Products between vectors, bivectors and trivectors in geometric algebra

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2 authors:

Debashis Sen
Independent Researcher

Deeprodyuti Sen
Bhabha Atomic Research Centre

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D. Sen
Salt Lake, Kolkata, India.
email: debashissen59@gmail.com

Deeprodyuti Sen
RSD, BARC, Mumbai, India.
email: deeprocool@gmail.com

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Abstract

Geometric algebra provides an immensely productive unification of different mathematical systems used for the representations of physical variables (including quaternion and spinor) by introducing the geometric product. Also, its coordinate free approach facilitates a seamless generalisation with a basic vector space of arbitrary finite dimensions. However, recourse to the coordinate system sometimes may be advantageous in working out various relations, which are not readily available in the literature. In this paper, following the multiplication rules, various products of exterior and geometric algebra are derived in terms of the components of the higher grade elements and their usefulness are indicated.

Keywords: Quaternion, spinor, exterior algebra, multivector, pseudoscalar and pseudovector.

I. **Geometric algebra – an associative multivector algebra for all finite dimensions:**
The inner (dot) and exterior (wedge) products of two vectors in Grassman's 'Algebra of Extension' [1] complement each other. While one lowers the grade of the product, the other raises it – one is commutative and the other is anticommutative. Obviously, these products are not invertible in general. In Geometric algebra (GA), introduced by Clifford, they are combined to form the geometric product of two vectors:

$$\mathbf{uw} = \mathbf{u}.\mathbf{w} + \mathbf{u} \wedge \mathbf{w} = \mathbf{w}.\mathbf{u} - \mathbf{w} \wedge \mathbf{u} = s + \mathbf{B} = \mathbf{C}, \tag{1}$$

producing a general *multivector* (also called *clif*) $\mathbf{C}$ as the sum of a scalar s ($= \mathbf{u}.\mathbf{w}$) and a *bivector* $\mathbf{B}$ ($= \mathbf{u} \wedge \mathbf{w}$). This combined product executes both lowering and raising operations simultaneously and thus allowing almost all nonnull elements of the algebra to be invertible[1]. The associativity and almost invertibility of geometric product makes GA a proficient tool of mathematical physics, providing solutions of equations involving multivectors. Since $\mathbf{v} \wedge \mathbf{v}$ is identically zero, the geometric product of a vector with itself is simply the scalar product giving the squared norm of the vector.

The geometric algebra (GA), defined on a vector space ($\mathcal{V}$) with a quadratic form and its associated scalar field ($\mathcal{S}$), creates a graded algebraic structure with its unique multiplication operation – the geometric product. The multivectors thus generated are elements of this algebra, which form a superset on both $\mathcal{V}$ and $\mathcal{S}$. Multivectors of definite grade k, like bivectors, trivectors etc. (k-blade, $k = 2, 3$) provide a more natural representation of pseudovectors and pseudoscalars

[1]All nonzero elements of the algebra, however, do not have multiplicative inverses. For example, consider an element $\mathbf{C} = \frac{1}{2}(1 + \hat{\mathbf{u}})$ involving an unit vector $\hat{\mathbf{u}}$. $\mathbf{C}^2 = \mathbf{C}$, $\mathbf{C}$ is idempotent and since $\mathbf{C}(1 - \hat{\mathbf{u}}) = \frac{1}{2}(1 - \hat{\mathbf{u}}^2) = (\frac{1}{2})(1 - 1) = 0$, it is a nonzero zero divisor and thus has no inverse.

of ordinary 3-D vector algebra and also of oriented area, oriented angle of rotation, oriented volume, and so on. More importantly, the algebra can be seamlessly extended to any dimension!

A bivector that can be written as the exterior product of two vectors is called *simple* and geometrically they represent oriented planes. In two and three dimensions all bivectors are simple. But in four and higher dimensions, not all bivectors are simple. For example, in 4-D the bivector $\mathbf{B} = \hat{\mathbf{e}}_1 \wedge \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 \wedge \hat{\mathbf{e}}_4$ $(= \hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3\hat{\mathbf{e}}_4$, with orthogonal basis vectors $\hat{\mathbf{e}}_i,\ i = 1, ..., 4)$ cannot be written as the exterior product of two vectors and therefore, cannot represent an oriented plane. In an n dimensional space, a simple k-blade, formed by taking wedge product of k independent vectors, represents an oriented hypervolume of a *k-parallelotope* and the one-component highest grade n-blade represents a pseudoscalar. Different elements of GA, in addition to providing natural representation of various physical quantities [2], also describe a number of basic physical operations. In the following, we first discuss some of these basic operations.

II. Projection, Rejection, Reflection - Verser, Rotor and Spinor:

Elements of GA represents both physical objects and operations and this algebra provides evocative names for its elements. Product of a vector with a unit vector can alter or reverse the direction of the vector. Hence a unit vector is often called a versor. Also, since a product of two unit vectors can rotate things, it is in general called a rotor and we will find that a special kind of rotor defines a spinor.

For any vector $\mathbf{v}$ and any invertible vector $\mathbf{u}$, (with $\mathbf{u}^{-1} = |\mathbf{u}|^{-2}\,\mathbf{u}$), we can write $\mathbf{v} = \mathbf{v}\,\mathbf{u}\,\mathbf{u}^{-1} = (\mathbf{v}.\mathbf{u} + \mathbf{v} \wedge \mathbf{u})\mathbf{u}^{-1} = \mathbf{v}_{\|\mathbf{u}} + \mathbf{v}_{\perp\mathbf{u}}$. Here $\mathbf{v}_{\|\mathbf{u}} = (\mathbf{v}.\mathbf{u})\mathbf{u}^{-1}$ is the parallel part of $\mathbf{v}$ along, or its projection onto $\mathbf{u}$ and $\mathbf{v}_{\perp\mathbf{u}} = (\mathbf{v} \wedge \mathbf{u})\mathbf{u}^{-1}$ represents the perpendicular part, or the rejection of $\mathbf{v}$ from $\mathbf{u}$. With unit vector $\hat{\mathbf{u}}$, the expression becomes even simpler as $\hat{\mathbf{u}}^{-1} = \hat{\mathbf{u}}$ and $\mathbf{v} = \mathbf{v}\,\hat{\mathbf{u}}\,\hat{\mathbf{u}} = (\mathbf{v}.\hat{\mathbf{u}} + \mathbf{v} \wedge \hat{\mathbf{u}})\hat{\mathbf{u}} = \mathbf{v}_{\|\mathbf{u}} + \mathbf{v}_{\perp\mathbf{u}}$.

Simple reflections are now readily be expressed through the conjugation with a unit vector. The reflection $\mathbf{v}'$ of a vector $\mathbf{v}$ along a vector $\hat{\mathbf{u}}$, or equivalently from the plane orthogonal to $\hat{\mathbf{u}}$, is the same as reverting the component of the vector parallel to $\hat{\mathbf{u}}$ and leaving $\mathbf{v}_{\perp\mathbf{u}}$ unaltered. Thus, $\mathbf{v}'$ can be expressed as:

$$\begin{aligned} \mathbf{v}' &= -\mathbf{v}_{\|\mathbf{u}} + \mathbf{v}_{\perp\mathbf{u}} = -(\mathbf{v}.\hat{\mathbf{u}})\hat{\mathbf{u}} + (\mathbf{v} \wedge \hat{\mathbf{u}})\hat{\mathbf{u}} \\ &= -(\hat{\mathbf{u}}.\mathbf{v})\hat{\mathbf{u}} - (\hat{\mathbf{u}} \wedge \mathbf{v})\hat{\mathbf{u}} \\ &= -\hat{\mathbf{u}}\,\mathbf{v}\,\hat{\mathbf{u}}\,. \end{aligned} \tag{2}$$

This is usually treated in matrix algebra by a 2×2 reflection matrix. However, the simplicity and power of the approach of GA will be realised in the following examples. For the product $\mathbf{v}_1\,\mathbf{v}_2\,...\,\mathbf{v}_k$ of k vectors we may write:

$$\begin{aligned} \mathbf{v}_1'\,\mathbf{v}_2'\,...\,\mathbf{v}_k' &= -(\hat{\mathbf{u}}\,\mathbf{v}_1\,\hat{\mathbf{u}})(-\hat{\mathbf{u}}\,\mathbf{v}_2\,\hat{\mathbf{u}})\,...\,(-\hat{\mathbf{u}}\,\mathbf{v}_k\,\hat{\mathbf{u}}) \\ &= (-1)^k\,\hat{\mathbf{u}}\,\mathbf{v}_1\,\mathbf{v}_2\,...\,\mathbf{v}_k\,\hat{\mathbf{u}} \\ &= (\mathbf{v}_1\,\mathbf{v}_2\,...\,\mathbf{v}_k)'\,. \end{aligned}$$

