

Albert 1.4: Which Lindbladians are the Focus of this Work?; and  
 Albert 1.6: A Technical Introduction (continued)

Thursday, August 22, 2019 03:13 PM

The form of  $\hat{\mathcal{L}}$  is not unique: we're free to apply a transformation (a sort of gauge transformation) that leaves  $\hat{\mathcal{L}}$  invariant but changes  $\hat{H}$  and  $\hat{F}$  by mixing  $\hat{H}$  and the jump operators. For  $g_e \in \mathbb{C}$ , this transformation is:

$$\hat{H} \mapsto \hat{H} - \frac{i}{2} \sum_e K_e (g_e^* \hat{F}^e - g_e \hat{F}^{e+}) ; \quad \hat{F}^e \mapsto \hat{F}^e + g_e \mathbb{1}$$

- There exists a unique set of  $\{g_e\}$ 's for which  $\text{Tr}\{\hat{F}^e\} = 0$  for all  $\hat{F}^e$ . Ref. 125, Theorem 2.2: review & reproduce.
- $\hat{\mathcal{L}}$  is also invariant under unitary transformations on jumps (another sort of gauge transformation).  $U(N)$  gauge transformation given for  $\hat{U}_f^e \in U(N)$  given by:

$$\sqrt{K_e} \hat{F}^e \mapsto \sum_f \hat{U}_f^e \sqrt{K_f} \hat{F}^f$$

The evolution of observables  $\hat{A} \in \mathcal{O}_p(\mathcal{H})$  in the Heisenberg picture,  $\frac{d\langle \hat{A}(t) \rangle}{dt} = \frac{d \text{Tr}\{\hat{p}_s \hat{A}(t)\}}{dt}$ , can be expressed in the double-ket language: from the inner product  $\text{Tr}\{\hat{A}^+(0) \hat{p}_s(t)\} = \langle\langle \hat{A}^+(0) | \hat{p}_s(t) \rangle\rangle$  and  $|\hat{p}_s(t)\rangle\rangle = e^{t\hat{\mathcal{L}}} |\hat{p}_s(0)\rangle\rangle$ , we have:

$$\langle\langle \hat{A}^+(0) | \hat{p}_s(t) \rangle\rangle = \langle\langle \hat{A}^+(0) | e^{t\hat{\mathcal{L}}} \hat{p}_s(0) \rangle\rangle = \langle\langle e^{t\hat{\mathcal{L}}^+} \hat{A}^+(0) | \hat{p}_s(0) \rangle\rangle = \langle\langle \hat{A}^+(t) | \hat{p}_s(0) \rangle\rangle$$

Here, since we have  $\langle\langle \hat{A}^+(t) | = \langle\langle \hat{A}^+(0) | e^{t\hat{\mathcal{L}}}$ , and thus  $e^{t\hat{\mathcal{L}}^+} |\hat{A}(0)\rangle\rangle = |\hat{A}(t)\rangle\rangle$ ,  $|\hat{A}(t)\rangle\rangle$  satisfies the diff. eq.  $\frac{d|\hat{A}(t)\rangle\rangle}{dt} = \hat{\mathcal{L}}^+ |\hat{A}(t)\rangle\rangle$ . From this the evolution of  $|\hat{A}(t)\rangle\rangle$  is the same as that derived in Section 1.2:

Here, since we have  $\langle\langle A(t) \rangle\rangle = \langle\langle A(0) \rangle\rangle e^{-iHt}$ , and thus  $e^{-iHt} |A(0)\rangle\rangle = |A(t)\rangle\rangle$ ,  $|A(t)\rangle\rangle$  satisfies the diff. eq.  $\frac{d}{dt} |A(t)\rangle\rangle = \mathcal{L} |A(t)\rangle\rangle$ . From this, the evolution of  $|\hat{A}(t)\rangle\rangle$  is the same as what we derived in Section 1.2:

$$\frac{d|\hat{A}(t)\rangle\rangle}{dt} = \hat{\mathcal{L}}^\pm |\hat{A}\rangle\rangle = -i[\hat{H}, \hat{A}] + \frac{1}{2} \sum_{e>0} K_e (2\hat{F}^{e+} \hat{A} \hat{F}^e - \hat{F}^{e+} \hat{F}^e \hat{A} - \hat{A} \hat{F}^{e+} \hat{F}^e)$$

$$\frac{d|\hat{A}(t)\rangle\rangle}{dt} = -i([\hat{H}, \hat{A}]) + \frac{1}{2} \sum_{e>0} K_e (2(\hat{F}^{e+} \hat{A} \hat{F}^e) - (\hat{F}^{e+} \hat{F}^e \hat{A} + \hat{A} \hat{F}^{e+} \hat{F}^e))$$

