> # Initial problem: $\frac{dy}{dx} = \frac{2y}{x} + x^2 \cos(x), \quad y(1) = 0$

> # range of x : [1;3], step-size : $h = \frac{x_n - x_0}{n}$, where n = 100

> # Explicit methods:

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ b_2 & c_{21} & 0 & 0 & 0 \\ b_3 & c_{31} & c_{32} & 0 & 0 \\ b_4 & c_{41} & c_{42} & c_{43} & 0 \\ \hline & a_1 & a_2 & a_3 & a_4 \end{array}$$

> # Classic fourth-order method:

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} \end{array}$$

> # 3/8—rule fourth—order method:

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{2}{3} & -\frac{1}{3} & 1 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 \\ \hline & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{array}$$

> # Runge-Kutta-Gill method:

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & \frac{1}{6} & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} \end{array}$$

> *#` `solving the problem using the classical method` `(the method can be changed* ***by*** *modifying the a, b vectors* ***and*** *c matrix)*

> $n := 100; X := Array(0..n); Y := Array(0..n);$

$$n := 100$$

$X :=$ [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,...,
⋯ 1 Array(0 .. 100) entries not shown]

$Y :=$ [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,...,
⋯ 1 Array(0 .. 100) entries not shown] **(1)**

> $X[0] := 1; X[n] := 3; h := \frac{X[n] - X[0]}{n};$

$$X_0 := 1$$

$$X_{100} := 3$$

$$h := \frac{1}{50} \tag{2}$$

> $f := (x, y) \rightarrow \frac{2 \cdot y}{x} + x^2 \cdot \cos(x);$

(3)

$$f := (x, y) \mapsto \frac{2 \cdot y}{x} + x^2 \cdot \cos(x) \quad \textbf{(3)}$$

> $a := \left[\frac{1}{6}, \frac{2}{6}, \frac{2}{6}, \frac{1}{6}\right]; b := \left[0, \frac{1}{2}, \frac{1}{2}, 1\right]; c := Matrix\left(\left[[0, 0, 0, 0], \left[\frac{1}{2}, 0, 0, 0\right], \left[0, \frac{1}{2}, 0, 0\right], [0, 0, 1, 0]\right]\right)$

$$a := \left[\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}\right]$$

$$b := \left[0, \frac{1}{2}, \frac{1}{2}, 1\right]$$

$$c := \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \textbf{(4)}$$

> $r := 4$ # fourth-order method

$$r := 4 \quad \textbf{(5)}$$

```
> for i from 0 to n − 1 do:
    k[1] := f(X[i], Y[i]);
    for j from 2 to r do:
      k[j] := f(X[i] + h·b[j], Y[i] + h·add(c[j, l]·k[l], l = 1 ..j − 1));
    end do;
    Y[i + 1] := evalf(X[i] + h);
    X[i + 1] := evalf(Y[i] + h·add(a[j]·k[j], j = 1 ..4));
  end do;
```

$$k_1 := \cos(1)$$

$$Y_1 := 1.020000000$$

$$X_1 := 0.01106677293$$

$$k_1 := 184.3357019$$

$$Y_2 := 0.03106677293$$

$$X_2 := 7.554831200$$

$$k_1 := 16.82885060$$

$$Y_3 := 7.574831200$$

$$X_3 := 0.3584394087$$

$$k_1 := 42.38592938$$

$$Y_4 := 0.3784394087$$

$$X_4 := 8.446397783$$

$$k_1 := -39.74521058$$

$$Y_5 := 8.466397783$$

$$X_5 := -0.4320793817$$
$$k_1 := -39.01954991$$
$$Y_6 := -0.4120793817$$
$$X_6 := 7.703854419$$
$$k_1 := 8.769544677$$
$$Y_7 := 7.723854419$$
$$X_7 := -0.2475771771$$
$$k_1 := -62.33610309$$
$$Y_8 := -0.2275771771$$
$$X_8 := 6.527363299$$
$$k_1 := 41.27287928$$
$$Y_9 := 6.547363299$$
$$X_9 := 0.6008248919$$
$$k_1 := 22.09235082$$
$$Y_{10} := 0.6208248919$$
$$X_{10} := 6.996824217$$
$$k_1 := 37.18711476$$
$$Y_{11} := 7.016824217$$
$$X_{11} := 1.362314643$$
$$k_1 := 10.68545146$$
$$Y_{12} := 1.382314643$$
$$X_{12} := 7.231898401$$
$$k_1 := 30.85923667$$
$$Y_{13} := 7.251898401$$
$$X_{13} := 1.994297798$$
$$k_1 := 5.638173042$$
$$Y_{14} := 2.014297798$$
$$X_{14} := 7.363994722$$
$$k_1 := 26.06773789$$
$$Y_{15} := 7.383994722$$
$$X_{15} := 2.528788148$$
$$k_1 := 0.608781629$$
$$Y_{16} := 2.548788148$$
$$X_{16} := 7.394230592$$