In fact, when the dimension $n \geq 4$, a more general reflection may be expressed as the composite of any *odd* number (r) of single-axis reflections as:

$$\mathbf{v}' = -(\mathbf{u}_r\,...\,\mathbf{u}_2\,\mathbf{u}_1)\mathbf{v}(\mathbf{u}_1\,\mathbf{u}_2\,...\,\mathbf{u}_r).$$

Next, we note that in 3-D, the clif $\mathbf{C}$ of eq.(1) is isomorphic with *quaternion* and forms an even subalgebra. While a quaternion produces a dilation in addition to rotation (unit quaternion generates pure rotation) of a vector in 3-D, GA generalises the description using the geometric product of two vectors [2, 3, 4]. For example, a successive second reflection of $\mathbf{v}$ of eq.(2) along $\hat{\mathbf{w}}$ (say) finally rotates $\mathbf{v}$ to a new vector $\mathbf{v}''$ through an angle $|\theta|$ – twice the angle between $\hat{\mathbf{w}}$

and $\hat{\mathbf{u}}$ and expressed as:

$$\begin{aligned}\mathbf{v}'' &= -\hat{\mathbf{w}}\,\mathbf{v}'\,\hat{\mathbf{w}} = -\hat{\mathbf{w}}(-\hat{\mathbf{u}}\,\mathbf{v}\,\hat{\mathbf{u}})\hat{\mathbf{w}} = \hat{\mathbf{w}}\,\hat{\mathbf{u}}\,\mathbf{v}\,\hat{\mathbf{u}}\,\hat{\mathbf{w}} \\ &= \mathbf{R}\,\mathbf{v}\,\tilde{\mathbf{R}}\,. \end{aligned} \tag{3}$$

where, $\mathbf{R} = \hat{\mathbf{w}}\hat{\mathbf{u}} = \hat{\mathbf{w}}.\hat{\mathbf{u}} + \hat{\mathbf{B}}\,|\hat{\mathbf{w}} \wedge \hat{\mathbf{u}}|$ (with unit bivector $\hat{\mathbf{B}} = \frac{\hat{\mathbf{w}} \wedge \hat{\mathbf{u}}}{|\hat{\mathbf{w}} \wedge \hat{\mathbf{u}}|}$), formed by taking the geometric product of two unit vectors, represents normalized element of an even subalgebra called *rotor*. Bivectors in an Euclidean space are characterized by a negative square. Consequently, the rotors are elliptic functions represented as $\mathbf{R} = \cos(\frac{|\theta|}{2}) + \hat{\mathbf{B}}\,\sin(\frac{|\theta|}{2})$, where, $\cos(\frac{|\theta|}{2}) = \hat{\mathbf{w}}.\hat{\mathbf{u}}$, $\sin(\frac{|\theta|}{2}) = |\hat{\mathbf{w}} \wedge \hat{\mathbf{u}}|$. Using the generalisation of the concept of exponential function of multivectors introduced by Hestenes [3], it may be finally expressed as: $\mathbf{R} = \exp(\frac{\hat{\mathbf{B}}|\theta|}{2}) = \exp(\mathbf{B})$. The bilinear expression of eq.(3) encodes pure rotation in GA. The rotor that generates this rotation is the exponential of the bivector $\mathbf{B}$ which encodes both the oriented plane $\hat{\mathbf{B}}$ and the angle rotation θ (i.e., the direction and the magnitude). The half-angle that appears in the expression is due to the bilinear form of the operation. It is also important to note that in the non-Euclidean space, the bivectors may possess a positive square. As a consequence, rotors are no longer elliptic, but hyperbolic. We will discuss this operation more explicitly in the following.

III. Geometric product as the basic product of GA:

In GA, the geometric product is regarded as the basic product of vectors. It has no definite symmetry in general and on reversing the order of multiplication, known as reversion, one gets: $\mathbf{wu} = \text{reverse}(\mathbf{uw}) = \tilde{\mathbf{uw}} = \tilde{\mathbf{C}}$. The inner and exterior products can be derived axiomatically as the symmetric and antisymmetric parts of the geometric product, given by, $\mathbf{u}.\mathbf{v} = \frac{\mathbf{uv}+\mathbf{vu}}{2}$, and $\mathbf{u} \wedge \mathbf{v} = \frac{\mathbf{uv}-\mathbf{vu}}{2}$. From the simple axioms of GA it follows that the symmetries for the products between a bivector and a vector, on the other hand, are just the opposite: $\mathbf{B}.\mathbf{v} = \frac{\mathbf{Bv}-\mathbf{vB}}{2}$, and $\mathbf{B} \wedge \mathbf{v} = \frac{\mathbf{Bv}+\mathbf{vB}}{2}$. All these[2] are well discussed in the literature [4, 5, 6]. The geometric product of a vector with a multivector $\mathbf{A}_r$ of definite grade-r, which also contains only two terms, can easily be obtained as: $\mathbf{v}\mathbf{A}_r = \mathbf{v}.\mathbf{A}_r + \mathbf{v} \wedge \mathbf{A}_r$, where the inner product $\mathbf{v}.\mathbf{A}_r =< \mathbf{v}\mathbf{A}_r >_{r-1}= \frac{1}{2}(\mathbf{v}\mathbf{A}_r + (-1)^{r-1}\mathbf{A}_r\mathbf{v})$ lowers the grade of $\mathbf{A}_r$ by one and the exterior product $\mathbf{v} \wedge \mathbf{A}_r =< \mathbf{v}\mathbf{A}_r >_{r+1}= \frac{1}{2}(\mathbf{v}\mathbf{A}_r - (-1)^{r+1}\mathbf{A}_r\mathbf{v})$ raises the grade of $\mathbf{A}_r$ by one. However, appropriate multiple inner (dot) products between two higher grade elements of exterior algebra are possible and consequently, their geometric product contains more terms. In n-dimension, the geometric product of two blades of grade s and r $(\leq s)$, contains $r+1$ blades of grades from $s-r$ to $s+r$ $(\leq n)$ in steps of $+2$. For example, with two bivectors $\mathbf{A}$ and $\mathbf{B}$, we get a clif composed of three terms:

$$\mathbf{AB} =< \mathbf{AB} >_0 + < \mathbf{AB} >_2 + < \mathbf{AB} >_4$$

[2]Consistent with the exterior algebra, similar inductive definition of grading for the entire algebra can be extended by introducing a third vector $\mathbf{w}$ as follows:

$$\begin{aligned}\mathbf{wu} \wedge \mathbf{v} &= \mathbf{w}\frac{1}{2}(\mathbf{uv} - \mathbf{vu}) = \frac{1}{2}[(\mathbf{wuv} + \mathbf{uwv}) - (\mathbf{wvu} + \mathbf{vwu})] - \frac{1}{2}(\mathbf{uwv} - \mathbf{vwu}) \\ &= (\mathbf{u}.\mathbf{w})\mathbf{v} - (\mathbf{v}.\mathbf{w})\mathbf{u} - \frac{1}{2}(\mathbf{uwv} - \mathbf{vwu}) \\ &= 2(\mathbf{u}.\mathbf{w})\mathbf{v} - 2(\mathbf{v}.\mathbf{w})\mathbf{u} + \frac{1}{2}(\mathbf{uv} - \mathbf{vu})\mathbf{w}\end{aligned}$$

$\Rightarrow \mathbf{wu} \wedge \mathbf{v} - \mathbf{u} \wedge \mathbf{vw} = 2(\mathbf{u}.\mathbf{w})\mathbf{v} - 2(\mathbf{v}.\mathbf{w})\mathbf{u}$.

