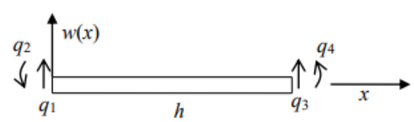
> *restart*;
with(*LinearAlgebra*) :

> **# TASK :**

- *To determine the shape function vector* **for** *a typical beam element of length h, allowing the representation of displacements w(x) at any point within the element using the nodal displacement vector q*
- *To determine the mass* **and** *stiffness matrices* **for** *the considered element*



$$w(x) = \mathbf{d}(x)^T \mathbf{q}, \quad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

Here, q1, q3 represent the transverse displacements at the nodes, and q2, q4 represent the nodal rotation angles.

> $n := 4$; # *number of nodal quantities*

$$n := 4 \qquad (1)$$

> $w := add(a[i] \cdot x^i, i = 0..n)$; # *Interpolation polynomial of degree n*
$w2 := diff(w, x)$;

$$w := a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$$

$$w2 := 4 x^3 a_4 + 3 x^2 a_3 + 2 x a_2 + a_1 \qquad (2)$$

> *#boundary conditions*
$con1 := eval(w, x = 0) = q[1]$;
$con2 := eval(w2, x = 0) = q[2]$;
$con3 := eval(w, x = h) = q[3]$;
$con4 := eval(w2, x = h) = q[4]$;

$$con1 := a_0 = q_1$$

$$con2 := a_1 = q_2$$

$$con3 := h^4 a_4 + h^3 a_3 + h^2 a_2 + h a_1 + a_0 = q_3$$

$$con4 := 4 h^3 a_4 + 3 h^2 a_3 + 2 h a_2 + a_1 = q_4 \tag{3}$$

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> solve({con1, con2, con3, con4}, {a[0], a[1], a[2], a[3]});
  assign(%);
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$$\left\{a_0 = q_1, a_1 = q_2, a_2 = \frac{h^4 a_4 - 2 h q_2 - h q_4 - 3 q_1 + 3 q_3}{h^2}, a_3 = -\frac{2 h^4 a_4 - h q_2 - h q_4 - 2 q_1 + 2 q_3}{h^3}\right\} \tag{4}$$

```
> d := Vector([coeff(w, q[1]), coeff(w, q[2]), coeff(w, q[3]), coeff(w, q[4])]);
     # shape function vector
  M := Matrix(1..n) : K := Matrix(1..n) : d_fun := unapply(d, x) : d_fun(x);
```

$$d := \begin{bmatrix} \frac{2 x^3}{h^3} - \frac{3 x^2}{h^2} + 1 \\ \frac{x^3}{h^2} - \frac{2 x^2}{h} + x \\ -\frac{2 x^3}{h^3} + \frac{3 x^2}{h^2} \\ \frac{x^3}{h^2} - \frac{x^2}{h} \end{bmatrix}$$

$$\begin{bmatrix} \frac{2 x^3}{h^3} - \frac{3 x^2}{h^2} + 1 \\ \frac{x^3}{h^2} - \frac{2 x^2}{h} + x \\ -\frac{2 x^3}{h^3} + \frac{3 x^2}{h^2} \\ \frac{x^3}{h^2} - \frac{x^2}{h} \end{bmatrix} \tag{5}$$

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> Mm := ρ·d_fun(x) • Transpose(d_fun(x))
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$$Mm := \left[\left[\rho\left(\frac{2 x^3}{h^3} - \frac{3 x^2}{h^2} + 1\right)^2, \rho\left(\frac{2 x^3}{h^3} - \frac{3 x^2}{h^2} + 1\right)\left(\frac{x^3}{h^2} - \frac{2 x^2}{h} + x\right), \rho\left(\frac{2 x^3}{h^3} - \frac{3 x^2}{h^2} + 1\right)\left(-\frac{2 x^3}{h^3} + \frac{3 x^2}{h^2}\right), \rho\left(\frac{2 x^3}{h^3} - \frac{3 x^2}{h^2} + 1\right)\left(\frac{x^3}{h^2} - \frac{x^2}{h}\right)\right], \right. \tag{6}$$

