Albert 1.2: What is a Lindbladian?

Thursday, August 22, 2019 03:12 PM

(also uses Landi 4.3: Lindblad Master Equations and Preskill 3.5: Master Equations for Open Quantum Systems)

Time evolution of $\hat{\rho}(t)$ over ∞simal time increment dt: $\hat{\rho}(t+dt)$. We assume this follows the Kraus map, so we assume $\hat{\rho}(t+dt)$ is:

$$\hat{\rho}(t+dt) = \sum_e{}' \hat{E}^e(dt)\,\hat{\rho}(t)\,\hat{E}^{e\dagger}(dt) =: \hat{\hat{\mathcal{E}}}_{dt}[\hat{\rho}(t)]$$

Relies on Markov approximation: assumption that $\hat{\rho}(t+dt)$ relies only on $\hat{\rho}(t)$ and not previous times $\tau < t$. (Alternately: Markov processes are AR-1 processes.) Taylor approximation for superoperator $\hat{\hat{\mathcal{E}}}_{dt} \in \mathcal{L}_1(O_p(\mathcal{H}))$:

$$\hat{\hat{\mathcal{E}}}_{dt} = \hat{\hat{\mathcal{I}}} + dt\left(\lim_{dt\to 0} \frac{\hat{\hat{\mathcal{E}}}_{dt} - \hat{\hat{\mathcal{I}}}}{dt}\right) + \ldots = \hat{\hat{\mathcal{I}}} + dt\,\hat{\hat{\mathcal{L}}}$$

$\hat{\hat{\mathcal{E}}}_{dt}$ is a linear map, at least L_1-finite, from $\hat{\rho} \in O_p(\mathcal{H})$ to $\hat{\sigma} \in O_p(\mathcal{H})$.

→ $\hat{\hat{\mathcal{I}}}[\hat{\rho}] = \hat{\rho}$ is the identity channel.

→ $\hat{\hat{\mathcal{L}}} = \lim_{dt\to 0} \frac{\hat{\hat{\mathcal{E}}}_{dt} - \hat{\hat{\mathcal{I}}}}{dt}$ is the Lindbladian.

→ $\hat{\hat{\mathcal{L}}}$ is the most general channel that's linear in $\hat{\rho}$ that's a CPTP map.

$\hat{\hat{\mathcal{L}}}$ is determined by the expansion of $\hat{E}^e(dt)$, kept to $\hat{O}(dt)$.

→ If evolution of $\hat{\rho}$ is generated by a Hamiltonian $\hat{H} \in O_p(\mathcal{H})$ alone, then $\hat{E}^{e=0} = \mathbb{1} - i\hat{H}\,dt$ and $\hat{E}^{e>0} = \emptyset$. Yields regular Liouville-von Neumann equation:

$$\hat{\rho}(t+dt) = \sum_e{}' \hat{E}^e(dt)\,\hat{\rho}(dt)\,\hat{E}^{e\dagger}(dt) = (\mathbb{1} - i\hat{H}\,dt)\,\hat{\rho}(t)\,(\mathbb{1} + i\hat{H}\,dt) = \hat{\rho}(t) + i\hat{\rho}(t)\hat{H}\,dt - i\hat{H}\hat{\rho}(t)\,dt + \hat{H}\hat{\rho}(t)\hat{H}\,(dt)^2$$

$$\left.\frac{\hat{\rho}(t+dt) - \hat{\rho}(t)}{dt}\right|_{\mathcal{O}(1)} =: \frac{d\hat{\rho}(t)}{dt} = i\hat{\rho}(t)\hat{H} - i\hat{H}\hat{\rho}(t)\,dt + \cancel{\hat{H}\hat{\rho}(t)\hat{H}\,dt} = -i[\hat{H}, \hat{\rho}(t)]$$

→ More general case: each $\hat{E}^e$ has an $\mathcal{O}(\sqrt{dt})$ dependence, so each $\hat{E}^e(dt)\,\hat{\rho}(t)\,\hat{E}^{e\dagger}(dt)$ term contributes an overall $\mathcal{O}(dt)$ term. WLOG, can write:

$$\hat{E}^{e=0}(dt) \sim \mathbb{1} + (-i\hat{H} + \hat{V})\,dt\,;\quad \hat{E}^{e>0}(dt) \sim \sqrt{K_e\,dt}\;\hat{F}^e$$

* $K_e \in \mathbb{R}$ is a nonzero rate.
* $\hat{V}$ is a to-be-determined perturbing potential.
* Right now, $\hat{F}^e$ is an unknown operator.
* This is the most general & lowest order expression.

