Albert 1.4: Which Lindbladians are the Focus of this Work?; and
Albert 1.6: A Technical Introduction (continued)

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The form of $\hat{\hat{\mathcal{L}}}$ is <u>not</u> unique: we're free to apply a transformation (a sort of gauge transformation) that leaves $\hat{\hat{\mathcal{L}}}$ invariant but changes $\hat{H}$ and $\hat{F}$ by mixing $\hat{H}$ and the jump operators. For $g_e \in \mathbb{C}$, this transformation is:

$$\hat{H} \mapsto \hat{H} - \tfrac{i}{2} \sum_e{}' \kappa_e \left(g_e^* \hat{F}^e - g_e \hat{F}^{e\dagger} \right); \quad \hat{F}^e \mapsto \hat{F}^e + g_e \mathbb{1}$$

→ There exists a unique set of $\{g_e\}$s for which $\mathrm{Tr}\{\hat{F}^e\} = \emptyset$ for all $\hat{F}^e$. Ref. 125, Theorem 2.2: review & reproduce.

→ $\hat{\hat{\mathcal{L}}}$ is also invariant under unitary transformations on jumps (another sort of gauge transformation). $U(N)$ gauge transformation given for $\hat{U}^e_f \in U(N)$ given by:

$$\sqrt{\kappa_e}\, \hat{F}^e \mapsto \sum_f{}' \hat{U}^e_f \sqrt{\kappa_f}\, \hat{F}^f$$

The evolution of observables $\hat{A} \in O_p(\mathcal{H})$ in the Heisenberg picture, $\frac{d\langle \hat{A}(t)\rangle}{dt} = \frac{d\,\mathrm{Tr}\{\hat{\rho}_s \hat{A}(t)\}}{dt}$, can be expressed in the double-ket language: from the inner product $\mathrm{Tr}\{\hat{A}^\dagger(\emptyset)\hat{\rho}_s(t)\} = \langle\langle \hat{A}^\dagger(\emptyset) | \hat{\rho}_s(t)\rangle\rangle$ and $|\hat{\rho}_s(t)\rangle\rangle = e^{t\hat{\hat{\mathcal{L}}}}$, we have:

$$\langle\langle \hat{A}^\dagger(\emptyset) | \hat{\rho}_s(t)\rangle\rangle = \langle\langle \hat{A}^\dagger(\emptyset) | e^{t\hat{\hat{\mathcal{L}}}} \hat{\rho}_s(\emptyset)\rangle\rangle = \langle\langle e^{t\hat{\hat{\mathcal{L}}}^\ddagger} \hat{A}^\dagger(\emptyset) | \hat{\rho}_s(\emptyset)\rangle\rangle = \langle\langle \hat{A}^\dagger(t) | \hat{\rho}_s(\emptyset)\rangle\rangle$$

Here, since we have $\langle\langle \hat{A}^\dagger(t)| = \langle\langle \hat{A}^\dagger(\emptyset)| e^{t\hat{\hat{\mathcal{L}}}}$, and thus $e^{t\hat{\hat{\mathcal{L}}}^\ddagger} |\hat{A}(\emptyset)\rangle\rangle = |\hat{A}(t)\rangle\rangle$, $|\hat{A}(t)\rangle\rangle$ satisfies the diff. eq. $\frac{d|\hat{A}(t)\rangle\rangle}{dt} = \hat{\hat{\mathcal{L}}}^\ddagger |\hat{A}(t)\rangle\rangle$.

From this, the evolution of $|\hat{A}(t)\rangle\rangle$ is the same as what we derived in Section 1.2:

Here, since we have $\langle\langle A(t)| = \langle\langle A(0)| e^{t\mathcal{L}}$, and thus $e^{t\mathcal{L}^\dagger} |A(0)\rangle\rangle = |A(t)\rangle\rangle$, $|A(t)\rangle\rangle$ satisfies the diff. eq. $\frac{d|A(t)\rangle\rangle}{dt} = \mathcal{L}^\dagger |A(t)\rangle\rangle$.
From this, the evolution of $|\hat{A}(t)\rangle\rangle$ is the same as what we derived in Section 1.2:

