Albert 1.6.1: The Playground and its Features (with Landi 4.3: Lindblad Master Equations)

Tuesday, May 5, 2020 11:41 PM

(also includes Caves 17-20: Superoperators & Completely Positive Maps and Jagadish & Petruccione)

Lindbladians & quantum channels generally are L_1-finite superoperators on $O_p(\mathcal{H})$: $\hat{\hat{\mathcal{L}}} \in \mathcal{L}_1(O_p(\mathcal{H}))$. Want to examine properties of superoperators in terms of operator algebra $O_p(\mathcal{H}) := \mathcal{H} \otimes \mathcal{H}^\dagger$ over the Hilbert space $\mathcal{H}$ of states.

$O_p(\mathcal{H})$ is a Hilbert space itself under the Hilbert-Schmidt inner product $(\hat{A}, \hat{B}) := \mathrm{Tr}\{\hat{A}^\dagger \hat{B}\}$ and the Frobenius / L_2-norm $\|\hat{A}\| := \sqrt{\mathrm{Tr}\{\hat{A}^\dagger \hat{A}\}}$ for $\hat{A}, \hat{B} \in O_p(\mathcal{H})$.

→ Thus, operators $\hat{A} \in O_p(\mathcal{H})$ in QM are both operators acting on vectors in a vector space, and themselves vectors in a vector space being acted on by superoperators.

→ $O_p(\mathcal{H})$ is also called a Liouville space in NMR literature. This is studied decently well there; it can serve as an additional resource.

Master equations & quantum operations are sometimes unwieldy, since sometimes you need to take products of $\hat{\rho}$ with operators on both sides (as in the jump term). This makes the power series expansion of $e^{t\mathcal{L}}$ especially nasty.

Instead, since every set $\mathbb{F}^{m\times n}$ of $m\times n$ matrices over the field $\mathbb{F}$ forms a vector space, and $\mathbb{F}^{n\times n}$ forms a Hilbert space under the Hilbert-Schmidt product & Frobenius norm, we can map $\mathbb{F}^{n\times n}$ to $\mathbb{F}^{n^2}$ ($= \mathbb{F}^{n^2\times 1}$). This is the Choi-Jamiołkowski isomorphism;

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{Insert discussion of quantum measurement models & rest of Caves 17-20. Eventually, start over using Attal, Hall Quantum GTM, Wilde Ch. 4, Preskill notes.}.

Going back (finally...) to the Lindbladian, we have the Lindbladian as:

$$\frac{d\hat{\rho}_s(t)}{dt} = \hat{\mathcal{L}}[\hat{\rho}_s] := -i[\hat{H}_s, \hat{\rho}_s(t)] + \frac{1}{2}\sum_{e>\emptyset} \kappa_e \left(2\hat{F}^e \hat{\rho}_s(t)\hat{F}^{e\dagger} - \hat{F}^{e\dagger}\hat{F}^e \hat{\rho}_s(t) - \hat{\rho}_s(t)\hat{F}^{e\dagger}\hat{F}^e\right)$$

Expressing this in vectorised form, and using the identity $|\hat{A}\hat{B}\hat{C}\rangle\rangle = (\hat{C}^T \otimes \hat{A})|\hat{B}\rangle\rangle$, we get:

$$\frac{d|\hat{\rho}_s(t)\rangle\rangle}{dt} = \hat{\mathcal{L}}|\hat{\rho}_s(t)\rangle\rangle = -i\left(\mathbb{1} \otimes \hat{H} - \hat{H}^T \otimes \mathbb{1}\right) + \frac{1}{2}\sum_{e>\emptyset} \kappa_e \left(2\hat{F}^{e*} \otimes \hat{F}^e - \mathbb{1} \otimes \hat{F}^{e\dagger}\hat{F}^e - (\hat{F}^{e\dagger}\hat{F}^e)^T \otimes \mathbb{1}\right)|\hat{\rho}_s(t)\rangle\rangle$$

→ The jump term has $\hat{F}^{e*}$, not $\hat{F}^{e\dagger}$, since $(\hat{C}^T \otimes \hat{A})$ already transposes.