To determine $\hat{V}$, plug $\hat{E}^{e=0}$ and $\hat{E}^{e>0}$ into Kraus operator unitarity constraint and Markov assumption:

* Kraus operator unitarity constraint $\sum_e{}' \hat{E}^{e\dagger}\hat{E}^e = \mathbb{1}$:

$$\sum_e{}' \hat{E}^{e\dagger}\hat{E}^e = (\mathbb{1} + (-i\hat{H} + \hat{V})\,dt)^\dagger(\mathbb{1} + (-i\hat{H} + \hat{V})\,dt) + \sum_{e>0}{}' (\sqrt{K_e\,dt}\,\hat{F}^e)^\dagger(\sqrt{K_e\,dt}\,\hat{F}^e) = \mathbb{1}$$

$$\sum_e{}' \hat{E}^{e+}\hat{E}^e = (\mathbb{1} + i\hat{H}\,dt + \hat{V}\,dt)(\mathbb{1} - i\hat{H}\,dt + \hat{V}\,dt) + \sum_{e>0}{}' \kappa_e \hat{F}^{e+}\hat{F}^e\,dt = \mathbb{1}$$

$$\sum_e{}' \hat{E}^{e+}\hat{E}^e = \mathbb{1} + i\hat{H}\,dt + \hat{V}\,dt - i\hat{H}\,dt - i^2\hat{H}^2(dt)^2 + i\hat{H}\hat{V}(dt)^2 + \hat{V}\,dt - i\hat{H}\hat{V}(dt)^2 + \hat{V}^2(dt)^2 + \sum_{e>0}{}' \kappa_e \hat{F}^{e+}\hat{F}^e\,dt = \mathbb{1}$$

Yellow, purple, and green terms all cancel with each other, and grey terms are higher-order in dt. Gives:

$$2\hat{V}\,dt = -\,dt \sum_{e>0}{}' \kappa_e \hat{F}^{e+}\hat{F}^e$$

$$\hat{V} = -\frac{1}{2}\sum_{e>0}{}' \kappa_e \hat{F}^{e+}\hat{F}^e$$

* Markov assumption: $\hat{\rho}(t+dt) = \sum_e{}' \hat{E}^e(dt)\,\hat{\rho}(t)\,\hat{E}^{e+}(dt)$

$$\hat{\rho}(t+dt) = (\mathbb{1} - i\hat{H}\,dt + \hat{V}\,dt)\,\hat{\rho}(t)\,(\mathbb{1} + i\hat{H}\,dt + \hat{V}\,dt) + \sum_{e>0}{}' (\sqrt{\kappa_e\,dt}\,\hat{F}^e)\,\hat{\rho}(t)\,(\sqrt{\kappa_e\,dt}\,\hat{F}^e)^+$$

$$\hat{\rho}(t+dt) = \hat{\rho}(t) - i\hat{H}\hat{\rho}(t)\,dt + \hat{V}\hat{\rho}(t)\,dt + i\hat{\rho}(t)\hat{H}\,dt - i^2\hat{H}\hat{\rho}(t)\hat{H}(dt)^2$$

$$- i\hat{H}\hat{\rho}(t)\hat{V}(dt)^2 + \hat{\rho}(t)\hat{V}\,dt - i\hat{H}\hat{\rho}(t)\hat{V}(dt)^2 + \hat{V}\hat{\rho}(t)\hat{V}(dt)^2 + \sum_{e>0}{}' \kappa_e\,dt\,\hat{F}^e\hat{\rho}(t)\hat{F}^{e+}.$$

As before, the grey terms are higher-order in dt. This gives:

$$\hat{\rho}(t+dt) - \hat{\rho}(t)\Big|_{\mathcal{O}(dt)} = -i\left(\hat{H}\hat{\rho}(t) - \hat{\rho}(t)\hat{H}\right)dt + dt\left(\hat{V}\hat{\rho}(t) + \hat{\rho}(t)\hat{V} + \sum_{e>\emptyset}' \kappa_e \hat{F}^e \hat{\rho}(t)\hat{F}^{e+}\right)$$

