Vector Space Model: Distance Functions

(Teaching Inspired by Research)

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1. Introduction
2. Distance Functions for Real Points
3. Distance Functions for Binary Points
4. Distance Functions for Sequences
5. Conclusion

1 Introduction

2 Distance Functions for Real Points

3 Distance Functions for Binary Points

4 Distance Functions for Sequences

5 Conclusion

Contents of the Course

Week	Lecture	Practical Exercises
1	**(05/04)** Introduction to Medical Information Retrieval (MIR)	**(05/04)** Introduction to Python
2	**(12/04)** Main Components and Classification of MIR Systems	**(12/04)** Introduction to Python
3	**(19/04)** Metadata in Medical Information Retrieval Systems	**(19/04)** CBIR in Medical Applications
4	***(26/04)*** *No Lecture due to a Business Trip*	**(26/04)** CBIR in Medical Applications
5	**(03/05)** Set Theoretic Model: Boolean Retrieval	**(03/05)** CBIR in Medical Applications
6	**(10/05)** Set Theoretic Model: Fuzzy Retrieval	**(10/05)** Flask Tutorial
7	**(17/05)** Vector Space Model: Similarity Measures	**(17/05)** Flask Tutorial

Contents of the Course

8	**(24/05)** Vector Space Model: Distance Functions	**(24/05)** HTML
9	**(31/05)** Vector Space Model: Latent Semantic Indexing	**(31/05)** HTML
10	**(07/06)** Probabilistic Model	**(07/06)** HTML
11	**(14/06)** Text-based Retrieval of Medical Information	**(14/06)** Deep Learning
12	**(21/06)** Audio-based Retrieval of Medical Information	**(21/06)** Deep Learning
13	**(28/06)** Image-based Retrieval of Medical Information	**(28/06)** Relevance Feedback
14	**(05/07)** Demonstrators from Current Research Projects	**(05/07)** Relevance Feedback
15	**(12/07)** Summary and Conclusions	**(12/07)** Evaluation

Generally about Distance Functions

A metric on a set $\mathbb{R}^l$ is a distance function

$$d : \mathbb{R}^l \times \mathbb{R}^l \longrightarrow [0, \infty) \quad ,$$

if for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^l$ all of the following conditions are satisfied:

(1) $d(\boldsymbol{x}, \boldsymbol{y}) \geq 0$;

(2) $d(\boldsymbol{x}, \boldsymbol{y}) = 0 \Leftrightarrow \boldsymbol{x} = \boldsymbol{y}$;

(3) $d(\boldsymbol{x}, \boldsymbol{y}) = d(\boldsymbol{y}, \boldsymbol{x})$;

(4) $d(\boldsymbol{x}, \boldsymbol{z}) \leq d(\boldsymbol{x}, \boldsymbol{y}) + d(\boldsymbol{y}, \boldsymbol{z})$.

1. Introduction
2. Distance Functions for Real Points
3. Distance Functions for Binary Points
4. Distance Functions for Sequences
5. Conclusion

Minkowski Distance – General Form

A popular metric extensively used for IR is the Minkowski distance:

$$d(\boldsymbol{x}, \boldsymbol{y}) = \left(\sum_{i=1}^{l} \omega_i |x_i - y_i|^p \right)^{\frac{1}{p}} \quad .$$

Minkowski Distance – Examples

Selected examples of the Minkowski distance for different values of p and $\omega_{i=1,\dots,l} = 1$:

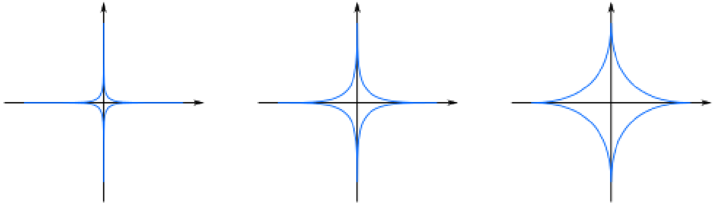
$$p = 1 \quad \Rightarrow \quad \sum_{i=1}^{l} |x_i - y_i|$$

$$\lim_{p \to \infty} \left(\sum_{i=1}^{l} |x_i - y_i|^p \right)^{\frac{1}{p}} = \max_{i=1,\dots,l} |x_i - y_i|$$

$$\lim_{p \to -\infty} \left(\sum_{i=1}^{l} |x_i - y_i|^p \right)^{\frac{1}{p}} = \min_{i=1,\dots,l} |x_i - y_i|$$

Minkowski Distance – Unit Circles

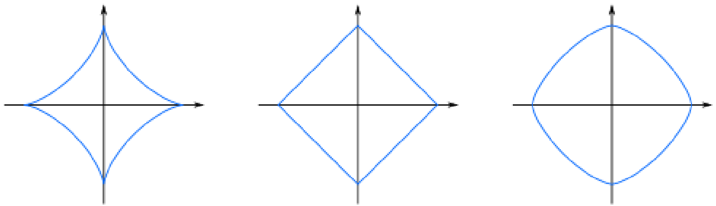
Assuming $\omega_{i=1,\ldots,l} = 1$:



$p = 2^{-2} = 0.25$ $\quad$ $p = 2^{-1.5} = 0.354$ $\quad$ $p = 2^{-1} = 0.5$

Minkowski Distance – Unit Circles

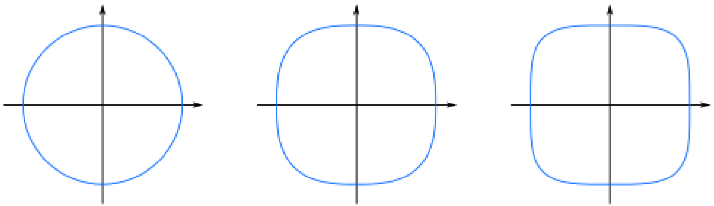
Assuming $\omega_{i=1,\ldots,l} = 1$:



