

Albert 1.2: What is a Lindbladian?

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(also uses Landi 4.3: Lindblad Master Equations and Preskill 3.5: Master Equations for Open Quantum Systems)

Time evolution of $\hat{\rho}(t)$ over small time increment dt : $\hat{\rho}(t+dt)$. We assume this follows the Kraus map, so we assume $\hat{\rho}(t+dt)$ is:

$$\hat{\rho}(t+dt) = \sum_e \hat{E}^e(dt) \hat{\rho}(t) \hat{E}^{e*}(dt) =: \hat{\mathcal{E}}_{dt} [\hat{\rho}(t)]$$

Relies on Markov approximation: assumption that $\hat{\rho}(t+dt)$ relies only on $\hat{\rho}(t)$ and not previous times $t < t$. (Alternately: Markov processes are AR-1 processes.) Taylor approximation for superoperator $\hat{\mathcal{E}}_{dt} \in \mathcal{L}_1(\mathcal{O}_p(\mathcal{H}))$:

$$\hat{\mathcal{E}}_{dt} = \hat{\mathbb{I}} + dt \left(\lim_{dt \rightarrow 0} \frac{\hat{\mathcal{E}}_{dt} - \hat{\mathbb{I}}}{dt} \right) + \dots = \hat{\mathbb{I}} + dt \hat{\mathcal{L}}$$

$\hat{\mathcal{E}}_{dt}$ is a linear map, at least L_1 -finite, from $\hat{\rho} \in \mathcal{O}_p(\mathcal{H})$ to $\hat{\sigma} \in \mathcal{O}_p(\mathcal{H})$.

$\rightarrow \hat{\mathbb{I}}[\hat{\rho}] = \hat{\rho}$ is the identity channel.

$\rightarrow \hat{\mathcal{L}} = \lim_{dt \rightarrow 0} \frac{\hat{\mathcal{E}}_{dt} - \hat{\mathbb{I}}}{dt}$ is the Lindbladian.

$\hat{\mathcal{L}}$ is determined by the expansion of $\hat{E}^e(dt)$, kept to $\hat{\mathcal{O}}(dt)$.

\rightarrow If evolution of $\hat{\rho}$ is generated by a Hamiltonian $\hat{H} \in \mathcal{O}_p(\mathcal{H})$ alone, then $\hat{E}^{e=0} = \hat{\mathbb{I}} - i\hat{H}dt$ and $\hat{E}^{e>0} = \emptyset$. Yields regular Liouville-won Neumann equation:

$$z(t+dt) = \sum_e \hat{E}^e(dt) z(t) \hat{E}^{e*}(dt) = (\hat{\mathbb{I}} - i\hat{H}dt) z(t) (\hat{\mathbb{I}} + i\hat{H}dt) = z(t) + i\hat{z}(t) \hat{H}dt - i\hat{H}\hat{z}(t) dt + \hat{H}z(t) \hat{H}(dt)^2$$

$$\hat{\rho}(t+dt) = \sum_e \hat{E}^e(dt) \hat{\rho}(t) \hat{E}^{e\dagger}(dt) = (\mathbb{1} - i\hat{H}dt) \hat{\rho}(t) (\mathbb{1} + i\hat{H}dt) = \hat{\rho}(t) + i\hat{\rho}(t)\hat{H}dt - i\hat{H}\hat{\rho}(t)dt + \hat{H}\hat{\rho}(t)\hat{H}(dt)^2$$

$$\left. \frac{\hat{\rho}(t+dt) - \hat{\rho}(t)}{dt} \right|_{O(1)} =: \frac{d\hat{\rho}(t)}{dt} = i\hat{\rho}(t)\hat{H} - i\hat{H}\hat{\rho}(t)dt + \cancel{\hat{H}\hat{\rho}(t)\hat{H}dt} = -i[\hat{H}, \hat{\rho}(t)]$$

→ More general case: each \hat{E}^e has an $O(\sqrt{dt})$ dependence, so each $\hat{E}^e(dt) \hat{\rho}(t) \hat{E}^{e\dagger}(dt)$ term contributes an overall $O(dt)$ term. WLOG, can write:

$$\hat{E}^{e=0}(dt) \sim \mathbb{1} + (-i\hat{H} + \hat{V})dt; \quad \hat{E}^{e>0}(dt) \sim \sqrt{K_e dt} \hat{F}^e$$

- * $K_e \in \mathbb{R}$ is a nonzero rate.
- * \hat{V} is a to-be-determined perturbing potential.
- * Right now, \hat{F}^e is an unknown operator.
- * This is the most general \leq lowest order expression.