→ As shown in Sec. 1.2, the Hamiltonian and Lindbladian evolution both preserve the trace of  $\hat{\rho}$ . In double-ket language, this corresponds to  $\text{Tr}\{\hat{\rho}\} = \langle\langle 1|\hat{\rho}\rangle\rangle = 1$  and  $\text{Tr}\{\hat{\mathcal{L}}[\hat{\rho}]\} = \langle\langle 1|2|\hat{\rho}\rangle\rangle = 1$ .

\* As always,  $\text{Tr}\{\hat{\rho}\}$  is the norm of the wavefunction.

\* Thus, analytic functions  $f[\hat{\mathcal{L}}]$  of  $\hat{\mathcal{L}}$  also preserve the trace of  $\hat{\rho}$ .

\* Hermiticity is preserved:  $\hat{\mathcal{L}}|\hat{A}^+\rangle\rangle = \hat{\mathcal{L}}^\pm|\hat{A}\rangle\rangle = (\hat{\mathcal{L}}|\hat{A}\rangle\rangle)^\pm$ .

\* Norm / purity of  $\hat{\rho}$ , given by  $\text{Tr}\{\hat{\rho}^2\} = \langle\langle \hat{\rho}|\hat{\rho}\rangle\rangle$ , is not always preserved.

Initial states undergoing Lindbladian evolution evolve into infinite-time states (a.k.a. asymptotic states)  $\hat{\rho}_{\infty,s}$ , given by:

$$\hat{\rho}_{\infty,s} := \lim_{t \rightarrow \infty} e^{t\hat{\mathcal{L}}} [\hat{\rho}_{in,s}] = e^{-i\hat{H}_\infty t} \hat{P}_\infty [\hat{\rho}_{in,s}] e^{i\hat{H}_\infty t}$$

→ Nonunitary effects of  $\hat{\mathcal{L}}$  on the system subspace are encapsulated in the asymptotic projection superoperator  $\hat{P}_\infty$ , with  $\hat{P}_\infty^2 = \hat{P}_\infty$ .

→ The Hamiltonian  $\hat{H}_\infty$  qualifies residual unitary evolution within the system, which persists for all time and does not exhibit further decoherence.

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Asymptotic states  $\{\hat{\rho}_{\infty}\}$  are elements of the asymptotic subspace  $As(\hat{\mathcal{L}})$ , which is a subspace of  $O_p(\mathcal{H})$  defined from  $\hat{P}_{\infty}$  by  
 $As(\mathcal{H}) := \hat{P}_{\infty} [O_p(\mathcal{H})]$ .

→  $As(\hat{\mathcal{L}})$  attracts all initial states, is free from non-unitary effects of  $\hat{\mathcal{L}}$ , and all time evolution within  $As(\hat{\mathcal{L}})$  is exclusively unitary.

\* Thus,  $As(\hat{\mathcal{L}})$  is a Hamiltonian-evolving subspace of the larger Lindbladian-evolving space  $O_p(\mathcal{H})$ .

\* If  $As(\hat{\mathcal{L}})$  has no further evolution, then  $\hat{H}_{\infty} = 0$  and all  $\{\hat{\rho}_{\infty,s}\}$  are stationary/steady.

### 1.4.1: Multiple Steady States

When  $\dim As(\hat{\mathcal{L}}) = 1$ , only one asymptotic state exists, and all  $\hat{\rho}_{in,s}$  converge to it. For  $\dim As(\hat{\mathcal{L}}) > 1$ , multiple asymptotic states.  $\hat{\rho}_{\infty,s}$  depends on  $\hat{\rho}_{in,s}$ .

→ Former happens generically (i.e. for any Lindbladian we choose at random):  $\hat{\mathcal{L}}$  with multiple steady states is a set of measure 0 in the set of all possible  $\hat{\mathcal{L}}$ .