$$\frac{d\hat{A}(t)}{dt} = \hat{\mathcal{L}}^\dagger[\hat{A}] = -i[\hat{H},\hat{A}] + \frac{1}{2}\sum_{e>\emptyset}' \kappa_e \left(2\hat{F}^{e\dagger}\hat{A}\hat{F}^e - \hat{F}^{e\dagger}\hat{F}^e\hat{A} - \hat{A}\hat{F}^{e\dagger}\hat{F}^e\right)$$

$$\frac{d\langle\hat{A}(t)\rangle}{dt} = -i\left(\langle[\hat{H},\hat{A}]\rangle\right) + \frac{1}{2}\sum_{e>\emptyset}' \kappa_e \left(2\langle\hat{F}^{e\dagger}\hat{A}\hat{F}^e\rangle - \langle\hat{F}^{e\dagger}\hat{F}^e\hat{A} + \hat{A}\hat{F}^{e\dagger}\hat{F}^e\rangle\right)$$

→ As shown in Sec. 1.2, the Hamiltonian and Lindbladian evolution both preserve the trace of $\hat{\rho}$. In double-ket language, this corresponds to $\mathrm{Tr}\{\hat{\rho}\} = \langle\langle \mathbb{1} | \hat{\rho}\rangle\rangle = 1$ and $\mathrm{Tr}\{\hat{\mathcal{L}}[\hat{\rho}]\} = \langle\langle \mathbb{1} | \hat{\mathcal{L}} | \hat{\rho}\rangle\rangle = 1$.
- * As always, $\mathrm{Tr}\{\hat{\rho}\}$ is the norm of the wavefunction.
- * Thus, analytic functions $f[\hat{\mathcal{L}}]$ of $\hat{\mathcal{L}}$ also preserve the trace of $\hat{\rho}$.
- * Hermiticity is preserved: $\hat{\mathcal{L}}|\hat{A}^\dagger\rangle\rangle = \hat{\mathcal{L}}^\dagger|\hat{A}\rangle\rangle = (\hat{\mathcal{L}}|\hat{A}\rangle\rangle)^\dagger$.
- * <u>Norm</u> / <u>purity</u> of $\hat{\rho}$, given by $\mathrm{Tr}\{\hat{\rho}^2\} = \langle\langle\hat{\rho}|\hat{\rho}\rangle\rangle$, is not always preserved.

Initial states undergoing Lindbladian evolution evolve into infinite-time states (a.k.a. <u>asymoptic states</u>) $\hat{\rho}_{\infty,s}$, given by:

$$\hat{\rho}_{\infty,s} := \lim_{t\to\infty} e^{t\hat{\mathcal{L}}}[\hat{\rho}_{in,s}] = e^{-i\hat{H}_\infty t}\, \hat{\mathcal{P}}_\infty[\hat{\rho}_{in,s}]\, e^{i\hat{H}_\infty t}$$

→ Nonunitary effects of $\hat{\mathcal{L}}$ <u>on the system subspace</u> are encapsulated in the <u>asymptotic projection superoperator</u> $\hat{\mathcal{P}}_\infty$, with $\hat{\mathcal{P}}_\infty^2 = \hat{\mathcal{P}}_\infty$.
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Asymptotic states $\{\hat{\rho}_\infty\}$ are elements of the asymptotic subspace $As(\mathcal{H})$, which is a subspace of $O_p(\mathcal{H})$ defined from $\hat{\hat{P}}_\infty$ by $As(\mathcal{H}) := \hat{\hat{P}}_\infty[O_p(\mathcal{H})]$.

→ $As(\mathcal{H})$ attracts all initial states, is free from non-unitary effects of $\hat{\hat{\mathcal{L}}}$, and all time evolution within $As(\mathcal{H})$ is exclusively unitary.

* Thus, $As(\mathcal{H})$ is a Hamiltonian-evolving subspace of the larger Lindbladian-evolving space $O_p(\mathcal{H})$.
* If $As(\mathcal{H})$ has no further evolution, then $\hat{H}_\infty = \emptyset$ and all $\{\hat{\rho}_{\infty,s,i}\}$ are stationary / steady.

1.4.1: Multiple Steady States

When $\dim As(\mathcal{H}) = 1$, only one asymptotic state exists, and all $\hat{\rho}_{in,s}$ converge to it. For $\dim As(\mathcal{H}) > 1$, multiple asymptotic states. $\hat{\rho}_{\infty,s}$ depends on $\hat{\rho}_{in,s}$.

