```
> #
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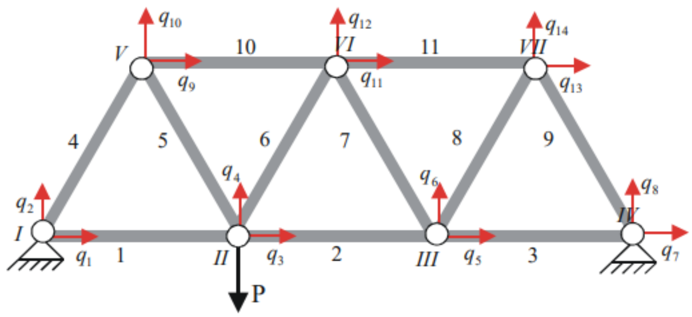
Static analysis of rod structures using the Finite Element Method $\cdot$ (FEM)



For the truss structure shown in the diagram, nodal displacements caused by the concentrated force P will be calculated using the Finite Element Method (FEM). The masses of the bars are treated as small and will be neglected. The stiffness matrix of the typical rod element used in the model is given by

$$\mathbf{k}^e = \begin{bmatrix} EA/h & 0 & -EA/h & 0 \\ 0 & 0 & 0 & 0 \\ -EA/h & 0 & EA/h & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where, h - lenght of the element, E - Young's modulus, A - cross-sectional area (treated as constant symbolic value)

```
> restart; with(LinearAlgebra) :
  # The stiffness matrix
```

$$ke := Matrix\left(\left[\left[\frac{E\cdot A}{h}, 0, -\frac{E\cdot A}{h}, 0\right], [0, 0, 0, 0], \left[-\frac{E\cdot A}{h}, 0, \frac{E\cdot A}{h}, 0\right], [0, 0, 0, 0]\right]\right);$$

$$ke := \begin{bmatrix} \frac{E\,A}{h} & 0 & -\frac{E\,A}{h} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{E\,A}{h} & 0 & \frac{E\,A}{h} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (1)$$