Albert Refs. 280, 281, 283: stabilisability of subspaces. Worth looking at.

Albert: interested in  $As(\hat{\mathcal{L}}) > 1$  which can support quantum information, especially when these states could exhibit topological protection; or can help experimental probes into driven-dissipative open systems.

### 1.4.2: Presence of Population Decay

Unlike Hamiltonian dynamics, Lagrangian dynamics exhibit population decay, defined as the existence of at least one state  $|q\rangle \in \mathcal{H}$  such that we have  $\langle q|e^{t\hat{\mathcal{L}}} [|\psi\rangle\langle\psi|] |q\rangle \xrightarrow{t \rightarrow \infty} 0$ .

Need to distinguish between decaying and nondecaying parts of  $\mathcal{H}$ , which we can do via matrix representation of  $\hat{A} \in \mathcal{O}_p(\mathcal{H})$ . Block diagonalise:

- Non-decaying parts of  $\hat{A}$ : . Decoherence-free subspace (DFS).
- Completely decaying parts of  $\hat{A}$ : .
- Coherences in off-diagonal pieces: .
- Generally,  $As(\mathcal{H}) \subset \boxed{\phantom{0}}$

\* Ex. of  $As(\mathcal{H}) \subset \boxed{\phantom{0}}$ : in NMR, have an asymptotic ground state subspace  $\{|q_k\rangle\langle q_\ell|\}_{k,\ell=0}^{2^{n-1}}$ . To get our  $As(\mathcal{H}) \subset \boxed{\phantom{0}}$ , let coherences between  $|q_k\rangle$  and  $|q_{k+\ell}\rangle$  go to 0.

Operator  $\hat{I}$ : superoperator projections on blocks:

→ Operator projections: used for projecting  $|\psi\rangle \in \mathcal{H}$  onto the relevant subspaces.

\*  $\hat{Y}$ : projector onto asymptotic subspace / maximal invariant subspace of  $\mathcal{H}$ . (Note that  $\hat{Y}\mathcal{H}$  is not  $As(\mathcal{H})$ , since  $As(\mathcal{H}) \subset \mathcal{O}_p(\mathcal{H})$ .)  
 $\hat{Y}$  is uniquely defined by:

◦  $\hat{P}_{\infty,s} = \hat{Y} \hat{P}_{\infty,s} \hat{Y}$  for all  $\hat{P}_{\infty,s} \in As(\mathcal{H})$ . (This condition ensures we project onto all non-decaying subspaces.)

◦  $\text{Tr } \hat{Y} = \max_{\hat{P}_{\infty,s}} \{\text{rank } \hat{P}_{\infty,s}\}$ . (This condition ensures we don't project onto any decaying subspace.)

\*  $\hat{S}$ : projector onto maximal decaying subspace of  $\mathcal{H}$ . Defined by:  $\hat{S} := \hat{I} - \hat{Y}$ , with  $\hat{Y}\hat{S} = \hat{S}\hat{Y} = 0$ .  
◦  $As(\mathcal{H}) \subset \hat{Y}$

\*  $\hat{S}$ : projector onto maximal decaying subspace of  $\mathcal{H}$ . Defined by:  $\hat{S} := \hat{\mathbb{I}} - \hat{Y}$ , with  $\hat{Y}\hat{S} = \hat{S}\hat{Y} = 0$ .

- Naturally,  $\hat{S} \perp \hat{Y}$ .

- $\hat{S}\hat{P}_s(t)\hat{S} \rightarrow 0$  as  $t \rightarrow \infty$ .

→ The projections  $\hat{Y}, \hat{S}$  on  $\mathcal{H}$  induce superoperator projections on  $\text{As}(\mathcal{H})$ , called the four-corners projections:

$$*\hat{A}_{\square} := \hat{P}_{\square} |\hat{A}\rangle\rangle = \hat{Q}\hat{A}\hat{Y}.$$

$$*\hat{A}_{\square} := \hat{P}_{\square} |\hat{A}\rangle\rangle = \hat{Q}\hat{A}\hat{S}.$$

$$*\hat{A}_{\square} := \hat{P}_{\square} |\hat{A}\rangle\rangle = \hat{S}\hat{A}\hat{Y}.$$

$$*\hat{A}_{\square} := \hat{P}_{\square} |\hat{A}\rangle\rangle = \hat{S}\hat{A}\hat{S}.$$

→ Properties of superoperators:

\*  $\hat{Y}$  projects onto the support of  $\hat{P}_{\infty}|\hat{\mathbb{I}}\rangle\rangle$ .

