CAS 741: SRS

Dynamical Systems: Multi-Pendulum Simulation

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Contents

Revision History

Table 1: Revision History

Date	Developer(s)	Change
October 4, 2018	Karol Serkis	First full draft
October 3, 2018	Karol Serkis	First revision and all content sections added
September 28, 2018	Karol Serkis	First draft of document in landscape orientation for presentation
September 26, 2018	Karol Serkis	SRS presentation slides discussed with Dr. Spencer Smith

Reference Material

This section records information for easy reference.

Notation

This section describes the notation conventions used in this document.

Mathematical Notation

The notation in this document follows the standard mathematical notation conventations. The standard mathematical spaces are used for the symbols in this document (see Table of Symbols). Example of mathematical notation use:

$$x_1 = l_1 \sin\theta_1 \qquad y_1 = -l_1 \cos\theta_1$$

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 \qquad y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2$$

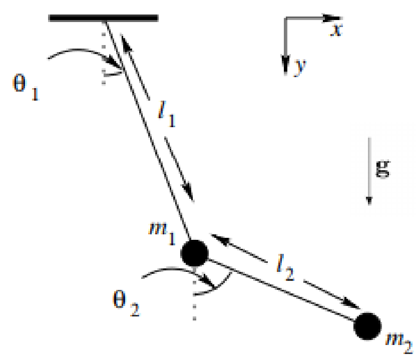

Figure 1: A simple gravity double pendulum (model assumes no friction or air resistance)[1]

Table of Units

Throughout this document SI (Système International d'Unités) is employedas the unit system. In addition to the basic units, several derived units are used as described below. For each unit, the symbol is given followed by a description of the unit and the SI name.

symbol	**unit**	**SI**
m	length	metre
kg	mass	kilogram
s	time	second

Table of Symbols

The table that follows summarizes the symbols used in this document along with their units. The choice of symbols was made to be consistent with calculus, ordinary differentials (ODE), the Lagrangian, kinematics etc. The standard mathematical spaces are used (e.g. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, etc.) as well as some additional spaces defined in the following table.

symbol	**space**	**unit**	**description**
g	$\mathbb{R}$	–	gravitational constant
m_1	$\mathbb{R}$	–	mass of the 1st pendulum weight
m_2	$\mathbb{R}$	–	mass of the 2nd pendulum weight
m_n	$\mathbb{R}$	–	mass of the nth pendulum weight
l_1	$\mathbb{R}$	–	length of the 1st pendulum rod
l_2	$\mathbb{R}$	–	length of the 2nd pendulum rod
l_n	$\mathbb{R}$	–	length of the nth pendulum rod
θ_1	$\mathbb{R}$	–	amplitude from the pivot point
θ_2	$\mathbb{R}$	–	amplitude from the 1st pendulum weight
θ_n	$\mathbb{R}$	–	amplitude from the nth pendulum weight
L	$\sum \mathbb{R}$	–	Pendulum system Lagrangian
T	$\sum \mathbb{R}$	–	Kinetic energy of system
V	$\sum \mathbb{R}$	–	Potential energy of system
$P(x)$	$\mathbb{Z} \times \mathbb{R} \implies \mathbb{R}$	–	Poincaré map P projects point x onto point P(x)

Abbreviations and Acronyms

The symbols are listed in alphabetical order.

symbol	description
A	Assumption
DD	Data Definition
GD	General Definition
GS	Goal Statement
IM	Instance Model
LC	Likely Change
NF	Non-Functional Requirement
PS	Physical System Description
R	Requirement
SRS	Software Requirements Specification
T	Theoretical Model

1 Introduction

This documents is an SRS for the Multi-Pendulum Simulation program. The directory for this project can be found at GitHub: /karolserkis/CAS-741-Pendula/

1.1 Purpose of Document

The purpose of this document is to describe the requirements for a Multi-Pendulum Simulation program solution that only focuses on multi-pendulum simulations (double & triple pendula and beyond)and tracking the chaotic motion of the system. It will allow users to generate diagrams (e.g. Poincare mapping) and plot trajectories over time using two different ODE/DAE initial value problem solvers. In the case of a double pendulum you have a new system that is dynamic and chaotic and requires a set of coupled ordinary differential equation solvers. Once one introduces multiple pendula the system becomes chaotic and interesting to model and simulate.

The theoretical models used in the Multi-Pendulum Simulation code will be provided, insuring assumptions and unambiguous definitions are identified. This document is intended to be used as a reference to provide all information necessary to understand and verify the inputs to outputs. The SRS is abstract: the contents describe the problem being solved, but not how to solve it.