To determine \hat{V} , plug $\hat{E}^{e=0}$ and $\hat{E}^{e>0}$ into Kraus operator unitarity constraint and Markov assumption:

- * Kraus operator unitarity constraint $\sum_e \hat{E}^{e\dagger} \hat{E}^e = \mathbb{1}$:

$$\sum_e \hat{E}^{e\dagger} \hat{E}^e = (\mathbb{1} + (-i\hat{H} + \hat{V})dt)^+ (\mathbb{1} + (-i\hat{H} + \hat{V})dt) + \sum_e (\sqrt{K_e dt} \hat{F}^e)^+ (\sqrt{K_e dt} \hat{F}^e) = \mathbb{1}$$

$$\sum_e \hat{E}^e \hat{E}^{e*} = (\mathbb{1} + (-i\hat{H} + \hat{V})dt)^+ (\mathbb{1} + (-i\hat{H} + \hat{V})dt) + \sum_{e>\delta} (\sqrt{K_e dt} \hat{F}^e)^+ (\sqrt{K_e dt} \hat{F}^e) = \mathbb{1}$$

$$\sum_e \hat{E}^e \hat{E}^{e*} = (\mathbb{1} + i\hat{H}dt + \hat{V}dt) (\mathbb{1} - i\hat{H}dt + \hat{V}dt) + \sum_{e>\delta} K_e \hat{F}^{e*} \hat{F}^e dt = \mathbb{1}$$

$$\sum_e \hat{E}^e \hat{E}^{e*} = \mathbb{1} + i\hat{H}dt + \hat{V}dt - i\hat{H}dt - i^2 \hat{H}^2(dt)^2 + i\hat{H}\hat{V}(dt)^2 + \hat{V}dt - i\hat{H}\hat{V}(dt)^2 + \hat{V}^2(dt)^2 + \sum_{e>\delta} K_e \hat{F}^{e*} \hat{F}^e dt = \mathbb{1}$$

Yellow, purple, and green terms all cancel with each other, and grey terms are higher-order in dt . Gives:

$$2\hat{V}dt = -dt \sum_{e>\delta} K_e \hat{F}^{e*} \hat{F}^e$$

$$\hat{V} = -\frac{1}{2} \sum_{e>\delta} K_e \hat{F}^{e*} \hat{F}^e$$

* Markov assumption: $\hat{\rho}(t+dt) = \sum_e \hat{E}^e(dt) \hat{\rho}(t) \hat{E}^{e*}(dt)$

$$\hat{\rho}(t+dt) = (\mathbb{1} - i\hat{H}dt + \hat{V}dt) \hat{\rho}(t) (\mathbb{1} + i\hat{H}dt + \hat{V}dt) + \sum_{e>\delta} (\sqrt{K_e dt} \hat{F}^e)^+ (\sqrt{K_e dt} \hat{F}^e)^*$$

$$\hat{\rho}(t+dt) = \hat{\rho}(t) - i\hat{H}\hat{\rho}(t)dt + \hat{V}\hat{\rho}(t)dt + i\hat{\rho}(t)\hat{H}dt - i^2 \hat{H}\hat{\rho}(t)\hat{H}(dt)^2$$

$$-i\hat{H}\hat{\rho}(t)\hat{V}(dt)^2 + \hat{\rho}(t)\hat{V}dt - i\hat{H}\hat{\rho}(t)\hat{V}(dt)^2 + \hat{V}\hat{\rho}(t)\hat{V}(dt)^2 + \sum_{e>\delta} K_e dt \hat{F}^e \hat{\rho}(t) \hat{F}^{e*}.$$

