Chapter 6

Temporal-Difference Learning

If one had to identify one idea as central and novel to reinforcement learning, it would undoubtedly be *temporal-difference* (TD) learning. TD learning is a combination of Monte Carlo ideas and dynamic programming (DP) ideas. Like Monte Carlo methods, TD methods can learn directly from raw experience without a model of the environment's dynamics. Like DP, TD methods update estimates based in part on other learned estimates, without waiting for a final outcome (they bootstrap). The relationship between TD, DP, and Monte Carlo methods is a recurring theme in the theory of reinforcement learning. This chapter is the beginning of our exploration of it. Before we are done, we will see that these ideas and methods blend into each other and can be combined in many ways. In particular, in Chapter 7 we introduce n-step algorithms, which provide a bridge from TD to Monte Carlo methods, and in Chapter 12 we introduce the TD(λ) algorithm, which seamlessly unifies them.

As usual, we start by focusing on the policy evaluation or *prediction* problem, that of estimating the value function v_π for a given policy π. For the *control* problem (finding an optimal policy), DP, TD, and Monte Carlo methods all use some variation of generalized policy iteration (GPI). The differences in the methods are primarily differences in their approaches to the prediction problem.

6.1 TD Prediction

Both TD and Monte Carlo methods use experience to solve the prediction problem. Given some experience following a policy π, both methods update their estimate v of v_π for the nonterminal states S_t occurring in that experience. Roughly speaking, Monte Carlo methods wait until the return following the visit is known, then use that return as a target for $V(S_t)$. A simple every-visit Monte Carlo method suitable for nonstationary environments is

$$V(S_t) \leftarrow V(S_t) + \alpha\Big[G_t - V(S_t)\Big], \tag{6.1}$$

where G_t is the actual return following time t, and α is a constant step-size parameter (c.f., Equation 2.4). Let us call this method *constant-α MC*. Whereas Monte Carlo

methods must wait until the end of the episode to determine the increment to $V(S_t)$ (only then is G_t known), TD methods need wait only until the next time step. At time $t+1$ they immediately form a target and make a useful update using the observed reward R_{t+1} and the estimate $V(S_{t+1})$. The simplest TD method, known as *TD(0)*, is

$$V(S_t) \leftarrow V(S_t) + \alpha\Big[R_{t+1} + \gamma V(S_{t+1}) - V(S_t)\Big]. \tag{6.2}$$

In effect, the target for the Monte Carlo update is G_t, whereas the target for the TD update is $R_{t+1} + \gamma V(S_{t+1})$. The box at the bottom of the page specifies TD(0) completely in procedural form. TD(0) is a special case of TD(λ), explored in Chapter 12.

Because the TD method bases its update in part on an existing estimate, we say that it is a *bootstrapping* method, like DP. We know from Chapter 3 that

$$\begin{aligned} v_\pi(s) &\doteq \mathbb{E}_\pi[G_t \mid S_t = s] && (6.3)\\ &= \mathbb{E}_\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \;\middle|\; S_t = s\right] \\ &= \mathbb{E}_\pi\left[R_{t+1} + \gamma \sum_{k=0}^{\infty} \gamma^k R_{t+k+2} \;\middle|\; S_t = s\right] \\ &= \mathbb{E}_\pi[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s]\,. && (6.4) \end{aligned}$$

Roughly speaking, Monte Carlo methods use an estimate of (6.3) as a target, whereas DP methods use an estimate of (6.4) as a target. The Monte Carlo target is an estimate because the expected value in (6.3) is not known; a sample return is used in place of the real expected return. The DP target is an estimate not because of the expected values, which are assumed to be completely provided by a model of the environment, but because $v_\pi(S_{t+1})$ is not known and the current estimate, $V(S_{t+1})$, is used instead. The TD target is an estimate for both reasons: it samples the expected values in (6.4) *and* it uses the current estimate V instead of the true v_π. Thus, TD methods combine the sampling of Monte Carlo with the bootstrapping of

Tabular TD(0) for estimating v_π

Input: the policy π to be evaluated
Initialize $V(s)$ arbitrarily (e.g., $V(s) = 0, \forall s \in \mathcal{S}^+$)
Repeat (for each episode):
  Initialize S
  Repeat (for each step of episode):
    $A \leftarrow$ action given by π for S
    Take action A, observe R, S'
    $V(S) \leftarrow V(S) + \alpha\big[R + \gamma V(S') - V(S)\big]$
    $S \leftarrow S'$
  until S is terminal

DP. As we shall see, with care and imagination this can take us a long way toward obtaining the advantages of both Monte Carlo and DP methods.


TD(0)

The diagram to the right is the backup diagram for tabular TD(0). The value estimate for the state node at the top of the backup diagram is updated on the basis of the one sample transition from it to the immediately following state. We refer to TD and Monte Carlo updates as *sample backups* because they involve looking ahead to a sample successor state (or state–action pair), using the value of the successor and the reward along the way to compute a backed-up value, and then changing the value of the original state (or state–action pair) accordingly. *Sample* backups differ from the *full* backups of DP methods in that they are based on a single sample successor rather than on a complete distribution of all possible successors.

