Path Ordering for the Wilczek-Zee Phase

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Date: *17 July 2022 (Sunday)*

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We start, following [1–3], by considering a nonrelativistic quantum mechanical system governed by a Hamiltonian $\hat{H}$ which depends on a set of k parameters $\{R_i\}$. If the time dependence of $\hat{H}$ is expressed exclusively by the time dependence of $\{R_i\}$, we can set up a k-dimensional parameter space in which $R^\mu = (R_1, \cdots, R_k)$ is now a vector.

We now consider the time evolution of this system under the adiabatic approximation. If we consider a state that starts in an N-fold degenerate eigensubspace at time t_i, the adiabatic theorem tells us that this subspace will be mapped to a corresponding subspace (representing the time-evolved version of the energy eigenvalue) at time t. The state is given by:

$$|\Psi_n(t)\rangle = \exp\left\{-\mathrm{i}\int_{t_i}^{t} \mathrm{d}\mathfrak{t}\, E_n(R^\mu(\mathfrak{t}))\right\} \sum_{a=1}^{N} U_{na}\, |n_a(t)\rangle \tag{0.1}$$

If we put this in the time-dependent Schrödinger equation and take the inner product with a generic basis vector $\langle n_b(t)|$ of the subspace, we can derive a differential equation for U_{na} analogous to the differential equation we used to derive the Berry-Pancharatnam phase in the nondegenerate case:

$$\frac{\partial \hat{U}}{\partial t} = \mathrm{i}\left\langle n_b \middle| \frac{\partial}{\partial t} \middle| n_a \right\rangle \hat{U} = \mathrm{i}\left\langle n_b \middle| \frac{\partial}{\partial R^\mu} \middle| n_a \right\rangle \hat{U} =: \mathrm{i}A_\mu \hat{U} \tag{0.2}$$

Here, we absorbed the exponential prefactor into $\hat{U}$, and used the definition of the Berry-Pancharatnam connection, unchanged from the Abelian case:

$$A_\mu := \left\langle n_b \middle| \frac{\partial}{\partial R^\mu} \middle| n_a \right\rangle \tag{0.3}$$

In particular, since $\hat{U} \in \mathrm{U}(\mathbf{N})$, A_μ takes the familiar role of a non-Abelian $\mathrm{U}(\mathbf{N})$ gauge potential. We then define the Wilczek-Zee curvature in exactly the same way as we do the Berry-Pancharatnam curvature, as the quantum geometric analogue of the field strength tensor:

$$\Omega_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}[A_\mu, A_\nu] = \mathrm{i}\left\langle \frac{\partial n}{\partial R^\mu} \middle| \frac{\partial n}{\partial R^\nu} \right\rangle - \mathrm{i}\left\langle \frac{\partial n}{\partial R^\nu} \middle| \frac{\partial n}{\partial R^\mu} \right\rangle - \mathrm{i}[A_\mu, A_\nu] \tag{0.4}$$

(Conventions vary in the definition of the non-Abelian field strength tensor; different sources put in different factors of i and minus signs in various places, especially in front of the commutator. Making sure the conventions are right when doing a calculation is absolutely critical.)

(0.2) is quite familiar to us from perturbation theory (whether quantum mechanical or perturbative quantum field theory): this is simply the Dyson series, with the solution given by [4]:

$$\begin{aligned} \hat{U} &= \sum_{j=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int_{t_i}^{t} \mathrm{d}\mathfrak{t}_1 \cdots \int_{t_i}^{t} \mathrm{d}\mathfrak{t}_j\, \mathscr{T}\left[\left\langle n_b \middle| \frac{\partial}{\partial \mathfrak{t}_1} \middle| n_a \right\rangle \cdots \left\langle n_b \middle| \frac{\partial}{\partial \mathfrak{t}_j} \middle| n_a \right\rangle\right] \\ &=: \mathscr{T} \exp\left\{\int_{t_i}^{t} \mathrm{d}\mathfrak{t} \left\langle n_b \middle| \frac{\partial}{\partial \mathfrak{t}} \middle| n_a \right\rangle\right\} \end{aligned} \tag{0.5}$$

([4] provides an excellent derivation of the Dyson series solution, as well as a more detailed introduction to the time-ordering operator.) We note that the series expansion in the first line *defines* the time-ordered exponential. The $\hat{U}$ that we get out of this is the non-Abelian equivalent of the Berry-Pancharatnam phase in the Abelian case.