By definition, $\mathbf{wu} \wedge \mathbf{v} = \mathbf{w}.(\mathbf{u} \wedge \mathbf{v}) + \mathbf{w} \wedge \mathbf{u} \wedge \mathbf{v}$ of which the first term $\mathbf{w}.(\mathbf{u} \wedge \mathbf{v})$ is a vector. It is, therefore, represented by the antisymmetric part of the product, i.e. $\mathbf{w}.(\mathbf{u} \wedge \mathbf{v}) = \frac{1}{2}(\mathbf{wu} \wedge \mathbf{v} - \mathbf{u} \wedge \mathbf{vw})$. The remaining trivector term represents the symmetric part of the product, consistent with the definition of exterior algebra. Thus, it follows from the simple axioms of GA that, the symmetries of the inner and exterior products between a bivector and a vector are just the opposite of those between two vectors.

Therefore, each of the terms on r.h.s. can not be expressed individually by either of the symmetric or the antisymmetric part of the geometric product. Manipulations are usually made by equating the terms of same grade, using the *grade preserving property* of the geometric products [7].

The last term of the above equation comes from the exterior (wedge) product of the two bivectors. Hestenes and Ziegler [5] have denoted the first (scalar product, grade-0) term by $\mathbf{A}.\mathbf{B}$ and the second (grade-2) term by the 'commutator product' $\mathbf{A} \times \mathbf{B}$ – the antisymmetric part of $\mathbf{AB}$. The authors have also observed that, the decomposition of $\mathbf{AB}$ into terms of homogeneous grade cannot be expressed in terms of inner and exterior products alone without decomposing the bivectors into vectors. However, this is not true and the three terms of the equation are manifestly denoted by [2]:

$$\mathbf{AB} = < \mathbf{AB} >_0 + < \mathbf{AB} >_2 + < \mathbf{AB} >_4 = \mathbf{A} : \mathbf{B} + \mathbf{A}.\mathbf{B} + \mathbf{A} \wedge \mathbf{B} \tag{4}$$

Using the appropriate multiple inner products or contractions together with the exterior product, all the three terms are directly deduced in the following, even with nonsimple bivectors.

In fact dispensing with the Descartes coordinate system, GA creates great flexibility and consistently generalises the formulation to arbitrary n-dimensional space. Notwithstanding the warning from Hestenes [8], one can still recourse to the coordinate system to advantage when required. For example, a term by term equality between the product of two quaternions and its analogue in GA can be easily derived using the coordinate representations [2]. Following the multiplication rules of exterior algebra, products between vectors, bivectors and trivectors are derived explicitly in the next section which are not readily available in the literature.

IV. Products between blades of different grades:

(i) The geometric product between a vector and any homogeneous multivector blade also contains only two terms, one inner and one wedge product. Using the summation over repeated index convention, the following inner products are expressed as :

a)

$$\begin{aligned} \mathbf{B}.\mathbf{v} &= B_{ij}\,\hat{e}_i \wedge \hat{e}_j.v_k\,\hat{e}_k = B_{ij}\,v_j\,\hat{e}_i - B_{ij}\,v_i\,\hat{e}_j \\ &= (B_{ij} - B_{ji})\,v_j\,\hat{e}_i;\;\; i \neq j \\ &= (B_{ji} - B_{ij})v_i\,\hat{e}_j;\; \text{interchanging } i \text{ and } j \\ &= -\mathbf{v}.\mathbf{B}\,, \end{aligned} \tag{5}$$

the product anticommutes and defines a vector transformation equation by the bivector operator $\mathbf{B}$. For a simple bivector $\mathbf{B} = \mathbf{u} \wedge \mathbf{w}$, $B_{ij} = u_i\,w_j$ etc.,

$$\mathbf{B}.\mathbf{v} = (B_{ij} - B_{ji})v_j\,\hat{e}_i = (u_i\,w_j - u_j\,w_i)v_j\,\hat{e}_i = (\mathbf{v}.\mathbf{w})\mathbf{u} - (\mathbf{u}.\mathbf{v})\mathbf{w} \tag{6}$$

Therefore, the transformed vector is coplaner with $\mathbf{u}$ and $\mathbf{w}$, i.e. lies on the oriented plane defined by $\mathbf{B}$.

b)

$$\begin{aligned} \mathbf{T}.\mathbf{v} &= T_{ijk}\,\hat{e}_i \wedge \hat{e}_j \wedge \hat{e}_k.v_l\,\hat{e}_l = T_{ijk}\,v_k\,\hat{e}_i \wedge \hat{e}_j + T_{ijk}\,v_i\,\hat{e}_j \wedge \hat{e}_k \\ &- T_{ijk}\,v_j\,\hat{e}_i \wedge \hat{e}_k \\ &= (T_{ijk} + T_{kij} - T_{ikj})\,v_k\,\hat{e}_i \wedge \hat{e}_j,\;\; i \neq j \neq k \\ &= \mathbf{v}.\mathbf{T}\,, \end{aligned} \tag{7}$$

the product commutes and produces a bivector.

(ii) Wedge products of bivector and trivector with a vector:

a)

$$\begin{aligned} \mathbf{B} \wedge \mathbf{v} &= B_{ij}\,\hat{e}_i \wedge \hat{e}_j \wedge v_k\,\hat{e}_k = B_{ij}\,v_k\,\hat{e}_i \wedge \hat{e}_j \wedge \hat{e}_k \;\; i \neq j \neq k \\ &= \mathbf{v} \wedge \mathbf{B}, \end{aligned} \tag{8}$$

the product commutes and produces a trivector. In three dimensions, the basis elements $\hat{e}_i\wedge\hat{e}_j\wedge\hat{e}_k$ of a trivector are all equivalent up to a sign. Hence, a trivector $(\mathbf{v}_1\wedge\mathbf{v}_2\wedge\mathbf{v}_3)$ in 3-D has only one component which is identical with the scalar triple product – representing volume, a pseudoscalar in VA.

$b)$

$$\begin{aligned}\mathbf{T}\wedge\mathbf{v} &= T_{ijK}\,\hat{e}_i\wedge\hat{e}_j\wedge\hat{e}_k\wedge v_l\,\hat{e}_l\\ &= T_{ijk}\,v_l\,\hat{e}_i\wedge\hat{e}_j\wedge\hat{e}_k\wedge\hat{e}_l \quad i\neq j\neq k\neq l\\ &= -\mathbf{v}\wedge\mathbf{T},\end{aligned}\tag{9}$$

the product anticommutes and produces a quadrivector. Due to the closure property of wedge product, the resulting quadrivector vanishes in 3-D and is non-vanishing for dimensions higher than 3. In 4-D quadrivectors represent pseudoscalars.

(iii) The geometric product between two bivectors (eq.4) contains two inner products and a wedge product. The successive terms are:

(a)

$$\begin{aligned}\mathbf{A}:\mathbf{B} &= A_{ij}\,\hat{e}_i\wedge\hat{e}_j : B_{kl}\,\hat{e}_k\wedge\hat{e}_l\\ &= A_{ij}(B_{ji}-B_{ij}),\quad i\neq j\\ &= \mathbf{B}:\mathbf{A}, \text{producing a scalar (grade 0).}\end{aligned}\tag{10}$$

Also, $\mathbf{B}:\mathbf{B}$ $(=B_{ij}(B_{ji}-B_{ij}))$ gives the squared norm of the bivector $\mathbf{B}$. For the simple bivector $\mathbf{B}=\mathbf{u}\wedge\mathbf{w}$, $\mathbf{B}:\mathbf{B}=-(|\mathbf{u}||\mathbf{w}|\sin\theta)^2$, where θ is the angle between $\mathbf{u}$ and $\mathbf{w}$. Denoting $|\mathbf{B}|^2=|\mathbf{B}:\mathbf{B}|$, the unit bivector is given by $\hat{\mathbf{B}}=|\mathbf{B}|^{-1}\,\mathbf{B}$ yielding: $\hat{\mathbf{B}}:\hat{\mathbf{B}}=-1$.

$$\begin{aligned}\text{(b) }\mathbf{A}.\mathbf{B} &= A_{ij}\,\hat{e}_i\wedge\hat{e}_j.\,B_{kl}\,\hat{e}_k\wedge\hat{e}_l\\ &= A_{ij}\,B_{jl}\;\hat{e}_i\wedge\hat{e}_l - A_{ij}\,B_{kj}\;\hat{e}_i\wedge\hat{e}_k - A_{ij}\,B_{il}\;\hat{e}_j\wedge\hat{e}_l + A_{ij}\,B_{ki}\,\hat{e}_j\wedge\hat{e}_k\\ &= A_{ij}\{(B_{jk}-B_{kj})\hat{e}_i\wedge\hat{e}_k + (B_{ki}-B_{ik})\hat{e}_j\wedge\hat{e}_k\}\\ &= (A_{ij}-A_{ji})(B_{jk}-B_{kj})\,\hat{e}_i\wedge\hat{e}_k,\quad i\neq j\neq k,\\ &= -\mathbf{B}.\mathbf{A}, \text{ giving a (grade 2) bivector.}\end{aligned}\tag{11}$$