Finally, note that the quantity in brackets in the TD(0) update is a sort of error, measuring the difference between the estimated value of S_t and the better estimate $R_{t+1}+\gamma V(S_{t+1})$. This quantity, called the *TD error*, arises in various forms throughout reinforcement learning:

$$\delta_t \doteq R_{t+1} + \gamma V(S_{t+1}) - V(S_t). \tag{6.5}$$

Notice that the TD error at each time is the error in the estimate *made at that time*. Because the TD error depends on the next state and next reward, it is not actually available until one time step later. That is, δ_t is the error in $V(S_t)$, available at time $t+1$. Also note that the Monte Carlo error can be written as a sum of TD errors:

$$\begin{aligned}
G_t - V(S_t) &= R_{t+1} + \gamma G_{t+1} - V(S_t) + \gamma V(S_{t+1}) - \gamma V(S_{t+1}) \\
&= \delta_t + \gamma\big(G_{t+1} - V(S_{t+1})\big) \\
&= \delta_t + \gamma\delta_{t+1} + \gamma^2\big(G_{t+2} - V(S_{t+2})\big) \\
&= \delta_t + \gamma\delta_{t+1} + \gamma^2\delta_{t+2} + \cdots + \gamma^{T-t+1}\delta_{T-t+1} + \gamma^{T-t}\big(G_T - V(S_T)\big) \\
&= \delta_t + \gamma\delta_{t+1} + \gamma^2\delta_{t+2} + \cdots + \gamma^{T-t-1}\delta_{T-t-1} + \gamma^{T-t}\big(0 - 0\big) \\
&= \sum_{k=0}^{T-t-1} \gamma^k \delta_{t+k}.
\end{aligned} \tag{6.6}$$

This fact and its generalizations play important roles in the theory of TD learning.

Example 6.1: Driving Home Each day as you drive home from work, you try to predict how long it will take to get home. When you leave your office, you note the time, the day of week, the weather, and anything else that might be relevant. Say on this Friday you are leaving at exactly 6pm, and you estimate that it will take 30 minutes to get home. As you reach your car it is 6:05, and you notice it is starting to rain. Traffic is often slower in the rain, so you reestimate that it will take 35 minutes from then, or a total of 40 minutes. Fifteen minutes later you have completed the highway portion of your journey in good time. As you exit onto a secondary road you cut your estimate of total travel time to 35 minutes. Unfortunately, at this point

you get stuck behind a slow truck, and the road is too narrow to pass. You end up having to follow the truck until you turn onto the side street where you live at 6:40. Three minutes later you are home. The sequence of states, times, and predictions is thus as follows:

State	*Elapsed Time (minutes)*	*Predicted Time to Go*	*Predicted Total Time*
leaving office, friday at 6	0	30	30
reach car, raining	5	35	40
exiting highway	20	15	35
2ndary road, behind truck	30	10	40
entering home street	40	3	43
arrive home	43	0	43

The rewards in this example are the elapsed times on each leg of the journey.[1] We are not discounting ($\gamma = 1$), and thus the return for each state is the actual time to go from that state. The value of each state is the *expected* time to go. The second column of numbers gives the current estimated value for each state encountered.

A simple way to view the operation of Monte Carlo methods is to plot the predicted total time (the last column) over the sequence, as in Figure 6.1 (left). The arrows show the changes in predictions recommended by the constant-α MC method (6.1), for $\alpha = 1$. These are exactly the errors between the estimated value (predicted time to go) in each state and the actual return (actual time to go). For example, when you exited the highway you thought it would take only 15 minutes more to get home, but in fact it took 23 minutes. Equation 6.1 applies at this point and determines an increment in the estimate of time to go after exiting the highway. The error, $G_t - V(S_t)$, at this time is eight minutes. Suppose the step-size parameter, α, is 1/2.

[1] If this were a control problem with the objective of minimizing travel time, then we would of course make the rewards the *negative* of the elapsed time. But since we are concerned here only with prediction (policy evaluation), we can keep things simple by using positive numbers.

