> # Initial problem: $\frac{dy}{dx} = \frac{2y}{x} + x^2\cos(x), \quad y(1) = 0$

> # range of x : [1;3], step-size : $h = \frac{x_n - x_0}{n}$, where n = 100

> # Explicit methods:

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ b_2 & c_{21} & 0 & 0 & 0 \\ b_3 & c_{31} & c_{32} & 0 & 0 \\ b_4 & c_{41} & c_{42} & c_{43} & 0 \\ \hline & a_1 & a_2 & a_3 & a_4 \end{array}$$

> # Classic fourth-order method:

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} \end{array}$$

> # 3/8—rule fourth—order method:

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{2}{3} & -\frac{1}{3} & 1 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 \\ \hline & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{array}$$

> # Runge-Kutta-Gill method:

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & \frac{1}{6} & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} \end{array}$$

> #` `solving the problem using the classical method` `(the method can be changed **by** modifying the a, b vectors **and** c matrix)

> $n := 100; X := Array(0..n); Y := Array(0..n);$

$$n := 100$$

$X := [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$
$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$
$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots,$
$\cdots$ 1 Array(0 .. 100) entries not shown]

$Y := [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$ **(1)**
$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$
$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots,$
$\cdots$ 1 Array(0 .. 100) entries not shown]

> $X[0] := 1; X[n] := 3; h := \frac{X[n] - X[0]}{n};$

$$X_0 := 1$$

$$X_{100} := 3$$

$$h := \frac{1}{50} \qquad \textbf{(2)}$$

> $f := (x, y) \to \frac{2 \cdot y}{x} + x^2 \cdot \cos(x);$

(3)

$$f := (x, y) \mapsto \frac{2 \cdot y}{x} + x^2 \cdot \cos(x) \tag{3}$$

> $a := \left[\frac{1}{6}, \frac{2}{6}, \frac{2}{6}, \frac{1}{6}\right]$; $b := \left[0, \frac{1}{2}, \frac{1}{2}, 1\right]$; $c := Matrix\left(\left[[0, 0, 0, 0], \left[\frac{1}{2}, 0, 0, 0\right], \left[0, \frac{1}{2}, 0, 0\right], [0, 0, 1, 0]\right]\right)$

$$a := \left[\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}\right]$$

$$b := \left[0, \frac{1}{2}, \frac{1}{2}, 1\right]$$

$$c := \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{4}$$

> $r := 4$ `# fourth-order method`

$$r := 4 \tag{5}$$

```
> for i from 0 to n − 1 do:
    k[1] := f(X[i], Y[i]) :
    for j from 2 to r do:
      k[j] := f(X[i] + h·b[j], Y[i] + h·add(c[j, l]·k[l], l = 1 ..j − 1));
    end do:
    Y[i + 1] := evalf(X[i] + h);
    X[i + 1] := evalf(Y[i] + h·add(a[j]·k[j], j = 1 ..4));
  end do:
  X; Y;
```

[1, 0.01106677293, 7.554831200, 0.3584394087, 8.446397783, −0.4320793817, 7.703854419, −0.2475771771, 6.527363299, 0.6008248919, 6.996824217, 1.362314643, 7.231898401, 1.994297798, 7.363994722, 2.528788148, 7.394230592, 3.040548538, 7.325529305, 3.612025348, 7.193138629, 4.282687537, 7.129791238, 4.996521089, 7.353055694, 5.556409973, 7.895109464, 5.540665103, 8.432136769, 4.794366429, 8.565377582, 3.863297800, 8.450101766, 3.084304323, 8.387790282, 2.387629726, 8.463934625, 1.582665426, 8.698587544, 0.463300448, 9.491729726, −1.318735880, 9.234146417, −2.989257941, 8.956719971, −4.428000983, 8.783057808, −5.678848360, 9.264500841, −7.387887271, 9.731887476, −9.205326247, 8.066406318, −9.519420735, 6.254174756, −8.773123594, 5.036637687, −8.655013093, 3.972902701, −8.934687051, 2.580280675, −9.168611257, 0.973979660, −9.517248529, −0.8079869450, −9.024229347, −2.271250889, −8.910905615, −3.610072545, −9.023813743, −5.060522935, −8.764225098, −6.215642202, −7.922643326, −6.237628298, −7.080075104, −5.478704475, −6.599662676, −4.600190541, −6.573716534, −3.726933510, −6.714144300, −2.866907608, −6.756641928, −2.018190537, −6.637609633, −1.161764630,

$-6.381115909, -0.3284827834, -5.608285554, 0.1774306836, -6.918326859,$
$0.9680237808, -7.175412268, 1.633195382, -7.335688591, 2.183722726,$
$-7.506971081, 2.583783972, -7.719076905, \ldots,$ ⋯ 1 Array(0 .. 100) entries not shown]