→ <u>Lindblad's theorem</u>: <u>every</u> quantum operation that satisfies the semigroup property satisfies the GKSL equation.

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* Proof in Landi. Include in typeset notes.

Solution, as always, is $|\hat{\rho}_s(t)\rangle\rangle = e^{t\hat{\mathcal{L}}}|\hat{\rho}_{in,s}(t)\rangle\rangle$. Using the Choi-Jamiołkowski isomorphism, we can expand $e^{t\hat{\mathcal{L}}}$ without having to worry about BCH nastiness by examining the spectral decomposition of $\hat{\mathcal{L}}$.

Assuming $\hat{\mathcal{L}}$ is diagonalisable, it's still generally (or usually) Hermitian, so generally the left eigenvectors $\{\langle\langle\hat{y}_{a,b}|\}$ satisfying $\langle\langle\hat{y}_{a,b}|\hat{\mathcal{L}} = \lambda_{a,b}\langle\langle\hat{y}_{a,b}|$ and the right eigenvectors $\{|\hat{x}_{a,b}\rangle\rangle\}$ satisfying $\hat{\mathcal{L}}|\hat{x}_{a,b}\rangle\rangle$ are different.

→ Eigenvalues are still the same, because the characteristic polynomials are the same.

→ If $\hat{\mathcal{S}}$ is a matrix whose columns are $\{|\hat{x}_{a,b}\rangle\rangle\}$, then the rows of $\hat{\mathcal{S}}^{-1}$ are given by $\{\langle\langle\hat{y}_{a,b}|\}$.

→ Diagonalisation of $\hat{\mathcal{L}}$: if $\hat{\mathcal{M}} = \mathrm{diag}(\{\lambda_{a,b}\}) = \begin{pmatrix} \lambda_{1,1} & & & \\ & \lambda_{1,2} & & \\ & & \ddots & \\ & & & \lambda_{a,b} \end{pmatrix}$, then we have $\hat{\mathcal{L}}$ given by:

$$\hat{\mathcal{L}} = \hat{\mathcal{S}}\hat{\mathcal{M}}\hat{\mathcal{S}}^{-1} = \sum_{a,b}{}' \lambda_{a,b}\,|\hat{x}_{a,b}\rangle\rangle\langle\langle\hat{y}_{a,b}|.$$

As always, the spectral decomposition gives us an easy way to write $f(\hat{\mathcal{L}})$: $f(\hat{\mathcal{L}}) = \hat{\mathcal{S}}\, f(\hat{\mathcal{M}})\,\hat{\mathcal{S}}^{-1} = \sum_{a,b}{}' f(\lambda_{a,b})\,|\hat{x}_{a,b}\rangle\rangle\langle\langle\hat{y}_{a,b}|$.

As always, the spectral decomposition gives us an easy way to write $f(\hat{\mathcal{L}})$: $f(\hat{\mathcal{L}}) = \hat{\mathcal{S}}\, f(\hat{\mathcal{M}})\, \hat{\mathcal{S}}^{-1} = \sum'_{a,b} f(\lambda_{a,b})\, |\hat{x}_{a,b}\rangle\!\rangle\langle\!\langle \hat{y}_{a,b}|$. Applying this to $e^{t\hat{\mathcal{L}}}$:

$$e^{t\hat{\mathcal{L}}} = \hat{\mathcal{S}}\, e^{t\hat{\mathcal{M}}}\, \hat{\mathcal{S}}^{-1} = \sum'_{a,b} e^{t\lambda_{a,b}}\, |\hat{x}_{a,b}\rangle\!\rangle\langle\!\langle \hat{y}_{a,b}|$$

