Tryggvason G, Bunner B, Esmaeeli A, Juric D, Al-Rawahi N *et al.* (2001) A front-tracking method for the computations of multiphase flow. *Journal of Computational Physics* 169: 708–759.

Zheng X, Lowengrub J, Anderson A, and Cristini V (2005) Adaptive unstructured volume remeshing II. Application to two- and three-dimensional level-set simulations of multiphase flow. *Journal of Computational Physics* 208: 626–650.

Intermittency in Turbulence

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Introduction

Intermittency has several meanings in turbulence. The oldest one, now most often labeled "external" or "large-scale" intermittency, refers to the coexistence of turbulent and laminar regions in inhomogeneous turbulent flows, such as in boundary layers or in free shear layers. In those cases, the interface between laminar irrotational flow and turbulent vortical fluid is typically sharp and corrugated. An observer sitting near the edge of the layer is immersed in turbulent fluid only part of the time.

The intermittency coefficient γ measures the fraction of turbulent fluid over the sampling universe over which the statistics are taken. For example, in a boundary layer such as that in Figure 1, the intermittency coefficient as a function of wall distance measures the fraction of turbulent fluid at a given distance from the wall. External intermittency is important in any attempt to model realistic turbulent flows, which are almost always inhomogeneous. Consider, for example, the classical homogeneous relation in eqn [1] between the mean kinetic energy K of the turbulent fluctuations and the energy dissipation rate ε:

$$\varepsilon = C\frac{K^{3/2}}{L} \qquad [1]$$

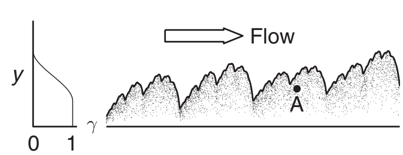

Figure 1 Sketch of a turbulent boundary layer, and of the associated intermittency factor. An observer such as A, at a distance y from the wall, only sees turbulent flow for a fraction γ of the time.

where L is the length scale of the largest eddies, and $C \approx 0.1$ is an experimentally determined constant. Such relations are often implicit in turbulent models, and they have to be modified to account for intermittency. Equation [1] only holds within the turbulent regions where the energy and the dissipation rates are K_T and ε_T, while the overall mean values used in the modeling conservation equations are $K = \gamma K_T$ and $\varepsilon = \gamma\varepsilon_T$. The true overall relation should therefore be

$$\varepsilon = C\gamma^{-1/2}\frac{K^{3/2}}{L} \qquad [2]$$

which may differ substantially from eqn [1], especially near the edge of the layer. Experimental values and rough theoretical estimates for the distribution of the intermittency coefficient are available for most practical turbulent flows.

Internal Intermittency

While the external intermittency just described is probably the most important one from the point of view of applications, it is not the most interesting from the theoretical point of view. Turbulence is a multiscale phenomenon which is inhomogeneous at all length scales, from the largest ones to the inner viscous cutoff (*see* Turbulence Theories). Moreover, this inhomogeneity goes beyond what could be expected just from the statistics of a random process. Consider, for example, the velocity difference Δu between two points separated by a distance r. The original Kolmogorov formulation of the energy cascade assumes that the probability density function (PDF), $p(\Delta u)$, is a universal function in the inertial range of scales, whose only parameter is a velocity scale depending on r. It then follows from Kolmogorov's analysis that

$$p(\Delta u) = F\left[\Delta u/(\bar{\varepsilon} r)^{1/3}\right] \qquad [3]$$

where $\bar{\varepsilon}$ is the average energy transfer rate across scales per unit mass, and the average $\overline{(\,)}$ is taken either over the whole flow or over a suitably designed ensemble of experiments. In an equilibrium system, global energy conservation implies that $\bar{\varepsilon}$ is equal to the average viscous dissipation per unit mass:

$$\bar{\varepsilon} = \nu\overline{|\nabla \boldsymbol{u}|^2} \qquad [4]$$

In eqn [4], the kinematic viscosity of the fluid is ν, and $|\nabla \boldsymbol{u}|$ is the L_2-norm of the velocity gradient tensor. Equation [3] is valid as long as the separation r is much larger than the Kolmogorov viscous cutoff $\eta = (\nu^3/\bar{\varepsilon})^{1/4}$, and much smaller than the integral scale of the largest eddies $L_\varepsilon = u'^3/\bar{\varepsilon}$, where u' is the root-mean-square value of the fluctuations of one velocity component. The extent of this inertial range is a function of the Reynolds number $Re_L = u'L_\varepsilon/\nu$:

