Zad 1

Sprawdzić, że

a) $\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$

b) $\sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = np$

a) Wzór dwumianowy Newtona: $\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = (x+y)^n$

Stąd $\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (1-p+p)^n = 1^n = 1$

b) $k\binom{n}{k} = n\binom{n-1}{k-1}$

$k=0 \Rightarrow \sum = 0$

$$\sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=1}^{n} n \binom{n-1}{k-1} p^k (1-p)^{n-k} =$$

$$= \sum_{k=1}^{n} n \binom{n-1}{k-1} p \cdot p^{k-1} (1-p)^{n-k} =$$

$$= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} =$$

$$= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-k-1} =$$

$$= np(1-p+p)^{n-1} = np \cdot 1^{n-1} = np$$

Zad 2

Sprawdzić, że

a) $\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = 1$ b) $\sum_{k=0}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{k!} = \lambda$

Szereg Maclaurina dla e^{λ}

a) $\sum_{k=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{k!} = e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} \cdot \left(\lambda^0 \cdot \frac{1}{0!} + \lambda^1 \cdot \frac{1}{1!} + \dots\right) =$

$= e^{-\lambda} \cdot e^{\lambda} = 1$

b) $\sum_{k=0}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \cdot \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} =$

$= \lambda \cdot e^{-\lambda} \cdot \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} =$

$= \lambda \cdot e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} =$

$= \lambda \cdot e^{-\lambda} \cdot e^{\lambda} = \lambda$

Zad 3

Funkcja Γ-Eulera:

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t} dt \qquad p > 0$$

Wykażemy, że $\Gamma(n) = (n-1)!$ dla $n \in \mathbb{N}$

Udowodnimy to indukcyjnie po n:

1. Baza indukcji

$$\Gamma(1) = \int_0^\infty t^0 e^{-t} dt = \int_0^\infty e^{-t} dt = -e^{-t}\Big|_0^\infty = 0 - (-1) = 1 = 0! \quad \checkmark$$

2. Krok indukcyjny

Załóżmy, że dla $n \in \mathbb{N}$ $\Gamma(n) = (n-1)!$. Pokażemy, że wtedy dla $\Gamma(n+1) = n! = n\Gamma(n)$

$$\Gamma(n+1) = \int_0^\infty t^n e^{-t} dt = \left| \begin{array}{l} u = t^n \\ du = nt^{n-1} \\ v = -e^{-t} \\ dv = e^{-t} dt \end{array} \right| = \overbrace{-\frac{t^n}{e^t}\Big|_0^\infty}^{0} + \int_0^\infty n t^{n-1} e^{-t} dt =$$

całk. mnożenia

$$= n\Gamma(n) = n \cdot (n-1)! = n! \qquad \blacksquare$$

Zad 4

$f(x) = \lambda \exp(-\lambda x), \quad \lambda > 0$ Oblicz:

a) $\int_0^\infty f(x)\,dx$ b) $\int_0^\infty x f(x)\,dx$

a) $$\int_0^\infty f(x)\,dx = \int_0^\infty \lambda \exp(-\lambda x)\,dx = \lambda \int_0^\infty \exp(-\lambda x)\,dx = \lambda \cdot \frac{-\exp(-\lambda x)}{\lambda}\Big|_0^\infty =$$

$$= -\exp(-\lambda x)\Big|_0^\infty = -\lim_{a\to\infty} \exp(-\lambda a) + \exp(0) = 0 + 1 = 1$$

b) $$\int_0^\infty x f(x)\,dx = \int_0^\infty x\lambda \exp(-\lambda x)\,dx = \lambda \int_0^\infty x \exp(-\lambda x)\,dx =$$

$$= \left| \begin{matrix} u = x & du = 1 \\ dv = \exp(-\lambda x) & v = -\frac{\exp(-\lambda x)}{\lambda} \end{matrix} \right| =$$

$$= \lambda \left[-\frac{x\exp(-\lambda x)}{\lambda}\Big|_0^\infty - \int_0^\infty -\frac{\exp(-\lambda x)}{\lambda} \right] =$$

$$= \lambda \left[-\frac{x\exp(-\lambda x)}{\lambda}\Big|_0^\infty - \left[\frac{\exp(-\lambda x)}{\lambda^2}\right]_0^\infty \right] =$$

$$= \left[-\frac{(\lambda x + 1)\exp(-\lambda x)}{\lambda} \right]_0^\infty = 0 - \left(-\frac{1}{\lambda}\right) = \frac{1}{\lambda}$$

Zad 5.