$\Rightarrow \mathbf{B}.\mathbf{B}=0$ i.e. a null bivector. This result can also be verified directly as follows:
$\mathbf{C}=\mathbf{B}.\mathbf{B}=(B_{ij}-B_{ji})(B_{jk}-B_{kj})\hat{e}_i\wedge\hat{e}_k \Rightarrow C_{ik}=(B_{ij}-B_{ji})(B_{jk}-B_{kj})$
$\Rightarrow C_{ki}=(B_{kj}-B_{jk})(B_{ji}-B_{ij})=(B_{ij}-B_{ji})(B_{jk}-B_{kj}), \Rightarrow C_{ik}-C_{ki}=0$
$\Rightarrow \mathbf{C}=0$, since $\hat{e}_i\wedge\hat{e}_k=-\hat{e}_k\wedge\hat{e}_i$.

(c) $\mathbf{A}\wedge\mathbf{B}=A_{ij}\,\hat{e}_i\wedge\hat{e}_j\wedge B_{kl}\,\hat{e}_k\wedge\hat{e}_l=A_{ij}\,B_{kl}\,\hat{e}_i\wedge\hat{e}_j\wedge\hat{e}_k\wedge\hat{e}_l=\mathbf{B}\wedge\mathbf{A}$.
As discussed above, the resulting quadrivector (of grade 4) vanishes in 3-D and is non-vanishing for dimensions higher than three. In 4-D the quadrivector represents a pseudoscalar.

Also, $\mathbf{B}\wedge\mathbf{B}=B_{ij}\,B_{kl}\,\hat{e}_i\wedge\hat{e}_j\wedge\hat{e}_k\wedge\hat{e}_l$ vanishes only for a simple bivector $\mathbf{B}$ $(=\mathbf{u}\wedge\mathbf{w}$, say):
$\mathbf{Q}=B_{ij}\,B_{kl}\,\hat{e}_i\wedge\hat{e}_j\wedge\hat{e}_k\wedge\hat{e}_l=Q_{ijkl}\,\hat{e}_i\wedge\hat{e}_j\wedge\hat{e}_k\wedge\hat{e}_l$
Now, $Q_{ijkl}-Q_{ijlk}+Q_{ikjl}-Q_{iklj}+\ldots-\ldots\,(4!\,\text{terms})=u_i\,u_k\,w_j\,w_l-u_i\,u_l\,w_j\,w_k+\ldots\;=0$,
$\Rightarrow \mathbf{B}\wedge\mathbf{B}=0$.

Therefore, for a simple bivector, the geometric product is identical with the scalar inner product $\mathbf{B}:\mathbf{B}$ and gives the squared norm of the bivector: $\mathbf{B}\mathbf{B}=\mathbf{B}:\mathbf{B}+\mathbf{B}.\mathbf{B}+\mathbf{B}\wedge\mathbf{B}=\mathbf{B}:\mathbf{B}$.

(iv) The three terms of the geometric product between a trivector and a bivector,
$\mathbf{T}\mathbf{B}=\mathbf{T}:\mathbf{B}+\mathbf{T}.\mathbf{B}+\mathbf{T}\wedge\mathbf{B}$ are given by:

$$\begin{aligned}\text{(a) }\mathbf{T}:\mathbf{B} &= T_{ijk}\,\hat{e}_i\wedge\hat{e}_j\wedge\hat{e}_k : B_{lm}\,\hat{e}_l\wedge\hat{e}_m\\ &= T_{ijk}\,B_{lm}\{\hat{e}_i\wedge\hat{e}_j\wedge\hat{e}_k : (\hat{e}_l\wedge\hat{e}_m-\hat{e}_m\wedge\hat{e}_l)\\ &+ \hat{e}_j\wedge\hat{e}_k\wedge\hat{e}_i : (\hat{e}_l\wedge\hat{e}_m-\hat{e}_m\wedge\hat{e}_l)+\hat{e}_k\wedge\hat{e}_i\wedge\hat{e}_j : (\hat{e}_l\wedge\hat{e}_m-\hat{e}_m\wedge\hat{e}_l)\}\\ &= T_{ijk}\{(B_{kj}-B_{jk})\hat{e}_i+(B_{ik}-B_{ki})\hat{e}_j+(B_{ji}-B_{ij})\hat{e}_k\}\\ &= (T_{ijk}-T_{jik}-T_{kji})(B_{kj}-B_{jk})\,\hat{e}_i\,.\end{aligned}\tag{12}$$

The product gives a vector (grade 1) and commutes i.e. $\mathbf{T} : \mathbf{B} = \mathbf{B} : \mathbf{T}$.

$$\begin{aligned}
\text{(b) } \mathbf{T}.\mathbf{B} &= T_{ijk}\, \hat{e}_i \wedge \hat{e}_j \wedge \hat{e}_k . B_{lm}\, \hat{e}_l \wedge \hat{e}_m \\
&= T_{ijk}\, B_{lm} \{\hat{e}_i \wedge \hat{e}_j \wedge \hat{e}_k .(\hat{e}_l \wedge \hat{e}_m - \hat{e}_m \wedge \hat{e}_l) - \hat{e}_i \wedge \hat{e}_k \wedge \hat{e}_j .(\hat{e}_l \wedge \hat{e}_m \\
&- \hat{e}_m \wedge \hat{e}_l) + \hat{e}_j \wedge \hat{e}_k \wedge \hat{e}_i .(\hat{e}_l \wedge \hat{e}_m - \hat{e}_m \wedge \hat{e}_l)\} \\
&= T_{ijk} \{(B_{kl} - B_{lk}) \hat{e}_i \wedge \hat{e}_j \wedge \hat{e}_l - (B_{jl} - B_{lj}) \hat{e}_i \wedge \hat{e}_k \wedge \hat{e}_l \\
&+ (B_{il} - B_{li}) \hat{e}_j \wedge \hat{e}_k \wedge \hat{e}_l\} \\
&= (T_{ijk} - T_{ikj} + T_{kij})(B_{kl} - B_{lk})\, \hat{e}_i \wedge \hat{e}_j \wedge \hat{e}_l\, . \qquad (13) \\
\text{Also, } \mathbf{B}.\mathbf{T} &= B_{lm}\, \hat{e}_l \wedge \hat{e}_m . T_{ijk}\, \hat{e}_i \wedge \hat{e}_j \wedge \hat{e}_k \\
&= B_{lm}\, T_{ijk} \{\hat{e}_l \wedge \hat{e}_m .(\hat{e}_i \wedge \hat{e}_j \wedge \hat{e}_k - \hat{e}_j \wedge \hat{e}_i \wedge \hat{e}_k + \hat{e}_k \wedge \hat{e}_i \wedge \hat{e}_j) \\
&- \hat{e}_m \wedge \hat{e}_l .(\hat{e}_i \wedge \hat{e}_j \wedge \hat{e}_k - \hat{e}_j \wedge \hat{e}_i \wedge \hat{e}_k + \hat{e}_k \wedge \hat{e}_i \wedge \hat{e}_j)\} \\
&= T_{ijk} \{(B_{li}\, \hat{e}_l \wedge \hat{e}_j \wedge \hat{e}_k - B_{lj}\, \hat{e}_l \wedge \hat{e}_i \wedge \hat{e}_k + B_{lk}\, \hat{e}_l \wedge \hat{e}_i \wedge \hat{e}_j) \\
&- (B_{im}\, \hat{e}_m \wedge \hat{e}_j \wedge \hat{e}_k - B_{jm}\, \hat{e}_m \wedge \hat{e}_i \wedge \hat{e}_k + B_{km}\, \hat{e}_m \wedge \hat{e}_i \wedge \hat{e}_j)\} \\
&= T_{ijk} \{(B_{li} - B_{il}) \hat{e}_j \wedge \hat{e}_k \wedge \hat{e}_l + (B_{jl} - B_{lj}) \hat{e}_i \wedge \hat{e}_k \wedge \hat{e}_l \\
&+ (B_{lk} - B_{kl}) \hat{e}_i \wedge \hat{e}_j \wedge \hat{e}_l\} \\
&= (T_{kij} - T_{ikj} + T_{ijk})(B_{lk} - B_{kl})\, \hat{e}_i \wedge \hat{e}_j \wedge \hat{e}_l \\
&= -\mathbf{T}.\mathbf{B}\, .
\end{aligned}$$

