Geometric Algebra

5. Spacetime Algebra

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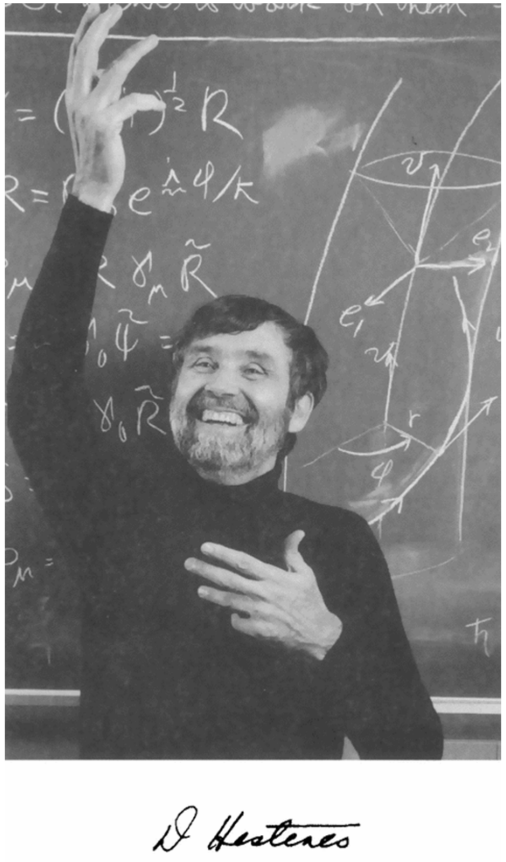
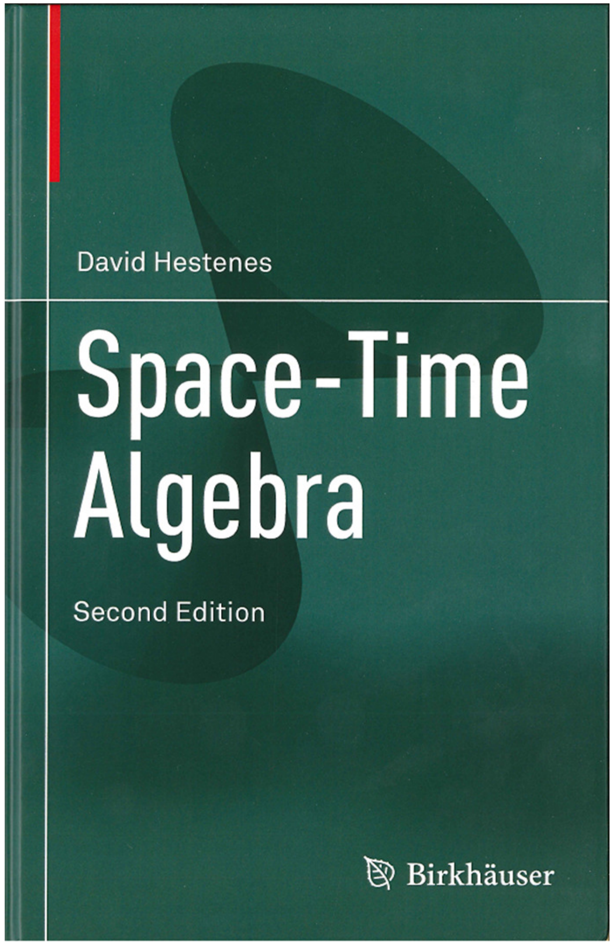
History


D Hestenes


David Hestenes
Space-Time
Algebra
Second Edition
Birkhäuser

A geometric algebra of spacetime

Invariant interval of spacetime is $s^2 = c^2t^2 - \mathrm{x}^2 - \mathrm{y}^2 - \mathrm{z}^2$

The "particle physics" convention

Need 4 generators $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$

$$\gamma_0^2 = 1, \quad \gamma_i^2 = -1$$

$$\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = \mathrm{diag}(+ - - -)$$