$$L_\varepsilon/\eta = Re_L^{3/4} \qquad [5]$$

The strict similarity hypothesis in eqn [3] is not well satisfied by experiments. While the velocity distribution at a given point is approximately Gaussian, Figure 2a shows that the velocity increments become increasingly non-Gaussian as the spatial separation is made much smaller than L_ε. It was also soon noted that the dependence of eqn [3] on a single parameter such as $\bar{\varepsilon}$ was theoretically suspect, since it is difficult to see how the PDFs of a whole set of local properties, such as the Δu for different intervals, could depend only on a single global property. Kolmogorov himself sought to bypass that difficulty by substituting eqn [3] by a "refined similarity" hypothesis,

$$p(\Delta u) = F\left[\Delta u/(\varepsilon_r r)^{1/3}\right] \qquad [6]$$

where ε_r is no longer a global average, but the mean value of the dissipation over a ball of radius of order r centered at the midpoint of the interval. This refined similarity is better satisfied by experiments (see Figure 2b), although, from the practical point of view, it just transfers the problem of characterizing Δu to that of characterizing the statistics of ε_r.

It has become customary to measure the behavior of $p(\Delta u)$ in terms of its structure functions,

$$S(n) = \int_{-\infty}^{\infty} \Delta u^n p(\Delta u)\mathrm{d}\Delta u \qquad [7]$$

which can be normalized as generalized flatness factors,

$$\sigma(n) = S(n)/S(2)^{n/2} \qquad [8]$$

It follows from the strict similarity hypothesis [3] that

$$S(n) \sim r^{n/3} \qquad [9]$$

and that all the $\sigma(n)$ should be independent of the separation.

For example, the fourth-order flatness of a Gaussian distribution is $\sigma(4) = 3$. Figure 3 shows that this is not true. The flatness increases as the separation decreases, and it only levels off at lengths of the order of the Kolmogorov viscous scale. For separations in that viscous range the flow is smooth, $\Delta u \approx (\partial_x u)r$, and

$$\sigma(n) \approx \overline{(\partial_x u)^n}/\overline{(\partial_x u)^2}^{\,n/2} \qquad [10]$$

It follows from eqn [10] and from Figure 3 that the velocity gradients become increasingly non-Gaussian as L_ε and η separate at high Reynolds numbers. The velocity differences across intervals which are large with respect to η also become very non-Gaussian when $r \ll L_\varepsilon$.

Because the velocity difference between two points which are not too close to each other can be expressed as the sum of velocity differences over subintervals, a loose application of the central limit

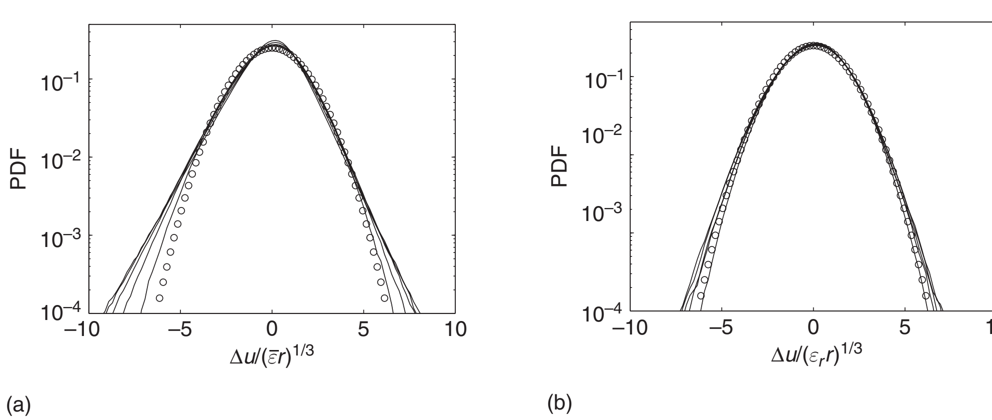

Figure 2 PDFs of the differences of the velocity component in the direction of the separation (for separations in the inertial range of scales). $r/L_\varepsilon = 0.02-0.36$, increasing by factors of 2; equivalent to $r/\eta = 180-3000$. Nominally isotropic turbulence at Reynolds number $\mathrm{Re}_L = 10^5$.. (a) Δu is normalized with the global energy dissipation rate $\bar{\varepsilon}$; distributions are wider as the separation decreases. (b) Δu is scaled with the locally averaged dissipation over the separation interval. Data courtesy of H Willaime and P Tabeling.