As before, the grey terms are higher-order in Δt . This gives:

$$\hat{\rho}(t+\Delta t) - \hat{\rho}(t) \Big|_{O(\Delta t)} = -i(\hat{H}\hat{\rho}(t) - \hat{\rho}(t)\hat{H})\Delta t + \Delta t \left(\hat{V}\hat{\rho}(t) + \hat{\rho}(t)\hat{V} + \sum_{e>0}^1 K_e \hat{F}^e \hat{\rho}(t) \hat{F}^{e+} \right)$$

Applying $\hat{V} = -\frac{i}{2} \sum_{e>0}^1 K_e \hat{F}^{e+} \hat{F}^e$ defines the Lindbladian / Liouillian / Lindblad master equation / Gorini-Kossakowski-Sudarshan-Lindblad master equation:

$$\frac{\hat{\rho}(t+\Delta t) - \hat{\rho}(t)}{\Delta t} \Big|_{O(\Delta t)} = \frac{d\hat{\rho}(t)}{dt} =: \mathcal{L}[\hat{\rho}] =: -i[\hat{H}, \hat{\rho}(t)] + \frac{1}{2} \sum_{e>0}^1 K_e (2\hat{F}^e \hat{\rho}(t) \hat{F}^{e+} - \hat{F}^{e+} \hat{F}^e \hat{\rho}(t) - \hat{\rho}(t) \hat{F}^{e+} \hat{F}^e)$$

- \hat{H} : Hamiltonian
- \hat{F}^e : jump operators
- $\hat{F}^e \hat{\rho}(t) \hat{F}^{e+}$: recycling term / sandwich term / jump term. Acts on $\hat{\rho}$ from both sides, so $\mathcal{L}[\hat{\rho}]$ can't simply be reduced to an L_2 -finite operator acting on a ket state $|n\rangle \in \mathcal{H}$.
- $K_e \in \mathbb{R}^+$: rates.
- $\hat{K} := \hat{H} - \frac{i}{2} \sum_{e>0}^1 K_e \hat{F}^{e+} \hat{F}^e$: deterministic / no-jump term. Defining the adjoint commutator $[\hat{A}, \hat{B}]^\# := \hat{A}\hat{B} - \hat{B}^+\hat{A}^+$, the deterministic term allows everything except the recycling term to be expressed in terms of a von Neumann equation:

$$\left(\frac{d\hat{\rho}(t)}{dt} \right)_\xi := -i[\hat{H}\hat{\rho}(t) + \hat{\rho}(t)\hat{H}] - \frac{1}{2} \sum_{e>0}^1 K_e \hat{F}^{e+} \hat{F}^e \hat{\rho}(t) - \frac{1}{2} \sum_{e>0}^1 K_e \hat{\rho}(t) \hat{F}^{e+} \hat{F}^e.$$

$$\left(\frac{d\hat{\rho}(t)}{dt}\right)_{\xi} = -i[\hat{H}, \hat{\rho}(t)] - i\left(-\frac{i}{2}\sum_{e>0} K_e \hat{F}^{e+} \hat{F}^e\right)\hat{\rho}(t) + i\hat{\rho}(t)[\hat{H}] + i\hat{\rho}(t)\left(\frac{i}{2}\sum_{e>0} K_e \hat{F}^{e+} \hat{F}^e\right)$$

$$\left(\frac{d\hat{\rho}(t)}{dt}\right)_{\xi} = -i\left([\hat{H}] - \frac{i}{2}\sum_{e>0} K_e \hat{F}^{e+} \hat{F}^e\right)\hat{\rho}(t) + i\hat{\rho}(t)\left([\hat{H}] + \frac{i}{2}\sum_{e>0} K_e \hat{F}^{e+} \hat{F}^e\right)$$

$$\left(\frac{d\hat{\rho}(t)}{dt}\right)_{\xi} = -i[\hat{K}, \hat{\rho}(t)] + i\hat{\rho}(t)\hat{K}^+ = -i[\hat{K}, \hat{\rho}(t)]^\#$$

