Albert 1.1.2: Open Quantum Systems

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Closed quantum systems: dynamics generated by Hamiltonian. Open quantum systems: dynamics derived from <u>general</u> Hamiltonian EOM, given by Liouville-von Neumann equation:

$$\frac{d\hat{\rho}_{SE}}{dt} = -i\left[\hat{H}_{SE}, \hat{\rho}_{SE}\right]$$

Solution: $\hat{\rho}_{SE}(t) = e^{-i\hat{H}_{SE}t}\,\hat{\rho}_{SE}(\emptyset)\,e^{i\hat{H}_{SE}t}$. Reduced DM:

$$\hat{\rho}_S(t) = \mathrm{Tr}_E\left\{\hat{\rho}_{SE}(t)\right\} = \mathrm{Tr}_E\left\{e^{-i\hat{H}_{SE}\hat{t}}\,\hat{\rho}_{SE}(t=\emptyset)\,e^{i\hat{H}_{SE}t}\right\} = \sum_e{}' \langle e|\,e^{-i\hat{H}_{SE}\hat{t}}\,\hat{\rho}_{SE}(t=\emptyset)\,e^{i\hat{H}_{SE}t}|e\rangle$$

Assumption: initial state factorises. $\hat{\rho}_{SE}(t=\emptyset) = \hat{\rho}_{SE}(\emptyset) = \hat{\rho}_{in,S} \otimes |e=\emptyset\rangle\langle e=\emptyset|_E = \hat{\rho}_{in,S} \otimes |\emptyset\rangle\langle\emptyset|_E$. Then, $\hat{\rho}_S(t)$ is:

$$\hat{\rho}_S(t) = \sum_e{}' \langle e|\,e^{-i\hat{H}_{SE}t}\,\hat{\rho}_{SE}(\emptyset)\,e^{i\hat{H}_{SE}t}|e\rangle = \sum_e{}' \langle e|\,e^{-i\hat{H}_{SE}t}\left(\hat{\rho}_{in,S} \otimes |\emptyset\rangle\langle\emptyset|_E\right)e^{i\hat{H}_{SE}t}|e\rangle = \sum_e{}' \langle e|\,e^{-i\hat{H}_{SE}t}|\emptyset\rangle\,\hat{\rho}_{in,S}\,\langle\emptyset|\,e^{i\hat{H}_{SE}t}|e\rangle$$

$$\hat{\rho}_S(t) = \sum_e{}' \hat{E}^e(t)\,\hat{\rho}_{in,S}\,\hat{E}^{e\dagger}$$

$\hat{E}^e(t) := \langle e|\,e^{-i\hat{H}_{SE}t}|\emptyset\rangle$ are <u>Kraus operators</u>.

→ One operator for each $|e\rangle$.

→ Time-dependent.

→ $\sum' \hat{E}^{e\dagger}\hat{E}^e = \mathbb{1}$ (P...

→ line-dependent.

→ $\sum'_e \hat{E}^{e\dagger}\hat{E}^e = \mathbb{1}_S$. (Proven below.)

→ Operate exclusively on $\mathcal{H}_S$; i.e. $\{\hat{E}^e(t)\}$ map $\mathcal{D}(\mathcal{H}_S) \to \mathcal{D}(\mathcal{H}_S)$; with $\{\hat{E}^e(t)\}: \hat{\rho}_{in,S} \mapsto \hat{\rho}_S(t)$.

→ Recall discussion from Gould paper: these are actually processes, not maps, since they change as we change $\{|e\rangle\}$.

* Maps, conversely, are fixed.