$$k_1 := 24.94990722$$
$$Y_{17} := 7.414230592$$
$$X_{17} := 3.040548538$$
$$k_1 := -4.320877263$$
$$Y_{18} := 3.060548538$$
$$X_{18} := 7.325529305$$
$$k_1 := 27.89252046$$
$$Y_{19} := 7.345529305$$
$$X_{19} := 3.612025348$$
$$k_1 := -7.562226978$$
$$Y_{20} := 3.632025348$$
$$X_{20} := 7.193138629$$
$$k_1 := 32.76773323$$
$$Y_{21} := 7.213138629$$
$$X_{21} := 4.282687537$$
$$k_1 := -4.272511229$$
$$Y_{22} := 4.302687537$$
$$X_{22} := 7.129791238$$
$$k_1 := 34.88592061$$
$$Y_{23} := 7.149791238$$
$$X_{23} := 4.996521089$$
$$k_1 := 9.860270498$$
$$Y_{24} := 5.016521089$$
$$X_{24} := 7.353055694$$
$$k_1 := 27.32970176$$
$$Y_{25} := 7.373055694$$
$$X_{25} := 5.556409973$$
$$k_1 := 25.72644875$$
$$Y_{26} := 5.576409973$$
$$X_{26} := 7.895109464$$
$$k_1 := -1.150264374$$
$$Y_{27} := 7.915109464$$
$$X_{27} := 5.540665103$$
$$k_1 := 25.47508117$$

$$Y_{28} := 5.560665103$$
$$X_{28} := 8.432136769$$
$$k_1 := -37.53630349$$
$$Y_{29} := 8.452136769$$
$$X_{29} := 4.794366429$$
$$k_1 := 5.408081474$$
$$Y_{30} := 4.814366429$$
$$X_{30} := 8.565377582$$
$$k_1 := -46.77571143$$
$$Y_{31} := 8.585377582$$
$$X_{31} := 3.863297800$$
$$k_1 := -6.759370854$$
$$Y_{32} := 3.883297800$$
$$X_{32} := 8.450101766$$
$$k_1 := -39.16978965$$
$$Y_{33} := 8.470101766$$
$$X_{33} := 3.084304323$$
$$k_1 := -4.004936501$$
$$Y_{34} := 3.104304323$$
$$X_{34} := 8.387790282$$
$$k_1 := -35.05755242$$
$$Y_{35} := 8.407790282$$
$$X_{35} := 2.387629726$$
$$k_1 := 2.887030436$$
$$Y_{36} := 2.407629726$$
$$X_{36} := 8.463934625$$
$$k_1 := -40.46751224$$
$$Y_{37} := 8.483934625$$
$$X_{37} := 1.582665426$$
$$k_1 := 10.69134216$$
$$Y_{38} := 1.602665426$$
$$X_{38} := 8.698587544$$
$$k_1 := -56.20726744$$
$$Y_{39} := 8.718587544$$

$$X_{39} := 0.463300448$$
$$k_1 := 37.82888184$$
$$Y_{40} := 0.4833004480$$
$$X_{40} := 9.491729726$$
$$k_1 := -89.78924995$$
$$Y_{41} := 9.511729726$$
$$X_{41} := -1.318735880$$
$$k_1 := -13.99180421$$
$$Y_{42} := -1.298735880$$
$$X_{42} := 9.234146417$$
$$k_1 := -84.00607374$$
$$Y_{43} := 9.254146417$$
$$X_{43} := -2.989257941$$
$$k_1 := -15.02378461$$
$$Y_{44} := -2.969257941$$
$$X_{44} := 8.956719971$$
$$k_1 := -72.25757893$$
$$Y_{45} := 8.976719971$$
$$X_{45} := -4.428000983$$
$$k_1 := -9.555716522$$
$$Y_{46} := -4.408000983$$
$$X_{46} := 8.783057808$$
$$k_1 := -62.79976855$$
$$Y_{47} := 8.803057808$$
$$X_{47} := -5.678848360$$
$$k_1 := 23.43699145$$
$$Y_{48} := -5.658848360$$
$$X_{48} := 9.264500841$$
$$k_1 := -85.95250808$$
$$Y_{49} := 9.284500841$$
$$X_{49} := -7.387887271$$
$$k_1 := 22.01524613$$
$$Y_{50} := -7.367887271$$
$$X_{50} := 9.731887476$$