This document will be used as a starting point for subsequent development phases, including writing the design specification and the software verification and validation plan. The verification and validation plan will show the steps inthe software documentation/implementation.

1.2 Scope of Requirements

The scope of the Multi-Pendulum Simulation program is limited to the generation of diagrams and plot trajectories that are possible to run and compute on a local system.

Assumptions: The Multi-Pendulum Simulation will be a closed system. Air resistance and friction will not be considered for the simulation. The Multi-Pendulum Simulation will be limited to the user initialized inputs and the output of the Multi-Pendulum Simulation will either plot trajectories over time, generate diagrams, like Poincare mapping and limit the user to a specific duration of the simulation, in order to allow diagrams and trajectory history to be saved. The user will be able to set a range of time and initialize the system.

1.3 Characteristics of Intended Reader

Simplification of some physical concepts are proposed to make the document technically accessible and also the software to be accessible. Nevertheless, the intended reader is expected to have a basic knowledge in mathematics (calculus, differentials/ODEs) and physics (kinematics, energy potential, Lagrangian, Poincare mapping) is recommended to get a deeper understanding of the document.

1.4 Organization of Document

- The organization of this document follows the template for an SRS for scientific computing software proposed by Dr. Spencer Smith.
- The presentation follows the standard pattern of presenting goals, theories, definitions, and assumptions.
- The goal statements are refined to the theoretical models, and the theoretical models to the instance models. The data definitions are used to support the definitions of the different models.

2 General System Description

This section identifies the interfaces between the system and its environment, describes the user characteristics and lists the system constraints.

2.1 System Context

$$\boxed{\textbf{USER}} \xLeftrightarrow[\textbf{Input with the User}]{\textbf{User enters input}} \boxed{\textbf{Multi-Pendulum Simulation}} \xLeftrightarrow[\textbf{Output to the User}]{\textbf{User sees results}} \boxed{\textbf{USER}}$$

- User Responsibilities:
  - Ensure that the input data is fits the system model.
  - Ensure that the input data is within scope.
- Multi-Pendulum Simulation program Responsibilities:
  - Detect data type mismatch, such as a string of characters instead of a floating point number.
  - Determine if the inputs satisfy the required physical and software constraints.
  - Solve the system of equations arising from the input data to generate the output data.
  - Generate a plot of the output data and generate diagrams to display to the user.

2.2 User Characteristics

The end user of Multi-Pendulum Simulation program should have an understanding of first year undergraduate math and physics. Less understanding of physics and math are required to use the software than understand this document or the inner workings of the software program.

2.3 System Constraints

There are no system constraints. The Multi-Pendulum Simulation software will be created with multi-platform support.

3 Specific System Description

This section first presents the problem description, which gives a high-level view of the problems to be solved and the motivation behind Multi-Pendulum Simulation software. This is followed by the solution characteristics specification, which presents the assumptions, theories, definitions and finally the instance models.

3.1 Problem Description

The Multi-Pendulum Simulation software will generate a plot trajectory in a 3D plot grid. A simple gravity pendulum has very easy to system to model and consists of a weight suspended from a pivot and the weight is given enough space to swing freely. To simplify the model we assume no air resistance with a frictionless pivot. The model and calculations for the simple gravity pendulum are well defined and only require simple derivations and differential solvers.

Multi-Pendulum Simulation program will produce a simulation given a set of equilibrium constantsand input. Terminologies and the physical system are described below.

3.1.1 Terminology and Definitions

This subsection provides a list of terms that are used in the subsequent sections and their meaning, with the purpose of reducing ambiguity and making it easier to correctly understand the requirements:

Equilibrium position: The pendulum rod and weight position in its resting state.

3D Cartesian coordinate system: The pendulum rod and weight swing from a pivot position origin (x, y, z)

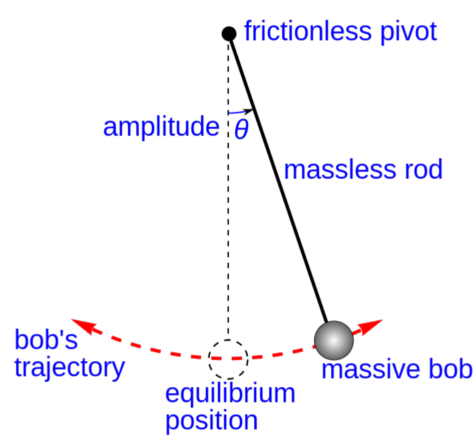

Figure 2: A simple gravity pendulum where the model assumes no friction or air resistance

Lagrangian: The $L = T - V$, where T and V are the kinetic and potential energies of the system respectively.