Shifting to parameter space and imposing the $|n(t)\rangle = |n(t_i)\rangle$ boundary condition, the time-ordered exponential becomes the *path*-ordered exponential in parameter space:

$$\begin{aligned}\hat{U} &= \sum_{j=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \oint_C \mathrm{d}R_1^\mu \wedge \cdots \wedge \oint_C \mathrm{d}R_j^\mu \; \mathscr{P}\left[\left\langle n_b \middle| \frac{\partial}{\partial R_1^\mu} \middle| n_a \right\rangle \cdots \left\langle n_b \middle| \frac{\partial}{\partial R_j^\mu} \middle| n_a \right\rangle\right] \\ &= \mathscr{P} \exp\left\{ -\mathrm{i} \oint_C \mathrm{d}R^\mu \left\langle n_b \middle| \frac{\partial}{\partial R^\mu} \middle| n_a \right\rangle \right\} = \mathscr{P} \exp\left\{ -\mathrm{i} \oint_C \mathrm{d}R^\mu \, A_\mu \right\}\end{aligned} \tag{0.6}$$

The last quantity, the path ordered exponential of the gauge connection (here, the Wilczek-Zee connection), is defined in gauge theories generally as the *Wilson loop*. In gauge theories, the path ordering is done by specifying an explicit parametrization with a well-ordering (since, of course, $d > 1$-dimensional spaces don't have a total ordering). In the case of the Wilczek-Zee phase, this is more straightforward; the path ordering corresponds to an existing time ordering, so we can simply carry over the preexisting time ordering.

As mentioned before, $\hat{U}$ (the Wilson loop) is the Wilczek-Zee equivalent of the Berry-Pancharatnam phase. $\hat{U}$, however, is not gauge-invariant, but is rather gauge-*co*variant. If we apply an arbitrary phase $|\Psi_n\rangle \mapsto \mathrm{e}^{\mathrm{i}\xi} |\Psi_n\rangle$ for some $\xi \in \mathbb{C}$, we have $\hat{U}$ transforming as:

$$\hat{U} \mapsto \mathrm{e}^{\mathrm{i}\xi} \hat{U} \mathrm{e}^{-\mathrm{i}\xi} \tag{0.7}$$

However, if we take the *trace* of $\hat{U}$, we get a gauge-invariant quantity:

$$\mathrm{Tr}\left[\hat{U}\right] = \mathrm{Tr}\left[\mathscr{P} \exp\left\{ -\mathrm{i} \oint_C \mathrm{d}R^\mu \, A_\mu \right\} \right] \tag{0.8}$$

(We can see the gauge invariance directly: $\mathrm{Tr}\left[\mathrm{e}^{\mathrm{i}\xi} \hat{U} \mathrm{e}^{-\mathrm{i}\xi}\right] = \mathrm{Tr}\left[\hat{U} \mathrm{e}^{-\mathrm{i}\xi} \mathrm{e}^{\mathrm{i}\xi}\right] = \mathrm{Tr}\left[\hat{U}\right]$.)

The gauge invariance of the trace of the Wilson loop, (0.8), makes this an essential quantity when calculating observable quantities in non-Abelian gauge theories. Indeed, *every* observable quantity in non-Abelian gauge theories is built out of products of Wilson loops at various points [5–8]. The necessity of the Wilson loop in calculating any meaningful quantities is a direct consequence of the geometric properties of the underlying space that our non-Abelian theory lives on, which we see by examining the gauge transformation properties of $|\Psi_n\rangle$, A_μ, and $\Omega_{\mu\nu}$.

First, starting with $|\Psi_n\rangle$ directly, we note that (0.1) *defines* the time evolution (*resp.*, path evolution in parameter space) of the state. Thus, the time evolution of the wavefunction is directly given by $\hat{U}$, with

the $\hat{U} \mapsto \mathrm{e}^{\mathrm{i}\xi}\hat{U}\,\mathrm{e}^{-\mathrm{i}\xi}$ transformation law we saw earlier. As a result, the state itself evolves according to the non-Abelian matrix $\hat{U}$, which must be ordered according to (0.5) and (0.6) in order to retrieve any meaningful values.