\* Projection acts before adjoint, so that taking the Hermitian conjugate of the upper-right part yields the lower-left part. Thus, in our notation,  $(\hat{A}_{\square})^+ = (\hat{A}^+)_{\square}$ .

- We define  $\hat{A}_{\square}^+ := (\hat{A}_{\square})^+$ . Note that this means that  $\hat{A}_{\square} = (\hat{A}_{\square}^+)^+$ : in our notation, when we see a corner  $\square$  and a dagger  $\dagger$  without parenthesis, that's the adjoint of an operator in the same corner. This is just a notation thing: Adjoints flip corners along the diagonal, always, but we start in our notation identifying an adjointed operator with the operator and corner it came from.

$$*\hat{P}_{\square}^2 = \hat{P}_{\square}. \text{ (These guys are projectors, after all, on } \text{Op}(\mathcal{H})\text{!)}$$

\*  $\hat{P}_{\square}$  partitions the identity on  $\text{Op}(\mathcal{H})$ :  $\hat{\mathbb{I}} = \hat{P}_{\square} + \hat{P}_{\square}^+ + \hat{P}_{\square}^{\perp} + \hat{P}_{\square}^{\perp\perp}$ .

- They're also additive as expected; e.g.:  $\hat{P}_{\square} + \hat{P}_{\square}^+ = \hat{P}_{\square}$ .

\* Subspace  $\mathbb{E} := \hat{P}_{\square} \text{Op}(\mathcal{H})$  is the space of all coherences between  $\hat{Q}\mathcal{H}$  and  $\hat{S}\mathcal{H}$ .

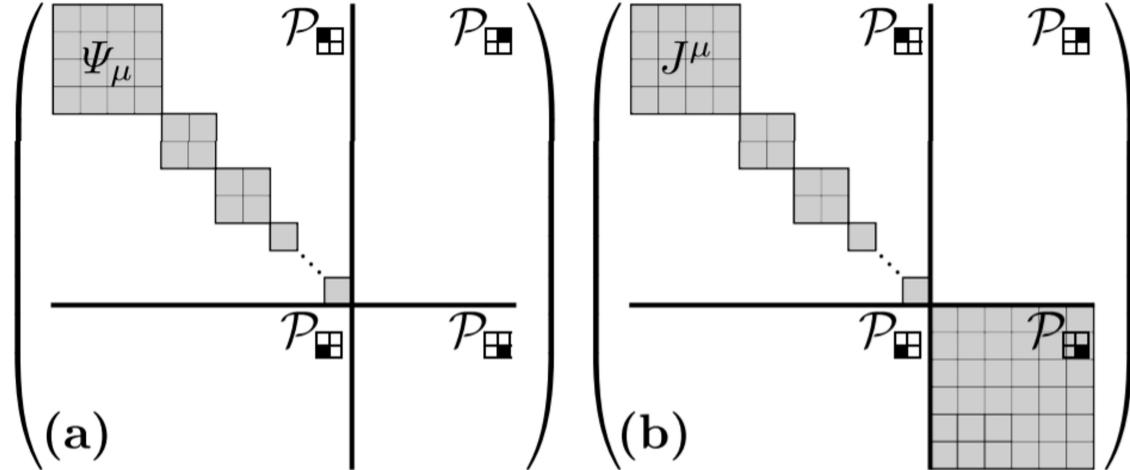


Figure 1.2: Decompositions of the space of matrices  $\text{Op}(H)$  acting on a Hilbert space  $H$  using the projections  $\{P, Q\}$  defined in (1.14) and their corresponding superoperator projections  $\{\mathcal{P}_{\square}, \mathcal{P}_{\square}, \mathcal{P}_{\square}, \mathcal{P}_{\square}\}$  defined in (1.16). Panel **(a)** depicts the block diagonal structure of the asymptotic subspace  $\text{As}(H)$ , which is spanned by steady-state basis elements  $\Psi_\mu$  (cf. [61], Fig. 3). Panel **(b)** depicts the subspace of  $\text{Op}(H)$ , spanned by conserved quantities  $J_\mu$ . These quantities determine dependence of the final (asymptotic) state  $\rho_\infty$  on the initial state  $\rho_{\text{in}}$ .