Position vector

$$x = x^\mu \gamma_\mu = ct\gamma_0 + x^i \gamma_i$$

Sometimes use the reciprocal frame

$$\gamma^0 = \gamma_0, \quad \gamma^i = -\gamma_i$$

So

$$\gamma_\mu \cdot \gamma^\nu = \delta_\mu^\nu$$

Recover components of a vector

$$a_\mu = a \cdot \gamma_\mu, \quad a^\mu = a \cdot \gamma^\mu$$

Spacetime algebra

1	$\{\gamma_\mu\}$	$\{\gamma_\mu \wedge \gamma_\nu\}$	$\{I\gamma_\mu\}$	I
1	4	6	4	1
scalar	vectors	bivectors	trivectors	pseudoscalar
$\tilde{\alpha} = \alpha$	$\tilde{a} = a$	$\tilde{B} = -B$	$(Ia)^{\sim} = -Ia$	$\tilde{I} = I$

The pseudoscalar is defined by $I = \gamma_0\gamma_1\gamma_2\gamma_3$

This satisfies
$$I^2 = \gamma_0\gamma_1\gamma_2\gamma_3\gamma_0\gamma_1\gamma_2\gamma_3 = \gamma_0\gamma_1\gamma_2\gamma_0\gamma_1\gamma_3$$
$$= -\gamma_0\gamma_1\gamma_0\gamma_1 = -1$$

The bivector algebra

Space-like

$$\{\gamma_1\gamma_2,\ \gamma_2\gamma_3,\ \gamma_3\gamma_1\}$$

$$(\gamma_i\gamma_j)^2 = -\gamma_i^2\gamma_j^2 = -1$$

- Generate rotations in a plane
- Form a closed algebra
- Same behaviour as the bivectors we have met already

Time-like

$$\{\gamma_1\gamma_0,\ \gamma_2\gamma_0,\ \gamma_3\gamma_0\}$$

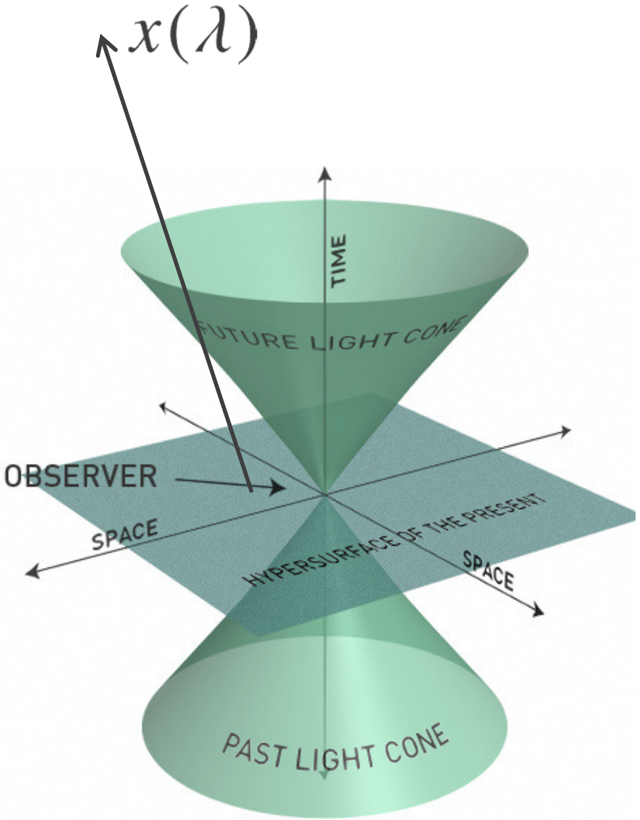
$$(\gamma_1\gamma_0)^2 = -\gamma_1^2\gamma_0^2 = +1$$

- A new type of bivector, with positive square
- Generate boosts
- Commutator of two time-like bivectors is a space-like one

