```
> with(RandomTools):with(plots):
> Task : To solve the Poisson equation
```

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \sin(\pi \cdot x)\sin(\pi \cdot y),\ x, y = [0, 3],\ where\ u(x, y) = 0 \ \textbf{for}\ x, y = 0, 3$$

using the collocation method utilizing global interpolation with radial basis functions (RBF). We choose the multiquadric radial basis function:

$$\phi_j(\mathbf{x}) = \sqrt{(x - x_j)^2 + (x - y_j)^2 + \sigma^2}$$

```
>                    # Step 1 : Discretization
> # Generation of source points

> N := 130:
  Xs_I := Generate(listlist(float(range = 0.1 .. 2.9), N, 2)); # internal source points
```

Xs_I := [[1.216563558, 0.7683894405], [0.1414604988, 1.786542192], [1.559597957, 0.9761316416], [2.498789390, 2.854454841], [1.014627637, 1.650027487], [1.841783793, 2.182327172], [2.871140352, 2.885292467], [2.276451851, 2.094739461], [0.9486133071, 2.225726350], [2.428238898, 2.542004087], [2.542211212, 1.386245962], [0.7165825295, 2.566940152], [0.2451916006, 1.114864595], [1.813641808, 0.8892030975], [0.2709549131, 1.754290724], [2.750683268, 2.412787762], [1.318017304, 2.655881026], [0.1274798275, 2.103398522], [1.890314128, 1.007613645], [1.883516713, 1.128866625], [0.2702847708, 2.440771035], [1.001317692, 2.018268552], [2.569232016, 0.4132445400], [1.275234885, 1.910673715], [0.9811093824, 0.1252108313], [2.428476501, 1.307062082], [2.813053157, 0.4797144724], [2.852690978, 0.6160690922], [1.828938940, 1.417026142], [1.679000400, 1.423311844], [1.959468095, 1.976720224], [2.589731328, 1.464290013], [0.1398188339, 0.2232254610], [0.3119746088, 0.7032509458], [0.9879313609, 0.2610718605], [1.752951908, 1.545131122], [2.858098957, 2.191203495], [2.416948607, 1.531197498], [0.1184820092, 0.4678705335], [0.6071038276, 0.6726549301], [1.573759768, 1.926473622], [0.5860034889, 2.286362429], [2.656426012, 0.2200634140], [1.636807555, 1.540106822], [0.2361342997, 2.151594257], [2.406474539, 1.041236273], [2.252218669, 2.415964254], [0.3739986313, 1.225734344], [1.785815368, 2.196265270], [1.396844897, 0.4094116720], [2.501349211, 2.271755003], [2.374817208, 1.158520451], [2.484536833, 2.463696731], [1.445757769, 2.215314398], [0.9175679917, 2.091272407], [1.105868042, 2.532520600], [2.530784154, 2.793132710], [1.234850669, 1.143356557], [1.733037916, 1.101665213], [1.693650460, 1.921347571], [2.642241498, 0.6940994106], [0.8645580788, 2.460086565], [1.768620826, 1.988921292], [2.366857011, 2.351647934], [1.128639186, 1.290597043], [1.602977508, 2.412629952], [2.174783129, 2.302440845], [0.2072303166, 2.732046766], [2.238329624, 2.559671642], [1.913936068, 0.8226613267], [2.869699007, 1.247640570], [0.4905695311, 0.4602455955], [0.9379782585, 0.4770515788], [2.552027039, 0.5169182346], [1.999625125, 0.4596074775], [1.428570522, 1.248227997], [1.430740043, 1.258563821], [2.340234032, **(1)** 2.581559096], [2.711656486, 0.7035722543], [1.907103384, 2.284004902], [2.040974119, 1.808747288], [0.6008319327, 0.4944817852], [0.6603866737, 1.555165951], [1.820569715, 0.9632539889], [1.903187397, 0.2689563162], [2.331888600, 2.176727634], [1.943150805, 1.506808038], [1.914803369, 1.510808949], [1.854336840, 1.529277960], [0.3623728343, 1.108417000], [2.614810453, 0.4878198506], [1.873308451, 0.3644918481], [1.745119150, 1.383263569], [2.095277987, 1.710755458], [1.284282221, 1.245314954], [2.049926964, 0.4136420082], [2.073004324, 2.662881355], [2.281146601, 0.9740984933], [1.977850472, 0.9924570134], [2.022908362, 0.7187554953], [2.798009604, 0.7514627911], [0.6874916524, 0.7866042074], [0.9661156729, 1.159902228], [1.004738331, 0.6749797774], [0.5668165519, 1.906101591], [0.7518811276, 0.7342718390], [2.226272463, 1.325326451], [0.5210357809, 0.8998549468], [0.9240428031, 0.8845936516], [2.048886995, 0.6015586810], [1.326857467, 1.320362007], [2.156510646, 2.560809473], [0.4154857385, 0.3518841063], [0.3419053996, 1.355948308], [2.507431036, 2.291001181], [0.2249681632, 2.813828714], [0.8049174100, 0.1238034272], [0.4403832724, 0.9252536567], [1.633136598, 1.150697867], [2.661718419, 0.5811208364], [0.5230215670, 0.1987712840], [1.644897988, 1.633452678], [2.146398882, 2.854011461], [2.271485987, 0.7468228084], [0.6033878930, 2.045201844], [0.8227076835, 1.161899632], [1.960070632, 0.6903496797], [1.447728906, 0.8427672331], [0.6709297458, 2.704358036], [0.9192159098, 2.292433980]]