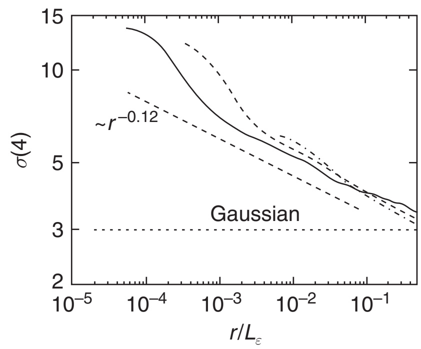

Figure 3 Fourth-order flatness of the differences of the velocity component in the direction of the separation, for separations in the inertial range of scales, $r/L_\varepsilon = 0.5$ to $r/\eta = 2$.. The Reynolds numbers of the different flows range from $\mathrm{Re}_L = 1800$ to 10^6.. Data in part courtesy of H Willaime, P Tabeling, and R A Antonia.

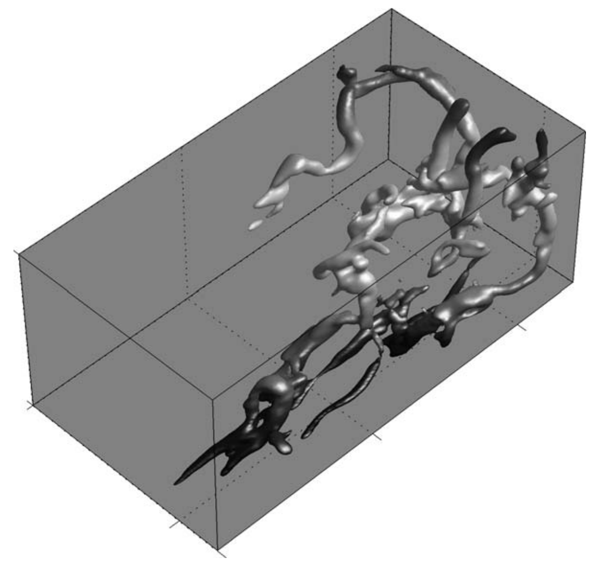

Figure 4 Intense vortex tangle in the logarithmic layer of a turbulent channel. The vortex diameters are of the order of 10η, and the size of the bounding box is of the order of the channel width. Reproduced with permission of J C del Álamo.

theorem would suggest that its PDF should be roughly Gaussian. The key conditions for that to happen are that the summands should be mutually independent, that their magnitudes should be comparable, and that each of them has a probability distribution with a finite variance. The first of those three conditions is probably a good approximation if the separation is much longer than the viscous cutoff, but the second one depends on the structure of the flow. The experimental non-Gaussian behavior suggests the existence of occasional very strong velocity jumps. In the viscous range of scales, those structures have been identified both experimentally and numerically as very strong linear vortices, in whose neighborhoods the strongest gradients are generated. An example of a tangle of such structures is shown in Figure 4.

In another example, the vorticity in decaying two-dimensional turbulence concentrates very quickly into relatively few strong compact vortices, which are stable except when they interact with each other. The velocity field is dominated by them, and the flatness of the velocity increments reaches values of the order of $\sigma(4) \approx 50$–100, even at moderate Reynolds numbers. That case is interesting because something can be said about the probability distribution of the velocity gradients. We have noted that the PDF of a sum of mutually comparable independent random variables with finite variances tends to Gaussian when the number of summands is large. This well-known theorem is a particular case of a more general result about sums of random variables whose incomplete second moments diverge as

$$\mu_2(s) = \int_{-s}^{s} x^2 p(x)\mathrm{d}x \sim s^{2-\alpha} \quad \text{when } s \to \infty \qquad [11]$$

When $0 < \alpha \leq 2$, the sums of such variables tend to a family of "stable" distributions parametrized by α. The Gaussian case is the limit of that family when $\alpha = 2$. In the case of two-dimensional vortices with very small cores, the velocity gradients at a distance R from the center of the vortex behave as $1/R^2$. If we take s in eqn [11] to be one of those velocity derivatives, its probability distribution is proportional to the area covered by gradients with a given magnitude, and

$$\mu_2(s) \sim \int_0^{s^{1/2}} R^{-4} 2\pi R\,\mathrm{d}R \sim s^{-1} \qquad [12]$$

The velocity derivatives at any point, which are the sums of the velocity derivatives induced by all the randomly distributed neighboring vortices, should therefore be distributed according to the stable distribution with $\alpha = 1$, which is Cauchy's

$$p(s) = \frac{c}{\pi(c^2 + s^2)} \qquad [13]$$

This distribution has no moments for $n > 1$. Its tails decay as s^{-2}, and the distribution of the gradients essentially reflects the properties of the closest vortex. In real two-dimensional turbulent flows, the distribution [13] is followed fairly well, but its extreme tails only reach to the maximum values of the velocity gradient found within the viscous vortex cores, which are not exactly point vortices.