→ Former happens generically (i.e. for any Lindbladian we choose at random): $\hat{\hat{\mathcal{L}}}$ with multiple steady states is a set of measure $\emptyset$ in the set of all possible $\hat{\hat{\mathcal{L}}}$.

Albert Refs. 28∅, 281, 283: stabilisability of subspaces. Worth looking at.

Albert: interested in $As(\mathcal{H}) > 1$ which can support quantum information, especially when these states could exhibit topological protection; or can help experimental probes into driven-dissipative open systems.

1.4.2: Presence of Population Decay

Unlike Hamiltonian dynamics, Lagrangian dynamics exhibit population decay, defined as the existence of at least one state $|\psi\rangle \in \mathcal{H}$ such that we have $\langle\psi| e^{t\hat{\mathcal{L}}}[|\psi\rangle\langle\psi|]|\psi\rangle \xrightarrow{t\to\infty} \emptyset$.

Need to distinguish between decaying and nondecaying parts of $\mathcal{H}$, which we can do via matrix representations of $\hat{A} \in O_p(\mathcal{H})$. Block diagonalise:

→ Non-decaying parts of $\hat{A}$: $\begin{bmatrix}\blacksquare & \square \\ \square & \square\end{bmatrix}$. <u>Decoherence-free subspace (DFS).</u>

→ Completely decaying parts of $\hat{A}$: $\begin{bmatrix}\square & \square \\ \square & \blacksquare\end{bmatrix}$.

→ Coherences in off-diagonal pieces: $\begin{bmatrix}\square & \blacksquare \\ \blacksquare & \square\end{bmatrix}$.

→ Generally, $As(\mathcal{H}) \subseteq \begin{bmatrix}\blacksquare & \square \\ \square & \square\end{bmatrix}$

* Ex. of $As(\mathcal{H}) \subset \begin{bmatrix}\blacksquare & \square \\ \square & \square\end{bmatrix}$: in NMR, have an asymptotic ground state subspace $\{|\psi_k\rangle\langle\psi_\ell|\}_{k,\ell=\emptyset}^{n-1}$. To get our $As(\mathcal{H}) \subset \begin{bmatrix}\blacksquare & \square \\ \square & \square\end{bmatrix}$, let coherences between $|\psi_k\rangle$ and $|\psi_{k\neq\ell}\rangle$ go to $\emptyset$.

Operator ⋮ superoperator projections on blocks:

→ Operator projections: used for projecting $|\psi\rangle \in \mathcal{H}$ onto relevant subspaces.

* $\hat{\mathcal{Y}}$: projector onto asymptotic subspace / maximal invariant subspace of $\mathcal{H}$. (<u>Not</u> $As(\mathcal{H})$, since $As(\mathcal{H}) \subset O_p(\mathcal{H})$.) <u>Uniquely</u> defined by:
  - $\hat{\rho}_{\infty,s} = \hat{\mathcal{Y}}\hat{\rho}_{\infty,s}\hat{\mathcal{Y}}$ for all $\hat{\rho}_{\infty,s} \in As(\mathcal{H})$. (This condition ensures we project onto <u>all</u> non-decaying subspaces.)
  - $\mathrm{Tr}\{\hat{\mathcal{Y}}\} = \max_{\hat{\rho}_{\infty,s}}\{\mathrm{rank}[\hat{\rho}_{\infty,s}]\}$. (This condition ensures we don't project onto any decaying subspace.)
* $\hat{\mathcal{S}}$: projector onto maximal decaying subspace of $\mathcal{H}$. Defined by: $\hat{\mathcal{S}} := \mathbb{1} - \hat{\mathcal{Y}}$, with $[\hat{\mathcal{Y}}, \hat{\mathcal{S}}] = \emptyset$.
  - Naturally, $\hat{\mathcal{S}} \perp \hat{\mathcal{Y}}$.
  - $\hat{\mathcal{S}}\hat{\rho}(t)\hat{\mathcal{S}} \to \hat{\mathbb{O}} \quad t\to\infty$

- Naturally, $\hat{s} \perp \hat{q}$.
- $\hat{s}\,\hat{\rho}_s(t)\,\hat{s} \to \hat{\emptyset}$ as $t \to \infty$.