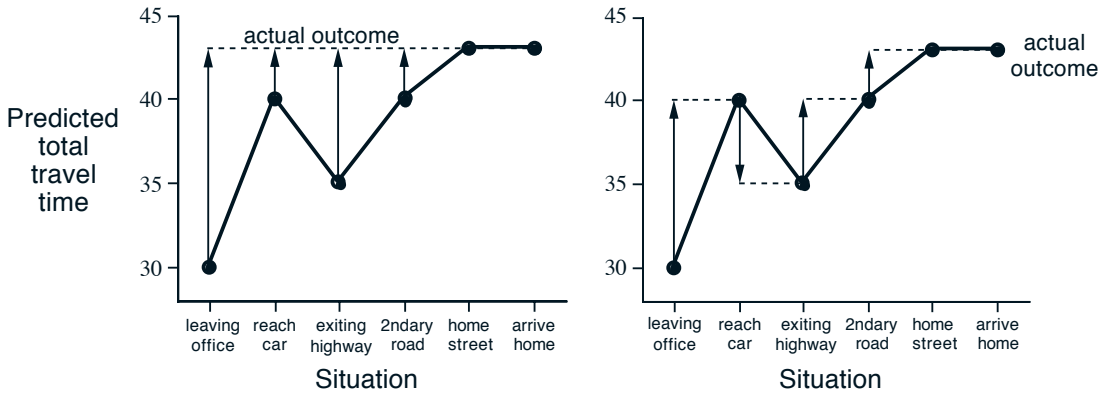

Figure 6.1: Changes recommended in the driving home example by Monte Carlo methods (left) and TD methods (right).

Then the predicted time to go after exiting the highway would be revised upward by four minutes as a result of this experience. This is probably too large a change in this case; the truck was probably just an unlucky break. In any event, the change can only be made off-line, that is, after you have reached home. Only at this point do you know any of the actual returns.

Is it necessary to wait until the final outcome is known before learning can begin? Suppose on another day you again estimate when leaving your office that it will take 30 minutes to drive home, but then you become stuck in a massive traffic jam. Twenty-five minutes after leaving the office you are still bumper-to-bumper on the highway. You now estimate that it will take another 25 minutes to get home, for a total of 50 minutes. As you wait in traffic, you already know that your initial estimate of 30 minutes was too optimistic. Must you wait until you get home before increasing your estimate for the initial state? According to the Monte Carlo approach you must, because you don't yet know the true return.

According to a TD approach, on the other hand, you would learn immediately, shifting your initial estimate from 30 minutes toward 50. In fact, each estimate would be shifted toward the estimate that immediately follows it. Returning to our first day of driving, Figure 6.1 (right) shows the changes in the predictions recommended by the TD rule (6.2) (these are the changes made by the rule if $\alpha = 1$). Each error is proportional to the change over time of the prediction, that is, to the *temporal differences* in predictions.

Besides giving you something to do while waiting in traffic, there are several computational reasons why it is advantageous to learn based on your current predictions rather than waiting until termination when you know the actual return. We briefly discuss some of these next. ■

6.2 Advantages of TD Prediction Methods

TD methods learn their estimates in part on the basis of other estimates. They learn a guess from a guess—they *bootstrap*. Is this a good thing to do? What advantages do TD methods have over Monte Carlo and DP methods? Developing and answering such questions will take the rest of this book and more. In this section we briefly anticipate some of the answers.

Obviously, TD methods have an advantage over DP methods in that they do not require a model of the environment, of its reward and next-state probability distributions.

The next most obvious advantage of TD methods over Monte Carlo methods is that they are naturally implemented in an on-line, fully incremental fashion. With Monte Carlo methods one must wait until the end of an episode, because only then is the return known, whereas with TD methods one need wait only one time step. Surprisingly often this turns out to be a critical consideration. Some applications have very long episodes, so that delaying all learning until an episode's end is too slow. Other applications are continuing tasks and have no episodes at all. Finally, as

we noted in the previous chapter, some Monte Carlo methods must ignore or discount episodes on which experimental actions are taken, which can greatly slow learning. TD methods are much less susceptible to these problems because they learn from each transition regardless of what subsequent actions are taken.

But are TD methods sound? Certainly it is convenient to learn one guess from the next, without waiting for an actual outcome, but can we still guarantee convergence to the correct answer? Happily, the answer is yes. For any fixed policy π, the TD algorithm described above has been proved to converge to v_π, in the mean for a constant step-size parameter if it is sufficiently small, and with probability 1 if the step-size parameter decreases according to the usual stochastic approximation conditions (2.7). Most convergence proofs apply only to the table-based case of the algorithm presented above (6.2), but some also apply to the case of general linear function approximation. These results are discussed in a more general setting in the next two chapters.

If both TD and Monte Carlo methods converge asymptotically to the correct predictions, then a natural next question is "Which gets there first?" In other words, which method learns faster? Which makes the more efficient use of limited data? At the current time this is an open question in the sense that no one has been able to prove mathematically that one method converges faster than the other. In fact, it is not even clear what is the most appropriate formal way to phrase this question! In practice, however, TD methods have usually been found to converge faster than constant-α MC methods on stochastic tasks, as illustrated in Example 6.2.

Exercise 6.1 This is an exercise to help develop your intuition about why TD methods are often more efficient than Monte Carlo methods. Consider the driving home example and how it is addressed by TD and Monte Carlo methods. Can you imagine a scenario in which a TD update would be better on average than a Monte Carlo update? Give an example scenario—a description of past experience and a current state—in which you would expect the TD update to be better. Here's a hint: Suppose you have lots of experience driving home from work. Then you move to a new building and a new parking lot (but you still enter the highway at the same place). Now you are starting to learn predictions for the new building. Can you see why TD updates are likely to be much better, at least initially, in this case? Might the same sort of thing happen in the original task?