Other similar general results can be derived that link the behavior of the structure functions with the properties of the stable distributions corresponding to the type of flow singularities expected in the limit of infinite Reynolds number.

The common feature of the two cases just described is the presence of strong structures that live for long times because viscosity stabilizes them. They are therefore more common than what could be expected on purely statistical grounds. They are responsible for the tails of the probability distributions of the velocity derivatives, but they are not the only intermittent features of turbulent flows. The increase of the flatness in Figure 3 below $r \approx 50\eta$ is clearly connected with the presence of the coherent vortices, but even for larger separations there is a smooth evolution of $\sigma(4)$ that suggests that the formation of intense structures is a gradual process that takes place across the inertial range. Much less is known about those hypothetical inertial structures than about the viscous ones.

We can now recast the problem of intermittency in Navier–Stokes turbulence into geometric terms. The defining empirical observation for that system is that the energy dissipation given by eqn [4] does not vanish even in the infinite Reynolds number limit in which $\nu \to 0$. This means that the flow has to become singular as $\overline{|\nabla \boldsymbol{u}|} L_\epsilon / u' \sim Re_L^{1/2}$. The strict similarity approximation assumes that those singularities are uniformly distributed across the flow, but the experimental evidence just discussed shows that this is not true. The singularities are distributed inhomogeneously, and the inhomogeneity develops across the inertial cascade. The problem of intermittency is to characterize the geometry of the support of the flow singularities in the limit of infinite Reynolds number.

In the absence of detailed physical mechanisms for the dynamics of the inertial range, most intermittency models are based on plausible processes compatible with the invariances of the inviscid Euler equations. The precise power law given in eqn [9] for the structure functions depends on the strict similarity hypothesis [3], but the fact that it is a power law only depends on the scaling invariances of the equations of motion. The energies and sizes of the eddies in the inertial range are too small for the integral scales of the flow to be relevant, and too large for the viscosity to be important. They therefore have no intrinsic velocity or length scales. Under those conditions, any function of the velocity which depends on a length has to be a power. Consider a quantity with dimensions of velocity, such as $u(r) = S(n)^{1/n}$, which is a function of a distance such as r. On dimensional grounds we should be able to write it as

$$u(r) = UF(\rho) \qquad [14]$$

where $\rho = r/L$, and L and $U(L)$ are arbitrary length and velocity scales. The value of $u(r)$ should not depend on the choice of units, and we can differentiate eqn [14] with respect to L to give

$$\partial_L u = (\mathrm{d}U/\mathrm{d}L)F(\rho) - U\rho L^{-1}(\mathrm{d}F/\mathrm{d}\rho) = 0 \qquad [15]$$

which can only be satisfied if

$$\frac{\mathrm{d}F}{\mathrm{d}\rho} = \zeta F \Rightarrow F \sim \rho^\zeta \qquad [16]$$

and $\zeta = L(\mathrm{d}U/\mathrm{d}L)/U$ is constant. This suggests generalizing eqn [9] to

$$S(n) \sim r^{\zeta(n)} \qquad [17]$$

where the exponents are empirically adjusted. Only $\zeta(3) = 1$ can be derived directly from the Navier–Stokes equations. Equation [17] implies that $\sigma(n)$ satisfies a power law with exponent $\zeta(n) - n\zeta(2)/2$. In Figure 3, for example, the flatness follows a reasonably good power law outside the viscous range, consistent with $\zeta(4) - 2\zeta(2) \approx -0.12$. The anomalous behavior near the viscous limit, and similar limitations at the largest scales, mean that only very high Reynolds number flows can be used to measure the scaling exponents, and that the range over which they are measured is never very large. Moreover, the integrand of the higher-order structure functions peaks at the extreme tails of the probability distributions of the velocity differences, which implies that very long experimental samples have to be used to accumulate enough statistics to measure the high-order exponents. For these and for other reasons, the scaling exponents above $n \gtrsim 8-10$ are poorly known. This is unfortunate because we will see later that some of the most interesting intermittency properties of the velocity field, such as the nature of the flow singularities in the infinite Reynolds number limit, depend on the behavior of the $\zeta(n)$ for large n.