Operator Notation	Superoperator Notation	Operator Notation	Superoperator Notation
$\mathcal{L}(\rho)$	$\mathcal{L} \rho\rangle\langle\rho $	$A_{\square} \equiv PAP$	$ A_{\square}\rangle\langle A  \equiv \mathcal{P}_{\square} A\rangle\langle A $
$\text{Tr}\{A^{\dagger}\mathcal{O}(\rho)\}$	$\langle\langle A \mathcal{O} \rho\rangle\rangle$	$A_{\square} \equiv PAQ$	$ A_{\square}\rangle\langle A  \equiv \mathcal{P}_{\square} A\rangle\langle A $
$-i[H, \rho]$	$\mathcal{H} \rho\rangle\langle\rho $	$A_{\square} \equiv QAP$	$ A_{\square}\rangle\langle A  \equiv \mathcal{P}_{\square} A\rangle\langle A $
$-i[V, \rho]$	$\mathcal{V} \rho\rangle\langle\rho $	$A_{\square} \equiv QAQ$	$ A_{\square}\rangle\langle A  \equiv \mathcal{P}_{\square} A\rangle\langle A $
$U\rho U^{\dagger}$	$\mathcal{U} \rho\rangle\langle\rho $	$P\mathcal{L}(Q\rho Q)P$	$\mathcal{P}_{\square}\mathcal{L}\mathcal{P}_{\square} \rho\rangle\langle\rho $
$S\rho S^{\dagger}$	$\mathcal{S} \rho\rangle\langle\rho $	$i\text{Tr}\{H[\Psi_{\mu}, \Psi_{\nu}]\}$	$\langle\langle\Psi_{\mu} \mathcal{H} \Psi_{\nu}\rangle\rangle$

Table 1.1: Comparison of operator and superoperator notations for symbols used throughout the text (cf. Table 3.2 in [199]).  $\mathcal{L}$  is a Lindbladian superoperator (1.8),  $\mathcal{O}$  is a superoperator,  $A$  is an operator, and  $\rho$  is a density matrix. Hamiltonians  $H$  and  $V$  have corresponding Hamiltonian superoperators  $\mathcal{H}$  and  $\mathcal{V}$ , respectively. Unitary operators  $U$  and  $S$  have corresponding unitary superoperators  $\mathcal{U}$  and  $\mathcal{S}$ , respectively. The projection  $P$  (3.34) projects onto the largest subspace whose states do not decay under  $\mathcal{L}$  and  $Q \equiv I - P$  with  $I$  the identity. The last two entries respectively represent the part  $\mathcal{P}_{\square}\mathcal{L}\mathcal{P}_{\square}$  of the projection decomposition of  $\mathcal{L}$  (1.8) acting on  $\rho$  and a (superoperator) matrix element of  $\mathcal{H}$  in terms of a Hermitian matrix basis  $\{\Psi_{\mu}\}$ .

Special cases of systems with no decaying subspaces  $\boxed{\square}$ .

→ Hamiltonian case: we can write a Hamiltonian  $\hat{H}$  such that  $\hat{\mathcal{L}}[\hat{A}] = -i[\hat{A}, \hat{A}]$ .

- Hamiltonian case: we can write a Hamiltonian  $\hat{H}$  such that  $\hat{\mathcal{L}}[\hat{A}] = -i[\hat{H}, \hat{A}]$ .
- Unique-state case: if there's one steady state  $\hat{p}_{\text{ss}}$  with a spectral decomposition  $\sum_{k=0}^{\dim(\hat{p}_{\text{ss}})-1} \lambda_k |\psi_k\rangle\langle\psi_k|$ , and all  $\lambda_k = 0$  (e.g. Gibbs state).