Thus, the product anticommutes and produces a trivector (grade 3). The result is valid for $n \geq 4$ and $\mathbf{T}.\mathbf{B} = 0 = \mathbf{B}.\mathbf{T}$, in 3-D.

$$\begin{aligned}
\text{(c) } \mathbf{T} \wedge \mathbf{B} &= T_{ijk}\, \hat{e}_i \wedge \hat{e}_j \wedge \hat{e}_k \wedge B_{lm}\, \hat{e}_l \wedge \hat{e}_m \\
&= T_{ijk}\, B_{lm}\, \hat{e}_i \wedge \hat{e}_j \wedge \hat{e}_k \wedge \hat{e}_l \wedge \hat{e}_m = \mathbf{B} \wedge \mathbf{T}\, .
\end{aligned}$$

The product commutes and gives a pentavector (of grade 5) in higher dimension ($n \geq 5$).

V. Associativity of Geometric products of vectors and bivectors:

In the following, the associativity of geometric products are demonstrated using the above equations.

(i) Geometric product of three vectors:

$$\begin{aligned}
(\mathbf{uv})\mathbf{w} &= (\mathbf{u}.\mathbf{v} + \mathbf{u} \wedge \mathbf{v})\mathbf{w} = (\mathbf{u}.\mathbf{v})\mathbf{w} + (\mathbf{u} \wedge \mathbf{v}).\mathbf{w} + (\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w} \\
&= (\mathbf{u}.\mathbf{v})\mathbf{w} + (\mathbf{v}.\mathbf{w})\mathbf{u} - (\mathbf{u}.\mathbf{w})\mathbf{v} + \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} \\
&= \mathbf{u}(\mathbf{v}.\mathbf{w}) + (\mathbf{u}.\mathbf{v})\mathbf{w} - (\mathbf{u}.\mathbf{w})\mathbf{v} + \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} \\
&= \mathbf{u}(\mathbf{v}.\mathbf{w} + \mathbf{v} \wedge \mathbf{w}) \equiv \mathbf{u}(\mathbf{vw})\, . \qquad (14)
\end{aligned}$$

(ii) Product involving bivector(s):

$$\begin{aligned}
(\mathbf{uv})\mathbf{B} &= (\mathbf{u}.\mathbf{v} + \mathbf{u} \wedge \mathbf{v})\mathbf{B} \\
&= (\mathbf{u}.\mathbf{v})\mathbf{B} + (\mathbf{u} \wedge \mathbf{v}) : \mathbf{B} + (\mathbf{u} \wedge \mathbf{v}).\mathbf{B} + (\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{B} \\
&= (\mathbf{u}.\mathbf{v})\mathbf{B} + u_i\, v_j\, \hat{e}_i \wedge \hat{e}_j : B_{kl}\, \hat{e}_k \wedge \hat{e}_l + u_i\, v_j\, \hat{e}_i \wedge \hat{e}_j . B_{kl}\, \hat{e}_k \wedge \hat{e}_l \\
&+ \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{B} \\
&= u_i\, v_j (B_{ji} - B_{ij}) + (\mathbf{u}.\mathbf{v})\mathbf{B} + (u_i\, v_j - u_j\, v_i)(B_{jk} - B_{kj}) \hat{e}_i \wedge \hat{e}_k \\
&+ \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{B} \qquad (15) \\
\text{Also, } \mathbf{u}(\mathbf{vB}) &= \mathbf{u}(\mathbf{v}.\mathbf{B} + \mathbf{v} \wedge \mathbf{B}) \\
&= \mathbf{u}.(\mathbf{v}.\mathbf{B}) + \mathbf{u} \wedge (\mathbf{v}.\mathbf{B}) + \mathbf{u}.(\mathbf{v} \wedge \mathbf{B}) + \mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{B}) \\
&= u_i\, v_j (B_{ji} - B_{ij}) + u_i\, v_j (B_{jk} - B_{kj}) \hat{e}_i \wedge \hat{e}_k + u_i\, \hat{e}_i . v_j\, B_{kl}\, \hat{e}_j \wedge \hat{e}_k \wedge \hat{e}_l + \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{B}
\end{aligned}$$

$$
\begin{aligned}
&= u_i\, v_j (B_{ji} - B_{ij}) + u_i\, v_j (B_{jk} - B_{kj}) \hat{e}_i \wedge \hat{e}_k + u_i\, v_i\, B_{kl}\, \hat{e}_k \wedge \hat{e}_l + u_i\, v_j\, B_{ki}\, \hat{e}_j \wedge \hat{e}_k \\
&- u_i\, v_j\, B_{il}\, \hat{e}_j \wedge \hat{e}_l + \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{B} \\
&= u_i\, v_j (B_{ji} - B_{ij}) + u_i\, v_j (B_{jk} - B_{kj}) \hat{e}_i \wedge \hat{e}_k + (\mathbf{u}.\mathbf{v})\mathbf{B} + u_j\, v_i\, B_{kj}\, \hat{e}_i \wedge \hat{e}_k \\
&- u_j\, v_i\, B_{jk}\, \hat{e}_i \wedge \hat{e}_k + \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{B} \\
&= u_i\, v_j (B_{ji} - B_{ij}) + (\mathbf{u}.\mathbf{v})\mathbf{B} + (u_i\, v_j - u_j\, v_i)(B_{jk} - B_{kj}) \hat{e}_i \wedge \hat{e}_k \\
&+ \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{B} \qquad (16)
\end{aligned}
$$

Hence, $(\mathbf{uv})\mathbf{B} = \mathbf{u}(\mathbf{vB}) \equiv \mathbf{uvB}$.

$$
\begin{aligned}
\text{(iii)}(\mathbf{vA})\mathbf{B} &= (\mathbf{v}.\mathbf{A} + \mathbf{v} \wedge \mathbf{A})\mathbf{B} \\
&= (\mathbf{v}.\mathbf{A}).\mathbf{B} + (\mathbf{v}.\mathbf{A}) \wedge \mathbf{B} + (\mathbf{v} \wedge \mathbf{A}) : \mathbf{B} + (\mathbf{v} \wedge \mathbf{A}).\mathbf{B} + (\mathbf{v} \wedge \mathbf{A}) \wedge \mathbf{B} \\
&= (\mathbf{v}.\mathbf{A}).\mathbf{B} + (\mathbf{v} \wedge \mathbf{A}) : \mathbf{B} + (\mathbf{v}.\mathbf{A}) \wedge \mathbf{B} + (\mathbf{v} \wedge \mathbf{A}).\mathbf{B} + \mathbf{v} \wedge \mathbf{A} \wedge \mathbf{B}. \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
\text{Also, } \mathbf{v}(\mathbf{AB}) &= \mathbf{v}(\mathbf{A} : \mathbf{B} + \mathbf{A}.\mathbf{B} + \mathbf{A} \wedge \mathbf{B}) \\
&= \mathbf{v}(\mathbf{A} : \mathbf{B}) + \mathbf{v}.(\mathbf{A}.\mathbf{B}) + \mathbf{v} \wedge (\mathbf{A}.\mathbf{B}) + \mathbf{v}.(\mathbf{A} \wedge \mathbf{B}) + \mathbf{v} \wedge (\mathbf{A} \wedge \mathbf{B}) \\
&= \mathbf{v}(\mathbf{A} : \mathbf{B}) + \mathbf{v}.(\mathbf{A}.\mathbf{B}) + \mathbf{v}.(\mathbf{A} \wedge \mathbf{B}) + \mathbf{v} \wedge (\mathbf{A}.\mathbf{B}) + \mathbf{v} \wedge \mathbf{A} \wedge \mathbf{B}. \qquad (18)
\end{aligned}
$$

The first two terms of both the equations (17) and (18) represent vectors; the third and fourth terms represent trivectors, and the last or fifth terms - pentavectors.