> $a := 0. : b := 3. : n := 10 : h := \frac{(b-a)}{(n-1)} :$

$boundary\_points := seq(a + i \cdot h, i = 0..n-1);$

$boundary\_points := 0., 0.3333333333, 0.6666666666, 0.9999999999, 1.333333333, 1.666666666, 2.000000000, 2.333333333, 2.666666666, 3.000000000$ **(2)**

```
> X_Bx1:=[seq([boundary_points[i],a],i=1..n)];# points on the boundary x1
  X_Bx2:=[seq([boundary_points[i],b],i=1..n)];# points on the boundary x2
  X_By1:=[seq([a,boundary_points[i]],i=2..n-1)];# points on the boundary y1
  X_By2:=[seq([b,boundary_points[i]],i=2..n-1)];# points on the boundary y2
```

X_Bx1 := [[0., 0.], [0.3333333333, 0.], [0.6666666666, 0.], [0.9999999999, 0.], [1.333333333, 0.], [1.666666666, 0.], [2.000000000, 0.], [2.333333333, 0.], [2.666666666, 0.], [3.000000000, 0.]]

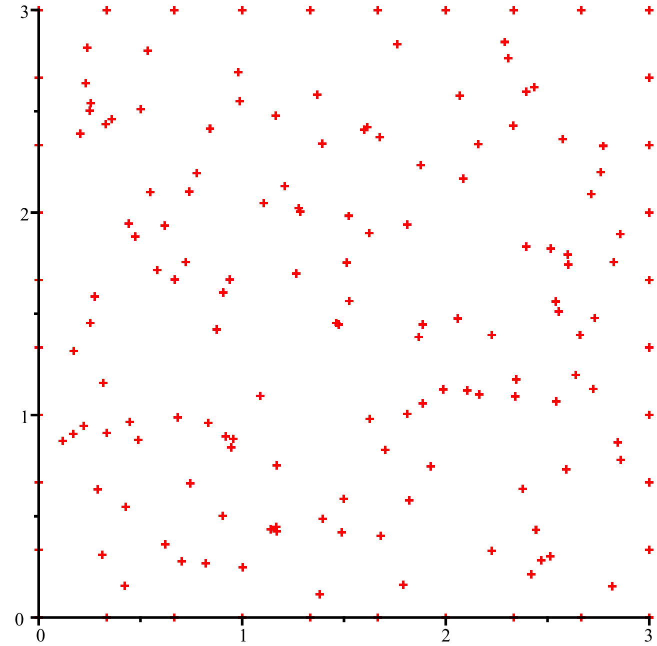
X_Bx2 := [[0., 3.], [0.3333333333, 3.], [0.6666666666, 3.], [0.9999999999, 3.], [1.333333333, 3.], [1.666666666, 3.], [2.000000000, 3.], [2.333333333, 3.], [2.666666666, 3.], [3.000000000, 3.]]