Exercise 6.2 From Figure 6.2 (left) it appears that the first episode results in a change in only $V(\mathsf{A})$. What does this tell you about what happened on the first episode? Why was only the estimate for this one state changed? By exactly how much was it changed?

Exercise 6.3 The specific results shown in Figure 6.2 (right) are dependent on the value of the step-size parameter, α. Do you think the conclusions about which algorithm is better would be affected if a wider range of α values were used? Is there a different, fixed value of α at which either algorithm would have performed significantly better than shown? Why or why not?

*__Exercise 6.4__ In Figure 6.2 (right) the RMS error of the TD method seems to go

Example 6.2: Random Walk In this example we empirically compare the prediction abilities of TD(0) and constant-α MC applied to the small Markov reward process shown in the upper part of the figure below. All episodes start in the center state, C, and proceed either left or right by one state on each step, with equal probability. This behavior can be thought of as due to the combined effect of a fixed policy and an environment's state-transition probabilities, but we do not care which; we are concerned only with predicting returns however they are generated. Episodes terminate either on the extreme left or the extreme right. When an episode terminates on the right, a reward of +1 occurs; all other rewards are zero. For example, a typical epsiode might consist of the following state-and-reward sequence: $\mathsf{C}, 0, \mathsf{B}, 0, \mathsf{C}, 0, \mathsf{D}, 0, \mathsf{E}, 1$. Because this task is undiscounted, the true value of each state is the probability of terminating on the right if starting from that state. Thus, the true value of the center state is $v_\pi(\mathsf{C}) = 0.5$. The true values of all the states, A through E, are $\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}$, and $\frac{5}{6}$. The left part of Figure 6.2 shows the values learned by TD(0) approaching the true values as more episodes are experienced. Averaging over many episode sequences, the right part of the figure shows the average error in the predictions found by TD(0) and constant-α MC, for a variety of values of α, as a function of number of episodes. In all cases the approximate value function was initialized to the intermediate value $V(s) = 0.5$, for all s. The TD method was consistently better than the MC method on this task.

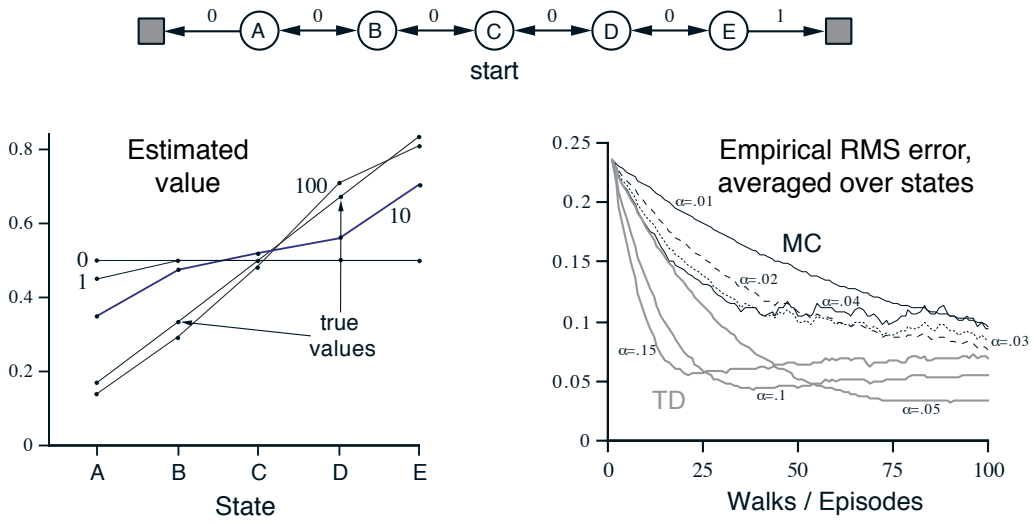

Figure 6.2: Results with the 5-state random walk. **Above:** The small Markov reward process generating the episodes. **Left:** Results from a single run after various numbers of episodes. The estimate after 100 episodes is about as close as they ever get to the true values; with a constant step-size parameter ($\alpha = 0.1$ in this example), the values fluctuate indefinitely in response to the outcomes of the most recent episodes. **Right:** Learning curves for TD(0) and constant-α MC methods, for various values of α. The performance measure shown is the root mean-squared (RMS) error between the value function learned and the true value function, averaged over the five states. These data are averages over 100 different sequences of episodes.

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down and then up again, particularly at high α's. What could have caused this? Do you think this always occurs, or might it be a function of how the approximate value function was initialized?

Exercise 6.5 Above we stated that the true values for the random walk task are $\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}$, and $\frac{5}{6}$, for states A through E. Describe at least two different ways that these could have been computed. Which would you guess we actually used? Why?