→ Superoperator projections in terms of $\hat{P}$ & $\hat{q}$. Four-corners projections acting on $\hat{A} \in O_p(\mathcal{H})$.

* $\hat{A}_{\text{◰}} := \hat{\hat{P}}_{\text{◰}} := \hat{q}\hat{A}\hat{q}$.
* $\hat{A}_{\text{◳}} := \hat{\hat{P}}_{\text{◳}} := \hat{q}\hat{A}\hat{s}$.
* $\hat{A}_{\text{◱}} := \hat{\hat{P}}_{\text{◱}} := \hat{s}\hat{A}\hat{q}$.
* $\hat{A}_{\text{◲}} := \hat{\hat{P}}_{\text{◲}} := \hat{s}\hat{A}\hat{s}$

→ Properties of superoperators:

* $\hat{q}$ projects onto support of $\hat{\hat{P}}_{\infty}(\mathbb{1})$.
* Projection acts before adjoint, so that taking Hermitian conjugate of upper-right part yields lower-left part: $\hat{A}_{\text{◳}}{}^{\dagger} := (\hat{A}_{\text{◳}})^{\dagger} = (\hat{A}^{\dagger})_{\text{◱}}$.
* $\hat{\hat{P}}_{\boxplus}{}^{2} = \hat{\hat{P}}_{\boxplus}$ (these guys are projectors, after all, on $O_p(\mathcal{H})$).
* $\hat{\hat{P}}_{\boxplus}$ partitions the identity on $O_p(\mathcal{H})$: $\hat{\hat{\mathcal{I}}} = \hat{\hat{P}}_{\text{◰}} + \hat{\hat{P}}_{\text{◳}} + \hat{\hat{P}}_{\text{◱}} + \hat{\hat{P}}_{\text{◲}}$.
  - They're also additive as expected; e.g. $\hat{\hat{P}}_{\text{◰}+\text{◳}} = \hat{\hat{P}}_{\text{◰}} + \hat{\hat{P}}_{\text{◳}}$.
* Subspace $\text{◳}+\text{◱} := \hat{\hat{P}}_{\text{◳}+\text{◱}}\, O_p(\mathcal{H})$ is all coherences between $\hat{q}\mathcal{H}$ & $\hat{s}\mathcal{H}$.