X_By1 := [[0., 0.3333333333], [0., 0.6666666666], [0., 0.9999999999], [0., 1.333333333], [0., 1.666666666], [0., 2.000000000], [0., 2.333333333], [0., 2.666666666]]

X_By2 := [[3., 0.3333333333], [3., 0.6666666666], [3., 0.9999999999], [3., 1.333333333], [3., 1.666666666], [3., 2.000000000], [3., 2.333333333], [3., 2.666666666]] **(3)**

```
> Xs:=[op(Xs_I),op(X_Bx1),op(X_Bx2),op(X_By1),op(X_By2)]: # list of all source points
  Ns:=nops(Xs); # number of source points
```

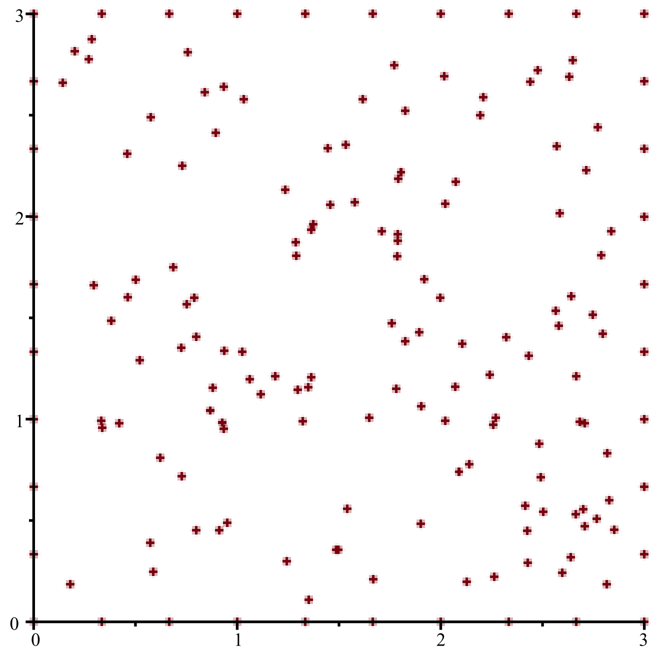
```
> p_Xs:=plot(Xs,style=point,symbol=cross,colour=red);
```



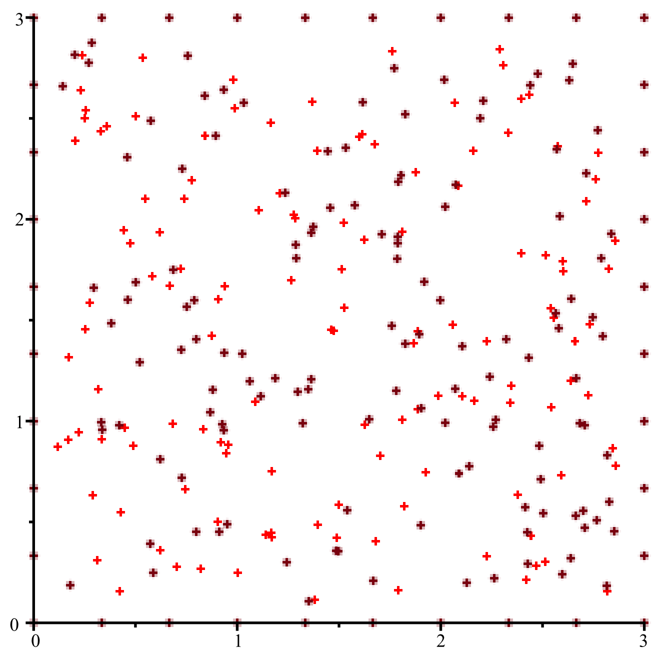
```
# Generation of collocation points
> Xc_I:= Generate(listlist(float(range = 0.1 .. 2.9), N, 2)): # list of internal collocation points
  Nc_I:=nops(Xc_I): # number of internal collocation points

> Xc_Bx1:=X_Bx1:
  Xc_Bx2:=X_Bx2:
  Xc_By1:=X_By1:
  Xc_By2:=X_By2:
> Xc_B:=[op(Xc_Bx1),op(Xc_Bx2),op(Xc_By1),op(Xc_By2)]; # list of collocation points on the boundaries
  Nc_B:=nops(Xc_B): # number of collocation points on the boundaries
```