6.3 Optimality of TD(0)

Suppose there is available only a finite amount of experience, say 10 episodes or 100 time steps. In this case, a common approach with incremental learning methods is to present the experience repeatedly until the method converges upon an answer. Given an approximate value function, V, the increments specified by (6.1) or (6.2) are computed for every time step t at which a nonterminal state is visited, but the value function is changed only once, by the sum of all the increments. Then all the available experience is processed again with the new value function to produce a new overall increment, and so on, until the value function converges. We call this *batch updating* because updates are made only after processing each complete *batch* of training data.

Under batch updating, TD(0) converges deterministically to a single answer independent of the step-size parameter, α, as long as α is chosen to be sufficiently small. The constant-α MC method also converges deterministically under the same conditions, but to a different answer. Understanding these two answers will help us understand the difference between the two methods. Under normal updating the methods do not move all the way to their respective batch answers, but in some sense they take steps in these directions. Before trying to understand the two answers in general, for all possible tasks, we first look at a few examples.

Example 6.3: Random walk under batch updating Batch-updating versions of TD(0) and constant-α MC were applied as follows to the random walk prediction example (Example 6.2). After each new episode, all episodes seen so far were treated as a batch. They were repeatedly presented to the algorithm, either TD(0) or constant-α MC, with α sufficiently small that the value function converged. The resulting value function was then compared with v_π, and the average root mean-squared error across the five states (and across 100 independent repetitions of the whole experiment) was plotted to obtain the learning curves shown in Figure 6.3. Note that the batch TD method was consistently better than the batch Monte Carlo method.

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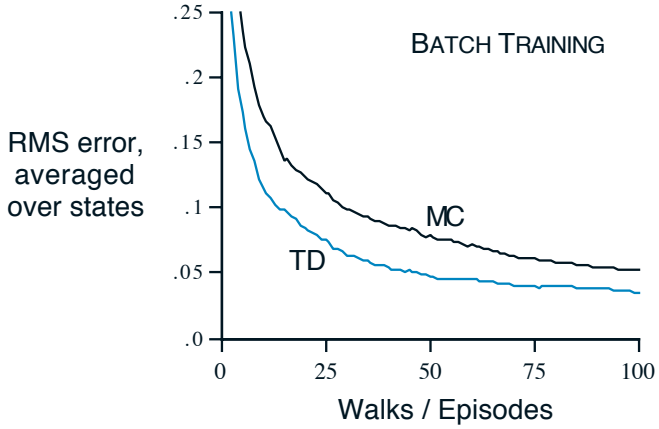


Figure 6.3: Performance of TD(0) and constant-α MC under batch training on the random walk task.

Under batch training, constant-α MC converges to values, $V(s)$, that are sample averages of the actual returns experienced after visiting each state s. These are optimal estimates in the sense that they minimize the mean-squared error from the actual returns in the training set. In this sense it is surprising that the batch TD method was able to perform better according to the root mean-squared error measure shown in Figure 6.3. How is it that batch TD was able to perform better than this optimal method? The answer is that the Monte Carlo method is optimal only in a limited way, and that TD is optimal in a way that is more relevant to predicting returns. But first let's develop our intuitions about different kinds of optimality through another example. Consider Example 6.4, on the next page.

Example 6.4 illustrates a general difference between the estimates found by batch TD(0) and batch Monte Carlo methods. Batch Monte Carlo methods always find the estimates that minimize mean-squared error on the training set, whereas batch TD(0) always finds the estimates that would be exactly correct for the maximum-likelihood model of the Markov process. In general, the *maximum-likelihood estimate* of a parameter is the parameter value whose probability of generating the data is greatest. In this case, the maximum-likelihood estimate is the model of the Markov process formed in the obvious way from the observed episodes: the estimated transition probability from i to j is the fraction of observed transitions from i that went to j, and the associated expected reward is the average of the rewards observed on those transitions. Given this model, we can compute the estimate of the value function that would be exactly correct if the model were exactly correct. This is called the *certainty-equivalence estimate* because it is equivalent to assuming that the estimate of the underlying process was known with certainty rather than being approximated. In general, batch TD(0) converges to the certainty-equivalence estimate.

This helps explain why TD methods converge more quickly than Monte Carlo methods. In batch form, TD(0) is faster than Monte Carlo methods because it computes the true certainty-equivalence estimate. This explains the advantage of TD(0)

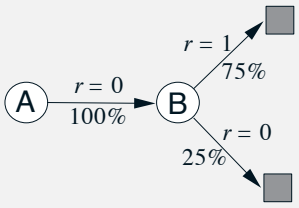
Example 6.4 You are the Predictor

Place yourself now in the role of the predictor of returns for an unknown Markov reward process. Suppose you observe the following eight episodes:

A, 0, B, 0	B, 1
B, 1	B, 1
B, 1	B, 1
B, 1	B, 0

This means that the first episode started in state A, transitioned to B with a reward of 0, and then terminated from B with a reward of 0. The other seven episodes were even shorter, starting from B and terminating immediately. Given this batch of data, what would you say are the optimal predictions, the best values for the estimates $V(\mathsf{A})$ and $V(\mathsf{B})$? Everyone would probably agree that the optimal value for $V(\mathsf{B})$ is $\frac{3}{4}$, because six out of the eight times in state B the process terminated immediately with a return of 1, and the other two times in B the process terminated immediately with a return of 0.