Experimental values for the scaling exponents are given in Table 1. They are generally smaller than the ones predicted by the strict similarity approximation, implying that the moments of the velocity differences decrease with the separation more slowly than they would if they were self-similar, and suggesting that new stronger structures become important as the scale decreases.

Note that we have included in the table values for odd-order powers. Up to now we have not specified

Table 1 Longitudinal scaling exponents

Order	*Experimental*	*Strict similarity*
2	0..70 ± ..01	0.667
3	1.00	1
4	1..30 ± ..03	1.333
5	1..56 ± ..04	1.667
6	1..79 ± ..03	2.000
7	1..99 ± ..10	2.333
8	2..22 ± ..05	2.667

The values on the second column are averages from different experiments, and the standard deviations reflect scatter among experiments. The third column is the value from the strict similarity equation [9].

which velocity component is being analyzed, but most experiments refer to the one in the direction of the separation. That is the easiest case to measure, specially if time is used as a surrogate for distance, and those PDFs are not symmetric even in isotropic turbulence. Negative increments are more common than positive ones because of the extra energy required to stretch a vortex, and the effect is clearly visible in the distributions in Figure 2. Those longitudinal odd-order structure functions do not vanish, and their scaling exponents are the ones given in the table. The transverse structure functions are those in which the velocity component is normal to the separation, and their odd-order moments vanish by symmetry in isotropic turbulence. There has been a lot of discussion about whether the longitudinal scaling exponents of even orders differ from the transverse ones. Early results suggested that the latter are lower than the former, undermining the case for intermittency theories based on similarity arguments, and suggesting that a more mechanistic approach was needed. The present consensus seems to be that both sets of exponents are equal, but that there are residual effects of low Reynolds numbers and of flow anisotropy that are difficult to avoid experimentally. The question is still open.

Multiplicative Models

The most successful phenomenological models for the geometry of intermittency are based on the concept of a multiplicative cascade. Consider some flow property v, such as the locally averaged energy transfer rate by eddies of size r_k, which cascades into smaller eddies of size r_{k+1} which is some fraction of r_k. Denote by $p_k(v_k)$ the probability distribution of the value of v at the step k of the cascade.

Assume that the cascade is Markovian in the sense that the probability distribution of v_k depends only on its value in the previous step,

$$p_{k+1}(v_{k+1}) = \int p_{\mathrm{T}}(v_{k+1}|v_k;\, k)p_k(v_k)\,\mathrm{d}v_k \qquad [18]$$

This is in contrast to some more complicated functional dependence, such as on the values of v_k in some extended spatial neighborhood, or on several previous cascade stages. This assumption intuitively implies that v_{k+1} evolves faster, or on a smaller scale, than v_k, and that it is in some kind of equilibrium with its precursor. If the cascade is deterministic in that sense, v_k can be represented as a product

$$v_k/v_0 = q_k q_{k-1} \ldots q_1 \qquad [19]$$

in which the factors $q_k = v_k/v_{k-1}$ are statistically independent of each other.

If the underlying process is invariant to scaling transformations, the transition probability density function has to have the form

$$p_{\mathrm{T}}(v_{k+1}|v_k) = v_k^{-1} w(q_{k+1};\, k) \qquad [20]$$

The multiplicative model works most naturally for positive variables, and we will assume that to be the case in the following, but most results can be generalized to arbitrary distributions. We will also assume for simplicity that all the cascade steps are equivalent, so that the distribution $w(q)$ of the multiplicative factors is independent of k, and depends only on our choice for r_{k+1}/r_k.

Local deterministic self-similar cascades lead naturally to intermittent distributions, in the sense that the high-order flatness factors for v_k become arbitrarily large as k increases. It follows from eqns [18]–[20] that the nth order moment for p_k can be written as

$$S_k(n) = \int \xi^n p_k(\xi)\,\mathrm{d}\xi = S_0(n)S_w(n)^k \qquad [21]$$

where $S_w(n)$ is the nth order moment of the multiplicative factor q, and n is any real number for which the integral exists. If we define flatness factors as in eqn [7], we can rewrite eqn [21] as

$$\sigma_k(n) = \sigma_0(n)\sigma_w(n)^k \qquad [22]$$

It follows from Chebichev's inequality that

$$S(n) \geq S(n-2)S(2) \geq S(n-4)S(2)^2 \ldots \qquad [23]$$

from where

$$1 \le \sigma(4) \le \sigma(6) \ldots \quad [24]$$

which is true for any distribution of positive numbers. Equality only holds for trivial distributions concentrated on a single value. The product in eqn [22] therefore increases without bound with the number of cascade steps, and the flatness factors diverge.