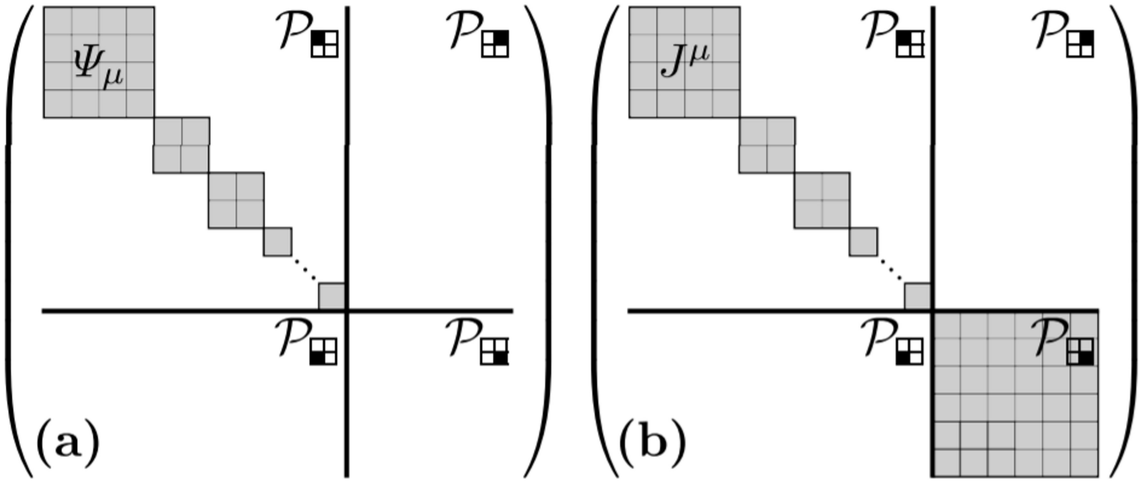

Figure 1.2: Decompositions of the space of matrices Op(H) acting on a Hilbert space H using the projections $\{P, Q\}$ defined in (1.14) and their corresponding superoperator projections $\{\mathcal{P}_{\boxplus}, \mathcal{P}_{\boxplus}, \mathcal{P}_{\boxplus}, \mathcal{P}_{\boxplus}\}$ defined in (1.16). Panel **(a)** depicts the block diagonal structure of the asymptotic subspace As(H), which is spanned by steady-state basis elements Ψ_μ (cf. [61], Fig. 3). Panel **(b)** depicts the subspace of Op(H), spanned by conserved quantities J_μ. These quantities determine dependence of the final (asymptotic) state ρ_∞ on the initial state ρ_{in}.

Operator Notation	Superoperator Notation	Operator Notation	Superoperator Notation
$\mathcal{L}(\rho)$	$\mathcal{L}\|\rho\rangle\rangle$	$A_{\text{▘}} \equiv PAP$	$\|A_{\text{▘}}\rangle\rangle \equiv \mathcal{P}_{\text{▘}}\|A\rangle\rangle$
$\mathrm{Tr}\{A^\dagger \mathcal{O}(\rho)\}$	$\langle\langle A\|\mathcal{O}\|\rho\rangle\rangle$	$A_{\text{▝}} \equiv PAQ$	$\|A_{\text{▝}}\rangle\rangle \equiv \mathcal{P}_{\text{▝}}\|A\rangle\rangle$
$-i[H,\rho]$	$\mathcal{H}\|\rho\rangle\rangle$	$A_{\text{▖}} \equiv QAP$	$\|A_{\text{▖}}\rangle\rangle \equiv \mathcal{P}_{\text{▖}}\|A\rangle\rangle$
$-i[V,\rho]$	$\mathcal{V}\|\rho\rangle\rangle$	$A_{\text{▗}} \equiv QAQ$	$\|A_{\text{▗}}\rangle\rangle \equiv \mathcal{P}_{\text{▗}}\|A\rangle\rangle$
$U\rho U^\dagger$	$\mathcal{U}\|\rho\rangle\rangle$	$P\mathcal{L}(Q\rho Q)P$	$\mathcal{P}_{\text{▘}}\mathcal{L}\mathcal{P}_{\text{▗}}\|\rho\rangle\rangle$
$S\rho S^\dagger$	$\mathcal{S}\|\rho\rangle\rangle$	$i\mathrm{Tr}\{H[\Psi_\mu,\Psi_\nu]\}$	$\langle\langle\Psi_\mu\|\mathcal{H}\|\Psi_\nu\rangle\rangle$

Table 1.1: Comparison of operator and superoperator notations for symbols used throughout the text (cf. Table 3.2 in [199]). $\mathcal{L}$ is a Lindbladian superoperator (1.8), $\mathcal{O}$ is a superoperator, A is an operator, and ρ is a density matrix. Hamiltonians H and V have corresponding Hamiltonian superoperators $\mathcal{H}$ and $\mathcal{V}$, respectively. Unitary operators U and S have corresponding unitary superoperators $\mathcal{U}$ and $\mathcal{S}$, respectively. The projection P (3.34) projects onto the largest subspace whose states do not decay under $\mathcal{L}$ and $Q \equiv I - P$ with I the identity. The last two entries respectively represent the part $\mathcal{P}_{\text{▘}}\mathcal{L}\mathcal{P}_{\text{▗}}$ of the projection decomposition of $\mathcal{L}$ (1.8) acting on ρ and a (superoperator) matrix element of $\mathcal{H}$ in terms of a Hermitian matrix basis $\{\Psi_\mu\}$.

Special cases of systems with no decaying subspaces ▗.

→ Hamiltonian case: we can write a Hamiltonian $\hat{H}$ such that $\hat{\mathcal{L}}[\hat{A}] = -i[\hat{H}, \hat{A}]$.

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→ Unique-state case: if there's one steady state $\hat{\rho}_{\infty,s}$ with a spectral decomposition $\sum_{k=0}^{\dim[\hat{\rho}_{\infty,s}]-1} \lambda_k |\psi_k\rangle\langle\psi_k|$, and all $\lambda_k = 0$ (e.g. Gibbs state).