### 1.6.2: More on Lindbladians

As discussed before, eigenvalues  $\{\lambda_{a,b}\}$  of  $\hat{\mathcal{L}}$  have  $\text{Re}\{\lambda_{a,b}\} \leq 0$ . As with the zero eigenvalue, all eigenvalues for which  $\text{Re}\{\lambda_{a,b}\} = 0$  don't decay (since we require  $\text{Re}\{\lambda_{a,b}\} < 0$  to decay); the corresponding states define  $\text{As}(\mathcal{L})$ .

- Dissipative gap / dissipation gap / damping gap / relaxation gap / asymptotic decay rate: slowest non-zero rate of converge towards  $\text{As}(\mathcal{L})$ . Defined by  $\Delta_{dg} := \min_{\text{Re}\{\lambda_{a,b}\} = 0} |\text{Re}\{\lambda_{a,b}\}|$

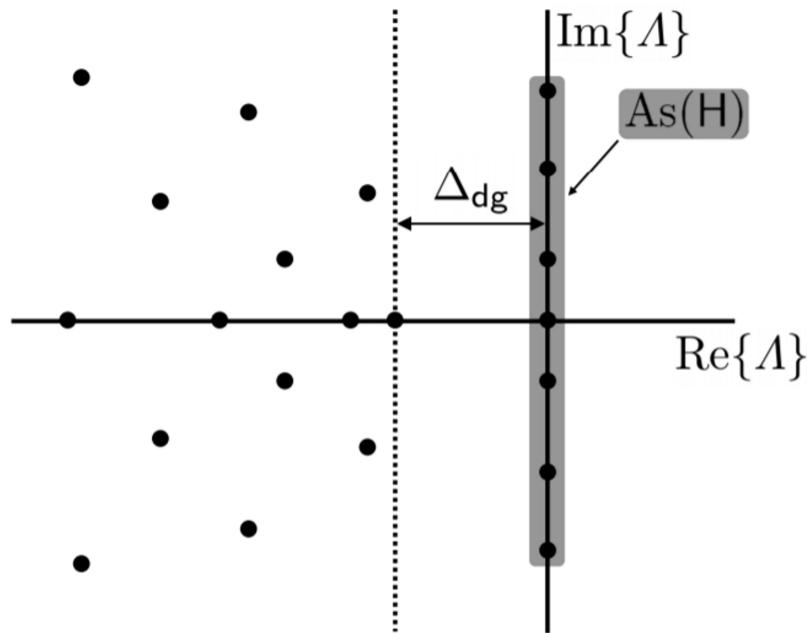


Figure 1.3: A plot of a spectrum of an example  $\mathcal{L}$  with 21 eigenvalues  $\Lambda$  in the complex plane.

As before, spectrally decompose  $e^{t\hat{\mathcal{L}}}$  into left  $\dagger$  right eigenstates. Now, label eigenstates with eigenvalue  $\lambda$  and intra-degeneracy index  $v$ .

$$e^{t\hat{\mathcal{L}}} |\hat{p}_{in,s}\rangle\rangle = \sum_{\lambda,v}^! e^{\lambda t} |\hat{x}_{\lambda v}\rangle\rangle \langle\langle \hat{y}_{\lambda v} | \hat{p}_{in,s}\rangle\rangle = \sum_{\lambda,v}^! c_{\lambda v} e^{\lambda t} |\hat{x}_{\lambda v}\rangle\rangle$$

If  $\hat{\mathcal{L}}$  is not diagonalisable, there exists at least one Jordan block of  $\hat{\mathcal{L}}$  in Jordan normal form with only one eigenstate (eigenmatrix). Generalised eigenmatrices: remaining matrix basis elements in  $\text{supp}\{\text{Jordan block}\}$ .

$\rightarrow$  Hall's Lie Groups, Lie Algebras, & Representations has a really good summary of Jordan normal forms (in the context of representations of the Heisenberg group). Also review Meyer Linear Algebra.

$\rightarrow \exp\{\hat{\mathcal{L}}\}$  due to this yields extra powers of  $t$  in front of  $e^{\lambda t} |\hat{x}_{\lambda v}\rangle\rangle \langle\langle \hat{y}_{\lambda v} | \hat{p}_{in,s}\rangle\rangle = c_{\lambda v} e^{\lambda t} |\hat{x}_{\lambda v}\rangle\rangle$ , as well as off-diagonal elements  $|\hat{x}_{\lambda v}\rangle\rangle \langle\langle \hat{y}_{\lambda u+v}|$ .