Observers and trajectories

NB will set c=1 from now on

$$x' = \frac{\partial x(\lambda)}{\partial \lambda}$$



Timelike

$$x'^2 > 0$$

Introduce the proper time τ:

$$v = \partial_\tau x = \dot{x}$$

$$v^2 = 1$$

Observers and massive particles follow timelike paths

Null

$$x'^2 = 0$$

- Photons follow null trajectories
- No concept of proper time for photons.
- They are ‘timeless’

Coordinate systems

Special relativity focuses on how different observers perceive the same events – passive transformations

Set $v = e_0$ Construct frame $e_i, \quad e_i \cdot v = 0$

General event can be written

$$x = x^\mu e_\mu = tv + x^i e_i$$

Time coordinate is

$$t = v \cdot x$$

Measures time on observer's clock

Spatial part is the remainder

$$\begin{aligned} x^i e_i &= x - x \cdot v\, v \\ &= (xv - x \cdot v)v \\ &= x \wedge v\, v \end{aligned}$$

Focus on the bivector part

Observer bivectors

Write $xv = x \cdot v + x \wedge v = t + \boldsymbol{x}$

Spatial generators

$$\boldsymbol{e}_i = e_i e_0$$

Satisfy

$$\begin{aligned} \boldsymbol{e}_i \cdot \boldsymbol{e}_j &= \langle e_i e_0 e_j e_0 \rangle \\ &= -\langle e_i e_j \rangle \\ &= \delta_{ij} \end{aligned}$$

Generate a spatial GA. The algebra of the relative space

Recover the spacetime metric by writing

$$\begin{aligned} x^2 &= xvvx \\ &= (t + \boldsymbol{x})(t + v \wedge x) \\ &= (t + \boldsymbol{x})(t - \boldsymbol{x}) \\ &= t^2 - \boldsymbol{x}^2 \end{aligned}$$

Metric properties flow naturally from the split

Observer splits

$$\begin{array}{ccccc} 1 & \{\gamma_\mu\} & \{\gamma_\mu \wedge \gamma_\nu\} & \{I\gamma_\mu\} & I \\ \downarrow & & \swarrow \quad \searrow & & \downarrow \\ 1 & \{\boldsymbol{e}_i\} & \{\boldsymbol{e}_i \wedge \boldsymbol{e}_j\} & & I = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 \end{array}$$

$$\boxed{\boldsymbol{e}_i = \gamma_i\gamma_0}$$

$$\boldsymbol{e}_1\boldsymbol{e}_2 = \gamma_1\gamma_0\gamma_2\gamma_0 = \gamma_0\gamma_1\gamma_2\gamma_3\gamma_0 = I\boldsymbol{e}_3$$

$$\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 = \gamma_1\gamma_0\gamma_2\gamma_0\gamma_3\gamma_0 = \gamma_0\gamma_1\gamma_2\gamma_3 = I$$

Relative space and spacetime share the same pseudoscalar

This projective split between spacetime and relative space is observer-dependent.
A very useful technique

Relative velocity

Observer with velocity $v = e_0$

Observes a trajectory $x(\tau)$

$$uv = \partial_\tau [x(\tau) v] = \partial_\tau (t + \boldsymbol{x})$$

$$\partial_\tau t = u \cdot v = \gamma$$

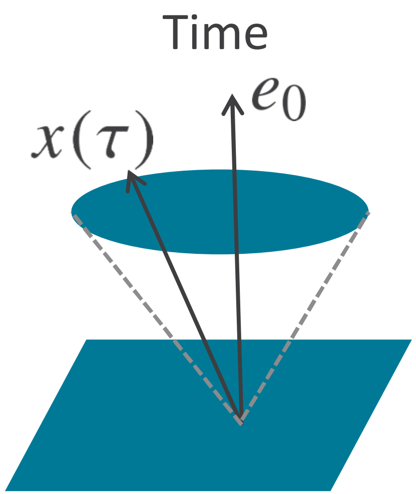
$$\boldsymbol{u} = \frac{\partial \boldsymbol{x}}{\partial t} = \frac{\partial \boldsymbol{x}}{\partial \tau}\frac{\partial \tau}{\partial t}$$

$$= \frac{u \wedge v}{u \cdot v}$$

Relative velocity

$$uv = \gamma(1 + \boldsymbol{u})$$

$$1 = \gamma^2 (1 - \boldsymbol{u}^2)$$



Note this is ‘textbook’ relativity
In reality you should focus on experiments and photon trajectories, not coordinate systems.