Poincaré map: Starting with the poincaré section of the Multi-Pendulum Simulation in 3D space, the poincaré map P is a projection from point x onto point P(x) transforming the 3D space into a 2D projection diagram plot.

3.1.2 Physical System Description

The physical system of Multi-Pendulum Simulation program includes the following elements:

PS1: Simulate an n-rod multi-pendulum system with no friction and no air resistance in a 3D space.

$$
\begin{aligned}
x_1 &= l_1 \sin\theta_1, & y_1 &= l_1 \cos\theta_1, \\
x_2 &= x_1 + l_2 \sin\theta_2, & y_2 &= y_1 + l_2 \cos\theta_2, \\
x_3 &= x_1 + x_2 + l_3 \sin\theta_3, & y_3 &= y_1 + y_2 + l_3 \cos\theta_3.
\end{aligned}
$$

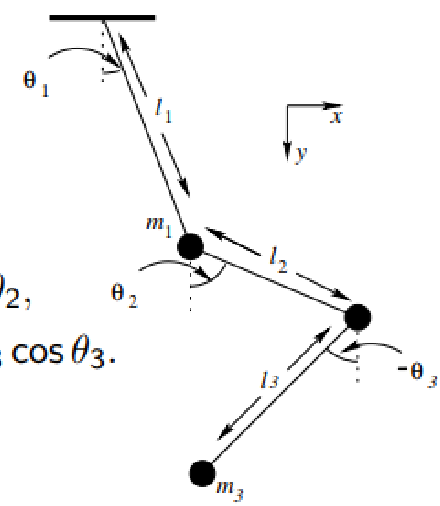

Figure 3: A triple pendulum example[1]

3.1.3 Goal Statements

Given the user input and the initial state of the Multi-Pendulum Simulation with reference to the table of symbols the goal statements are:

GS1: Generate a trajectory plot and a poincaré map plot of the movement of the pendula from equilibrium state of rest and show logged statistics over time to the user.

3.2 Solution Characteristics Specification

3.2.1 Assumptions

This section simplifies the original problem and helps in developing the theoretical model by filling in the missing information for the physical system. The numbers given in the square brackets refer to the theoretical model [T], general definition [GD], data definition [DD], instance model [IM], or likely change [LC], in which the respective assumption is used.

A1: All generated simulation diagrams will fit the mathematical model and scope.

A2: The user knows what the purpose of the simulation model and inputs weights and lengths according to possible simulation characteristics.

A3: In the model we assume no air resistance with a frictionless pivot.

3.2.2 Theoretical Models

This section focuses on the general equations and laws that Multi-Pendulum Simulation program is based on.

Number	T1
Label	**Double Pendulum Pivot rod**
Equation	$$x_1 = l_1 \sin\theta_1 \qquad y_1 = -l_1 \cos\theta_1$$ $$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 \qquad y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2$$
Description	Simple coordinate model system solution
Source	[6]
Ref. By	–

Number	T2
Label	**Double Pendulum Potential Energy**
Equation	$$\begin{aligned} T &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \\ &= \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) \\ &= \frac{1}{2}m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)\right] \end{aligned}$$
Description	Potential Energy model system solution
Source	[6]
Ref. By	–

Number	T3
Label	**Double Pendulum Potential Energy**
Equation	$$\begin{aligned} V &= m_1 g y_1 + m_2 g y_2 \\ &= -m_1 g l_1 \cos\theta_1 - m_2 g(l_1 \cos\theta_1 + l_2 \cos\theta_2) \\ &= -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2 \end{aligned}$$
Description	Potential Energy model system solution
Source	[6]
Ref. By	–

Number	T4
Label	**Double Pendulum Lagrangian ($L = T - V$)**
Equation	$$\begin{aligned} L = \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \\ + (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2 \end{aligned}$$
Description	Lagrangian model system solution
Source	[6]
Ref. By	–

Number	T5
Label	**Poincaré map($P(x)$)**
Equation	$$P(x) : \mathbb{Z} \times \mathbb{R} \implies \mathbb{R}$$
Description	Poincaré map P projects point x onto point P(x)
Source	[6]
Ref. By	–

3.2.3 General Definitions

We will use the Lagrangian and ODEs. No need for general definitions in current documentation.