$$k_1 := -91.79247304$$
$$Y_{51} := 9.751887476$$
$$X_{51} := -9.205326247$$
$$k_1 := -84.82450419$$
$$Y_{52} := -9.185326247$$
$$X_{52} := 8.066406318$$
$$k_1 := -15.99552954$$
$$Y_{53} := 8.086406318$$
$$X_{53} := -9.519420735$$
$$k_1 := -91.91275182$$
$$Y_{54} := -9.499420735$$
$$X_{54} := 6.254174756$$
$$k_1 := 36.06045782$$
$$Y_{55} := 6.274174756$$
$$X_{55} := -8.773123594$$
$$k_1 := -62.62591037$$
$$Y_{56} := -8.753123594$$
$$X_{56} := 5.036637687$$
$$k_1 := 4.606291909$$
$$Y_{57} := 5.056637687$$
$$X_{57} := -8.655013093$$
$$k_1 := -54.95889854$$
$$Y_{58} := -8.635013093$$
$$X_{58} := 3.972902701$$
$$k_1 := -14.98389556$$
$$Y_{59} := 3.992902701$$
$$X_{59} := -8.934687051$$
$$k_1 := -71.32580797$$
$$Y_{60} := -8.914687051$$
$$X_{60} := 2.580280675$$
$$k_1 := -12.54610971$$
$$Y_{61} := 2.600280675$$
$$X_{61} := -9.168611257$$
$$k_1 := -81.88751680$$

$$Y_{62} := -9.148611257$$
$$X_{62} := 0.973979660$$
$$k_1 := -18.25289640$$
$$Y_{63} := 0.9939796600$$
$$X_{63} := -9.517248529$$
$$k_1 := -90.39991752$$
$$Y_{64} := -9.497248529$$
$$X_{64} := -0.8079869450$$
$$k_1 := 23.95950585$$
$$Y_{65} := -0.7879869450$$
$$X_{65} := -9.024229347$$
$$k_1 := -74.81613444$$
$$Y_{66} := -9.004229347$$
$$X_{66} := -2.271250889$$
$$k_1 := 4.603830496$$
$$Y_{67} := -2.251250889$$
$$X_{67} := -8.910905615$$
$$k_1 := -68.64370917$$
$$Y_{68} := -8.890905615$$
$$X_{68} := -3.610072545$$
$$k_1 := -6.702822280$$
$$Y_{69} := -3.590072545$$
$$X_{69} := -9.023813743$$
$$k_1 := -74.17497446$$
$$Y_{70} := -9.003813743$$
$$X_{70} := -5.060522935$$
$$k_1 := 12.29477956$$
$$Y_{71} := -5.040522935$$
$$X_{71} := -8.764225098$$
$$k_1 := -59.50430451$$
$$Y_{72} := -8.744225098$$
$$X_{72} := -6.215642202$$
$$k_1 := 41.35973467$$
$$Y_{73} := -6.195642202$$

$$X_{73} := -7.922643326$$
$$k_1 := -2.742356456$$
$$Y_{74} := -7.902643326$$
$$X_{74} := -6.237628298$$
$$k_1 := 41.40149969$$
$$Y_{75} := -6.217628298$$
$$X_{75} := -7.080075104$$
$$k_1 := 36.79218512$$
$$Y_{76} := -7.060075104$$
$$X_{76} := -5.478704475$$
$$k_1 := 23.39307660$$
$$Y_{77} := -5.458704475$$
$$X_{77} := -6.599662676$$
$$k_1 := 43.04671307$$
$$Y_{78} := -6.579662676$$
$$X_{78} := -4.600190541$$
$$k_1 := 0.491267122$$
$$Y_{79} := -4.580190541$$
$$X_{79} := -6.573716534$$
$$k_1 := 42.79622601$$
$$Y_{80} := -6.553716534$$
$$X_{80} := -3.726933510$$
$$k_1 := -8.060725190$$
$$Y_{81} := -3.706933510$$
$$X_{81} := -6.714144300$$
$$k_1 := 42.06210950$$
$$Y_{82} := -6.694144300$$
$$X_{82} := -2.866907608$$
$$k_1 := -3.241087772$$
$$Y_{83} := -2.846907608$$
$$X_{83} := -6.756641928$$
$$k_1 := 41.47305227$$
$$Y_{84} := -6.736641928$$
$$X_{84} := -2.018190537$$