It is tempting to substitute k in [21] by a continuous variable, in which case the PDFs form a continuous semigroup generated by infinitesimal scaling steps. This leads to beautiful theoretical developments, but it is not necessarily a good idea from the physical point of view. For example, while it might be reasonable to assume that the properties of an eddy of size r depend only on those of the eddy of size $2r$ from which it derives, the same argument is weaker when applied to eddies of almost equal sizes. We will restrict ourselves here to the discrete case.

Limiting Distributions

The multiplicative process just described can be summarized as a family of distributions $p_k(v_k)$ such that the probability density for the product of two variables is

$$p(v_{k_1} v_{k_2}) = p_{k_1+k_2}(v_{k_1+k_2}) \quad [25]$$

and it is natural to ask whether there is a limiting distribution for large k. We know that, in the case of sums, rather than products, such distributions tend to be Gaussian under fairly general conditions, and the first attempt to analyze [25] was to reduce it to a sum by defining

$$z = k^{-1} \log(v_k/v_0) \quad [26]$$

The argument was that z would tend to a Gaussian distribution, and that the limiting distribution for v_k would be lognormal. This was soon shown to be incorrect. The central part of the distribution approaches lognormality, but the tails do not, because the central limit theorem says nothing about their behavior. The family of lognormal distributions is a fixed point of eqn [25], but it is unstable, and it is only attained if the individual generating distributions are themselves lognormal.

The lognormal distribution has moments

$$S_w(n) = \exp(an + bn^2) \quad [27]$$

which are conserved under [21], so that the product of lognormally distributed variables stays lognormal. The moments in eqn [27] are generated by the recursive relation

$$Q_w(n) = \frac{S_w(n+3) S_w^3(n+1)}{S_w(n) S_w^3(n+2)} = 1 \quad [28]$$

with suitable conditions for $n < 2$. Under [21], $Q_k(n) = Q_w^k(n)$, and it is clear that only when all the $Q_w(n)$ are exactly equal to 1 do they continue to be so under multiplication. Otherwise, any Q_w initially larger than 1 tends to infinity after enough cascade steps, while any one initially smaller than 1 tends to 0. Only an exactly lognormal distribution of the generating factors results in a lognormal limiting distribution, and even small errors lead to very different patterns of moments. This contrasts with the situation for sums of random variables, in which the Gaussian distribution is not only a fixed point, but also has a very large basin of attraction.

Multifractals

The problem with using the transformation [26] to find the limiting distribution of a multiplicative process is not so much the technique of analyzing the statistics of products in terms of those of sums, but the inappropriate use of the central limit theorem. It can be bypassed by using instead the theory of large deviations of sums of random variables. The key result is obtained by expanding the characteristic function of p_k when $k \gg 1$, and states that

$$p_k(v_k) \approx \left(\frac{-\phi_0''}{2\pi k}\right)^{1/2} \mathrm{e}^{k[\phi(z)-z]} \quad [29]$$

where z is defined as in [26] and ϕ, which plays the role of an entropy, is a smooth function of z. Primes stand for derivatives with respect to z. Let us define z_n as the point where

$$\phi_n' \equiv \phi'(z_n) = -n \quad [30]$$

which corresponds to the location of the maximum of $\phi + nz$. The entropy ϕ can be computed from the moments of the transition probability density. Using Laplace's method to expand the nth moment of p_k, we obtain

$$\begin{aligned} S_k(n) &= \int_{-\infty}^{\infty} k\mathrm{e}^{k(n+1)z} p_k(v_k)\, \mathrm{d}z \\ &\approx \left(\frac{\phi_0''}{\phi_n''}\right)^{1/2} \mathrm{e}^{k(\phi_n + nz_n)} \end{aligned} \quad [31]$$

from where, using [21],

$$\lambda_n \equiv \log S_w(n) = \phi(z_n) + nz_n \quad [32]$$

The essence of Laplace's approximation is that, for $k \gg 1$, most of the contribution to the integral in eqn [31] comes from the neighborhood of z_n, so that it makes sense to consider each such neighborhood as a separate "component" of the cascade.