→ We can apply this decomposition to manipulations involving $\hat{\mathcal{L}}$. Taking the inner product in $O_p(\mathcal{H})$ of $d|\hat{\rho}_s(t)\rangle\!\rangle/dt = \hat{\mathcal{L}}|\hat{\rho}_s(t)\rangle\!\rangle$ with $\langle\!\langle \mathbb{1}|$:

$$\langle\!\langle \mathbb{1}| \left(\frac{d|\hat{\rho}_s(t)\rangle\!\rangle}{dt}\right) = \langle\!\langle \mathbb{1}|\hat{\mathcal{L}}|\hat{\rho}_s(t)\rangle\!\rangle$$

Since $\langle\!\langle \mathbb{1}|$ is not time-dependent, we have $\langle\!\langle \mathbb{1}|\left(d|\hat{\rho}_s(t)\rangle\!\rangle/dt\right) = d/dt\left[\langle\!\langle \mathbb{1}|\hat{\rho}_s(t)\rangle\!\rangle\right]$; and from the density matrix normalisation condition, $\mathrm{Tr}\{\hat{\rho}\} = \langle\!\langle \mathbb{1}|\hat{\rho}\rangle\!\rangle = 1$. Thus, we have $\langle\!\langle \mathbb{1}|\left(d|\hat{\rho}_s(t)\rangle\!\rangle/dt\right) = 0$:

$$\langle\!\langle \mathbb{1}| \left(\frac{d|\hat{\rho}_s(t)\rangle\!\rangle}{dt}\right) = \langle\!\langle \mathbb{1}|\hat{\mathcal{L}}|\hat{\rho}_s(t)\rangle\!\rangle = 0.$$

However, since we didn't pick any $|\hat{\rho}_s(t)\rangle\!\rangle$ to start with, $|\hat{\rho}_s(t)\rangle\!\rangle$ is arbitrary, so this gives $\langle\!\langle \mathbb{1}|\hat{\mathcal{L}} = 0$. Thus, $\langle\!\langle \mathbb{1}|$ is <u>always</u> a left eigenstate of $\hat{\mathcal{L}}$, with left eigenvalue $\lambda_{\mathbb{1}} = 0$. The right eigenvector corresponding to this eigenvalue is defined to be the <u>steady state</u>:

$$\hat{\mathcal{L}}|\hat{\rho}_{ss}\rangle\!\rangle := 0.$$

$$\hat{\mathcal{L}}|\hat{\rho}_{ss,s}\rangle\rangle := \emptyset.$$

Thus, any trace-preserving Liouvillian has a zero eigenvalue, with right eigenstate $|\hat{\rho}_{ss,s}\rangle\rangle$ and left eigenstate $\langle\langle \mathbb{1}|$.

→ Now, we'll apply $\sum'_{a,b} e^{t\lambda_{a,b}}|\hat{x}_{a,b}\rangle\rangle\langle\langle\hat{y}_{a,b}|$ to $|\hat{\rho}_s(t)\rangle\rangle$. Defining $\langle\langle\hat{y}_{a,b}|\hat{\rho}_{in,s}\rangle\rangle = Tr\{\hat{y}_{a,b}^{+}\hat{\rho}_{in,s}\} =: c_{a,b} \in \mathbb{C}$ as the coefficients coming from the inner product in $Op(\mathcal{H})$, we have $|\hat{\rho}_s(t)\rangle\rangle$ as:

$$|\hat{\rho}_s(t)\rangle\rangle = e^{t\hat{\mathcal{L}}}|\hat{\rho}_{in,s}\rangle\rangle = \sum_{a,b}{}' e^{t\lambda_{a,b}}|\hat{x}_{a,b}\rangle\rangle\langle\langle\hat{y}_{a,b}|\hat{\rho}_{in,s}\rangle\rangle = \sum_{a,b}{}' c_{a,b}\, e^{t\lambda_{a,b}}|\hat{x}_{a,b}\rangle\rangle$$

Since we have a positive exponential, we <u>require</u> $\mathcal{Re}\{\lambda_{a,b}\} \leq \emptyset$: $\mathcal{Re}\{\lambda_{a,b}\} > \emptyset$ would blow up.

* One steady state: only eigenvalue is $\lambda_{a,b} = \emptyset$. Multiple steady states: can have pure imaginary eigenvalues.