Applying $\hat{V} = -\tfrac{1}{2}\sum_e' \kappa_e \hat{F}^{e+}\hat{F}^e$ defines the <u>Lindbladian</u> / <u>Liouvillian</u> / <u>Lindblad master equation</u> / <u>Gorini-Kossakowski-Sudarshan Lindblad master equation</u>:

$$\frac{\hat{\rho}(t+dt) - \hat{\rho}(t)}{dt}\Bigg|_{\mathcal{O}(dt)} =: \frac{d\hat{\rho}(t)}{dt} =: \mathcal{L}[\hat{\rho}] =: -i\left[\hat{H}, \hat{\rho}(t)\right] + \frac{1}{2}\sum_{e>\emptyset}' \kappa_e \left(2\hat{F}^e\hat{\rho}(t)\hat{F}^{e+} - \hat{F}^{e+}\hat{F}^e\hat{\rho}(t) - \hat{\rho}(t)\hat{F}^{e+}\hat{F}^e\right)$$

- $\hat{H}$: Hamiltonian
- $\hat{F}^e$: <u>jump operators</u>
- $\hat{F}^e\hat{\rho}(t)\hat{F}^{e+}$: <u>recycling term</u> / <u>sandwich term</u> / <u>jump term</u>. Acts on $\hat{\rho}$ from both sides, so $\hat{\mathcal{L}}[\hat{\rho}]$ can't simply be reduced to an L_2-finite operator acting on a ket state $|\psi\rangle \in \mathcal{H}$.
- $\kappa_e \in \mathbb{R}^+$: <u>rates</u>.
- $\hat{K} := \hat{H} - \tfrac{i}{2}\sum_{e>\emptyset}' \kappa_e \hat{F}^{e+}\hat{F}^e$: <u>deterministic</u> / <u>no-jump term</u>. Defining the adjoint commutator $[\hat{A}, \hat{B}]^{\#} := \hat{A}\hat{B} - \hat{B}^+\hat{A}^+$, the deterministic term allows everything except the recycling term to be expressed in terms of a von Neumann equation:

$$\left(\frac{d\hat{\rho}(t)}{dt}\right)_{\hat{K}} := -i\hat{H}\hat{\rho}(t) + i\hat{\rho}(t)\hat{H} - \frac{1}{2}\sum_{e>\emptyset}' \kappa_e \hat{F}^{e+}\hat{F}^e\hat{\rho}(t) - \frac{1}{2}\sum_{e>\emptyset}' \kappa_e \hat{\rho}(t)\hat{F}^{e+}\hat{F}^e.$$

$$\left(\frac{d\hat{\rho}(t)}{dt}\right)_{\xi} = -i\hat{H}\hat{\rho}(t) - i\left(-\frac{i}{2}\sum_{e>\emptyset}' \kappa_e \hat{F}^{e+}\hat{F}^{e}\right)\hat{\rho}(t) + i\hat{\rho}(t)\hat{H} + i\hat{\rho}(t)\left(\frac{i}{2}\sum_{e>\emptyset} \kappa_e \hat{F}^{e+}\hat{F}^{e}\right)$$

$$\left(\frac{d\hat{\rho}(t)}{dt}\right)_{\xi} = -i\left(\hat{H} - \frac{i}{2}\sum_{e>\emptyset}' \kappa_e \hat{F}^{e+}\hat{F}^{e}\right)\hat{\rho}(t) + i\hat{\rho}(t)\left(\hat{H} + \frac{i}{2}\sum_{e>\emptyset}' \kappa_e \hat{F}^{e+}\hat{F}^{e}\right)$$

$$\left(\frac{d\hat{\rho}(t)}{dt}\right)_{\xi} = -i\hat{K}\hat{\rho}(t) + i\hat{\rho}(t)\hat{K}^{+} = -i\left[\hat{K},\hat{\rho}(t)\right]^{\#}$$