The geometric interpretation of this classification into components as a multifractal was developed in the context of three-dimensional homogeneous turbulence. We have up to now assumed very little about the nature of each cascade step, but it is natural in turbulence to interpret it as the process in which eddies decay to a smaller geometric scale. The argument works for any variable for which scale similarity can be invoked, but we have seen that most experiments are done for the magnitude of the velocity increments across a distance r. If we assume for simplicity that $r_k/r_{k+1} = \mathrm{e}$, so that $r_k/r_0 = \exp(-k)$, eqns [26] and [29] can be written as

$$v_k/v_0 = (r_k/r_0)^{-z_n}, \qquad p_k(z_n) \sim (r_k/r_0)^{-\phi_n} \qquad [33]$$

The multifractal interpretation is that the "component" indexed by n, whose velocity increments are "singular" in terms of r with exponent z_n, lies on a fractal whose volume is proportional to its probability, and which therefore has a dimension $D(z_n) = 3 + \phi_n$.

Note that eqn [32] implies that the scaling exponents in eqn [17] can now be expressed as

$$\zeta(n) = -\log S_w(n) = -\lambda_n \qquad [34]$$

There was an enumeration there of several things which are equivalent: the exponents, the spectra, the distribution, and the limiting distribution $p_\infty(v)$ – univocally determine each other. Note however that different quantities have different scaling exponents. For example, it follows from eqn [6] that, if the scaling exponents for the local dissipation are $\zeta_\varepsilon(n)$, the exponents for Δu would be $\zeta_{\Delta u}(n) = n/3 + \zeta_\varepsilon(n/3)$.

Some properties can be easily derived from the previous discussion. If we assume, for example, that the multiplicative factor q is bounded above by q_b, which is reasonable for many physical systems, eqn [26] implies that $z_n \leq \log q_\mathrm{b}$. In fact, if the transition probability behaves near q_b as $w(q) \sim (q_\mathrm{b} - q)^\beta$, the scaling exponents tend to

$$\lambda_n = n \log q_\mathrm{b} - (\beta + 1) \log n + \mathrm{O}(1) \qquad [35]$$

for $n \gg 1$. In the case in which $w(q)$ has a concentrated component at $q = q_\mathrm{b}$, the $\log n$ is missing in eqn [35]. In all cases, the singularity exponent of the set associated with $n \to \infty$ is $z_\infty = \log q_\mathrm{b}$, because the very high moments are dominated by the largest possible multiplier. In the case of a concentrated distribution the dimension of this set approaches a finite limit, but otherwise

$$D(n) \approx -(\beta + 1) \log n \qquad [36]$$

which becomes infinitely negative. This should not be considered a flaw. The set of events which only happen at isolated points and at isolated instants has dimension $D = -1$ in three-dimensional space, and those which only happen at isolated instants, and only under certain circumstances, have still lower negative dimensions. Sets with very negative dimensions are however extremely sparse, and are difficult to characterize experimentally.

The multifractal spectrum of the velocity differences in three-dimensional Navier–Stokes turbulence has been measured for several flows in terms of the scaling exponents, and appears to be universal. The probability distribution $w(q)$ of the multipliers has also been measured directly, and agrees well with the values implied by the exponents. It is also approximately independent of r, although not completely, perhaps due to the same experimental problems of anisotropy and limited Reynolds number which plague the measurement of the scaling exponents. There has been extensive theoretical work on the consequences of imposing various physical constraints on the multipliers, specially the conservation requirement that the average value of the dissipation has to be conserved across each cascade step. Several simple models have been proposed for the transition distribution which approximate the experimental exponents well, but the relation lacks specificity. Models that are very different give very similar results, and it is impossible to choose among them using the available data.

Multiplicative cascades and the resulting intermittency are not limited to Navier–Stokes turbulence. The equations of motion have only entered the discussion in this section through the assumption of scaling invariance. Multifractal models have in fact been proposed for many chaotic systems, from social sciences to economics, although the geometric interpretation is hard to justify in most of them. It is also important to realize that the fact that a given process can in principle be described as a cascade does not necessarily mean that such a description is a good one. Neither does a cascade imply a multiplicative process. For each particular case, we need to provide a dynamical mechanism that implements both the cascade and the transition multipliers. In three-dimensional Navier–Stokes turbulence, the basic transport of energy to smaller scales and to higher gradients is vortex stretching. The differential strengthening and weakening of the vorticity under axial stretching and compression also provide a natural way of introducing the self-similar transition probabilities of the local dissipation.