(a) First two terms of eq.(17):

$(\mathbf{v}.\mathbf{A}).\mathbf{B} + (\mathbf{v} \wedge \mathbf{A}) : \mathbf{B} = v_i(A_{ij} - A_{ji})\, \hat{e}_j.\mathbf{B} + (v_i\ A_{jk}\ \hat{e}_i \wedge \hat{e}_j \wedge \hat{e}_k) : \mathbf{B}$

$= v_i(A_{ij} - A_{ji})(B_{jk} - B_{kj})\hat{e}_k + v_i\ A_{jk}\{(B_{kj} - B_{jk})\hat{e}_i + (B_{ik} - B_{ki})\hat{e}_j + (B_{ji} - B_{ij})\hat{e}_k\}$

and the first two terms of eq.(18):

$\mathbf{v}(\mathbf{A} : \mathbf{B}) + \mathbf{v}.(\mathbf{A}.\mathbf{B}) = v_i A_{jk}(B_{kj} - B_{jk})\, \hat{e}_i + v_i\ \hat{e}_i.A_{jk}\{(B_{kl} - B_{lk})\hat{e}_j \wedge \hat{e}_l + (B_{lj} - B_{jl})\hat{e}_k \wedge \hat{e}_l\}$

$= v_i A_{jk}(B_{kj} - B_{jk})\, \hat{e}_i + v_i\{A_{ik}\, (B_{kl} - B_{lk})\hat{e}_l - A_{jk}\, (B_{ki} - B_{ik})\hat{e}_j + A_{ji}\, (B_{lj} - B_{jl})\hat{e}_l - A_{jk}\, (B_{ij} - B_{ji})\hat{e}_k\}$

$= v_i A_{jk}(B_{kj} - B_{jk})\, \hat{e}_i + v_i\{A_{ij}\, (B_{jk} - B_{kj})\hat{e}_k - A_{jk}(B_{ki} - B_{ik})\hat{e}_j + A_{ji}\, (B_{kj} - B_{jk})\hat{e}_k - A_{jk}\, (B_{ij} - B_{ji})\hat{e}_k\}$

$= v_i A_{jk}(B_{kj} - B_{jk})\, \hat{e}_i + v_i(A_{ij} - A_{ji})(B_{jk} - B_{kj})\hat{e}_k + v_i\ A_{jk}(B_{ik} - B_{ki})\hat{e}_j + v_i\ A_{jk}\, (B_{ji} - B_{ij})\hat{e}_k\}$. This is identical with the corresponding expressions of eq.(17).

(b) Third and fourth terms of eq.(17):

$(\mathbf{v}.\mathbf{A}) \wedge \mathbf{B} + (\mathbf{v} \wedge \mathbf{A}).\mathbf{B} = v_i(A_{ij} - A_{ji})\hat{e}_j \wedge B_{kl}\, \hat{e}_k \wedge \hat{e}_l + (v_i\ A_{jk}\ \hat{e}_i \wedge \hat{e}_j \wedge \hat{e}_k).B_{lm}\ \hat{e}_l \wedge \hat{e}_m$
$= v_i(A_{ij} - A_{ji})B_{kl}\, \hat{e}_j \wedge \hat{e}_k \wedge \hat{e}_l + v_i\ A_{jk}\{(B_{kl} - B_{lk})\hat{e}_i \wedge \hat{e}_j \wedge \hat{e}_l + (B_{lj} - B_{jl})\hat{e}_i \wedge \hat{e}_k \wedge \hat{e}_l + (B_{il} - B_{li})\hat{e}_j \wedge \hat{e}_k \wedge \hat{e}_l\}$.

and the Third and fourth terms of eq.(18):

$\mathbf{v}.(\mathbf{A} \wedge \mathbf{B}) + \mathbf{v} \wedge (\mathbf{A}.\mathbf{B}) = v_i\, \hat{e}_i.A_{jk}\, B_{lm}\, \hat{e}_j \wedge \hat{e}_k \wedge \hat{e}_l \wedge \hat{e}_m + (v_i\ \hat{e}_i \wedge A_{jk}\{(B_{kl} - B_{lk})\hat{e}_j \wedge \hat{e}_l + (B_{lj} - B_{jl})\hat{e}_k \wedge \hat{e}_l$
$= v_i(A_{ik}\, B_{lm}\, \hat{e}_k \wedge \hat{e}_l \wedge \hat{e}_m - A_{ji}\, B_{lm}\, \hat{e}_j \wedge \hat{e}_l \wedge \hat{e}_m + A_{jk}\, B_{im}\, \hat{e}_j \wedge \hat{e}_k \wedge \hat{e}_m - A_{jk}\, B_{li}\, \hat{e}_j \wedge \hat{e}_k \wedge \hat{e}_l) + v_i\ A_{jk}\{(B_{kl} - B_{lk})\hat{e}_i \wedge \hat{e}_j \wedge \hat{e}_l + (B_{lj} - B_{jl})\hat{e}_i \wedge \hat{e}_k \wedge \hat{e}_l\}$
$= v_i(A_{ij} - A_{ji})B_{kl}\hat{e}_j \wedge \hat{e}_k \wedge \hat{e}_l + v_i\ A_{jk}(B_{il} - B_{li})\hat{e}_j \wedge \hat{e}_k \wedge \hat{e}_l + v_i\ A_{jk}\, \{(B_{kl} - B_{lk})\hat{e}_i \wedge \hat{e}_j \wedge \hat{e}_l + (B_{lj} - B_{jl})\hat{e}_i \wedge \hat{e}_k \wedge \hat{e}_l\}$, which is identical to the corresponding expressions of eq.(17). Thus, the associativity: $\mathbf{v}(\mathbf{AB}) = (\mathbf{vA})\mathbf{B} \equiv \mathbf{vAB}$ is established. Also, in both (a) and (b) above, equality of the selected grades confirm the grade preserving property of the geometric products.

We can also write,

$$\mathbf{v}(\mathbf{B}\mathbf{B}) = (\mathbf{v}\mathbf{B})\mathbf{B} \tag{19}$$

and for a simple bivector $\mathbf{B}$, $\mathbf{B}\mathbf{B}(\equiv \mathbf{B} : \mathbf{B})$, is the squared magnitude of $\mathbf{B}$. Therefore, the l.h.s. of eq.(19) is a vector and with the unit bivector $\hat{\mathbf{B}}$, $(\mathbf{B}\mathbf{B} = |\mathbf{B}|^2\,\hat{\mathbf{B}}\hat{\mathbf{B}} = -|\mathbf{B}|^2)$, the equation finally gets simplified to:

$$\mathbf{v} = (\hat{\mathbf{B}}.\mathbf{v}).\hat{\mathbf{B}} - (\hat{\mathbf{B}} \wedge \mathbf{v}) : \hat{\mathbf{B}}. \tag{20}$$

For a general bivector $\mathbf{B} = B_{ij}\,\hat{e}_i \wedge \hat{e}_j$, from eq.(5) we get:

$$\begin{aligned} (\mathbf{B}.\mathbf{v}).\mathbf{B} &= (B_{ij} - B_{ji})\,v_j\,\hat{e}_i.B_{kl}\,\hat{e}_k \wedge \hat{e}_l && (21)\\ &= (B_{ij} - B_{ji})(B_{ik} - B_{ki})\,v_j\,\hat{e}_k && (22)\end{aligned}$$