- <u>Dissipator</u> can be defined by extra terms from the Liouville-von Neumann equation. Define dissipator $\mathcal{D}[\hat{\rho}]$ by:

$$\mathcal{D}[\hat{\rho}] := \frac{1}{2}\sum_{e>\emptyset}' \kappa_e \left(2\hat{F}^{e}\hat{\rho}(t)\hat{F}^{e+} - \hat{F}^{e+}\hat{F}^{e}\hat{\rho}(t) - \hat{\rho}(t)\hat{F}^{e+}\hat{F}^{e}\right)$$

This gives $d\hat{\rho}(t)/dt$ as simply:

$$\frac{d\hat{\rho}(t)}{dt} = -i\left[\hat{H},\hat{\rho}(t)\right] + \frac{1}{2}\sum_{e>\emptyset}' \kappa_e \left(2\hat{F}^{e}\hat{\rho}(t)\hat{F}^{e+} - \hat{F}^{e+}\hat{F}^{e}\hat{\rho}(t) - \hat{\rho}(t)\hat{F}^{e+}\hat{F}^{e}\right) = -i\left[\hat{H},\hat{\rho}(t)\right] + \mathcal{D}[\hat{\rho}].$$

Then, the time derivative of the operator expectation value is just:

$$\frac{d\langle\hat{A}\rangle}{dt} = \frac{d\,\mathrm{Tr}\{\hat{A}\hat{\rho}(t)\}}{dt} = \mathrm{Tr}\left\{-i\hat{A}\left[\hat{H},\hat{\rho}(t)\right] + \hat{A}\mathcal{D}[\hat{\rho}]\right\} = i\langle[\hat{H},\hat{A}]\rangle + \mathrm{Tr}\{\hat{A}\mathcal{D}[\hat{\rho}]\}.$$

The first term is obviously just the Ehrenfest theorem. The second term is:

The first term is obviously just the Ehrenfest theorem. The second term is:

$$\mathrm{Tr}\{\hat{A}\hat{\mathcal{D}}[\hat{\rho}]\} = \sum_{e>0}' \frac{\kappa_e}{2}\left(\mathrm{Tr}\{2\hat{A}\hat{F}^e\hat{\rho}(t)\hat{F}^{e+}\} - \mathrm{Tr}\{\hat{A}\hat{F}^{e+}\hat{F}^e\hat{\rho}(t)\} - \mathrm{Tr}\{\hat{A}\hat{\rho}(t)\hat{F}^{e+}\hat{F}^e\}\right)$$

$$\mathrm{Tr}\{\hat{A}\hat{\mathcal{D}}[\hat{\rho}]\} = \mathrm{Tr}\{\sum_{e>0}' \kappa_e\hat{F}^{e+}\hat{A}\hat{F}^e\hat{\rho}(t)\} - \frac{1}{2}\mathrm{Tr}\{\sum_{e>0}' \kappa_e\hat{A}\hat{F}^{e+}\hat{F}^e\hat{\rho}(t)\} - \frac{1}{2}\mathrm{Tr}\{\sum_{e>0}' \kappa_e\hat{A}\hat{\rho}(t)\hat{F}^{e+}\hat{F}^e\}$$

$$\mathrm{Tr}\{\hat{A}\hat{\mathcal{D}}[\hat{\rho}]\} = \left\langle\sum_{e>0}' \kappa_e\hat{F}^{e+}\hat{A}\hat{F}^e\right\rangle - \frac{1}{2}\left\langle\left\{\sum_{e>0}' \kappa_e\hat{F}^{e+}\hat{F}^e, \hat{A}\right\}\right\rangle$$

Thus, we now have $\frac{d\langle\hat{A}\rangle}{dt}$ as:

$$\frac{d\langle\hat{A}\rangle}{dt} = i\langle[\hat{H},\hat{A}]\rangle + \mathrm{Tr}\{\hat{A}\hat{\mathcal{D}}[\hat{\rho}]\} = i\langle[\hat{H},\hat{A}]\rangle + \left\langle\sum_{e>0}' \kappa_e\hat{F}^{e+}\hat{A}\hat{F}^e\right\rangle - \frac{1}{2}\left\langle\left\{\sum_{e>0}' \kappa_e\hat{F}^{e+}\hat{F}^e, \hat{A}\right\}\right\rangle$$

Time evolution in Heisenberg picture is <u>unital</u> rather than <u>trace-preserving</u>: in the Heisenberg picture, the identity operator $\mathbb{1}$ does not evolve.