Wykazać, że $D_n = n$, gdzie

$$D_n = \begin{vmatrix} 1 & -1 & -1 & \dots & -1 \\ 1 & 1 & & & \\ 1 & & 1 & & \\ \vdots & & & \ddots & \\ 1 & & & & 1 \end{vmatrix} =$$

Dodajemy wszystkie kolejne wiersze do pierwszego

$$= \begin{vmatrix} n & 0 & 0 & \dots & 0 \\ 1 & 1 & & & \\ 1 & & 1 & & \\ \vdots & & & \ddots & \\ 1 & & & & 1 \end{vmatrix}$$

Otrzymaliśmy macierz dolnoprzekątniową. Jej wyznacznik jest równy iloczynowi elementów na przekątnej, czyli $n \cdot 1 \cdot 1 \cdot \ldots \cdot 1 = n$.

Zad 7

Symbol $\bar{s}$ oznacza średnią ciągu $s_1, \dots, s_n$. Uowodnić, że:

a) $\sum_{k=1}^{n} (x_k - \bar{x})^2 = \sum_{k=1}^{n} x_k^2 - n \cdot \bar{x}^2$

b) $\sum_{k=1}^{n} (x_k - \bar{x})(y_k - \bar{y}) = \sum_{k=1}^{n} x_k y_k - n\bar{x}\bar{y}$

a) $\sum_{k=1}^{n} (x_k - \bar{x})^2 = \sum_{k=1}^{n} (x_k^2 - 2x_k\bar{x} + \bar{x}^2) =$

$= \left[\sum_{k=1}^{n} x_k^2\right] - 2x_1\bar{x} - 2x_2\bar{x} - \dots - 2x_n\bar{x} + n \cdot \bar{x}^2 =$

$= \left[\sum_{k=1}^{n} x_k^2\right] - 2\bar{x} \, \frac{x_1 + x_2 + \dots + x_n}{n} \cdot n + n \cdot \bar{x}^2 =$

$= \left[\sum_{k=1}^{n} x_k^2\right] - 2\bar{x}^2 \cdot n + n \cdot \bar{x}^2 = \sum_{k=1}^{n} x_k^2 - n \cdot \bar{x}^2$

b) $\sum_{k=1}^{n} (x_k - \bar{x})(y_k - \bar{y}) = \sum_{k=1}^{n} (x_k y_k - \bar{x} y_k - \bar{y} x_k + \bar{x}\bar{y}) =$

$= \sum_{k=1}^{n} x_k y_k - \sum_{k=1}^{n} \bar{x} y_k - \sum_{k=1}^{n} x_k \bar{y} + \sum_{k=1}^{n} \bar{x}\bar{y} =$

$= \sum_{k=1}^{n} x_k y_k - \bar{x} \sum_{k=1}^{n} y_k - \bar{y} \sum_{k=1}^{n} x_k + \sum_{k=1}^{n} \bar{x}\bar{y} =$

$= \sum_{k=1}^{n} x_k y_k - \bar{x}\bar{y}n - \bar{y}\bar{x}n + n\bar{x}\bar{y} = \sum_{k=1}^{n} x_k y_k - n\bar{x}\bar{y}$

Zad 8

Dane są wektory $\vec{\mu}, X \in \mathbb{R}^n$ oraz macierz $\Sigma \in \mathbb{R}^{n \times n}$. Niech $S = (X-\vec{\mu})^T \cdot \Sigma^{-1}(X-\vec{\mu})$ oraz $Y = A \cdot X$, gdzie macierz A jest odwracalna.
Sprawdzić, że $S = (Y - A\vec{\mu})^T (A\Sigma A^T)^{-1} (Y - A\vec{\mu})$

Własności macierzy odwracalnych i transponowanych:

$(AB)^T = B^T A^T, \quad (ABC)^T = C^T B^T A^T$

$(AB)^{-1} = B^{-1} A^{-1} \quad (ABC)^{-1} = C^{-1} B^{-1} A^{-1}$

$$S = (Y - A\vec{\mu})^T (A\Sigma A^T)^{-1} (Y - A\vec{\mu}) =$$
$$= (AX - A\vec{\mu})^T (A\Sigma A^T)^{-1} (AX - A\vec{\mu}) =$$
$$= (A(X-\vec{\mu}))^T (A\Sigma A^T)^{-1} (A(X-\vec{\mu})) =$$
$$= (X-\vec{\mu})^T \underbrace{A^T (A^T)^{-1}} \Sigma^{-1} \underbrace{A^{-1} A} (X-\vec{\mu}) =$$
$$= (X-\vec{\mu})^T \Sigma^{-1} (X-\vec{\mu}) \quad \blacksquare$$