In the case of a simple bivector $\mathbf{B} = \mathbf{u} \wedge \mathbf{w} = u_i\,\hat{e}_i \wedge w_j\,\hat{e}_j = u_i\,w_j\,\hat{e}_i \wedge \hat{e}_j = B_{ij}\,\hat{e}_i \wedge \hat{e}_j$ with $B_{ij} = u_i\,w_j$,

$$\begin{aligned} (\mathbf{B}.\mathbf{v}).\mathbf{B} &= (B_{ij} - B_{ji})\,(B_{ik} - B_{ki})v_j\,\hat{e}_k = (u_i\,w_j - u_j\,w_i)\,(u_i\,w_k - u_k\,w_i)v_j\,\hat{e}_k \\ &= (u_i\,w_j\,u_i\,w_k - u_i\,w_j\,u_k\,w_i - u_j\,w_i\,u_i\,w_k + u_j\,w_i\,u_k\,w_i)v_j\,\hat{e}_k \\ &= \{(\mathbf{u}.\mathbf{v})\,|\mathbf{w}|^2 - (\mathbf{u}.\mathbf{w})(\mathbf{v}.\mathbf{w})\}\,\mathbf{u} + \{(\mathbf{v}.\mathbf{w})\,|\mathbf{u}|^2 - (\mathbf{u}.\mathbf{v})(\mathbf{u}.\mathbf{w})\}\,\mathbf{w} && (23)\end{aligned}$$

and like $\mathbf{B}.\mathbf{v}$, is a linear combination of $\mathbf{u}$ and $\mathbf{w}$ and lie on the plane containing the two vector factors of $\mathbf{B}$ i.e. on the plane of the bivector. Also, from eq.(6), it is obvious that $\mathbf{v}.(\mathbf{B}.\mathbf{v}) = 0$ and since $\mathbf{v}$, $(\mathbf{B}.\mathbf{v}).\mathbf{B}$ and $(\mathbf{B} \wedge \mathbf{v}) : \mathbf{B}$ all lie on the same plane, $\mathbf{B}.\mathbf{v}$ is normal to all these three vectors. The first term on the r.h.s. of eq.(20) is, therefore, represented as $\mathbf{v}_{\|\hat{\mathbf{B}}}$ – projection of $\mathbf{v}$ on to $\mathbf{B}$. Consequently using eq.(6) and eq.(23), the identity $\mathbf{B}.\mathbf{v}_{\|\hat{\mathbf{B}}} = \mathbf{B}.\mathbf{v}$ can also be verified easily.[3]

On the other hand, for the second term on the r.h.s. of eq.(20), consider:

$$\begin{aligned} (\mathbf{B} \wedge \mathbf{v}) : \mathbf{B} &= (B_{ij}\,v_k\,\hat{e}_i \wedge \hat{e}_j \wedge \hat{e}_k) : B_{lm}\,\hat{e}_l \wedge \hat{e}_m \\ &= B_{ij}\,v_k\{(B_{kj} - B_{jk})\hat{e}_i + (B_{ik} - B_{ki})\hat{e}_j + (B_{ji} - B_{ij})\hat{e}_k\} && (24)\end{aligned}$$

For the simple bivector $\mathbf{B} = \mathbf{u} \wedge \mathbf{w}$, this vector becomes:

$$\begin{aligned} &u_i\,w_j\,v_k\{(u_k\,w_j - u_j\,w_k)\hat{e}_i + (u_i\,w_k - u_k\,w_i)\hat{e}_j + (u_j\,w_i - u_i\,w_j)\hat{e}_k\} \\ &= \{(\mathbf{u}.\mathbf{v})|\mathbf{w}|^2 - (\mathbf{u}.\mathbf{w})(\mathbf{v}.\mathbf{w})\}\mathbf{u} + \{(\mathbf{v}.\mathbf{w})|\mathbf{u}|^2 - (\mathbf{u}.\mathbf{v})(\mathbf{u}.\mathbf{w})\}\mathbf{w} + \{(\mathbf{u}.\mathbf{w})^2 - |\mathbf{u}|^2|\mathbf{w}|^2\}\mathbf{v} \\ &\qquad (= (\mathbf{B}.\mathbf{v}).\mathbf{B} + \{(\mathbf{u}.\mathbf{w})^2 - |\mathbf{u}|^2|\mathbf{w}|^2\}\mathbf{v})\end{aligned}$$

Both the scalar products of this term with $\mathbf{u}$ and $\mathbf{w}$, respectively, vanish. Hence, it is perpendicular to both $\mathbf{u}$ and $\mathbf{w}$ and also to the plane of $\hat{\mathbf{B}}$ (representing rejection of $\mathbf{v}$ from $\hat{\mathbf{B}}$). Denoting the second term (of eq.20) with $\mathbf{v}_{\perp\hat{\mathbf{B}}}$, we note that all the three vectors $\mathbf{B}.\mathbf{v}$, $\mathbf{v}_{\|\hat{\mathbf{B}}}$ and $\mathbf{v}_{\perp\hat{\mathbf{B}}}$ are mutually perpendicular and finally we can verify :

$$\begin{aligned} \mathbf{v}_{\|\hat{\mathbf{B}}} + \mathbf{v}_{\perp\hat{\mathbf{B}}} &= (\hat{\mathbf{B}}.\mathbf{v}).\hat{\mathbf{B}} - (\hat{\mathbf{B}} \wedge \mathbf{v}) : \hat{\mathbf{B}} = |\mathbf{B}|^{-2}\{(\mathbf{B}.\mathbf{v}).\mathbf{B} - (\mathbf{B} \wedge \mathbf{v}) : \mathbf{B}\} \\ &= |\mathbf{B}|^{-2}\{|\mathbf{u}|^2|\mathbf{w}|^2 - (\mathbf{u}.\mathbf{w})^2\}\mathbf{v} = |\mathbf{B}|^{-2}|\mathbf{u}|^2|\mathbf{w}|^2(1 - \cos^2\theta)\mathbf{v} \\ &= |\mathbf{B}|^{-2}(|\mathbf{u}||\mathbf{w}|\sin\theta)^2\mathbf{v} = \mathbf{v}.\end{aligned}$$

All these will be elaborated further with new results and their interesting simple applications in a subsequent paper [7].

---

[3] $\mathbf{B}.\mathbf{v}_{\|\hat{\mathbf{B}}} = \mathbf{B}.\{(\hat{\mathbf{B}}.\mathbf{v}).\hat{\mathbf{B}}\} = |\mathbf{B}|^{-2}\mathbf{B}.\{(\mathbf{B}.\mathbf{v}).\mathbf{B}\} = |\mathbf{B}|^{-2}[(\mathbf{v}.\mathbf{w})\{|\mathbf{u}|^2|\mathbf{w}|^2 - (\mathbf{u}.\mathbf{w})^2\}\mathbf{u} - (\mathbf{u}.\mathbf{v})\{|\mathbf{u}|^2|\mathbf{w}|^2 - (\mathbf{u}.\mathbf{w})^2\}\mathbf{w}] = |\mathbf{B}|^{-2}|\mathbf{B}|^2\{(\mathbf{v}.\mathbf{w})\mathbf{u} - (\mathbf{u}.\mathbf{v})\mathbf{w}\} = \mathbf{B}.\mathbf{v}$.

VI. Rotor and Spinor:

Let us now consider the transformation equation (3) of $\mathbf{v}$ in view of the expression (19). Since, $\mathbf{R}\,\tilde{\mathbf{R}} = 1$ and $\mathbf{v}^2 = \mathbf{vv} = \mathbf{Rvv\tilde{R}} = \mathbf{Rv\tilde{R}Rv\tilde{R}}$, the transformation evidently preserves the magnitude of $\mathbf{v}$. Now expanding the expression, we get:

$$
\begin{aligned}
\mathbf{Rv\tilde{R}} &= \{\cos(\frac{|\theta|}{2}) + \hat{\mathbf{B}}\,\sin(\frac{|\theta|}{2})\}\,\mathbf{v}\,\{\cos(\frac{|\theta|}{2}) - \hat{\mathbf{B}}\,\sin(\frac{|\theta|}{2})\} \\
&= \mathbf{v}\cos^2(\frac{|\theta|}{2}) - \mathbf{v}\hat{\mathbf{B}}\,\cos(\frac{|\theta|}{2})\sin(\frac{|\theta|}{2}) + \hat{\mathbf{B}}\mathbf{v}\,\cos(\frac{|\theta|}{2})\sin(\frac{|\theta|}{2}) - (\hat{\mathbf{B}}\mathbf{v})\,\hat{\mathbf{B}}\,\sin^2(\frac{|\theta|}{2}) \\
&= \{(\hat{\mathbf{B}}.\mathbf{v}).\hat{\mathbf{B}} - (\mathbf{v}\wedge\hat{\mathbf{B}}) : \hat{\mathbf{B}}\}\cos^2(\frac{|\theta|}{2}) + 2(\hat{\mathbf{B}}.\mathbf{v})\cos(\frac{|\theta|}{2})\sin(\frac{|\theta|}{2}) - (\hat{\mathbf{B}}.\mathbf{v} + \hat{\mathbf{B}}\wedge\mathbf{v})\,\hat{\mathbf{B}} \\
&\quad \sin^2(\frac{|\theta|}{2}) \\
&= (\hat{\mathbf{B}}.\mathbf{v}).\hat{\mathbf{B}}\cos^2(\frac{|\theta|}{2}) - (\hat{\mathbf{B}}\wedge\mathbf{v}) : \hat{\mathbf{B}}\,cos^2(\frac{|\theta|}{2}) + (\hat{\mathbf{B}}.\mathbf{v})\sin|\theta| - \{(\hat{\mathbf{B}}.\mathbf{v}).\hat{\mathbf{B}} + (\hat{\mathbf{B}}\wedge\mathbf{v}) : \hat{\mathbf{B}}\} \\
&\quad \sin^2(\frac{|\theta|}{2}),\ \text{since for a simple bivector } \hat{\mathbf{B}}.\mathbf{v}\wedge\hat{\mathbf{B}},\ \hat{\mathbf{B}}\wedge\mathbf{v}.\hat{\mathbf{B}} \text{ and } \hat{\mathbf{B}}\wedge\mathbf{v}\wedge\hat{\mathbf{B}} \text{ all vanish,} \\
&= (\hat{\mathbf{B}}.\mathbf{v}).\hat{\mathbf{B}}\cos|\theta| + (\hat{\mathbf{B}}.\mathbf{v})\sin|\theta| - (\hat{\mathbf{B}}\wedge\mathbf{v}) : \hat{\mathbf{B}} \qquad (25)
\end{aligned}
$$