- From $\frac{d\langle\hat{A}\rangle}{dt}$ definition, can define <u>adjoint dissipator</u> as $\bar{\mathcal{D}}[\hat{F}^e](\hat{A}) := \sum_{e>0}' \kappa_e\hat{F}^{e+}\hat{A}\hat{F}^e - \frac{1}{2}\sum_{e>0}' \{\kappa_e\hat{F}^{e+}\hat{F}^e, \hat{A}\}$
    - $\bar{\mathcal{D}}$ is a superoperator acting on observables $\hat{A}$; as opposed to Lindbladian & dissipator, which are superoperators acting on density matrices.

- In $\hat{\hat{\mathcal{L}}}[\hat\rho]$ and $\mathcal{D}[\hat\rho]$, have both jump terms and <u>deterministic</u> terms. Conversely, in $\bar{\mathcal{D}}$, everything is always of the same order, with $\hat F^{e\dagger}$ always on the left. This allows us to factorise $\bar{\mathcal{D}}$ as:

$$\bar{\mathcal{D}}[\hat F^e](\hat A) := \sum_{e>\emptyset}{}' K_e \hat F^{e\dagger} \hat A \hat F^e - \frac{1}{2}\sum_{e>\emptyset}{}' \{K_e \hat F^{e\dagger}\hat F^e, \hat A\} = \frac{1}{2}\sum_{e>\emptyset} K_e \hat F^{e} [\hat A, \hat F^e] + \frac{1}{2}\sum_{e>\emptyset}{}' K_e [\hat F^{e\dagger}, \hat A]\hat F^e .$$

  - As always with Kraus operators, change of environment basis changes operators. In particular, we can change the basis for the Kraus operators $\hat E^{e>\emptyset}$ while leaving the operator $\hat E^{e=\emptyset}$ fixed, resulting in a transformation on $\hat F^{e>\emptyset}$. We can thus replace $\hat F^{e>\emptyset}$ with $\hat G^{\mu}$ defined by $\hat G^{\mu} = \sum_{e>\emptyset} M_{\mu e} \hat F^e$, where $M_{\mu e}$ is a unitary matrix. Different ways of choosing $\hat F^e$ are different <u>unravellings</u> of the same Markovian dynamics.

<u>Together</u>, recycling term $\hat F^e \hat\rho(t) \hat F^{e\dagger}$ and deterministic term $\hat K$ allow time evolution of $\hat\rho(t)$ under $\hat{\hat{\mathcal{L}}}$ to preserve CPTP properties, which are violated by the non-Hermitian operator (e.g. a non-Hermitian Hamiltonian, as discussed in some other open quantum systems literature) alone. Preservation of CPTPness can be demonstrated by trace preservation of $\hat\rho(t)$ under time evolution; or, equivalently, by showing that $\frac{d}{dt}\mathrm{Tr}\{\hat\rho(t)\} = \mathrm{Tr}\{\frac{d\hat\rho(t)}{dt}\} = \emptyset$:

$$\mathrm{Tr}\left\{\frac{d\hat\rho(t)}{dt}\right\} = \mathrm{Tr}\{-i[\hat H, \hat\rho(t)]\} + \frac{1}{2}\sum_{e>\emptyset}{}' 2K_e \mathrm{Tr}\{\hat F^e \hat\rho(t) \hat F^{e\dagger}\} - \sum_{e>\emptyset}{}' K_e \mathrm{Tr}\{\hat F^{e\dagger}\hat F^e \hat\rho(t)\} - \sum_{e>\emptyset}{}' K_e \mathrm{Tr}\{\hat\rho(t) \hat F^{e\dagger}\hat F^e\}$$

$$\mathrm{Tr}\left\{\frac{d\hat\rho(t)}{dt}\right\} = \cancelto{\emptyset}{-i\,\mathrm{Tr}\{\hat H \hat\rho(t)\} + i\,\mathrm{Tr}\{\hat\rho(t)\hat H\}} + \sum_{e>\emptyset}{}' \cancelto{\emptyset}{\left(K_e \mathrm{Tr}\{\hat F^{e\dagger}\hat F^e \hat\rho(t)\} - \frac{K_e}{2}\mathrm{Tr}\{\hat F^{e\dagger}\hat F^e \hat\rho(t)\} - \frac{K_e}{2}\mathrm{Tr}\{\hat F^{e\dagger}\hat F^e \hat\rho(t)\}\right)} = \emptyset$$

Then, the jump term accounts for the decay in probability caused by the deterministic term $\hat{K}$.