$Xc\_B := [[0., 0.], [0.3333333333, 0.], [0.6666666666, 0.], [0.9999999999, 0.], [1.333333333, 0.], [1.666666666, 0.], [2.000000000, 0.], [2.333333333, 0.], [2.666666666, 0.], [3.000000000, 0.], [0., 3.], [0.3333333333, 3.], [0.6666666666, 3.], [0.9999999999, 3.], [1.333333333, 3.], [1.666666666, 3.], [2.000000000, 3.], [2.333333333, 3.], [2.666666666, 3.], [3.000000000, 3.], [0., 0.3333333333], [0., 0.6666666666], [0., 0.9999999999], [0., 1.333333333], [0., 1.666666666], [0., 2.000000000], [0., 2.333333333], [0., 2.666666666], [3., 0.3333333333], [3., 0.6666666666], [3., 0.9999999999], [3., 1.333333333], [3., 1.666666666], [3., 2.000000000], [3., 2.333333333], [3., 2.666666666]]$ **(4)**

```
> p_Xc:=plot([op(Xc_I),op(Xc_B)],style=point);
```



> display(p_Xs,p_Xc)



> # **Step 2 : Determination of the matrix ϕL**

> sigma := 1.9 :
phi := sqrt($(x-X)^2+(y-Y)^2+\sigma^2$);
L_phi := *diff* (phi, *x*, *x*) + *diff* (phi, *y*, *y*);
phi_L := (*Xc*, *Xs*) → *subs*({*x*=*Xc*[1], *y*=*Xc*[2], *X*=*Xs*[1], *Y*=*Xs*[2]}, *L_phi*);

$$\phi := \sqrt{(x-X)^2+(y-Y)^2+3.61}$$

$$L\_phi := -\frac{(2\,x-2\,X)^2}{4\left((x-X)^2+(y-Y)^2+3.61\right)^{3/2}}+\frac{2}{\sqrt{(x-X)^2+(y-Y)^2+3.61}}$$

$$- \frac{(2\,y - 2\,Y)^2}{4\,\left((x - X)^2 + (y - Y)^2 + 3.61\right)^{3/2}}$$

$$phi\_L := (Xc, Xs) \mapsto subs\left(\left\{X = Xs_1, Y = Xs_2, x = Xc_1, y = Xc_2\right\}, L\_phi\right) \qquad \textbf{(5)}$$

```
> ϕL := Matrix(Nc_I, Ns);
```

$$\phi L := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \qquad \textbf{(6)}$$

130 × 166 Matrix

```
> for i from 1 to Nc_I do:
   for j from 1 to Ns do :
    ϕL[i, j] := phi_L(Xc_I[i], Xs[j]);
   end do;
   end do;
```

```
> ϕL;
```