Dissipator can be defined by extra terms from the Liouville-von Neumann equation. Define dissipator $\mathcal{D}[\hat{\rho}]$ by:

$$\mathcal{D}[\hat{\rho}] := \frac{1}{2}\sum_{e>0} K_e (2\hat{F}^e \hat{\rho}(t))\hat{F}^{e+} - \hat{F}^{e+} \hat{F}^e \hat{\rho}(t) - \hat{\rho}(t)\hat{F}^{e+} \hat{F}^e$$

This gives $d\hat{\rho}(t)/dt$ as simply:

$$\frac{d\hat{\rho}(t)}{dt} = -i[\hat{H}, \hat{\rho}(t)] + \frac{1}{2}\sum_{e>0} K_e (2\hat{F}^e \hat{\rho}(t))\hat{F}^{e+} - \hat{F}^{e+} \hat{F}^e \hat{\rho}(t) - \hat{\rho}(t)\hat{F}^{e+} \hat{F}^e = -i[\hat{H}, \hat{\rho}(t)] + \mathcal{D}[\hat{\rho}].$$

Then, the time derivative of the operator expectation value is just:

$$\frac{d\langle \hat{A} \rangle}{dt} = \frac{d\text{Tr}\{\hat{A}\hat{\rho}(t)\}}{dt} = \text{Tr}\left\{-i[\hat{A}[\hat{H}, \hat{\rho}(t)]] + \hat{A}\mathcal{D}[\hat{\rho}]\right\} = i\langle [\hat{H}, \hat{A}] \rangle + \text{Tr}\{\hat{A}\mathcal{D}[\hat{\rho}]\}.$$

The first term is obviously just the Ehrenfest theorem. The second term is:

The first term is obviously just the Liouville theorem. The second term is:

$$\text{Tr}\{\hat{A}\bar{\partial}[\hat{\rho}]\} = \sum_{e>0} \frac{1}{2} \left(\text{Tr}\{2\hat{A}\hat{F}^e \hat{\rho}(t)\hat{F}^{e+}\} - \text{Tr}\{\hat{A}\hat{F}^{e+}\hat{F}^e \hat{\rho}(t)\} - \text{Tr}\{\hat{A}\hat{\rho}(t)\hat{F}^{e+}\hat{F}^e\} \right)$$

$$\text{Tr}\{\hat{A}\bar{\partial}[\hat{\rho}]\} = \text{Tr}\left\{\sum_{e>0} K_e \hat{F}^{e+} \hat{A} \hat{F}^e \hat{\rho}(t)\right\} - \frac{1}{2} \text{Tr}\left\{\sum_{e>0} K_e \hat{A} \hat{F}^{e+} \hat{F}^e \hat{\rho}(t)\right\} - \frac{1}{2} \text{Tr}\left\{\sum_{e>0} K_e \hat{A} \hat{\rho}(t) \hat{F}^{e+} \hat{F}^e\right\}$$

$$\text{Tr}\{\hat{A}\bar{\partial}[\hat{\rho}]\} = \left\langle \sum_{e>0} K_e \hat{F}^{e+} \hat{A} \hat{F}^e \right\rangle - \frac{1}{2} \left\langle \left\{ \sum_{e>0} K_e \hat{F}^{e+} \hat{F}^e, \hat{A} \right\} \right\rangle$$

Thus, we now have $\frac{d(\hat{A})}{dt}$ as:

$$\frac{d(\hat{A})}{dt} = i([H, \hat{A}]) + \text{Tr}\{\hat{A}\bar{\partial}[\hat{\rho}]\} = i([H, \hat{A}]) + \left\langle \sum_{e>0} K_e \hat{F}^{e+} \hat{A} \hat{F}^e \right\rangle - \frac{1}{2} \left\langle \left\{ \sum_{e>0} K_e \hat{F}^{e+} \hat{F}^e, \hat{A} \right\} \right\rangle$$

Time evolution in Heisenberg picture is unital rather than trace-preserving: in the Heisenberg picture, the identity operator \mathbb{I} does not evolve.