\* Ex.: if  $\lambda$  has a 2D Jordan block with right eigenmatrix  $|\hat{x}_{\lambda 0}\rangle\rangle$  (so that we have  $\hat{\mathcal{L}}|\hat{x}_{\lambda 0}\rangle\rangle = \lambda |\hat{x}_{\lambda 0}\rangle\rangle$ ) and generalised right eigenmatrix  $|\hat{x}_{\lambda 1}\rangle\rangle$  (so that we have  $\hat{\mathcal{L}}|\hat{x}_{\lambda 1}\rangle\rangle = |\hat{x}_{\lambda 0}\rangle\rangle + \lambda |\hat{x}_{\lambda 1}\rangle\rangle$ ), then  $e^{t\hat{\mathcal{L}}}$  on this 2D Jordan block is given by:

$$e^{t\lambda} \left( |\hat{x}_{\lambda 0}\rangle\rangle \langle\langle \hat{y}_{\lambda 0} | + t |\hat{x}_{\lambda 1}\rangle\rangle \langle\langle \hat{y}_{\lambda 0} | + |\hat{x}_{\lambda 0}\rangle\rangle \langle\langle \hat{y}_{\lambda 1} | \right) = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

We see that when we partition the Jordan normal form of  $\hat{\mathcal{L}}$  into these blocks, we have both diagonal sub-blocks (i.e.  $|\hat{x}_{\lambda v}\rangle\rangle$ ),  $\langle\langle \hat{y}_{\lambda v} |$ ), off-diagonal sub-blocks with upper diagonals of all 1s (i.e.  $|\hat{x}_{\lambda 0}\rangle\rangle \langle\langle \hat{y}_{\lambda 1} |$ ), and terms that mix the two.

As in the example, if we partition the Jordan normal form of  $\hat{\mathcal{L}}$  into blocks that are either diagonal or have an upper diagonal of all 1s, we

As in the example, if we partition the Jordan normal form of  $\hat{J}$  into blocks that are either diagonal or have an upper diagonal of all 1s, we note that for any given  $\lambda$ , we might have terms with both diagonal & off-diagonal sub-blocks. Thus, the sum over  $\lambda$  has to include each sub-block separately. This gives the full expansion of  $e^{t\hat{J}}$  in terms of the Jordan normal form as:

$$e^{t\hat{J}} |\hat{\rho}_{in,s}\rangle\rangle = \sum_{a,b} e^{t\lambda_{a,b}} |\hat{x}_{a,b}\rangle\rangle \langle\langle \hat{y}_{a,b} | \hat{\rho}_{in,s}\rangle\rangle = \begin{cases} \sum_{\lambda,v} e^{t\lambda} |\hat{x}_{\lambda v}\rangle\rangle \langle\langle \hat{y}_{\lambda v} | \hat{\rho}_{in,s}\rangle\rangle & (\text{diagonal Jordan blocks}) \\ \sum_{\lambda,v} \sum_{\mu \geq v} \frac{e^{t\lambda} t^{v-\mu}}{(\mu-v)!} |\hat{x}_{\lambda v}\rangle\rangle \langle\langle \hat{y}_{\lambda v} | \hat{\rho}_{in,s}\rangle\rangle & (\text{nondiagonal Jordan blocks}) \end{cases}$$

$\rightarrow v, \mu \in \mathbb{N}$  indexes the generalised eigenmatrices of the Jordan block of  $\lambda$ .

All Jordan blocks with pure imaginary eigenvalues are diagonal.

$\rightarrow$  Proof: to be done later. Outline: AFSOC  $\hat{J}$  is not diagonalisable in subspace of Jordan normal form whose diagonals all have  $\text{Re } \lambda = 0$ . Then, if we take the exponential of those Jordan blocks, the dynamics diverges as  $t \rightarrow \infty$  (presumably by giving us an overall factor of  $e^a$  for  $\text{Re } a > 0$ ).

$\rightarrow$  Kinda obvious from the form of nondiagonal Jordan blocks.