$$\begin{bmatrix} 0.5072247882 & 0.5347735493 & 0.6079815540 & 0.9640888628 & 0.6798796418 & 0.7581687522 & 0.68216446 \\ 0.6154835670 & 0.4091368104 & 0.6377520724 & 0.5233104739 & 0.4637874527 & 0.8805391988 & 0.40554263 \\ 1.049211731 & 0.7250137117 & 1.005722555 & 0.6203595948 & 0.7385161571 & 0.7057085526 & 0.60923565 \\ 0.9878815242 & 0.7216122503 & 0.8722575818 & 0.5125539587 & 0.6663408527 & 0.5475622829 & 0.55599763 \\ 0.8692676266 & 0.5389649435 & 0.8638378940 & 0.5887090226 & 0.5917245204 & 0.8561742495 & 0.49736982 \\ 0.4924154475 & 0.8438891937 & 0.5349487350 & 0.5787545906 & 0.7726579067 & 0.3861170330 & 0.86063415 \\ 0.5834982010 & 0.8402612676 & 0.6844045403 & 0.9070478869 & 0.9618622538 & 0.5598006467 & 1.03558961 \\ 1.040526869 & 0.8166383084 & 1.027345683 & 0.6617501757 & 0.8214105472 & 0.6823184155 & 0.68168794 \\ 0.5537231752 & 0.4968121013 & 0.6533121025 & 0.8625509022 & 0.6216976887 & 0.9385450908 & 0.58321580 \\ 0.6418045784 & 0.8469050942 & 0.7598305691 & 0.9720224766 & 0.9961184621 & 0.6324265830 & 1.01547654 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

```
>          # Step 3 : Determination of the matrix ϕB
```

> $B\_phi := diff(\text{phi}, [\ ]);$
$phi\_B := (Xc, Xs) \rightarrow subs(\ \{x = Xc[1], y = Xc[2], X = Xs[1], Y = Xs[2]\}, B\_phi\ );$

$$B\_phi := \sqrt{(x - X)^2 + (y - Y)^2 + 3.61}$$

$$phi\_B := (Xc, Xs) \mapsto subs\left(\{X = Xs_1, Y = Xs_2, x = Xc_1, y = Xc_2\}, B\_phi\right) \tag{8}$$

> $\phi B := Matrix(Nc\_B, Ns);$

$$\phi B := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \tag{9}$$

36 × 166 Matrix

> **for** i **from** 1 **to** Nc_B **do**:
  for j **from** 1 **to** Ns **do** :
   $\phi B[i, j] := phi\_B(Xc\_B[i], Xs[j]);$
  end do;
 end do;

> $\phi B;$

$$\begin{bmatrix} 3.481563843 & 3.207485445 & 3.138729588 & 2.282747791 & 2.810344827 & 2.755224425 & 2.693771227 & 2.64 \\ 3.313852038 & 2.972787013 & 2.979998005 & 2.166095465 & 2.608173069 & 2.721614136 & 2.480756658 & 2.55 \\ 3.172406988 & 2.758475048 & 2.851556595 & 2.096478922 & 2.435019034 & 2.728612761 & 2.296546535 & 2.49 \\ 3.060872276 & 2.569655382 & 2.757641046 & 2.078629452 & 2.297453124 & 2.775913168 & 2.148562451 & 2.48 \\ 2.982605287 & 2.412321537 & 2.701854083 & 2.113858829 & 2.202154908 & 2.861517510 & 2.044685170 & 2.52 \\ 2.940263889 & 2.292963985 & 2.686571990 & 2.199618176 & 2.154739856 & 2.982128975 & 1.991827428 & 2.60 \\ 2.935403105 & 2.217722862 & 2.712479465 & 2.330335452 & 2.158365973 & 3.133708061 & 1.994050598 & 2.71 \\ 2.968207080 & 2.191147593 & 2.778424529 & 2.498965533 & 2.212782348 & 3.312005649 & 2.051175588 & 2.86 \\ 3.037455739 & 2.214990536 & 2.881659782 & 2.698409971 & 2.314409210 & 3.512955855 & 2.158848540 & 3.04 \\ 3.140739355 & 2.287675807 & 3.018361403 & 2.922366548 & 2.457396282 & 3.732902155 & 2.310011898 & 3.25 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

```
>          # Step 4 : Composition of the ϕLB matrix
```
> $\phi LB := \langle \phi L, \phi B \rangle;$