Lorentz Transformations

Expressed in terms of coordinate transformations

$$\mathrm{x}' = \gamma(\mathrm{x} - \beta t) \qquad t' = \gamma(t - \beta \mathrm{x})$$

$$\mathrm{x} = \gamma(\mathrm{x}' + \beta t') \qquad t = \gamma(t' + \beta \mathrm{x}')$$

These are passive. The same event expressed in two different coordinate systems

$$x = x^{\mu} e_{\mu} = x^{\mu\prime} e'_{\mu}$$

$$t = e^{0} \cdot x, \quad t' = e^{0\prime} \cdot x$$

Understand the transformation in terms of the frame transforming

Focus on the 0,1 components

$$t e_0 + \mathrm{x} e_1 = t' e'_0 + \mathrm{x}' e'_1$$

$$e'_0 = \gamma(e_0 + \beta e_1)$$

$$e'_1 = \gamma(e_1 + \beta e_0)$$

Hyperbolic geometry

Introduce the hyperbolic angle $\tanh\alpha = \beta, \quad (\beta < 1)$

$$\gamma = (1 - \tanh^2\alpha)^{-1/2} = \cosh\alpha$$

$$\begin{aligned} e_0' &= \mathrm{ch}(\alpha)e_0 + \mathrm{sh}(\alpha)e_1 \\ &= (\mathrm{ch}(\alpha) + \mathrm{sh}(\alpha)e_1e_0)e_0 \\ &= e^{\alpha\, e_1 e_0} e_0 \end{aligned}$$

Power series still works for exponential, but now $(e_1e_0)^2 = 1$

Also find $e_1' = e^{\alpha\, e_1 e_0} e_1$

Other two directions unaffected

$$e_\mu' = Re_\mu\tilde{R}, \qquad e^{\mu\,\prime} = Re^\mu\tilde{R}$$

$$R = e^{\alpha\, e_1 e_0/2}$$

Addition of velocities

$$v_1 = e^{\alpha_1 e_1 e_0} e_0 \qquad v_2 = e^{-\alpha_2 e_1 e_0} e_0$$

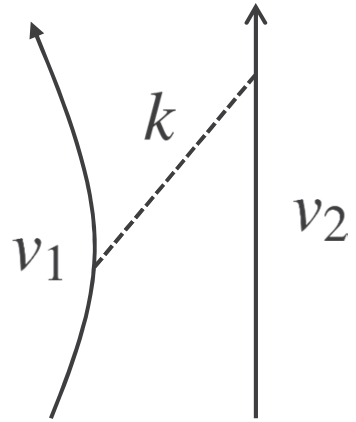
What is the relative velocity between the trains that the drivers agree on?

$$\frac{v_1 \wedge v_2}{v_1 \cdot v_2} = \frac{\langle e^{(\alpha_1+\alpha_2)e_1 e_0} \rangle_2}{\langle e^{(\alpha_1+\alpha_2)e_1 e_0} \rangle_0} = \frac{\sinh(\alpha_1 + \alpha_2) e_1 e_0}{\cosh(\alpha_1 + \alpha_2)}$$

Relative velocity is hyperbolic addition

$$\tanh(\alpha_1 + \alpha_2) = \frac{\tanh\alpha_1 + \tanh\alpha_2}{1 - \tanh\alpha_1 \tanh\alpha_2}$$