Examples of nonintermittent cascades abound. We have already mentioned that the vorticity in decaying two-dimensional turbulence gets concentrated into stable vortex cores which eventually block the decay. The resulting enstrophy distribution is highly intermittent, but it is not well described by a multifractal. Conversely, forced two-dimensional turbulence is dominated by an inverse energy cascade to larger scales, which is not intermittent.

In addition, the intermittency of some systems is not a small-scale effect. Turbulent mixing of a passive scalar, which is the key process in turbulent heat transfer and in the atmospheric dispersion of pollutants, is an extremely intermittent phenomenon. The gradients of the scalar tend to be very localized, but they concentrate in sheets, narrow in thickness but otherwise extended. Some progress has recently been made on a simplified model due to Kraichnan for this problem, which is the linear stirring of a passive scalar by a random noise with delta correlation. Its statistics have been computed analytically, but the constraints of linearity and of uncorrelated forcing are strong, and the same methods do not appear to be extensible to mixing by real turbulence (*see* Lagrangian Dispersion (Passive Scalar)). Another problem in which intermittency is confined to large-scale surfaces is the motion of a three-dimensional pressureless gas, which has been used as a model for hypersonic turbulence and for the large-scale evolution of dark matter in the early universe.

In summary, intermittency is a fascinating property of many random systems, including three-dimensional Navier–Stokes turbulence, which interferes, sometimes strongly, with their description by simple cascade models. Significant advances have been made in its quantitative kinematic analysis. In some cases we also have a qualitative understanding of its roots. But in very few cases do we understand it well enough to make quantitative predictions.

See also: Ergodic Theory; Incompressible Euler Equations: Mathematical Theory; Lagrangian Dispersion (Passive Scalar); Turbulence Theories; Vortex Dynamics; Wavelets: Applications.

Further Reading

Feller W (1971) *An Introduction to Probability Theory and Its Applications*, 2nd edn., vol. 2, pp. 169–172 and 574–581. New York: Wiley.
Frisch U (1995) *Turbulence. The Legacy of A.N. Kolmogorov*, pp. 159–192. Cambridge: Cambridge University Press.
Jiménez J (1998) Small scale intermittency in turbulence. *European Journal of Mechanics B: Fluids* 17: 405–419.
Jiménez J (2001) Intermittency and cascades. *Journal of Fluid Mechanics* 409: 99–120.
Lanford OE (1973) Entropy and equilibrium states in classical mechanics. In: Lenard A (ed.) *Statistical Mechanics and Mathematical Problems*, Lecture Notes in Physics, vol. 20, pp. 1–113. Berlin: Springer.
Nelkin M (1994) Universality and scaling in fully developed turbulence. *Advances in Physics* 43: 143–181.
Paladin G and Vulpiani A (1987) Anomalous scaling laws in multifractal objects. *Physics Reports* 156: 147–225.
Pope SB (2000) *Turbulent Flows*, pp. 167–173 and 254–263. Cambridge: Cambridge University Press.
Schroeder M (1991) *Fractals, Chaos, Power Laws*, Sect. 9. New York: W.H. Freeman.
Sreenivasan KR and Stolovitzky G (1995) Turbulent cascades. *Journal of Statistical Physics* 78: 311–333.
Vassilicos JC (ed.) (2001) *Intermittency in Turbulent Flows*. Cambridge: Cambridge University Press.

Intersection Theory

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Introduction

Intersection theory is the theory that governs the rigorous definition of intersections of cycles. This can take place in a variety of mathematical contexts, for instance, the intersections of two cycles on an oriented manifold in algebraic topology, of two currents on a differentiable manifold in differential geometry, or of two subvarieties on a nonsingular algebraic variety in algebraic geometry.

In algebraic geometry the theory is especially well developed (Fulton 1998). A cycle on an algebraic variety (or scheme) is a formal linear combination of irreducible closed subvarieties. These are subject to an equivalence relation called rational equivalence. For every rational function on every subvariety, its zero set is deemed rationally equivalent to its poles (with appropriate multiplicities).

As an example, in the complex projective plane $\mathbb{CP}^2$, any two lines are rationally equivalent since the ratio of two linear forms will vanish on one line and have a pole along the other. Similarly, a curve of degree d is rationally equivalent to d lines. Any two points in $\mathbb{CP}^2$ can be joined by a line (a copy of