But what is the optimal value for the estimate $V(\mathsf{A})$ given this data? Here there are two reasonable answers. One is to observe that 100% of the times the process was in state A it traversed immediately to B (with a reward of 0); and since we have already decided that B has value $\frac{3}{4}$, therefore A must have value $\frac{3}{4}$ as well. One way of viewing this answer is that it is based on first modeling the Markov process, in this case as shown to the right, and then computing the correct estimates given the model, which indeed in this case gives $V(\mathsf{A}) = \frac{3}{4}$. This is also the answer that batch TD(0) gives.



The other reasonable answer is simply to observe that we have seen A once and the return that followed it was 0; we therefore estimate $V(\mathsf{A})$ as 0. This is the answer that batch Monte Carlo methods give. Notice that it is also the answer that gives minimum squared error on the training data. In fact, it gives zero error on the data. But still we expect the first answer to be better. If the process is Markov, we expect that the first answer will produce lower error on *future* data, even though the Monte Carlo answer is better on the existing data. ■

shown in the batch results on the random walk task (Figure 6.3). The relationship to the certainty-equivalence estimate may also explain in part the speed advantage of nonbatch TD(0) (e.g., Figure 6.2, right). Although the nonbatch methods do not achieve either the certainty-equivalence or the minimum squared-error estimates, they can be understood as moving roughly in these directions. Nonbatch TD(0) may be faster than constant-α MC because it is moving toward a better estimate, even though it is not getting all the way there. At the current time nothing more definite can be said about the relative efficiency of on-line TD and Monte Carlo methods.

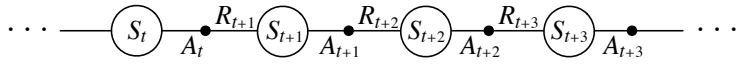
Finally, it is worth noting that although the certainty-equivalence estimate is in some sense an optimal solution, it is almost never feasible to compute it directly. If N is the number of states, then just forming the maximum-likelihood estimate of the process may require N^2 memory, and computing the corresponding value function requires on the order of N^3 computational steps if done conventionally. In these terms it is indeed striking that TD methods can approximate the same solution using memory no more than N and repeated computations over the training set. On tasks with large state spaces, TD methods may be the only feasible way of approximating the certainty-equivalence solution.

*__Exercise 6.6__ Design an off-policy version of the TD(0) update that can be used with arbitrary target policy π and covering behavior policy μ, using at each step t the importance sampling ratio ρ_t^{t+1} (5.3).

6.4 Sarsa: On-Policy TD Control

We turn now to the use of TD prediction methods for the control problem. As usual, we follow the pattern of generalized policy iteration (GPI), only this time using TD methods for the evaluation or prediction part. As with Monte Carlo methods, we face the need to trade off exploration and exploitation, and again approaches fall into two main classes: on-policy and off-policy. In this section we present an on-policy TD control method.

The first step is to learn an action-value function rather than a state-value function. In particular, for an on-policy method we must estimate $q_\pi(s,a)$ for the current behavior policy π and for all states s and actions a. This can be done using essentially the same TD method described above for learning v_π. Recall that an episode consists of an alternating sequence of states and state–action pairs:



In the previous section we considered transitions from state to state and learned the values of states. Now we consider transitions from state–action pair to state–action pair, and learn the values of state–action pairs. Formally these cases are identical: they are both Markov chains with a reward process. The theorems assuring the convergence of state values under TD(0) also apply to the corresponding algorithm for action values:

$$Q(S_t,A_t) \leftarrow Q(S_t,A_t) + \alpha\Big[R_{t+1} + \gamma Q(S_{t+1},A_{t+1}) - Q(S_t,A_t)\Big]. \tag{6.7}$$

This update is done after every transition from a nonterminal state S_t. If S_{t+1} is terminal, then $Q(S_{t+1}, A_{t+1})$ is defined as zero. This rule uses every element of the quintuple of events, $(S_t, A_t, R_{t+1}, S_{t+1}, A_{t+1})$, that make up a transition from one state–action pair to the next. This quintuple gives rise to the name *Sarsa* for the algorithm. The backup diagram for Sarsa is as shown to the right.


Sarsa

It is straightforward to design an on-policy control algorithm based on the Sarsa prediction method. As in all on-policy methods, we continually estimate q_π for the behavior policy π, and at the same time change π toward greediness with respect to q_π. The general form of the Sarsa control algorithm is given in the box below.