[0, 1.020000000, 0.03106677293, 7.574831200, 0.3784394087, 8.466397783, −0.4120793817, 7.723854419, −0.2275771771, 6.547363299, 0.6208248919, 7.016824217, 1.382314643, 7.251898401, 2.014297798, 7.383994722, 2.548788148, 7.414230592, 3.060548538, 7.345529305, 3.632025348, 7.213138629, 4.302687537, 7.149791238, 5.016521089, 7.373055694, 5.576409973, 7.915109464, 5.560665103, 8.452136769, 4.814366429, 8.585377582, 3.883297800, 8.470101766, 3.104304323, 8.407790282, 2.407629726, 8.483934625, 1.602665426, 8.718587544, 0.4833004480, 9.511729726, −1.298735880, 9.254146417, −2.969257941, 8.976719971, −4.408000983, 8.803057808, −5.658848360, 9.284500841, −7.367887271, 9.751887476, −9.185326247, 8.086406318, −9.499420735, 6.274174756, −8.753123594, 5.056637687, −8.635013093, 3.992902701, −8.914687051, 2.600280675, −9.148611257, 0.9939796600, −9.497248529, −0.7879869450, −9.004229347, −2.251250889, −8.890905615, −3.590072545, −9.003813743, −5.040522935, −8.744225098, −6.195642202, −7.902643326, −6.217628298, −7.060075104, −5.458704475, −6.579662676, −4.580190541, −6.553716534, −3.706933510, −6.694144300, −2.846907608, −6.736641928, −1.998190537, −6.617609633, −1.141764630, −6.361115909, −0.3084827834, −5.588285554, 0.1974306836, −6.898326859, 0.9880237808, −7.155412268, 1.653195382, −7.315688591, 2.203722726, −7.486971081, 2.603783972, …, ⋯ 1 Array(0 .. 100) entries not shown] **(6)**

```
> # exact solution
```

```
> unassign('x','y');
  ODE := diff(y(x), x) = 2·y(x)/x + x^2·cos(x);
```

$$ODE := \frac{d}{dx}\, y(x) = \frac{2\, y(x)}{x} + x^2 \cos(x) \qquad \textbf{(7)}$$

```
> ICs := y(1) = 0;
```

$$ICs := y(1) = 0 \qquad \textbf{(8)}$$

```
> sol := dsolve({ODE, ICs}, y(x))
```

$$sol := y(x) = (\sin(x) - \sin(1))\, x^2 \qquad \textbf{(9)}$$

```
> # Relative perccent error between the obtained solution (
     for each set of coefficients) and the exact solution
```

```
> for i from 1 to n do:
    result := abs( (evalf(eval(rhs(sol), x=X[i])) - Y[i]) / evalf(eval(rhs(sol), x=X[i])) )·100 :
   printf("%.8f %%\n", result);
   end do:
```

```
1003025.12700000 %
99.52302104 %
12116.07991000 %
144.66293220 %
3698.49940400 %
```

104.71430640 %
11697.71564000 %
99.10934347 %
6667.93543900 %
106.78578190 %
2662.23504100 %
191.75686040 %
2497.94447000 %
9.11064887 %
533.59451670 %
14.76383924 %
208.28772970 %
157.86130120 %
143.48484670 %
235.00285610 %
122.46540300 %
191.56933480 %
115.89840180 %
160.13388010 %
115.85816560 %
43.26481846 %
116.98911280 %
2055.82279200 %
120.00469480 %
178.10244160 %
138.29425170 %
489.83082360 %
213.53757140 %
127.38833510 %
1039.76643000 %
254.19402380 %
2037.48623600 %
111.93662800 %
10394.34960000 %
100.59055780 %
402.20113710 %
97.66393408 %
204.27143480 %
90.51728974 %
286.80129600 %
76.47505674 %
199.89548330 %
90.33111131 %
109.80546750 %
93.19844880 %
110.86541910 %
203.76018190 %
111.94625230 %
72.10029797 %
105.62972890 %
80.71660092 %
104.39065120 %
65.38124119 %
103.81186570 %
333.08157740 %
102.82527280 %
67142.26940000 %

```
101.46486040 %
829.92944480 %
99.21421898 %
8.68392379 %
97.87312385 %
74.95074403 %
96.42076017 %
456.79876960 %
95.48998237 %
70.75712873 %
94.63292944 %
74.48129273 %
92.03187427 %
94.38699064 %
89.12741268 %
304.22989040 %
90.60322995 %
63.26961451 %
93.46968460 %
26.80459543 %
95.19353818 %
5.11235414 %
96.18401695 %
178.74340230 %
97.01458736 %
4964.37545700 %
95.47302473 %
26594.34156000 %
100.28749640 %
41481.42783000 %
101.18460640 %
1813.22439100 %
101.79643490 %
6427.37444600 %
102.19458680 %
259.28626910 %
102.38483020 %
114.60059220 %

>
```