3.2.4 Data Definitions

This section collects and defines all the data needed to build the instance models. The models here are to satisfy the theoretical models constrained and closed 3D space.

Label	**Closed, Real intervals**
Equation	$$\mathbb{R}(\texttt{for Langrangian equation})$$ $$P(x) : \mathbb{Z} \times \mathbb{R} \implies \mathbb{R}$$
Description	Simple cartesian coordinate model system solution
Source	[6]
Ref. By	T1,T2,T3,T4,T5

3.2.5 Instance Models

This section transforms the problem defined in problem description into one which is expressed in mathematical terms.

Label	**Addition of closed, real intervals**
Equation	$$\sum \mathbb{R}(\texttt{for Langrangian equation})$$ $$P(x) : \mathbb{Z} \times \mathbb{R} \implies \mathbb{R}$$
Description	Simple cartesian coordinate model system solution
Source	[6]
Ref. By	T1,T2,T3,T4,T5

3.2.6 Data Constraints

The data constraints on the input and output variables, respectively. The column for physical constraints gives the physical limitations on the range of values that can be taken by the variable. The column for software constraints restricts the range of inputs to reasonable values. The constraints are conservative, to give the user of the model the flexibility to experiment with unusual situations. The column of typical values is intended to provide a feel for a common scenario. The uncertainty column provides an estimate of the confidence with which the physical quantities can be measured. This information would be part of the input if one were performing an uncertainty quantification exercise.

- Constraint on gravity: g = $9.8 m/s^2$

3.2.7 Properties of a Correct Solution

A correct solution must satisfy the system of non-linear equations described. The user will also be able to judge the results based on the knowledge about the model and input.

4 Requirements

This section provides the functional requirements, the business tasks that the software is expected to complete, and the nonfunctional requirements, the qualities that the software is expected to exhibit.

4.1 Functional Requirements

R1: Multi-Pendulum Simulation program will take the following inputs:

   1. The initial mass of the weights.
   2. The inital length of the rods.

R2: Multi-Pendulum Simulation program will ensure that the inputs do not violate the constraints specified in the Data Contraints section:

   1. Multi-Pendulum Simulation program will generate diagrams with and plot lines and timeline of logged movement.
   2. The timeline of swings of the pendulum will be logged and eventually return to a resting state in equilibrium

R3: Multi-Pendulum Simulation program will take the following inputs:

   1. The initial mass of the weights.
   2. The inital length of the rods.

4.2 Nonfunctional Requirements

Multi-Pendulum Simulation program will be try to be small and simple, so performance is not a priority. Any reasonable implementation will be very quick and use minimal storage. Rather than performance, the non-functional requirement priorities are correctness, understandability, reusability, maintainability, and portability.

NF1: Multi-Pendulum Simulation program access axis labels & 3D cartesian coordinates.

Correctness

- The Multi-Pendulum Simulation tool must be correct in its generation of plot trajectories.

Reliability

The Multi-Pendulum Simulation should run successfully on all platforms.

Robustness

- The Multi-Pendulum Simulation must be able to recognize violated data constraints and report them to the user.
- The Multi-Pendulum Simulation tool must inform the user when it encounters any unspecified state.

Performance

Performance is a priority in the Multi-Pendulum Simulation specification. It needs to be able to generate a plot reasonable amount of time.

Verifiability

- The Multi-Pendulum Simulation must be verifiable with respect to the correctness of its calculations. The calculation procedures used by the Multi-Pendulum Simulation tool must be implemented such that they can be verified using mathematical proofs.

Usability

- The user must be able to enter values using standard mathematical notation.
- The plot should generate and be large enough for the user's display.

Maintainability

- The evolvability of the Multi-Pendulum Simulation must allow the addition of real intervals.

Reusability

Reusability is not a priority because there are currently no future products that will rely on Multi-Pendulum Simulation

Portability

The portability of the Multi-Pendulum Simulation will be multi-platform.

5 Likely Changes

LC1: Generation of diagrams using distributed/parallel computing

References

- [1] Dynamics of multiple pendula
  http://wmii.uwm.edu.pl/~doliwa/IS-2012/Szuminski-2012-Olsztyn.pdf
- [2] Pendulum
  https://en.wikipedia.org/wiki/Pendulum
- [3] Pendulum (mathematics)
  https://en.wikipedia.org/wiki/Pendulum_(mathematics)
- [4] Double Pendulum
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- [4] Differential-Algebraic Equations by Taylor Series
  http://www.cas.mcmaster.ca/~nedialk/daets/
- [5] Multi-body Lagrangian Simulations
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