Sarsa: An on-policy TD control algorithm

Initialize $Q(s,a), \forall s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily, and $Q(\textit{terminal-state}, \cdot) = 0$
Repeat (for each episode):
  Initialize S
  Choose A from S using policy derived from Q (e.g., ϵ-greedy)
  Repeat (for each step of episode):
    Take action A, observe R, S'
    Choose A' from S' using policy derived from Q (e.g., ϵ-greedy)
    $Q(S,A) \leftarrow Q(S,A) + \alpha\big[R + \gamma Q(S',A') - Q(S,A)\big]$
    $S \leftarrow S'$; $A \leftarrow A'$;
  until S is terminal

The convergence properties of the Sarsa algorithm depend on the nature of the policy's dependence on Q. For example, one could use ε-greedy or ε-soft policies. According to Satinder Singh (personal communication), Sarsa converges with probability 1 to an optimal policy and action-value function as long as all state–action pairs are visited an infinite number of times and the policy converges in the limit to the greedy policy (which can be arranged, for example, with ε-greedy policies by setting $\varepsilon = 1/t$), but this result has not yet been published in the literature.

Example 6.5: Windy Gridworld Shown inset in Figure 6.4 is a standard gridworld, with start and goal states, but with one difference: there is a crosswind upward through the middle of the grid. The actions are the standard four—`up`, `down`, `right`, and `left`—but in the middle region the resultant next states are shifted upward by a "wind," the strength of which varies from column to column. The strength of the wind is given below each column, in number of cells shifted upward. For example, if you are one cell to the right of the goal, then the action `left` takes you to the cell just above the goal. Let us treat this as an undiscounted episodic task, with constant rewards of -1 until the goal state is reached.

The graph in Figure 6.4 shows the results of applying ε-greedy Sarsa to this task, with $\varepsilon = 0.1$, $\alpha = 0.5$, and the initial values $Q(s,a) = 0$ for all s, a. The increasing slope of the graph shows that the goal is reached more and more quickly over time. By 8000 time steps, the greedy policy was long since optimal (a trajectory from it is shown inset); continued ε-greedy exploration kept the average episode length at

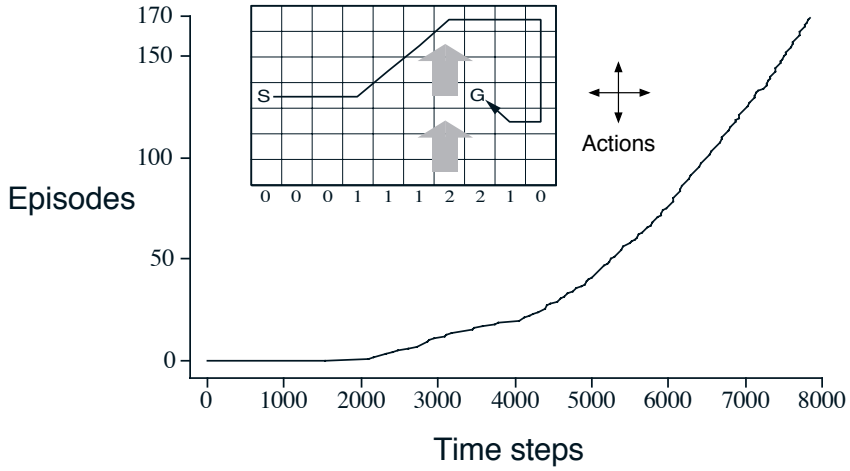


Figure 6.4: Results of Sarsa applied to a gridworld (shown inset) in which movement is altered by a location-dependent, upward "wind." A trajectory under the optimal policy is also shown.

about 17 steps, two more than the minimum of 15. Note that Monte Carlo methods cannot easily be used on this task because termination is not guaranteed for all policies. If a policy was ever found that caused the agent to stay in the same state, then the next episode would never end. Step-by-step learning methods such as Sarsa do not have this problem because they quickly learn *during the episode* that such policies are poor, and switch to something else. ■

Exercise 6.7: Windy Gridworld with King's Moves Re-solve the windy gridworld task assuming eight possible actions, including the diagonal moves, rather than the usual four. How much better can you do with the extra actions? Can you do even better by including a ninth action that causes no movement at all other than that caused by the wind?

Exercise 6.8: Stochastic Wind Re-solve the windy gridworld task with King's moves, assuming that the effect of the wind, if there is any, is stochastic, sometimes varying by 1 from the mean values given for each column. That is, a third of the time you move exactly according to these values, as in the previous exercise, but also a third of the time you move one cell above that, and another third of the time you move one cell below that. For example, if you are one cell to the right of the goal and you move `left`, then one-third of the time you move one cell above the goal, one-third of the time you move two cells above the goal, and one-third of the time you move to the goal.

6.5 Q-learning: Off-Policy TD Control

One of the early breakthroughs in reinforcement learning was the development of an off-policy TD control algorithm known as *Q-learning* (Watkins, 1989), defined by

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha\Big[R_{t+1} + \gamma \max_a Q(S_{t+1}, a) - Q(S_t, A_t)\Big]. \tag{6.8}$$

In this case, the learned action-value function, Q, directly approximates q_*, the optimal action-value function, independent of the policy being followed. This dramatically simplifies the analysis of the algorithm and enabled early convergence proofs. The policy still has an effect in that it determines which state–action pairs are visited and updated. However, all that is required for correct convergence is that all pairs continue to be updated. As we observed in Chapter 5, this is a minimal requirement in the sense that any method guaranteed to find optimal behavior in the general case must require it. Under this assumption and a variant of the usual stochastic approximation conditions on the sequence of step-size parameters, Q has been shown to converge with probability 1 to q_*. The Q-learning algorithm is shown in procedural form in the box below.