$$\phi LB := \begin{bmatrix} 0.5072247882 & 0.5347735493 & 0.6079815540 & 0.9640888628 & 0.6798796418 & 0.7581687522 & 0.68 \\ 0.6154835670 & 0.4091368104 & 0.6377520724 & 0.5233104739 & 0.4637874527 & 0.8805391988 & 0.40 \\ 1.049211731 & 0.7250137117 & 1.005722555 & 0.6203595948 & 0.7385161571 & 0.7057085526 & 0.60 \\ 0.9878815242 & 0.7216122503 & 0.8722575818 & 0.5125539587 & 0.6663408527 & 0.5475622829 & 0.55 \\ 0.8692676266 & 0.5389649435 & 0.8638378940 & 0.5887090226 & 0.5917245204 & 0.8561742495 & 0.49 \\ 0.4924154475 & 0.8438891937 & 0.5349487350 & 0.5787545906 & 0.7726579067 & 0.3861170330 & 0.86 \\ 0.5834982010 & 0.8402612676 & 0.6844045403 & 0.9070478869 & 0.9618622538 & 0.5598006467 & 1.0 \\ 1.040526869 & 0.8166383084 & 1.027345683 & 0.6617501757 & 0.8214105472 & 0.6823184155 & 0.68 \\ 0.5537231752 & 0.4968121013 & 0.6533121025 & 0.8625509022 & 0.6216976887 & 0.9385450908 & 0.58 \\ 0.6418045784 & 0.8469050942 & 0.7598305691 & 0.9720224766 & 0.9961184621 & 0.6324265830 & 1.0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

```
> # Step 5 : Determination of the vector F of the right
    -hand side of the system of equations
```
> $f := Vector(Nc\_I)$:

> **for** i **from** 1 **to** Nc_I **do**:
$f[i] := \sin(\mathrm{Pi} \cdot Xc\_I[i][1]) \cdot \sin(\mathrm{Pi} \cdot Xc\_I[i][2]);$
end do:

> f;

$$\begin{bmatrix} 0.6892360394 \\ 0.4971291150 \\ -0.4952669515 \\ 0.8303036623 \\ -0.2273678161 \\ 0.4807027750 \\ -0.9314729678 \\ -0.03096922237 \\ -0.4702423923 \\ -0.8103108170 \\ \vdots \end{bmatrix} \tag{12}$$

130 element Vector[column]

> $g := Vector(Nc\_B);$

$$g := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \textbf{(13)}$$

36 element Vector[column]

> $\boldsymbol{F} := \langle \boldsymbol{f}, \boldsymbol{g} \rangle;$

$$F := \begin{bmatrix} 0.6892360394 \\ 0.4971291150 \\ -0.4952669515 \\ 0.8303036623 \\ -0.2273678161 \\ 0.4807027750 \\ -0.9314729678 \\ -0.03096922237 \\ -0.4702423923 \\ -0.8103108170 \\ \vdots \end{bmatrix} \quad \textbf{(14)}$$

166 × 1 Matrix

> ***# Step 6 : Solving the linear system of equations to determine the coefficients of the interpolating function, the sought solution c = ϕLB^{-1} F***

> $with(LinearAlgebra):$

> $c := convert\left(LinearSolve\left(\phi LB, F\right), Vector\right);$

$$c := \begin{bmatrix} -54078.8911423096 \\ -153044.935407393 \\ -40350.8988349145 \\ 97957.0807564921 \\ -32875.5014571857 \\ -63549.0742715161 \\ -44401.4684942760 \\ -8099.55216664741 \\ 14042.2267693890 \\ 376885.337977825 \\ \vdots \end{bmatrix} \quad (15)$$