```
> n := 11; # number of elements
```

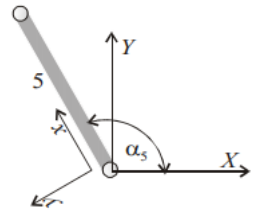
$$n := 11 \quad (2)$$

STEP 1 : *Transform the stiffness matrices of individual elements into the global coordinate system using the following relationship:*

$$\mathbf{K}_i^e = \mathbf{R}_i^T \, \mathbf{k}^e \, \mathbf{R}_i$$

where Ri denotes the element number i=1,...,n, and Ri is the transformation matrix in the form:

$$R_i = \begin{bmatrix} \cos(\alpha_i) & \sin(\alpha_i) & 0 & 0 \\ -\sin(\alpha_i) & \cos(\alpha_i) & 0 & 0 \\ 0 & 0 & \cos(\alpha_i) & \sin(\alpha_i) \\ 0 & 0 & -\sin(\alpha_i) & \cos(\alpha_i) \end{bmatrix}$$



in which αi represents the angle formed between the local reference frame of the i-th element and the global coordinate system. For example, this angle for element number 5 is $\alpha = \frac{\pi}{3}$.

> $\alpha[1] := 0; \alpha[2] := 0; \alpha[3] := 0; \alpha[4] := \frac{\mathrm{Pi}}{3}; \alpha[5] := \frac{2}{3}\mathrm{Pi}; \alpha[6] := \frac{\mathrm{Pi}}{3}; \alpha[7] := \frac{2}{3}\mathrm{Pi};$
> $\alpha[8] := \frac{\mathrm{Pi}}{3}; \alpha[9] := \frac{2}{3}\mathrm{Pi}; \alpha[10] := 0; \alpha[11] := 0;$

$$\alpha_1 := 0$$
$$\alpha_2 := 0$$
$$\alpha_3 := 0$$
$$\alpha_4 := \frac{\pi}{3}$$
$$\alpha_5 := \frac{2\pi}{3}$$
$$\alpha_6 := \frac{\pi}{3}$$
$$\alpha_7 := \frac{2\pi}{3}$$
$$\alpha_8 := \frac{\pi}{3}$$
$$\alpha_9 := \frac{2\pi}{3}$$
$$\alpha_{10} := 0$$
$$\alpha_{11} := 0 \qquad \textbf{(3)}$$

> **for** i **from** 1 **to** n **do** :
> $R[i] := Matrix([[\cos(\alpha[i]), \sin(\alpha[i]), 0, 0], [-\sin(\alpha[i]), \cos(\alpha[i]), 0, 0], [0, 0, \cos(\alpha[i]), \sin(\alpha[i])], [0, 0, -\sin(\alpha[i]), \cos(\alpha[i])]]);$
> **end do**;

$$R_1 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_2 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_3 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_4 := \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$R_5 := \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

$$R_6 := \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$R_7 := \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

$$R_8 := \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$R_9 := \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

$$R_{10} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_{11} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

> **for** i **from** 1 **to** n **do**:
> $Ke[i] := Transpose(R[i]) \cdot ke \cdot R[i]$;
> **end do**;

$$Ke_1 := \begin{bmatrix} \frac{E\,A}{h} & 0 & -\frac{E\,A}{h} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{E\,A}{h} & 0 & \frac{E\,A}{h} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Ke_2 := \begin{bmatrix} \frac{E\,A}{h} & 0 & -\frac{E\,A}{h} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{E\,A}{h} & 0 & \frac{E\,A}{h} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Ke_3 := \begin{bmatrix} \frac{E\,A}{h} & 0 & -\frac{E\,A}{h} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{E\,A}{h} & 0 & \frac{E\,A}{h} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Ke_4 := \begin{bmatrix} \frac{EA}{4h} & \frac{\sqrt{3}\,EA}{4h} & -\frac{EA}{4h} & -\frac{\sqrt{3}\,EA}{4h} \\ \frac{\sqrt{3}\,EA}{4h} & \frac{3EA}{4h} & -\frac{\sqrt{3}\,EA}{4h} & -\frac{3EA}{4h} \\ -\frac{EA}{4h} & -\frac{\sqrt{3}\,EA}{4h} & \frac{EA}{4h} & \frac{\sqrt{3}\,EA}{4h} \\ -\frac{\sqrt{3}\,EA}{4h} & -\frac{3EA}{4h} & \frac{\sqrt{3}\,EA}{4h} & \frac{3EA}{4h} \end{bmatrix}$$

