrow[ur, "\alpha"] \arrow[d, "\alpha \otimes id_Z"'] & & U \otimes (V \otimes (W \otimes Z)) \\
(U \otimes (V \otimes W)) \otimes Z \arrow[rr, "\alpha"'] & & U \otimes ((V \otimes W) \otimes Z) \arrow[u, "id_U \otimes \alpha"']
\end{tikzcd}$$

This is a coherence equation. We also need a coherence equation for:

$$\rho : A \otimes k \to A$$

$$\lambda : k \otimes A \to A$$

Here is the equation for these isos:

$$\begin{tikzcd}
(A \otimes k) \otimes B \arrow[rr, "\alpha"] \arrow[dr] & & A \otimes (k \otimes B) \arrow[dl] \\
& A \otimes B &
\end{tikzcd}$$

<u>Unit triangle</u>

It is a consequence of these two equations that any diagram built entirely of the iso's α, λ, ρ commutes. This is the <u>Mac Lane coherence thm</u>, stated somewhat imprecisely.

For more details, see Chapter 7.2 of Mac Lane - <u>Categories For The Working Mathematician</u>

Another excellent book is Kassel - <u>Quantum Groups</u>. He calls monoidal categories <u>tensor categories</u>. Coherence issues are discussed in Chapter 11.

Note: Mac Lane has an extra equation:

I have $\lambda_I, \rho_I : I \otimes I \longrightarrow I$. Mac Lane has as part of his definition that $\lambda_I = \rho_I$. As Kassel shows, this is a consequence of everything else.

Finally:

Defn: A monoidal category is a category $\mathcal{C}$ equipped with a bifunctor $\otimes$ and an object I

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

and natural isomorphisms

$$\alpha : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$$
$$\lambda : I \otimes A \longrightarrow A$$
$$\rho : A \otimes I \longrightarrow A$$

satisfying the Mac Lane pentagon and the unit triangle. I is called the unit.

Warning: This definition is frequently mis-stated.

Ex: 1) Vec, with above structure.

2) Any category with finite products.

Remember the terminal object 1

is considered as a 0-ary product.

<u>Note to class</u>

Make sure you understand this example. Why do the appropriate iso's exist? Why do they satisfy the equations? Everything follows from universality.

(3) Any category with finite coproducts.

(4) Let $\mathcal{C}$ be any category. Let $\mathrm{Func}(\mathcal{C})$ be the category whose objects are endofunctors

$$F : \mathcal{C} \to \mathcal{C}$$

and whose arrow are natural transformations. The composition makes this category monoidal. Here α, λ, ρ are all equalities. Such a monoidal category is called <u>strict</u>

(5) Rel, the category of sets and relations.

An object is a set. An arrow is a binary relation. So $R : X \to Y$ is a subset $R \subseteq X \times Y$. Relations can be composed just like functions:

So if $R: X \to Y$ & $S: Y \to Z$, say

$(x,z) \in S \circ R$ if $\exists y$ s.t. $(x,y) \in R$ and $(y,z) \in S$.

This is exactly how one composes functions.

We write $R: X \nrightarrow Y$ and $S: Y \nrightarrow Z$

for relations. We also write

$(x,y) \in R$ as xRy.

This is a category. What are identities?

This is also a monoidal category. Define

$X \otimes Y = X \times Y$, the cartesian product of the sets.

Note: This is not the categorical product in Rel.

What is the categorical product?

Thm: The above makes Rel a monoidal category.

Lots of details to check.

(6) Representations of groups & Hopf algebras

Difference between $\otimes$ and $\times$

In any category with finite products, we have canonical maps

$$X \xrightarrow{\Delta} X \times X \qquad \text{diagonal}$$

$$\left.\begin{array}{l} X \times Y \to X \\ X \times Y \to Y \end{array}\right\} \text{projections}$$

$$X \to 1 \qquad \text{terminal map.}$$

Do these maps exist in general monoidal categories? Vector spaces?