### 1.6.3: Double-Bra/Ket Basis for Steady States

More exposition on bases of  $\mathcal{D}_p(\mathcal{H})$  for basis of  $\mathcal{H}$ . As before, from the ON basis  $\{|t_a\rangle\}_{a=0}^{N-1}$  for  $\mathcal{H}$  ( $N$ -dim.), the ON basis for  $\mathcal{D}_p(\mathcal{H})$  ( $N^2 - \text{dim.}$ ) is  $\{|\Phi_{ab}\rangle\rangle\}_{a,b=0}^{N-1}$  with  $|\Phi_{ab}\rangle\rangle := |\psi_a\rangle\langle\psi_b|$ .

$\rightarrow |\Phi_{ab}\rangle\rangle$  is a "physical" ON basis for  $\mathcal{D}_p(\mathcal{H})$ , since  $|\Phi_{ab}\rangle\rangle := |\psi_a\rangle\langle\psi_b|$ .

(N-dim.) is  $\{|\psi_{ab}\rangle\}_{a,b=0}^N$  with  $|\psi_{ab}\rangle = |\psi_a\rangle \otimes |\psi_b\rangle$ .

$\rightarrow |\hat{\rho}_{ab}\rangle$  is a "physical" ON basis for  $O_p(\mathcal{H})$ , since  $|\hat{\rho}_{ab}\rangle := |\psi_a\rangle \langle \psi_b|$ .

$\rightarrow$  Normalised Hermitian ON basis  $\{|\tilde{\rho}_\alpha\rangle\}_{\alpha=0}^{N^2-1}$  with  $\tilde{\rho}_\alpha^\dagger = \tilde{\rho}_\alpha$  can also be constructed.

$$* \langle\langle \tilde{\rho}_\alpha | \tilde{\rho}_\beta \rangle\rangle = \text{Tr} \{ \tilde{\rho}_\alpha^\dagger \tilde{\rho}_\beta \} = \text{Tr} \{ \tilde{\rho}_\alpha \tilde{\rho}_\beta \} = \delta_{\alpha\beta}.$$

\* These are linear superpositions of  $|\hat{\rho}_{ab}\rangle$ s.

\* These are not density matrices.

\* They are the basis matrices for N-dim. irreps of Lie algebra  $su(N)$ .

o Ex.:  $\dim \mathcal{H} = N = 2$  gives  $\dim O_p(\mathcal{H}) = N^2 - 1 = 3$ . Basis: Pauli matrices.

o Ex.:  $\dim \mathcal{H} = N = 3$  gives  $\dim O_p(\mathcal{H}) = N^2 - 1 = 8$ . Basis: Gell-Mann matrices.

\* Coefficients of any Hermitian operator in this basis are real.

o Ex.: for a density matrix,  $|\hat{\rho}\rangle = \sum_{\alpha=0}^{N^2-1} c_\alpha |\tilde{\rho}_\alpha\rangle$ .  $c_\alpha = \langle\langle \tilde{\rho}_\alpha | \hat{\rho} \rangle\rangle \in \mathbb{R}$  represent components of generalised Bloch / coherence vector. (This is why the basis is N-dim irreps of  $su(N)$ .)

\* Superoperator  $\hat{A} = \sum_{\alpha,\beta=0}^{N^2-1} A_{\alpha\beta} |\tilde{\rho}_\alpha\rangle \langle\langle \tilde{\rho}_\beta |$  has matrix elements  $A_{\alpha\beta} := \langle\langle \tilde{\rho}_\alpha | \hat{A} | \tilde{\rho}_\beta \rangle\rangle = \text{Tr} \{ \tilde{\rho}_\alpha^\dagger \hat{A} [\tilde{\rho}_\beta] \}$

o Observables: Hermitian superoperators over  $O_p(\mathcal{H})$ . Matrix elements are real. (Proof in Albert Pg. 28, include in typeset notes (but it's exactly what you'd expect).)

Theorem: (when Lindbladians generate unitary evolution.) Lindbladian matrix elements  $\mathcal{L}_{\alpha\beta} \in \mathbb{R}$ . Furthermore,  $\mathcal{L}_{\alpha\beta} = -\mathcal{L}_{\beta\alpha}$  iff  $\hat{\mathcal{L}}[\hat{\rho}] = -i[\hat{H}, \hat{\rho}]$ .

Proof: Albert Pgs. 28-29. Include in typeset notes.