The formal solution to the equation $\frac{d\hat{\rho}(t)}{dt} =: \hat{\hat{\mathcal{L}}}[\hat{\rho}] := -i[\hat{H}, \hat{\rho}(t)] + \frac{1}{2}\sum'_{e>\emptyset} \kappa_e \left(2\hat{F}^e \hat{\rho}(t) \hat{F}^{e\dagger} - \hat{F}^{e\dagger}\hat{F}^e \hat{\rho}(t) - \hat{\rho}(t)\hat{F}^{e\dagger}\hat{F}^e\right)$ is given by $\hat{\rho}(t) := e^{t\hat{\hat{\mathcal{L}}}}[\hat{\rho}(t=\emptyset)]$. Expand in formal power series solution in $t\hat{\mathcal{L}}$ and $\hat{\rho}(t=\emptyset) =: \hat{\rho}_\emptyset$. Two ways to expand power series:

→ Re-express the $n \times n$ density matrices $\hat{\rho}$ as vectors of size $n^2 \times 1$. Called vectorisation / the Choi-Jamiołkowski isomorphism: stack columns from left to right. This is the technique used in the thesis.

* <u>Ex.</u>: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$.

* Lets $\hat{\mathcal{L}}$ be expressed as an $n^2 \times n^2$ matrix acting on $\hat{\rho}_\emptyset$ from the left. Discussed much more in detail in Sec. 1.6, and used in the rest of the thesis.

→ <u>If</u> $\hat{\rho}_\emptyset$ is initially pure in S (i.e. $\hat{\rho}_\emptyset = \sum'_i p_i |e\rangle_i\langle e|_i = |\psi\rangle\langle\psi|$ for $|e\rangle_i \in \mathcal{B}(\mathcal{H})$ and $|\psi\rangle \in \mathcal{H}$), we can determine $\hat{\mathcal{L}}$ as the asymptotic limit of the ensemble average (where p_i denotes ensemble probabilities) of jump terms applied to $|\psi\rangle$.

During the time evolution of a specific instance, system is acted on by one instance of the jump term, and then re-normalised (not renormalised) at a discrete set of randomly generated times $\{\tau_n\}$; otherwise (at $t \neq \tau_n$), evolves deterministically under K for all $\tau \in [t_i, t_f] \setminus \{\tau_n\}$. This process is called <u>unravelling</u>. (<u>Not</u> the same unravelling as with Kraus operator redefinition.)

* <u>Ex.</u>: Start at time $t_i = 0$, with one jump operator. Generate a random number $r \in [0, 1]$, and let the system evolve under K until $\| |\psi(t)\rangle \|^2 = |e^{i\hat{K}\tau_1} |\psi_i\rangle|^2 = r$. Then, apply jump term & re-normalise, yielding state $|\psi_1\rangle = \hat{F} e^{i\hat{K}\tau_1} |\psi_i\rangle / |\hat{F} e^{i\hat{K}\tau_1} |\psi_i\rangle|^2$. Repeat the same procedure with $|\psi_1\rangle$ (and subsequent $|\psi_n\rangle$s) until we reach t_f. In limit of ∞ # of such trajectories, average over final states is exactly $\hat{\rho}(t)$.

* Might be relevant to our case. Covered in RMP **70**, 191 (1998).

* Unravelling: useful numerically, and has a simple physical interpretation as the evolution of a continuously measured state conditioned on the measurement state.

   ◦ <u>Ex.</u>: If we have a photon detector measuring photon counts; jump operator represents photon loss, applying jump term to $\hat{\rho}$ represents detector click, and $e^{it\hat{K}}$ represents intra-detection time evolution of the detector.