- From $\frac{d(\hat{A})}{dt}$ definition, can define adjoint dissipator as $\bar{\partial}[\hat{F}^e](\hat{A}) := \sum_{e>0} K_e \hat{F}^{e+} \hat{A} \hat{F}^e - \frac{1}{2} \sum_{e>0} \{K_e \hat{F}^{e+} \hat{F}^e, \hat{A}\}$
- $\bar{\partial}$ is a superoperator acting on observables \hat{A} , as opposed to Lindbladian i dissipator, which are superoperators acting on density matrices.

$$T \cdot \widehat{\bar{\partial}} \Gamma \gamma \quad 1 \quad \text{par} \gamma \quad 1 \quad 1 \quad \dots \quad 1 \quad \dots \quad \dots \quad n \quad 1 \quad \overline{n} \quad \dots \quad 1 \quad \overline{n} \quad \dots$$

- In $\hat{\mathcal{L}}[\hat{\rho}]$ and $\hat{\mathcal{D}}[\hat{\rho}]$, have both jump terms and deterministic terms. Conversely, in $\overline{\mathcal{D}}$, everything is always of the same order, with \hat{F}^{e+} always on the left. This allows us to factorise $\overline{\mathcal{D}}$ as:

$$\overline{\mathcal{D}}[\hat{F}^e](\hat{A}) = \sum_{e>0} K_e \hat{F}^{e+} \hat{A} \hat{F}^e - \frac{1}{2} \sum_{e>0} \{ K_e \hat{F}^{e+} \hat{F}^e, \hat{A} \} = \frac{1}{2} \sum_{e>0} K_e \hat{F}^e [\hat{A}, \hat{F}^e] + \frac{1}{2} \sum_{e>0} K_e [\hat{F}^{e+}, \hat{A}] \hat{F}^e.$$

- As always with Kraus operators, change of environment basis changes operators. In particular, we can change the basis for the Kraus operators $\hat{E}^{e>0}$ while leaving the operator $\hat{E}^{e=0}$ fixed, resulting in a transformation on $\hat{F}^{e>0}$. We can thus replace $\hat{F}^{e>0}$ with \hat{F}'^e defined by $\hat{F}'^e = \sum_{e>0} M_{pe} \hat{F}^e$, where M_{pe} is a unitary matrix. Different ways of choosing \hat{F}^e are different unravelings of the same Markovian dynamics.

Together, recycling term $\hat{F}^e \hat{\rho}(t) \hat{F}^{e+}$ and deterministic term \hat{R} allow time evolution of $\hat{\rho}(t)$ under $\hat{\mathcal{L}}$ to preserve CPTP properties, which are violated by the non-Hermitian operator (e.g. a non-Hermitian Hamiltonian, as discussed in some other open quantum systems literature) alone. Preservation of CPTPness can be demonstrated by trace preservation of $\hat{\rho}(t)$ under time evolution; or, equivalently, by showing that $\frac{d}{dt} \text{Tr}\{\hat{\rho}(t)\} = \text{Tr}\left\{\frac{d\hat{\rho}(t)}{dt}\right\} = 0$:

$$\text{Tr}\left\{\frac{d\hat{\rho}(t)}{dt}\right\} = \text{Tr}\{-i[\hat{H}, \hat{\rho}(t)]\} + \frac{1}{2} \sum_{e>0} 2K_e \text{Tr}\{\hat{F}^e \hat{\rho}(t) \hat{F}^{e+}\} - \sum_{e>0} K_e \text{Tr}\{\hat{F}^{e+} \hat{F}^e \hat{\rho}(t)\} - \sum_{e>0} K_e \text{Tr}\{\hat{\rho}(t) \hat{F}^{e+} \hat{F}^e\}$$

$$\text{Tr}\left\{\frac{d\hat{\rho}(t)}{dt}\right\} = -i \cancel{\text{Tr}\{\hat{H} \hat{\rho}(t)\}} + i \cancel{\text{Tr}\{\hat{\rho}(t) \hat{H}\}} + \sum_{e>0} \left(K_e \text{Tr}\{\hat{F}^{e+} \hat{F}^e \hat{\rho}(t)\} - \frac{K_e}{2} \text{Tr}\{\hat{F}^{e+} \hat{F}^e \hat{\rho}(t)\}^2 - \frac{K_e}{2} \text{Tr}\{\hat{F}^{e+} \hat{F}^e \hat{\rho}(t)\} \right) = 0$$