$$\left[\rho\left(\frac{2 x^3}{h^3} - \frac{3 x^2}{h^2} + 1\right)\left(\frac{x^3}{h^2} - \frac{2 x^2}{h} + x\right), \rho\left(\frac{x^3}{h^2} - \frac{2 x^2}{h} + x\right)^2, \rho\left(\frac{x^3}{h^2} - \frac{2 x^2}{h}\right.\right.$$

$$+x\right)\left(-\frac{2x^3}{h^3}+\frac{3x^2}{h^2}\right),\ \rho\left(\frac{x^3}{h^2}-\frac{2x^2}{h}+x\right)\left(\frac{x^3}{h^2}-\frac{x^2}{h}\right)\Big],$$

$$\Big[\rho\left(\frac{2x^3}{h^3}-\frac{3x^2}{h^2}+1\right)\left(-\frac{2x^3}{h^3}+\frac{3x^2}{h^2}\right),\ \rho\left(\frac{x^3}{h^2}-\frac{2x^2}{h}+x\right)\left(-\frac{2x^3}{h^3}+\frac{3x^2}{h^2}\right),\ \rho\left(-\frac{2x^3}{h^3}+\frac{3x^2}{h^2}\right)^2,\ \rho\left(-\frac{2x^3}{h^3}+\frac{3x^2}{h^2}\right)\left(\frac{x^3}{h^2}-\frac{x^2}{h}\right)\Big],$$

$$\Big[\rho\left(\frac{2x^3}{h^3}-\frac{3x^2}{h^2}+1\right)\left(\frac{x^3}{h^2}-\frac{x^2}{h}\right),\ \rho\left(\frac{x^3}{h^2}-\frac{2x^2}{h}+x\right)\left(\frac{x^3}{h^2}-\frac{x^2}{h}\right),\ \rho\left(-\frac{2x^3}{h^3}+\frac{3x^2}{h^2}\right)\left(\frac{x^3}{h^2}-\frac{x^2}{h}\right),\ \rho\left(\frac{x^3}{h^2}-\frac{x^2}{h}\right)^2\Big]\Big]$$

```
> # Determine the mass matrix M and stiffness matrix  K of a beam
    element based on the obtained shape functions
  for i from 1 to n do :
   for j from 1 to n do:
   M[i, j] := int(Mm[i, j], x = 0 ..h);
   K[i, j] := int(E·Ii·diff(d[i], x, x)·diff(d[j], x, x), x = 0 ..h);
   end do;
  end do;
> M, K;
```

$$\begin{bmatrix} \frac{13\rho h}{35} & \frac{11\rho h^2}{210} & \frac{9\rho h}{70} & -\frac{13\rho h^2}{420} \\ \frac{11\rho h^2}{210} & \frac{\rho h^3}{105} & \frac{13\rho h^2}{420} & -\frac{\rho h^3}{140} \\ \frac{9\rho h}{70} & \frac{13\rho h^2}{420} & \frac{13\rho h}{35} & -\frac{11\rho h^2}{210} \\ -\frac{13\rho h^2}{420} & -\frac{\rho h^3}{140} & -\frac{11\rho h^2}{210} & \frac{\rho h^3}{105} \end{bmatrix}, \tag{7}$$

$$\begin{bmatrix} \frac{12\,E\,Ii}{h^3} & \frac{6\,E\,Ii}{h^2} & -\frac{12\,E\,Ii}{h^3} & \frac{6\,E\,Ii}{h^2} \\ \frac{6\,E\,Ii}{h^2} & \frac{4\,E\,Ii}{h} & -\frac{6\,E\,Ii}{h^2} & \frac{2\,E\,Ii}{h} \\ -\frac{12\,E\,Ii}{h^3} & -\frac{6\,E\,Ii}{h^2} & \frac{12\,E\,Ii}{h^3} & -\frac{6\,E\,Ii}{h^2} \\ \frac{6\,E\,Ii}{h^2} & \frac{2\,E\,Ii}{h} & -\frac{6\,E\,Ii}{h^2} & \frac{4\,E\,Ii}{h} \end{bmatrix}$$

ρ(x) - mass density per unit length, E - Young's modulus, I(x) - moment of inertia of the cross-sectional area