$p = 2^{-0.5} = 0.707$ $\quad$ $p = 2^{0} = 1$ $\quad$ $p = 2^{0.5} = 1.414$

Minkowski Distance – Unit Circles

Assuming $\omega_{i=1,\ldots,l} = 1$:

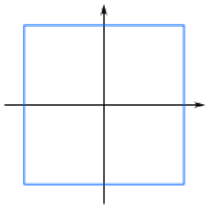


$p = 2^1 = 2$ | $p = 2^{1.5} = 2.828$ | $p = 2^2 = 4$

Minkowski Distance – Unit Circles

Assuming $\omega_{i=1,\ldots,l} = 1$:

$\ldots$



$p = 2^{\infty}$
$= \infty$

1. Introduction
2. Distance Functions for Real Points
3. Distance Functions for Binary Points
4. Distance Functions for Sequences
5. Conclusion

General Information

- Binary points store information about fulfilling or not fulfilling of properties.
- Graphically, such points can be represented as corner of a hypercube.

Comparison of Properties

$p \in P$	p fulfilled for $\boldsymbol{x}_1$	p not fulfilled for $\boldsymbol{x}_1$
p fulfilled for $\boldsymbol{x}_2$	$n_{1/1}$	$n_{0/1}$
p not fulfilled for $\boldsymbol{x}_2$	$n_{1/0}$	$n_{0/0}$

Example:

$$\boldsymbol{x}_1 = (0,0,0,0,1,1,1,1)^{\mathrm{T}} \quad \boldsymbol{x}_2 = (1,1,0,1,1,1,0,0)^{\mathrm{T}}$$

$$\Downarrow$$

$$n_{0/0} = 1 \quad n_{0/1} = 3 \quad n_{1/0} = 2 \quad n_{1/1} = 2$$

Minkowski Distance for Binary Points

General form assuming $\omega_{i=1,\ldots,l} = 1$:

$$d(\boldsymbol{x}_1, \boldsymbol{x}_2) = \left(\sum_{i=1}^{l} |x_{1,i} - x_{2,i}|^p \right)^{\frac{1}{p}} \quad .$$

For binary points:

$$d = (n_{1/0} + n_{0/1})^{1/p}$$

1 Introduction

2 Distance Functions for Real Points

3 Distance Functions for Binary Points

4 Distance Functions for Sequences

5 Conclusion

General Information

- A sequence is just a list of data elements of the same type.
- The number of data elements describing a document can vary.

Earth Mover's Distance – Data Format

- The following data type is considered here:
  $\mathrm{tuple}(\boldsymbol{x}_i \;:\; \mathrm{array}[1 \ldots l](\mathrm{real}) \;;\; w_{\boldsymbol{x}_i} \;:\; \mathrm{real})$.
- It describes the i-th element of the sequence $\boldsymbol{X}$.

Earth Mover's Distance – Story Behind

- Computing the distance between the sequence $\boldsymbol{X}$ with m elements and the sequence $\boldsymbol{Y}$ with n elements, we consider the elements of $\boldsymbol{X}$ to be mounds and the elements of $\boldsymbol{Y}$ to be holes in the ground.
- The points $\boldsymbol{x}_i$ and $\boldsymbol{y}_j$ can be interpreted as the positions of the mounds and the holes in a l-dimensional space.
- The volumes of the mounds/holes are given by $w_{\boldsymbol{x}_i}$ and $w_{\boldsymbol{y}_j}$ respectively.
- The distance between $\boldsymbol{X}$ and $\boldsymbol{Y}$ is defined as the minimum cost of transporting the earth from the mounds to the holes.

Earth Mover's Distance – Minimisation of Costs

- Thus, the goal is to minimise the transportation costs.
- A particular constellation of the whole earth transportation process between mounds and holes can be described by a matrix $F = [f_{ij}]$ with f_{ij} standing for the earth volume moved from the mound $\boldsymbol{x}_i$ into the hole $\boldsymbol{y}_j$.
- The overall transportation costs can be now computed as follows:

$$T_{\mathrm{costs}}(\boldsymbol{X}, \boldsymbol{Y}, F) = \sum_{i=1}^{m} \sum_{j=1}^{n} d(\boldsymbol{x}_i, \boldsymbol{y}_j) f_{ij} \qquad .$$

Earth Mover's Distance – Final Result

The final distance value:

$$d_{\text{EMD}}(\boldsymbol{X}, \boldsymbol{Y}) = \frac{\min\limits_{|f_{ij}|} \left(\sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} d(\boldsymbol{x}_i, \boldsymbol{y}_j) f_{ij} \right)}{\sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} f_{ij}} .$$

Earth Mover’s Distance

1. Introduction
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3. Distance Functions for Binary Points
4. Distance Functions for Sequences
5. Conclusion

Final Statements

- Most frequently, documents to be compared with each other for retrieval are described by feature vectors of the same dimensionality. In this case, well-known distance functions defined for real points, e.g., the Minkowski distance, can be applied.

- In more heterogeneous IR scenarios, the documents are represented by data structures with less consistency. In this case other methods (e.g., distance functions for sequences) are necessary for their comparison.