1.6.2: More on Lindbladians

As discussed before, eigenvalues $\{\lambda_{a,b}\}$ of $\hat{\hat{\mathcal{L}}}$ have $Re\{\lambda_{a,b}\} \leq 0$. As with the zero eigenvalue, all eigenvalues for which $Re\{\lambda_{a,b}\} = 0$ don't decay (since we require $Re\{\lambda_{a,b}\} < 0$ to decay); the corresponding states define $As(\mathcal{H})$.

→ Dissipative gap / dissipation gap / damping gap / relaxation gap / asymptotic decay rate: slowest non-zero rate of converge towards $As(\mathcal{H})$. Defined by $\Delta_{dg} := \min_{Re\{\lambda_{a,b}\}} |Re\{\lambda_{a,b}\}|$

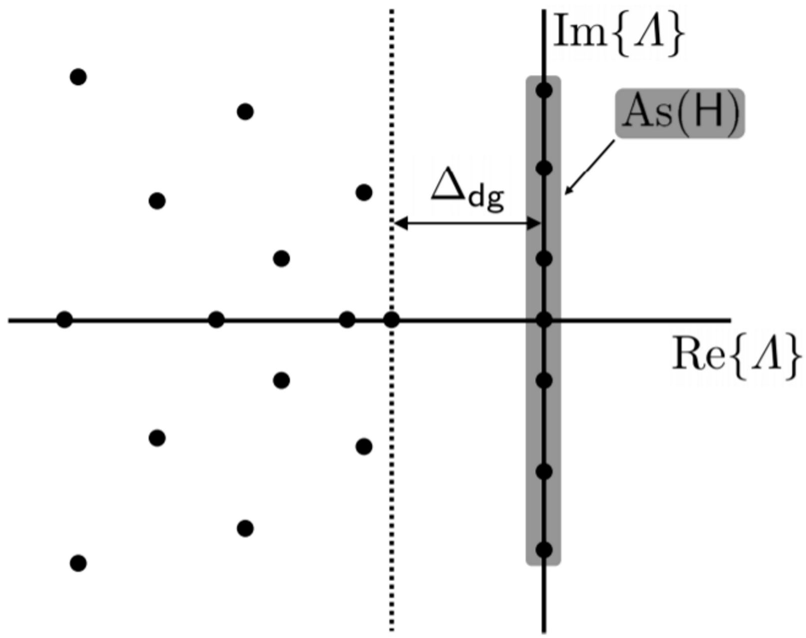

Figure 1.3: A plot of a spectrum of an example $\mathcal{L}$ with 21 eigenvalues Λ in the complex plane.

As before, spectrally decompose $e^{t\hat{\mathcal{L}}}$ into left & right eigenstates. Now, label eigenstates with eigenvalue λ and intra-degeneracy index ν.

$$e^{t\hat{\mathcal{L}}}|\hat{\rho}_{in,s}\rangle\rangle = \sum_{\lambda,\nu}{}' e^{\lambda t}|\hat{x}_{\lambda\nu}\rangle\rangle\langle\langle\hat{y}_{\lambda\nu}|\hat{\rho}_{in,s}\rangle\rangle = \sum_{\lambda,\nu}{}' c_{\lambda\nu} e^{\lambda t}|\hat{x}_{\lambda\nu}\rangle\rangle$$

If $\hat{\mathcal{L}}$ is not diagonalisable, there exists at least one Jordan block of $\hat{\mathcal{L}}$ in Jordan normal form with only one eigenstate (eigenmatrix). <u>Generalised eigenmatrices</u>: remaining matrix basis elements in supp{Jordan block}.

→ Hall's <u>Lie Groups, Lie Algebras, & Representations</u> has a really good summary of Jordan normal forms (in the context of representations of the Heisenberg group). Also review Meyer <u>Linear Algebra</u>.

→ $\exp\{\hat{\mathcal{L}}\}$ due to this yields <u>extra</u> powers of t in front of $e^{\lambda t}|\hat{x}_{\lambda\nu}\rangle\rangle\langle\langle\hat{y}_{\lambda\nu}|\hat{\rho}_{in,s}\rangle\rangle = c_{\lambda\nu}e^{\lambda t}|\hat{x}_{\lambda\nu}\rangle\rangle$, as well as off-diagonal elements $|\hat{x}_{\lambda\nu}\rangle\rangle\langle\langle\hat{y}_{\lambda,\mu\neq\nu}|$.