$$Ke_5 := \begin{bmatrix} \frac{EA}{4h} & -\frac{\sqrt{3}\,EA}{4h} & -\frac{EA}{4h} & \frac{\sqrt{3}\,EA}{4h} \\ -\frac{\sqrt{3}\,EA}{4h} & \frac{3EA}{4h} & \frac{\sqrt{3}\,EA}{4h} & -\frac{3EA}{4h} \\ -\frac{EA}{4h} & \frac{\sqrt{3}\,EA}{4h} & \frac{EA}{4h} & -\frac{\sqrt{3}\,EA}{4h} \\ \frac{\sqrt{3}\,EA}{4h} & -\frac{3EA}{4h} & -\frac{\sqrt{3}\,EA}{4h} & \frac{3EA}{4h} \end{bmatrix}$$

$$Ke_6 := \begin{bmatrix} \frac{EA}{4h} & \frac{\sqrt{3}\,EA}{4h} & -\frac{EA}{4h} & -\frac{\sqrt{3}\,EA}{4h} \\ \frac{\sqrt{3}\,EA}{4h} & \frac{3EA}{4h} & -\frac{\sqrt{3}\,EA}{4h} & -\frac{3EA}{4h} \\ -\frac{EA}{4h} & -\frac{\sqrt{3}\,EA}{4h} & \frac{EA}{4h} & \frac{\sqrt{3}\,EA}{4h} \\ -\frac{\sqrt{3}\,EA}{4h} & -\frac{3EA}{4h} & \frac{\sqrt{3}\,EA}{4h} & \frac{3EA}{4h} \end{bmatrix}$$

$$Ke_7 := \begin{bmatrix} \frac{EA}{4h} & -\frac{\sqrt{3}\,EA}{4h} & -\frac{EA}{4h} & \frac{\sqrt{3}\,EA}{4h} \\ -\frac{\sqrt{3}\,EA}{4h} & \frac{3EA}{4h} & \frac{\sqrt{3}\,EA}{4h} & -\frac{3EA}{4h} \\ -\frac{EA}{4h} & \frac{\sqrt{3}\,EA}{4h} & \frac{EA}{4h} & -\frac{\sqrt{3}\,EA}{4h} \\ \frac{\sqrt{3}\,EA}{4h} & -\frac{3EA}{4h} & -\frac{\sqrt{3}\,EA}{4h} & \frac{3EA}{4h} \end{bmatrix}$$

$$Ke_8 := \begin{bmatrix} \frac{E\,A}{4\,h} & \frac{\sqrt{3}\,E\,A}{4\,h} & -\frac{E\,A}{4\,h} & -\frac{\sqrt{3}\,E\,A}{4\,h} \\ \frac{\sqrt{3}\,E\,A}{4\,h} & \frac{3\,E\,A}{4\,h} & -\frac{\sqrt{3}\,E\,A}{4\,h} & -\frac{3\,E\,A}{4\,h} \\ -\frac{E\,A}{4\,h} & -\frac{\sqrt{3}\,E\,A}{4\,h} & \frac{E\,A}{4\,h} & \frac{\sqrt{3}\,E\,A}{4\,h} \\ -\frac{\sqrt{3}\,E\,A}{4\,h} & -\frac{3\,E\,A}{4\,h} & \frac{\sqrt{3}\,E\,A}{4\,h} & \frac{3\,E\,A}{4\,h} \end{bmatrix}$$

$$Ke_9 := \begin{bmatrix} \frac{E\,A}{4\,h} & -\frac{\sqrt{3}\,E\,A}{4\,h} & -\frac{E\,A}{4\,h} & \frac{\sqrt{3}\,E\,A}{4\,h} \\ -\frac{\sqrt{3}\,E\,A}{4\,h} & \frac{3\,E\,A}{4\,h} & \frac{\sqrt{3}\,E\,A}{4\,h} & -\frac{3\,E\,A}{4\,h} \\ -\frac{E\,A}{4\,h} & \frac{\sqrt{3}\,E\,A}{4\,h} & \frac{E\,A}{4\,h} & -\frac{\sqrt{3}\,E\,A}{4\,h} \\ \frac{\sqrt{3}\,E\,A}{4\,h} & -\frac{3\,E\,A}{4\,h} & -\frac{\sqrt{3}\,E\,A}{4\,h} & \frac{3\,E\,A}{4\,h} \end{bmatrix}$$

$$Ke_{10} := \begin{bmatrix} \frac{E\,A}{h} & 0 & -\frac{E\,A}{h} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{E\,A}{h} & 0 & \frac{E\,A}{h} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Ke_{11} := \begin{bmatrix} \frac{E\,A}{h} & 0 & -\frac{E\,A}{h} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{E\,A}{h} & 0 & \frac{E\,A}{h} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

STEP 2 : *Assembly of the global stiffness matrix and force vector*

 # Numbering of nodal coordinates in individual elements

> $W := Matrix([[1, 2, 3, 4], [3, 4, 5, 6], [5, 6, 7, 8], [1, 2, 9, 10], [3, 4, 9, 10], [3, 4, 11, 12], [5, 6, 11, 12], [5, 6, 13, 14], [7, 8, 13, 14], [9, 10, 11, 12], [11, 12, 13, 14]]);$

$$W := \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 8 \\ 1 & 2 & 9 & 10 \\ 3 & 4 & 9 & 10 \\ 3 & 4 & 11 & 12 \\ 5 & 6 & 11 & 12 \\ 5 & 6 & 13 & 14 \\ 7 & 8 & 13 & 14 \\ 9 & 10 & 11 & 12 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{6}$$

11 × 4 Matrix

