Tarea 6

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1. 2D Navier-Stokes equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = w$$
$$\nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = \frac{\partial u}{\partial y}\frac{\partial w}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial w}{\partial y}$$

- Aplicando el método de diferencias finitas muestre que estas ecuaciones quedan discretizadas como:

$$u_{i,j} = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} + h^2 w_{i,j})$$

$$\begin{aligned} w_{i,j} = &\frac{1}{4}(w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1}) \\ &-\frac{R}{16}([u_{i,j+1} - u_{i,j-1}][w_{i+1,j} - w_{i-1,j}]) \\ &+\frac{R}{16}([u_{i+1,j} - u_{i-1,j}][w_{i,j+1} - w_{i,j-1}]) \end{aligned}$$

donde $R = \frac{V_o h}{\nu}$ es el número de Reynolds del lattice, V_o es la velocidad del fluido en una de las fronteras y h es el paso de discretización.

Se toma inicialmente la primera ecuación, de w, y se resuelven $\frac{\partial^2 u}{\partial x^2}$ y $\frac{\partial^2 u}{\partial y^2}$ de la siguiente forma:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{1}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) \\ \frac{\partial^2 u}{\partial y^2} &= \frac{1}{h^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) \end{aligned}$$

Entonces:

$$\begin{aligned} w_{i,j} &= \frac{1}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + \frac{1}{h^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) \\ w_{i,j} &= \frac{1}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) \\ w_{i,j} &= \frac{1}{h^2}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}) \\ h^2 w_{i,j} &= u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} \\ 4u_{i,j} &= u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - h^2 w_{i,j} \\ u_{i,j} &= \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - h^2 w_{i,j}) \end{aligned}$$

Ahora para la segunda ecuación se resuelve cada término; $\frac{\partial^2 w}{\partial x^2}$, $\frac{\partial^2 w}{\partial y^2}$, $\frac{\partial u}{\partial y}\frac{\partial w}{\partial x}$ y $\frac{\partial u}{\partial x}\frac{\partial w}{\partial y}$:

$$\begin{aligned}
\frac{\partial^2 w}{\partial x^2} &= \frac{1}{h^2}(w_{i+1,j} - 2w_{i,j} + u_{i-1,j}) \\
\frac{\partial^2 w}{\partial y^2} &= \frac{1}{h^2}(w_{i,j+1} - 2w_{i,j} + w_{i,j-1}) \\
\frac{\partial u}{\partial y}\frac{\partial w}{\partial x} &= \frac{1}{h^2}[u_{i,j+1} - u_{i,j-1}][w_{i+1,j} - u_{i-1,j}] \\
\frac{\partial u}{\partial x}\frac{\partial w}{\partial y} &= \frac{1}{h^2}[u_{i+1,j} - u_{i-1,j}][w_{i,j+1} - u_{i,j-1}]
\end{aligned}$$

Entonces:

$$\begin{aligned}
&\nu\left(\frac{1}{h^2}(w_{i+1,j} - 2w_{i,j} + w_{i-1,j}) + \frac{1}{h^2}(w_{i,j+1} - 2w_{i,j} + w_{i,j-1})\right) \\
&= \frac{1}{h^2}[u_{i,j+1} - u_{i,j-1}][w_{i+1,j} - w_{i-1,j}] - \frac{1}{h^2}[u_{i+1,j} - u_{i-1,j}][w_{i,j+1} - w_{i,j-1}] \\
&\frac{\nu}{\cancel{h^2}}(w_{i+1,j} - 2w_{i,j} + w_{i-1,j} + w_{i,j+1} - 2w_{i,j} + w_{i,j-1}) \\
&= \frac{1}{\cancel{h^2}}([u_{i,j+1} - u_{i,j-1}][w_{i+1,j} - w_{i-1,j}] - [u_{i+1,j} - u_{i-1,j}][w_{i,j+1} - w_{i,j-1}]) \\
&w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1} - 4w_{i,j} \\
&= \frac{1}{\nu}([u_{i,j+1} - u_{i,j-1}][w_{i+1,j} - w_{i-1,j}] - [u_{i+1,j} - u_{i-1,j}][w_{i,j+1} - w_{i,j-1}]) \\
&\Rightarrow 4w_{i,j} = w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1} \\
&\quad -\frac{1}{\nu}[u_{i,j+1} - u_{i,j-1}][w_{i+1,j} - w_{i-1,j}] \\
&\quad +\frac{1}{\nu}[u_{i+1,j} - u_{i-1,j}][w_{i,j+1} - w_{i,j-1}] \\
&w_{i,j} = \frac{1}{4}(w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1}) \\
&\quad -\frac{1}{4\nu}[u_{i,j+1} - u_{i,j-1}][w_{i+1,j} - w_{i-1,j}] \\
&\quad +\frac{1}{4\nu}[u_{i+1,j} - u_{i-1,j}][w_{i,j+1} - w_{i,j-1}]
\end{aligned}$$

Como $\nu = \frac{V_o h}{R}$ se tomará la velocidad del fluido como $V_o = \frac{4}{h}$, de modo que, $\nu = \frac{4}{R}$. Así se obtiene $w_{i,j}$:

$$\begin{aligned}
w_{i,j} = &\frac{1}{4}(w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1}) \\
&-\frac{R}{16}[u_{i,j+1} - u_{i,j-1}][w_{i+1,j} - w_{i-1,j}] \\
&+\frac{R}{16}[u_{i+1,j} - u_{i-1,j}][w_{i,j+1} - w_{i,j-1}]
\end{aligned}$$

2. Condiciones a la frontera

- Muestre que la vorticidad en las fronteras del obstáculo se puede escribir como:

$$w_{i,j}|_{right} = -2\frac{u_{i,j+1} - u_{i,j}}{h^2}$$

$$w_{i,j}|_{left} = -2\frac{u_{i,j-1} - u_{i,j}}{h^2}$$