Q-learning: An off-policy TD control algorithm

Initialize $Q(s, a), \forall s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily, and $Q(\textit{terminal-state}, \cdot) = 0$
Repeat (for each episode):
  Initialize S
  Repeat (for each step of episode):
    Choose A from S using policy derived from Q (e.g., ϵ-greedy)
    Take action A, observe R, S'
    $Q(S, A) \leftarrow Q(S, A) + \alpha\big[R + \gamma \max_a Q(S', a) - Q(S, A)\big]$
    $S \leftarrow S'$
  until S is terminal

What is the backup diagram for Q-learning? The rule (6.8) updates a state–action pair, so the top node, the root of the backup, must be a small, filled action node. The backup is also *from* action nodes, maximizing over all those actions possible in the next state. Thus the bottom nodes of the backup diagram should be all these action nodes. Finally, remember that we indicate taking the maximum of these "next action" nodes with an arc across them (Figure 3.7-right). Can you guess now what the diagram is? If so, please do make a guess before turning to the answer in Figure 6.6.

Example 6.6: Cliff Walking This gridworld example compares Sarsa and Q-learning, highlighting the difference between on-policy (Sarsa) and off-policy (Q-learning) methods. Consider the gridworld shown in the upper part of Figure 6.5. This is a standard undiscounted, episodic task, with start and goal states, and the usual actions causing movement up, down, right, and left. Reward is -1 on all transitions except those into the region marked "The Cliff." Stepping into this region incurs a reward of -100 and sends the agent instantly back to the start.

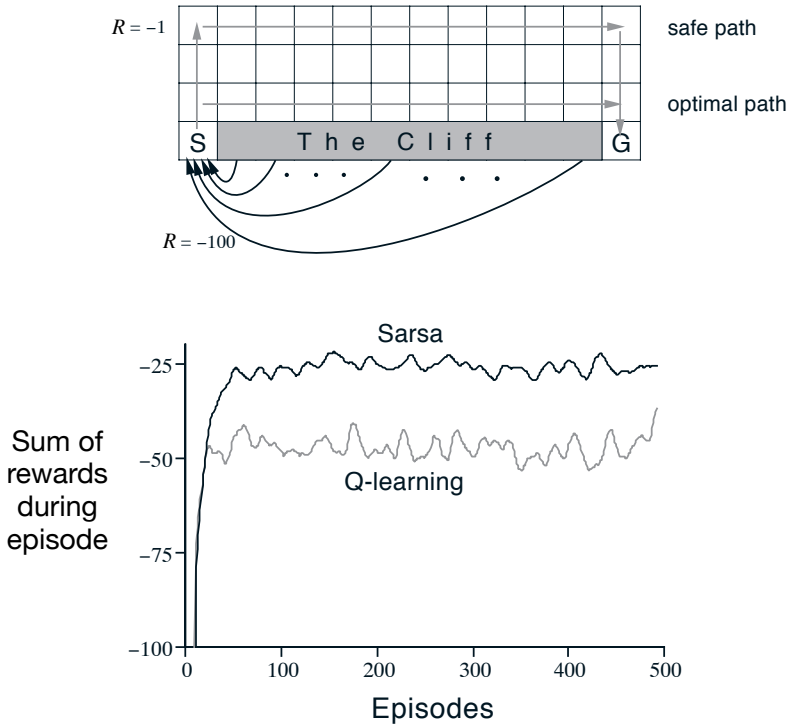


Figure 6.5: The cliff-walking task. The results are from a single run, but smoothed by averaging the reward sums from 10 successive episodes.

The lower part of Figure 6.5 shows the performance of the Sarsa and Q-learning methods with ε-greedy action selection, $\varepsilon = 0.1$. After an initial transient, Q-learning learns values for the optimal policy, that which travels right along the edge of the cliff. Unfortunately, this results in its occasionally falling off the cliff because of the ε-greedy action selection. Sarsa, on the other hand, takes the action selection into account and learns the longer but safer path through the upper part of the grid. Although Q-learning actually learns the values of the optimal policy, its online performance is worse than that of Sarsa, which learns the roundabout policy. Of course, if ε were gradually reduced, then both methods would asymptotically converge to the optimal policy. ■

Exercise 6.9 Why is Q-learning considered an *off-policy* control method?

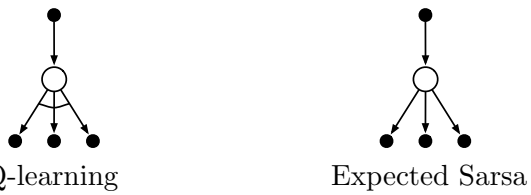


Figure 6.6: The backup diagrams for Q-learning and expected Sarsa.