```
> m_e := 4;
    # number of coordinatesnumber of nodal coordinates in the
    element
  m_g := 14; # number of nodal coordinates in the entire truss.
```

$$m\_e := 4$$
$$m\_g := 14 \tag{7}$$

 # Creating logical matrices.

```
> for i from 1 to n do:
  b := Matrix(1 ..m_e, 1 ..m_g) :
  for j from 1 to m_e do:
  b[j, W[i,j]] := 1 :
  B[i] := b;
  end do;
  end do:
> B[1];
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix} \tag{8}$$

4 × 14 Matrix

 # Assembly of the global stiffness matrix

```
> K := add(Transpose(B[i]) · Ke[i] · B[i], i = 1 ..n);
```

$$K := \left[\begin{array}{ccccccccc}
\frac{5\,E\,A}{4\,h} & \frac{\sqrt{3}\,E\,A}{4\,h} & -\frac{E\,A}{h} & 0 & 0 & 0 & 0 & 0 & -\frac{E\,A}{4\,h} \\
\frac{\sqrt{3}\,E\,A}{4\,h} & \frac{3\,E\,A}{4\,h} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}\,E\,A}{4\,h} \\
-\frac{E\,A}{h} & 0 & \frac{5\,E\,A}{2\,h} & 0 & -\frac{E\,A}{h} & 0 & 0 & 0 & -\frac{E\,A}{4\,h} \\
0 & 0 & 0 & \frac{3\,E\,A}{2\,h} & 0 & 0 & 0 & 0 & \frac{\sqrt{3}\,E\,A}{4\,h} \\
0 & 0 & -\frac{E\,A}{h} & 0 & \frac{5\,E\,A}{2\,h} & 0 & -\frac{E\,A}{h} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{3\,E\,A}{2\,h} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{E\,A}{h} & 0 & \frac{5\,E\,A}{4\,h} & -\frac{\sqrt{3}\,E\,A}{4\,h} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}\,E\,A}{4\,h} & \frac{3\,E\,A}{4\,h} & 0 \\
-\frac{E\,A}{4\,h} & -\frac{\sqrt{3}\,E\,A}{4\,h} & -\frac{E\,A}{4\,h} & \frac{\sqrt{3}\,E\,A}{4\,h} & 0 & 0 & 0 & 0 & \frac{3\,E\,A}{2\,h} \\
-\frac{\sqrt{3}\,E\,A}{4\,h} & -\frac{3\,E\,A}{4\,h} & \frac{\sqrt{3}\,E\,A}{4\,h} & -\frac{3\,E\,A}{4\,h} & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right.$$

 # Introduction of the vector F

```
> F := Vector(1..m_g);
```

(10)

$$F := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \text{14 element Vector[column]} \tag{10}$$

Introduction of the load P

> F[4] := -P;

$$F_4 := -P \tag{11}$$

> F

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \text{14 element Vector[column]} \tag{12}$$

STEP 3 : Solution of the system of equations

Incorporating boundary conditions by removing rows and columns in matrix K corresponding to the constrained coordinates