Then, the jump term accounts for the decay in probability caused by the deterministic term \hat{K} .

The formal solution to the equation $\frac{d\hat{\rho}(t)}{dt} = \hat{\mathcal{L}}[\hat{\rho}] = -i[\hat{H}, \hat{\rho}(t)] + \frac{1}{2} \sum_{e>\nu} K_e (2\hat{F}^e \hat{\rho}(t) \hat{F}^{e+} - \hat{F}^{e+} \hat{F}^e \hat{\rho}(t) - \hat{\rho}(t) \hat{F}^{e+} \hat{F}^e)$ is given by $\hat{\rho}(t) := e^{t\hat{\mathcal{L}}} [\hat{\rho}(t=0)]$. Expand in formal power series solution in $t\hat{\mathcal{L}}$ and $\hat{\rho}(t=0) =: \hat{\rho}_0$. Two ways to expand power series:

→ Re-express the $n \times n$ density matrices $\hat{\rho}$ as vectors of size $n^2 \times 1$. Called vectorisation / the Choi-Jamiolkowski isomorphism: stack columns from left to right. This is the technique used in the thesis.

$$*\text{ Ex.: } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

* Lets $\hat{\mathcal{L}}$ be expressed as an $n^2 \times n^2$ matrix acting on $\hat{\rho}_0$ from the left. Discussed much more in detail in Sec. 1.6, and used in the rest of the thesis.

→ If $\hat{\rho}_0$ is initially pure in S (i.e. $\hat{\rho}_0 = \sum_i p_i |e\rangle\langle e|$, $|e\rangle = |1\rangle\langle 1|$ for $|e\rangle \in \mathcal{B}(\mathcal{H})$ and $|1\rangle \in \mathcal{H}$), we can determine $\hat{\mathcal{L}}$ as the asymptotic limit of the ensemble average (where p_i denotes ensemble probabilities) of jump terms applied to $|1\rangle$.

During the time evolution of a specific instance, system is acted on by one instance of the jump term, and then re-normalised (not renormalised) at a discrete set of randomly generated times $\{T_n\}$; otherwise (at $t \neq T_n$), evolves deterministically under K for all $T \in [t_i, t_f] \setminus \{T_n\}$. This process is called unravelling. (Not the same unravelling as with Kraus operator redefinition.)

* Ex.: Start at time $t_i = 0$, with one jump operator. Generate a random number $r \in [0, 1]$, and let the system evolve under K until $\||\psi(t)\rangle\|^2 = |e^{iKt_1}|\psi_i\rangle|^2 = r$. Then, apply jump term \hat{J} ; re-normalise, yielding state $|\psi_1\rangle = \tilde{F}_c e^{iKt_1} |\psi_i\rangle / |\tilde{F}_c e^{iKt_1} |\psi_i\rangle|^2$. Repeat the same procedure with $|\psi_1\rangle$ (and subsequent $|\psi_m\rangle$'s) until we reach t_f . In limit of ∞ # of such trajectories, average over final states is exactly $\hat{\rho}(t)$.

* Might be relevant to our case. Covered in RMP 70, 191 (1998).

* Unravelling: useful numerically, and has a simple physical interpretation as the evolution of a continuously measured state conditioned on the measurement state.

o Ex.: If we have a photon detector measuring photon counts; jump operator represents photon loss, applying jump term to $\hat{\rho}$ represents detector click, and $e^{it\hat{K}}$ represents intra-detection time evolution of the detector.