We might hope that something like Stokes' theorem can allow us to bypass the path ordering, by evaluating A_μ or $\Omega_{\mu\nu}$ and then integrating over the entire region bounded by the contour, without having to worry about ordering. Unfortunately, this simply isn't possible. If we examine, for instance, the gauge transformation of $\partial_\mu |n\rangle$ (which, after all, is quite important for A_μ, we have:

$$\partial_\mu |n\rangle \mapsto \partial_\mu |n\rangle + i\xi\partial_\mu |n\rangle \tag{0.9}$$

This is due to the nature of the derivative itself: the derivative is not a local operator, but properly speaking is a bilocal operator [7, 8]. Hearkening back to the difference quotient definition from our first exposure to calculus, we defined the derivative by:

$$\frac{\mathrm{d}f}{\mathrm{d}x} := \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon} \tag{0.10}$$

Every type of derivative relies on this type of bilocal definition in some sense, whether it be via difference quotients or neighbourhoods, and whether it be one dimensional or multivariable or functional. As a result, even though we take the limit as the size of the neighbourhood around x go to zero, the gauge transformation itself is still affected by this lingering dependence on the other point(s).

For any gauge theory, the gauge transformation at $f(x+\epsilon)$ is different from that at $f(x)$. This is because these reside on two different points of the manifold that represents the gauge theory [9, 10] (more precisely, on completely separate fibre bundles located at these different manifold points [11]).

As a result, in order to evaluate a bilocal quantity like a derivative, we need to parallel transport $f(x)$ to $f(x+\epsilon)$. In fact, *any* quantity that we want to meaningfully evaluate at more than a single point, and indeed every quantity whose definition relies on more than a single point (including derivatives like A_μ and $\Omega_{\mu\nu}$; and indeed, from the adiabatic time evolution, like $|\Psi_n\rangle$ itself), must be parallel transported from x to $x+\epsilon$ in order to properly evaluate these quantities. We can parallel transport every quantity of interest, as always, via the (gauge-)covariant derivative D_μ, which is defined in terms of a bilocal *comparator function* $U(y,x)$. The covariant derivative is given by:

$$D_\mu f := \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - U(x+\epsilon, x)f(x)}{\epsilon} \tag{0.11}$$

Meanwhile, the comparator function $U(y,x)$ is defined [7, 9, 10] such that it transforms under a U(1) transformation as:

$$U(y,x) \mapsto \mathrm{e}^{\mathrm{i}\xi(y)}\,U(y,x)\,\mathrm{e}^{-\mathrm{i}\xi(x)} \tag{0.12}$$

We require the comparator function to have a number of properties, which we can find in [7, 9, 10]. It comes as no surprise, however, that for the theory we're considering, the comparator function in question is none other than the $\hat{U}$ we found in (0.5) and (0.6), and thus we need to use that function to parallel

transport any quantities. As a result, for *any* meaningful quantities to be calculated in the non-Abelian gauge theory, we *must* calculate the Wilson loop. (To take one pointed example, a few types of non-Abelian generalizations of Stokes' theorem have been developed; these are discussed in [12]. All take, as their starting point, the Wilson loop expression given in the last equality of (0.6).)

Before finishing, we might ask why, if the Wilson loop is essential for *any* gauge theory, these complications didn't show up in the (Abelian) Berry-Pancharatnam phase. In fact, all of this machinery is present in the Berry-Pancharatnam phase, connection, and curvature as well; we simply didn't see them because the Abelian nature of the theory massively simplified the expressions. Because the theory is Abelian, the time-ordering (*resp.*, path ordering) is trivial. Imposing an explicit time-ordering procedure is a necessary consequence of the commutator $\left[A_\mu(x), A_\nu(y)\right]$ not vanishing; in the Abelian case, since this commutator vanishes, the Baker-Campbell-Hausdorff expansion is trivial (and thus, so is the time-ordering procedure.)

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