The r.h.s. represents a vector – a transformed vector $\mathbf{v}''$ (say) and the transformation actually rotates $\mathbf{v}_{\|\hat{\mathbf{B}}}$ ($= (\hat{\mathbf{B}}.\mathbf{v}).\hat{\mathbf{B}}$) through an angle $|\theta|$ in the plane of $\hat{\mathbf{B}}$, leaving the part $\mathbf{v}_{\perp\hat{\mathbf{B}}}$ ($= -(\hat{\mathbf{B}}\wedge\mathbf{v}) : \hat{\mathbf{B}}$) unaltered. This can be easily perceived from the following analysis. For a simple bivector $\hat{\mathbf{B}}$, since $\mathbf{v}_{\|\hat{\mathbf{B}}}$ is in the plane of rotation generated by the bivector, this rotation can also be described as a single sided operation, analogous to the conventional formula for the rotation of ordinary complex number rotation of a 2-D vector as:

$$
\begin{aligned}
\mathbf{v}'' &= \mathbf{v}_{\|\hat{\mathbf{B}}}\exp(|\theta|\hat{\mathbf{B}}) + \mathbf{v}_{\perp\hat{\mathbf{B}}} \\
&= (\hat{\mathbf{B}}.\mathbf{v}).\hat{\mathbf{B}}(\cos|\theta| + \hat{\mathbf{B}}\sin|\theta|) - (\hat{\mathbf{B}}\wedge\mathbf{v}) : \hat{\mathbf{B}} \\
&= (\hat{\mathbf{B}}.\mathbf{v}).\hat{\mathbf{B}}\cos|\theta| + \hat{\mathbf{B}}.\mathbf{v}\sin|\theta| - (\hat{\mathbf{B}}\wedge\mathbf{v}) : \hat{\mathbf{B}} \qquad (26)
\end{aligned}
$$

as $\{(\hat{\mathbf{B}}.\mathbf{v}).\hat{\mathbf{B}}\}\hat{\mathbf{B}} = \{(\hat{\mathbf{B}}.\mathbf{v}).\hat{\mathbf{B}}\}.\hat{\mathbf{B}} = \hat{\mathbf{B}}.\mathbf{v}$. The vector $\hat{\mathbf{B}}.\mathbf{v}$ is orthogonal to $\mathbf{v}_{\|\hat{\mathbf{B}}}$ and also lies in the plane of $\hat{\mathbf{B}}$, since $\hat{\mathbf{B}}.\mathbf{v}\wedge\hat{\mathbf{B}} = 0$.

In 3-D, instead of the plane of rotation, rotation is usually specified by the axis of rotation $\hat{\mathbf{n}}$ – the unit normal to the plane of rotation. But such a specification is not possible for other dimensions. Representing the plane of rotation by a unit bivector $\hat{\mathbf{B}}$, geometric algebra (GA) appropriately generalises the description. Together with the size or angle of rotation θ, the unit bivector provides unambiguous specification of rotation in any dimension. In two and three dimensions the plane of rotation is uniquely defined and together with the angle of rotation it fully describes the rotation. A rotation with only one plane of rotation is a simple rotation, i.e. all rotations in two and three dimensions are simple. The dual of the bivectors representing the planes of rotation are scalar and a vector and represent the centre and the axis of rotation respectively in 2 and 3-D rotations. Since the dual of a bivector is a bivector, in 4-D the rotation can be said to take place about this plane, so points as they rotate do not change their distance from this plane. The plane of rotation is orthogonal to this plane.

In addition to simple rotations, nontrivial rotations may involve multiple planes of rotation in four and higher dimensions. For example, in 4-D there are also double and isoclinic rotations. In a double rotation there are two orthogonal planes of rotation, having no vectors in common. So every vector in one plane is at right angles to the vectors in the other plane. The two rotation planes span the four dimensional space, so this rotation has the origin as the only fixed point and no fixed, invariant plane. Thus, a double rotation is generally specified by two unique planes and two unique angles, θ and ϕ through which the points on the respective planes rotate. All other points rotate through an angle between θ and ϕ. If either angle is zero the rotation is simple.

A special case of the double rotation, called an isoclinic rotation, is when the angles θ and ϕ are equal ($\theta = \phi \neq 0$). In an isoclinic rotation, except the fixed origin, all other points rotate through the same angle, α. It differs from the simple rotation also due to the fact that the planes of rotation are not uniquely defined. There are instead an infinite number of pairs of orthogonal planes which may be treated as planes of rotation.

In even dimensions ($n = 2, 4, 6, ...$) there are up to $\frac{n}{2}$ rotational planes, having no vectors in common, span the space. So, a general rotation rotates all points except the origin which is the only fixed point. Similarly, in odd dimensions ($n = 3, 5, 7, ...$) there are $\frac{n-1}{2}$ planes and angles of rotation, the same as that in the, just one lower, even dimensional space. These do not span the space completely, but leave a line which does not rotate - like the axis of rotation in three dimensions, except rotations do not take place about this line but in multiple planes orthogonal to it [9].

Given a rotor, the bivector associated with it can be recovered by taking the logarithm of the rotor, which can then be split into simple bivectors to determine the planes of rotation. Although in practice, for all but the simplest of cases this may be impractical. But given the simple bivectors, geometric algebra is a useful tool for studying planes of rotation.

Rotors in higher dimensions turn out to act a lot like rotors in 3 dimensions. Moreover, in 4-D space-time continuum rotors for $x - t$, $y - t$ and $z - t$ space-time surfaces produce Lorentz boost in addition to the usual rotations on three orthogonal spatiatial planes $x - y$, $x - z$ and $y - z$). Bivectors containing a time-like component have different (positive) signs of their squares and the corresponding rotors follow hyperbolic geometry. It turns out that boost is sort of a generalized rotation and the same rotor prescription introduced for rotation in Euclidean space also works for boosts in relativity! This is dramatically simpler than having to work with 4×4 Lorentz transformation matrices.

It was Clifford [1] who first pointed out that the quaternion algebra is just a special case of Grassmann algebra. Hestenes [10] defined a rotor to be any element $\mathbf{R}$ of a geometric algebra that can be written as the product of an even number of unit vectors and satisfies $\mathbf{R}\tilde{\mathbf{R}} = 1$, where $\tilde{\mathbf{R}}$ is the reverse of $\mathbf{R}$ – that is, the product of the same vectors, but in the reverse order.

Writing $\mathbf{R} \equiv \mathbf{R}(\theta)$ for the generator of rotation through an angle θ, we note that $\mathbf{R}(\theta + 2\pi) = -\mathbf{R}(\theta)$ and $\mathbf{R}(\theta + 4\pi) = \mathbf{R}(\theta)$. Thus $\mathbf{R}(\theta)$ is periodic with a period 4π and shares identical algebra with spinors. Also, rotors can be combined and form a group and multiple rotors compose single-sidedly. Thus spinors may be regarded as non-normalised rotors in GA, which also provides an explicit construction. Spinors allow a more general treatment of the notion of invariance under rotation and Lorentz boosts. They can be used without reference to relativity, but arise naturally in the discussions of Lorentz group.

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