Zad 9

Udowodnić, że $\Gamma(p)\cdot\Gamma(q)=\Gamma(p+q)\cdot B(p,q)$, gdzie $p,q\in\mathbb{R}^+$ (czyli wszystkie potrzebne całki istnieją)

$$B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}\,dt,\qquad p,q>0$$

Dowód.

$$\Gamma(p+q)\cdot B(p,q)=\Gamma(p+q)\int_0^1 t^{q-1}(1-t)^{p-1}\,dt=\left|\begin{array}{l}\text{Symetryczność}\\ \text{funkcji beta}\\ B(p,q)=B(q,p)\end{array}\right.$$

$$=\Gamma(p+q)\left|\begin{array}{l}t=\frac{u}{1+u}\\ dt=\frac{1}{(1+u)^2}du\end{array}\right|=\Gamma(p+q)\int_0^\infty\left(\frac{u}{1+u}\right)^{q-1}\left(1-\frac{u}{1+u}\right)^{p-1}\frac{1}{(1+u)^2}\,du=$$

$$=\Gamma(p+q)\int_0^\infty u^{q-1}\left(\frac{1}{1+u}\right)^{p+q}du=\int_0^\infty v^{p+q-1}e^{-v}\,dv\int_0^\infty u^{q-1}\left(\frac{1}{1+u}\right)^{p+q}du=$$

$$=\int_0^\infty\int_0^\infty u^{q-1}\left(\frac{1}{1+u}\right)^{p+q}v^{p+q-1}e^{-v}\,dv\,du=\qquad\left|\begin{array}{l}\text{podst.}\\ s=\frac{v}{1+u}\\ ds=\frac{dv}{1+u}\end{array}\right.\quad du=ds(1+u)$$

$$=\int_0^\infty\int_0^\infty u^{q-1}s^{p+q-1}e^{-s(u+1)}\,ds\,du=$$

$$=\int_0^\infty s^p e^{-s}\int_0^\infty (us)^{q-1}e^{-su}\,du\,ds=\qquad\left|\begin{array}{l}\text{podst.}\\ u=\frac{t}{s}\\ du=\frac{1}{s}\cdot dt\end{array}\right.$$

$$=\int_0^\infty s^p e^{-s}\cdot\frac{1}{s}\Gamma(q)\,ds=$$

$$=\int_0^\infty s^{p-1}e^{-s}\Gamma(q)\,ds=$$

$$=\Gamma(p)\cdot\Gamma(q)$$

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