> K_conds_included := K([3, 4, 5, 6, 9, 10, 11, 12, 13, 14], [3, 4, 5, 6, 9, 10, 11, 12, 13, 14]);

$$K\_conds\_included := \left[\left[\frac{5\,E\,A}{2\,h}, 0, -\frac{E\,A}{h}, 0, -\frac{E\,A}{4\,h}, \frac{\sqrt{3}\,E\,A}{4\,h}, -\frac{E\,A}{4\,h}, -\frac{\sqrt{3}\,E\,A}{4\,h}, 0, 0\right],\right. \tag{13}$$

$$\left[0, \frac{3\,E\,A}{2\,h}, 0, 0, \frac{\sqrt{3}\,E\,A}{4\,h}, -\frac{3\,E\,A}{4\,h}, -\frac{\sqrt{3}\,E\,A}{4\,h}, -\frac{3\,E\,A}{4\,h}, 0, 0\right],$$

$$\left[-\frac{EA}{h}, 0, \frac{5EA}{2h}, 0, 0, 0, -\frac{EA}{4h}, \frac{\sqrt{3}\,EA}{4h}, -\frac{EA}{4h}, -\frac{\sqrt{3}\,EA}{4h}\right],$$

$$\left[0, 0, 0, \frac{3EA}{2h}, 0, 0, \frac{\sqrt{3}\,EA}{4h}, -\frac{3EA}{4h}, -\frac{\sqrt{3}\,EA}{4h}, -\frac{3EA}{4h}\right],$$

$$\left[-\frac{EA}{4h}, \frac{\sqrt{3}\,EA}{4h}, 0, 0, \frac{3EA}{2h}, 0, -\frac{EA}{h}, 0, 0, 0\right],$$

$$\left[\frac{\sqrt{3}\,EA}{4h}, -\frac{3EA}{4h}, 0, 0, 0, \frac{3EA}{2h}, 0, 0, 0, 0\right],$$

$$\left[-\frac{EA}{4h}, -\frac{\sqrt{3}\,EA}{4h}, -\frac{EA}{4h}, \frac{\sqrt{3}\,EA}{4h}, -\frac{EA}{h}, 0, \frac{5EA}{2h}, 0, -\frac{EA}{h}, 0\right],$$

$$\left[-\frac{\sqrt{3}\,EA}{4h}, -\frac{3EA}{4h}, \frac{\sqrt{3}\,EA}{4h}, -\frac{3EA}{4h}, 0, 0, 0, \frac{3EA}{2h}, 0, 0\right],$$

$$\left[0, 0, -\frac{EA}{4h}, -\frac{\sqrt{3}\,EA}{4h}, 0, 0, -\frac{EA}{h}, 0, \frac{3EA}{2h}, 0\right],$$

$$\left[0, 0, -\frac{\sqrt{3}\,EA}{4h}, -\frac{3EA}{4h}, 0, 0, 0, 0, 0, \frac{3EA}{2h}\right]\Bigg]$$

```
> # Similar procedure with the force vector F
```

> $F\_conds\_included := F([3, 4, 5, 6, 9, 10, 11, 12, 13, 14]);$

$$F\_conds\_included := \begin{bmatrix} 0 \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \textbf{(14)}$$

```
# Solving the system of equations Kq=F, where q is the vector of
unknown nodal displacements.
```

> $q := LinearSolve(K\_conds\_included, F\_conds\_included);$

$$q := \begin{bmatrix} 0 \\ -\frac{70\,P\,h}{27\,E\,A} \\ \frac{\sqrt{3}\,P\,h}{9\,E\,A} \\ -\frac{41\,P\,h}{27\,E\,A} \\ \frac{11\sqrt{3}\,P\,h}{27\,E\,A} \\ -\frac{35\,P\,h}{27\,E\,A} \\ -\frac{\sqrt{3}\,P\,h}{27\,E\,A} \\ -\frac{19\,P\,h}{9\,E\,A} \\ -\frac{7\sqrt{3}\,P\,h}{27\,E\,A} \\ -\frac{19\,P\,h}{27\,E\,A} \end{bmatrix} \tag{15}$$

>