→ Recall discussion from Gould paper: Griffiths' & Harrow's lecture notes have $\hat{A}_\ell : \mathcal{H}_S \otimes \mathcal{H}_E \to \mathcal{H}_E$ and $\hat{A}_\ell^\dagger : \mathcal{H}_E \to \mathcal{H}_S \otimes \mathcal{H}_E$, which makes sense from an operator perspective. This is technically correct, but Gould & Albert use the Kraus operators differently. Specifically, Gould uses $\hat{\rho}_E(t) = \mathrm{Tr}_S\{\hat{U}(\hat{\rho}_{in,S} \otimes \hat{\rho}_{in,E})\hat{U}^\dagger\}$ as a machine to get $\hat{\rho}_E(t)$ from $\hat{\rho}_{in,S} \otimes \hat{\rho}_{in,E}$; so for him we have $\hat{A}_\ell, \hat{A}_\ell^\dagger : \mathcal{H}_S \otimes \mathcal{H}_E \to \mathcal{H}_E$. Similarly, Albert uses $\hat{\rho}_S(t) = \sum'_e \hat{E}^e(t)\,\hat{\rho}_{in,S}\,\hat{E}^{e\dagger}(t)$ as a machine to get $\hat{\rho}_S(t)$ from $\hat{\rho}_{in,S}$. Since the use is different, the type of mapping is different too.

→ Also called Kraus maps, quantum channels, & completely positive trace preserving (CPTP) maps. Properties:

* Completely positive: $\hat{\rho}_S(t) \geq \hat{\emptyset}$ for all t. (Preserves positivity.)
* Preserves positivity when acting on a larger system: $\hat{\rho}_S(t) \otimes \hat{\rho}_A(t) \geq \hat{\emptyset}$ for all t.
* Trace preserving: $\mathrm{Tr}\{\hat{\rho}_S(t)\} = \mathrm{Tr}_S\{\hat{\rho}(t)\} = 1$ for all t.

Unitarity of $e^{-i\hat{H}_{SE}t}$ and completeness of $\{|e\rangle\}_E$ give $\sum'_e \hat{E}^{e\dagger}\hat{E}^e = \mathbb{1}_S$:

$$\sum_e{}' \hat{E}^{e\dagger}\hat{E}^e = \sum_e{}' \langle\emptyset| e^{i\hat{H}_{SE}t}|e\rangle\langle e| e^{-i\hat{H}_{SE}t}|\emptyset\rangle = \sum_e{}' (\mathbb{1}_S \otimes \langle\emptyset|_E)\, e^{i\hat{H}_{SE}t}\,(\mathbb{1}_S \otimes |e\rangle_E)(\mathbb{1}_S \otimes \langle e|_E)\, e^{-i\hat{H}_{SE}t}\,(\mathbb{1}_S \otimes |\emptyset\rangle_E)$$

$$\sum_e{}' \hat{E}^{e\dagger}\hat{E}^{e} = \left(\mathbb{1}_S \otimes \langle \emptyset |_E\right) e^{i\hat{H}_{SE}t} \sum_e{}' \left(\mathbb{1}_S \otimes | e\rangle_E\right)\left(\mathbb{1}_S \otimes \langle e|_E\right) e^{-i\hat{H}_{SE}t} \left(\mathbb{1}_S \otimes |\emptyset\rangle_E\right) = \left(\mathbb{1}_S \otimes \langle \emptyset |_E\right) e^{i\hat{H}_{SE}t} \left(\mathbb{1}_S \otimes \mathbb{1}_E\right) e^{-i\hat{H}_{SE}t} \left(\mathbb{1}_S \otimes |\emptyset\rangle_E\right)$$

$$\sum_e{}' \hat{E}^{e\dagger}\hat{E}^{e} = \left(\mathbb{1}_S \otimes \langle \emptyset |_E\right) e^{i\hat{H}_{SE}t} e^{-i\hat{H}_{SE}t} \left(\mathbb{1}_S \otimes |\emptyset\rangle_E\right) = \left(\mathbb{1}_S \otimes \langle \emptyset |_E\right) \mathbb{1}_{SE} \left(\mathbb{1}_S \otimes |\emptyset\rangle_E\right) = \mathbb{1}_S \otimes \langle \emptyset | \emptyset \rangle_E = \mathbb{1}_S \otimes 1 = \mathbb{1}_S .$$