* <u>Ex.</u>: if λ has a 2D Jordan block with right eigenmatrix $|x_{\lambda 0}\rangle\rangle$ (so that we have $\hat{\mathcal{L}}|x_{\lambda 0}\rangle\rangle = \lambda|x_{\lambda 0}\rangle\rangle$) and generalised right eigenmatrix $|x_{\lambda 1}\rangle\rangle$ (so that we have $\hat{\mathcal{L}}|\hat{x}_{\lambda 1}\rangle\rangle = |\hat{x}_{\lambda 0}\rangle\rangle + \lambda|\hat{x}_{\lambda 1}\rangle\rangle$), then $e^{t\hat{\mathcal{L}}}$ on this 2D Jordan block is given by:

$$e^{t\lambda}\left(|\hat{x}_{\lambda 0}\rangle\rangle\langle\langle\hat{y}_{\lambda 0}| + t|\hat{x}_{\lambda 1}\rangle\rangle\langle\langle\hat{y}_{\lambda 0}| + |\hat{x}_{\lambda 1}\rangle\rangle\langle\langle\hat{y}_{\lambda 1}|\right) = e^{\lambda t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

We see that when we partition the Jordan normal form of $\hat{\mathcal{L}}$ into these blocks, we have both diagonal sub-blocks (i.e. $|\hat{x}_{\lambda\nu}\rangle\rangle\langle\langle\hat{y}_{\lambda\nu}|$), off-diagonal sub-blocks with upper diagonals of all 1s (i.e. $|\hat{x}_{\lambda 1}\rangle\rangle\langle\langle\hat{y}_{\lambda 1}|$), and terms that mix the two.

As in the example, if we partition the Jordan normal form of $\hat{\mathcal{L}}$ into blocks that are <u>either</u> diagonal or have an upper diagonal of all 1s, we

As in the example, if we partition the Jordan normal form of $\mathcal{L}$ into blocks that are either diagonal or have an upper diagonal of all 1s, we note that for any given λ, we might have terms with both diagonal & off-diagonal sub-blocks. Thus, the sum over λ has to include each sub-block separately. This gives the full expansion of $e^{t\hat{\hat{\mathcal{L}}}}$ in terms of the Jordan normal form as:

$$e^{t\hat{\hat{\mathcal{L}}}}|\hat{\rho}_{in,s}\rangle\rangle = \sum_{a,b}' e^{t\lambda_{a,b}} |\hat{x}_{a,b}\rangle\rangle\langle\langle\hat{y}_{a,b}|\hat{\rho}_{in,s}\rangle\rangle = \begin{cases} \sum'_{\lambda,\nu} e^{t\lambda}|\hat{x}_{\lambda\nu}\rangle\rangle\langle\langle\hat{y}_{\lambda\nu}|\hat{\rho}_{in,s}\rangle\rangle & \text{(diagonal Jordan blocks)} \\ \sum'_{\lambda,\nu}\sum'_{\mu\geq\nu} \frac{e^{t\lambda}t^{\nu-\mu}}{(\mu-\nu)!}|\hat{x}_{\lambda\nu}\rangle\rangle\langle\langle\hat{y}_{\lambda\nu}|\hat{\rho}_{in,s}\rangle\rangle & \text{(nondiagonal Jordan blocks)} \end{cases}$$

→ $\nu,\mu \in \mathbb{N}$ indexes the generalised eigenmatrices of the Jordan block of λ.

All Jordan blocks with pure imaginary eigenvalues are diagonal.

→ Proof: to be done later. Outline: AFSOC $\hat{\hat{\mathcal{L}}}$ is not diagonalisable in subspace of Jordan normal form whose diagonals all have $\mathrm{Re} = 0$. Then, if we take the exponential of those Jordan blocks, the dynamics diverges as $t\to\infty$ (presumably by giving us an overall factor of e^a for $\mathrm{Re}\, a > 0$).

→ Kinda obvious from the form of nondiagonal Jordan blocks.