Photons and redshifts



Particle 1 emits a photon which is received by particle 2

$\omega_1 = v_1 \cdot k$ Frequency for particle 1

$\omega_2 = v_2 \cdot k$ Frequency for particle 2

Assume $v_2 = e_0$ $\quad k = \omega_2(e_0 + e_1)$ Unique form of null vector

$v_1 = \cosh\alpha \, e_0 - \sinh\alpha \, e_1$ Particle 1 is receding

$$1 + z = \frac{\omega_1}{\omega_2} = \frac{\omega_2(\cosh\alpha + \sinh\alpha)}{\omega_2} = e^{\alpha}$$

Compact expression for redshift

Spacetime rotor dynamics

Trajectory $x(\tau)$

Future-pointing velocity $v = \partial_\tau x, \quad v^2 = 1$

Put the dynamics into a rotor

$$v = R\gamma_0\tilde{R}$$

$$\begin{aligned}\dot{v} &= \partial_\tau(R\gamma_0\tilde{R}) \\ &= \dot{R}\gamma_0\tilde{R} + R\gamma_0\dot{\tilde{R}} \\ &= \dot{R}\tilde{R}v - v\dot{R}\tilde{R} \\ &= 2(\dot{R}\tilde{R})\cdot v\end{aligned}$$

Define the acceleration bivector

$$\dot{v}v = 2(\dot{R}\tilde{R})\cdot v\, v$$

Bivector projected into the instantaneous rest frame

Determines bivector up to a pure rotation in the IRF

Motion in an electromagnetic field

$$\frac{d\boldsymbol{p}}{dt} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

Famous equation in 3D, using the cross product. Quantities all in some rest frame

$$\boldsymbol{p} = p\wedge\gamma_0 \qquad \dot{t} = v\cdot\gamma_0 \qquad \boldsymbol{v} = v\wedge\gamma_0/v\cdot\gamma_0$$

Also have the energy equation $$\frac{d\epsilon}{dt} = q\boldsymbol{E}\cdot\boldsymbol{v}, \qquad \epsilon = p\cdot\gamma_0$$

$$\dot{p}\gamma_0 = q\big(\boldsymbol{E}v\cdot\gamma_0 + \boldsymbol{E}\cdot(v\wedge\gamma_0) + (I\boldsymbol{B})\times(v\wedge\gamma_0)\big)$$

Now think of both ***E*** and ***IB*** as spacetime bivectors

Note, this is the GA commutator now

Motion in an electromagnetic field

Electric term

$$\boldsymbol{E}v\cdot\gamma_0 + \boldsymbol{E}\cdot(v\wedge\gamma_0) = \tfrac{1}{2}(v\gamma_0\boldsymbol{E} + \boldsymbol{E}v\gamma_0)$$
$$= \tfrac{1}{2}(\boldsymbol{E}v\gamma_0 - v\boldsymbol{E}\gamma_0) = \boldsymbol{E}\cdot v\,\gamma_0$$

Magnetic term

$$(I\boldsymbol{B})\times v(v\wedge\gamma_0) = \tfrac{1}{2}(I\boldsymbol{B}v\gamma_0 - v\gamma_0 I\boldsymbol{B})$$
$$= (I\boldsymbol{B})\cdot v\,\gamma_0$$

Define the Faraday bivector $F = \boldsymbol{E} + I\boldsymbol{B}$

A true spacetime quantity

Motion in an electromagnetic field

Now have $\dot{p}\gamma_0 = qF\cdot v\,\gamma_0$

Remove the observer dependence to get a spacetime equation

$$\dot{v} = \frac{q}{m}F\cdot v$$

A particle responds to the electric field in its instantaneous rest frame
The Lorentz force law

Acceleration bivector is

$$\dot{v}v = \frac{q}{m}(F\cdot v)v$$

$$\dot{v}v = 2(\dot{R}\tilde{R})\cdot v\,v$$

Most natural to set

$$2\dot{R}\tilde{R} = \frac{q}{m}F$$

$$\dot{R} = \frac{q}{2m}FR$$

Spacetime dynamics in one simple equation!

Unification

The natural form of the relativistic rotor equation for a particle in an electromagnetic field predicts a gyromagnetic ratio of 2

Resources

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github.com/ga