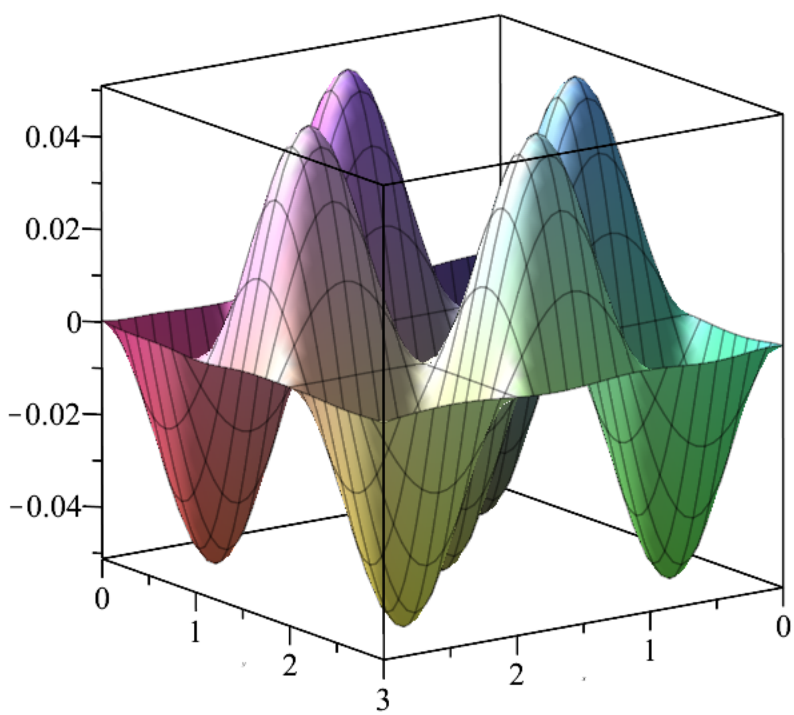
166 element Vector[column]

```
> # Step 7 : The creation of the interpolating function
     for the sought solution
```

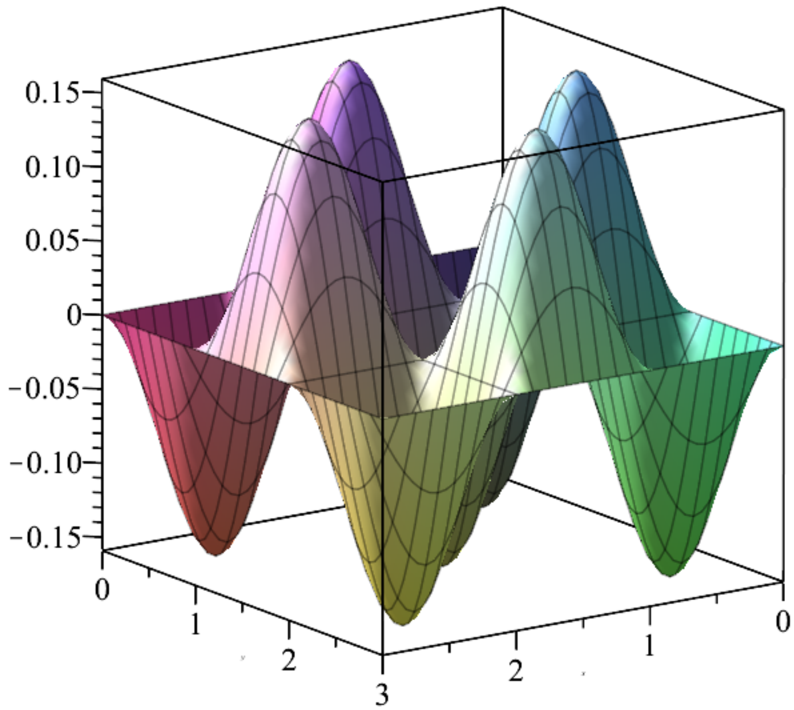
> $uh := add(c[j] \cdot subs(\{X = Xs[j][1], Y = Xs[j][2]\}, \phi), j = 1..Ns)$:

```
>   # Step 8 : The graphical assessment of solution quality
```

> $plot3d(uh, x = 0..3, y = 0..3);$



> $plot3d\left(-\frac{1}{2\cdot \mathrm{Pi}}\cdot \sin(\mathrm{Pi}\cdot x)\cdot \sin(\mathrm{Pi}\cdot y), x=0..3, y=0..3\right);$ `# exact solution`



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