1.6.3: Double-Bra/Ket Basis for Steady States

More exposition on bases of $O_p(\mathcal{H})$ from basis of $\mathcal{H}$. As before, from the ON basis $\{|\phi_a\rangle\}_{a=0}^{N-1}$ for $\mathcal{H}$ (N-dim.), the ON basis for $O_p(\mathcal{H})$ (N^2-dim.) is $\{|\Phi_{ab}\rangle\rangle\}_{a,b=0}^{N-1}$ with $|\Phi_{ab}\rangle\rangle := |\phi_a\rangle\langle\phi_b|$.

→ $|\Phi_{ab}\rangle\rangle$ is a "physical" ON basis for $O_p(\mathcal{H})$, since $|\Phi_{ab}\rangle\rangle := |\phi_a\rangle\langle\phi_b|$.

$(N^2$-dim.) is $\{|\Phi_{ab}\rangle\rangle\}_{a,b=0}$ with $|\Phi_{ab}\rangle\rangle = |a\rangle\langle b|$.

→ $|\Phi_{ab}\rangle\rangle$ is a "physical" ON basis for $O_p(\mathcal{H})$, since $|\Phi_{ab}\rangle\rangle := |\phi_a\rangle\langle\phi_b|$.

→ Normalised Hermitian ON basis $\{|\hat{\Gamma}_\alpha\rangle\rangle\}_{\alpha=0}^{N^2-1}$ with $\hat{\Gamma}_\alpha^\dagger = \hat{\Gamma}_\alpha$ can also be constructed.

- $\langle\langle\hat{\Gamma}_\alpha|\hat{\Gamma}_\beta\rangle\rangle = \mathrm{Tr}\{\hat{\Gamma}_\alpha^\dagger\hat{\Gamma}_\beta\} = \mathrm{Tr}\{\hat{\Gamma}_\alpha\hat{\Gamma}_\beta\} = \delta_{\alpha\beta}$.
- These are linear superpositions of $|\Phi_{ab}\rangle\rangle$s.
- These are <u>not</u> density matrices.
- They <u>are</u> the basis matrices for N-dim. irreps of Lie algebra $\mathfrak{su}(N)$.
  - <u>Ex.</u>: $\dim\mathcal{H} = N = 2$ gives $\dim O_p(\mathcal{H}) = N^2 - 1 = 3$. Basis: Pauli matrices.
  - <u>Ex.</u>: $\dim\mathcal{H} = N = 3$ gives $\dim O_p(\mathcal{H}) = N^2 - 1 = 8$. Basis: Gell-Mann matrices.
- Coefficients of any Hermitian operator in this basis are real.
  - <u>Ex.</u>: for a density matrix, $|\hat{\rho}\rangle\rangle = \sum_{\alpha=0}^{N^2-1} c_\alpha |\hat{\Gamma}_\alpha\rangle\rangle$. $c_\alpha = \langle\langle\hat{\Gamma}_\alpha|\hat{\rho}\rangle\rangle \in \mathbb{R}$ represent components of generalised Bloch / coherence vector. (This is why the basis is N-dim irreps of $\mathfrak{SU}(N)$.)
- Superoperator $\hat{\hat{A}} = \sum_{\alpha,\beta=0}^{N^2-1} A_{\alpha\beta}|\hat{\Gamma}_\alpha\rangle\rangle\langle\langle\hat{\Gamma}_\beta|$ has matrix elements $A_{\alpha\beta} := \langle\langle\hat{\Gamma}_\alpha|\hat{\hat{A}}|\hat{\Gamma}_\beta\rangle\rangle = \mathrm{Tr}\{\hat{\Gamma}_\alpha^\dagger\hat{\hat{A}}[\hat{\Gamma}_\beta]\}$
  - Observables: Hermitian superoperators over $O_p(\mathcal{H})$. Matrix elements are real. (Proof in Albert Pg. 28, include in typeset notes (but it's exactly what you'd expect).)

<u>Theorem</u>: (when Lindbladians generate unitary evolution.) Lindbladian matrix elements $\mathcal{L}_{\alpha\beta} \in \mathbb{R}$. Furthermore, $\mathcal{L}_{\alpha\beta} = -\mathcal{L}_{\beta\alpha}$ iff $\hat{\hat{\mathcal{L}}}[\hat{\rho}] = -i[\hat{H}, \hat{\rho}]$.

<u>Proof</u>: Albert Pgs. 28-29. Include in typeset notes.