$$k_1 := 4.913830540$$
$$Y_{85} := -1.998190537$$
$$X_{85} := -6.637609633$$
$$k_1 := 41.92158972$$
$$Y_{86} := -6.617609633$$
$$X_{86} := -1.161764630$$
$$k_1 := 11.92914436$$
$$Y_{87} := -1.141764630$$
$$X_{87} := -6.381115909$$
$$k_1 := 40.88139951$$
$$Y_{88} := -6.361115909$$
$$X_{88} := -0.3284827834$$
$$k_1 := 38.83241679$$
$$Y_{89} := -0.3084827834$$
$$X_{89} := -5.608285554$$
$$k_1 := 24.66745166$$
$$Y_{90} := -5.588285554$$
$$X_{90} := 0.1774306836$$
$$k_1 := -62.96020940$$
$$Y_{91} := 0.1974306836$$
$$X_{91} := -6.918326859$$
$$k_1 := 38.47225115$$
$$Y_{92} := -6.898326859$$
$$X_{92} := 0.9680237808$$
$$k_1 := -13.72113952$$
$$Y_{93} := 0.9880237808$$
$$X_{93} := -7.175412268$$
$$k_1 := 32.04167858$$
$$Y_{94} := -7.155412268$$
$$X_{94} := 1.633195382$$
$$k_1 := -8.928800093$$
$$Y_{95} := 1.653195382$$
$$X_{95} := -7.335688591$$
$$k_1 := 26.20779706$$

$$Y_{96} := -7.315688591$$

$$X_{96} := 2.183722726$$

$$k_1 := -9.443427122$$

$$Y_{97} := 2.203722726$$

$$X_{97} := -7.506971081$$

$$k_1 := 18.57841876$$

$$Y_{98} := -7.486971081$$

$$X_{98} := 2.583783972$$

$$k_1 := -11.45933492$$

$$Y_{99} := 2.603783972$$

$$X_{99} := -7.719076905$$

$$k_1 := 7.339187916$$

$$Y_{100} := -7.699076905$$

$$X_{100} := 2.761492160 \quad \textbf{(6)}$$

```
> # exact solution
> unassign('x','y');
  ODE := diff(y(x), x) = 2·y(x)/x + x^2·cos(x);
```

$$ODE := \frac{d}{dx} y(x) = \frac{2\, y(x)}{x} + x^2 \cos(x) \quad \textbf{(7)}$$

```
> ICs := y(1) = 0;
```

$$ICs := y(1) = 0 \quad \textbf{(8)}$$

```
> sol := dsolve({ODE, ICs}, y(x))
```

$$sol := y(x) = (\sin(x) - \sin(1))\, x^2 \quad \textbf{(9)}$$

```
> # Relative perccent error between the obtained solution (
     for each set of coefficients) and the exact solution
> for i from 1 to n do:
    result := (evalf(eval(rhs(sol), x=X[i])) - Y[i]) / evalf(eval(rhs(sol), x=X[i])) · 100 :
   printf("%.8f %%\n", result);
   end do:
1003025.12700000 %
99.52302104 %
12116.07991000 %
144.66293220 %
3698.49940400 %
104.71430640 %
11697.71564000 %
99.10934347 %
6667.93543900 %
106.78578190 %
-2662.23504100 %
191.75686040 %
-2497.94447000 %
```

9.11064887 %
533.59451670 %
14.76383924 %
208.28772970 %
-157.86130120 %
143.48484670 %
235.00285610 %
122.46540300 %
191.56933480 %
115.89840180 %
-160.13388010 %
115.85816560 %
43.26481846 %
116.98911280 %
2055.82279200 %
120.00469480 %
178.10244160 %
138.29425170 %
489.83082360 %
213.53757140 %
-127.38833510 %
1039.76643000 %
254.19402380 %
-2037.48623600 %
111.93662800 %
10394.34960000 %
100.59055780 %
402.20113710 %
97.66393408 %
204.27143480 %
90.51728974 %
-286.80129600 %
76.47505674 %
199.89548330 %
90.33111131 %
109.80546750 %
93.19844880 %
110.86541910 %
203.76018190 %
111.94625230 %
72.10029797 %
105.62972890 %
80.71660092 %
104.39065120 %
65.38124119 %
103.81186570 %
-333.08157740 %
102.82527280 %
-67142.26940000 %
101.46486040 %
-829.92944480 %
99.21421898 %
-8.68392379 %
97.87312385 %
-74.95074403 %
96.42076017 %
456.79876960 %

```
95.48998237 %
70.75712873 %
94.63292944 %
74.48129273 %
92.03187427 %
-94.38699064 %
89.12741268 %
304.22989040 %
90.60322995 %
-63.26961451 %
93.46968460 %
26.80459543 %
95.19353818 %
5.11235414 %
96.18401695 %
-178.74340230 %
97.01458736 %
-4964.37545700 %
95.47302473 %
-26594.34156000 %
100.28749640 %
-41481.42783000 %
101.18460640 %
1813.22439100 %
101.79643490 %
-6427.37444600 %
102.19458680 %
-259.28626910 %
102.38483020 %
-114.60059220 %
```

```
>
```