Albert 1.1.2: Open Quantum Systems

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(also uses Caves 17-20: Superoperators & Completely Positive Maps)

Closed quantum systems: dynamics generated by Hamiltonian. Open quantum systems: dynamics derived from general Hamiltonian EOM, given by Liouville-von Neumann equation:

$$\frac{d\hat{\rho}_{SE}}{dt} = -i\left[\hat{H}_{SE}, \hat{\rho}_{SE}\right]$$

Solution: $\hat{\rho}_{SE}(t) = e^{-i\hat{H}_{SE}t}\,\hat{\rho}_{SE}(\emptyset)\,e^{i\hat{H}_{SE}t}$. Reduced DM:

$$\hat{\rho}_S(t) = \mathrm{Tr}_E\{\hat{\rho}_{SE}(t)\} = \mathrm{Tr}_E\{e^{-i\hat{H}_{SE}t}\,\hat{\rho}_{SE}(t=\emptyset)\,e^{i\hat{H}_{SE}t}\} = \sum_e{}' \left(\mathbb{1}_S \otimes \langle e|_E\right) e^{-i\hat{H}_{SE}t}\,\hat{\rho}_{SE}(t=\emptyset)\,e^{i\hat{H}_{SE}t}\left(\mathbb{1}_S \otimes |e\rangle_E\right).$$

Assumption: initial state factorises. $\hat{\rho}_{SE}(t=\emptyset) = \hat{\rho}_{SE}(\emptyset) = \hat{\rho}_{in,S} \otimes |e=\emptyset\rangle\langle e=\emptyset|_E = \hat{\rho}_{in,S} \otimes |\emptyset\rangle\langle\emptyset|_E$. Then, $\hat{\rho}_S(t)$ is:

$$\hat{\rho}_S(t) = \sum_e{}' \left(\mathbb{1}_S \otimes \langle e|_E\right) e^{-i\hat{H}_{SE}t}\,\hat{\rho}_{SE}(\emptyset)\,e^{i\hat{H}_{SE}t}\left(\mathbb{1}_S \otimes |e\rangle_E\right) = \sum_e{}' \left(\mathbb{1}_S \otimes \langle e|_E\right) e^{-i\hat{H}_{SE}t}\left(\hat{\rho}_{in,S} \otimes |\emptyset\rangle\langle\emptyset|_E\right) e^{i\hat{H}_{SE}t}\left(\mathbb{1}_S \otimes |e\rangle_E\right)$$

$$\hat{\rho}_S(t) = \sum_e{}' \left(\mathbb{1}_S \otimes \langle e|_E\right) e^{-i\hat{H}_{SE}t}\left(\mathbb{1}_S \otimes |\emptyset\rangle_E\right)\left(\hat{\rho}_{in,S} \otimes \mathbb{1}_E\right)\left(\mathbb{1}_S \otimes \langle\emptyset|_E\right) e^{i\hat{H}_{SE}t}\left(\mathbb{1}_S \otimes |e\rangle_E\right)$$

Write $\left(\mathbb{1}_S \otimes \langle a|_E\right) f(\hat{A}_{SE})\left(\mathbb{1}_S \otimes |b\rangle_E\right)$ as $\langle a| f(\hat{A}_{SE})|b\rangle_E$ and $\hat{\rho}_{in,S} \otimes \mathbb{1}_E$ as $\hat{\rho}_{in,S}$ to get:

$$\hat{\rho}_S(t) = \sum_e{}' \langle e| e^{-i\hat{H}_{SE}t}|\emptyset\rangle_E\,\hat{\rho}_{in,S}\,\langle\emptyset| e^{i\hat{H}_{SE}t}|e\rangle_E = \sum{}' \hat{F}^e(t)\,\hat{\rho}_{in,S}\,\hat{F}^{e\dagger}(t)$$

$$\hat{\rho}_S(t) = \sum_e{}' \langle e| e^{-i\hat{H}_{SE}t} |\emptyset\rangle_E \, \hat{\rho}_{in,S} \, \langle \emptyset| e^{i\hat{H}_{SE}t} |e\rangle_E = \sum_e{}' \hat{E}^e(t) \, \hat{\rho}_{in,S} \, \hat{E}^{e+}(t).$$

$\hat{E}^e(t) := \langle e| e^{-i\hat{H}_{SE}t} |\emptyset\rangle_E$ are Kraus operators.

- One operator for each $|e\rangle_E$.
- Time-dependent.
- $\sum_e{}' \hat{E}^{e+}\hat{E}^e = \mathbb{1}_S$. (Proven below.)
- Operate exclusively on $\mathcal{H}_S$; i.e. $\{\hat{E}^e(t)\}$ map $\mathcal{D}(\mathcal{H}_S) \to \mathcal{D}(\mathcal{H}_S)$; with $\{\hat{E}^e(t)\}: \hat{\rho}_{in,S} \mapsto \hat{\rho}_S(t)$.
- Recall discussion from Gould paper: these are actually processes, not maps, since they change as we change $\{|e\rangle_E\}$.
  * Maps, conversely, are fixed.
- Recall discussion from Gould paper: Griffiths' & Harrow's lecture notes have $\hat{A}_\ell : \mathcal{H}_S \otimes \mathcal{H}_E \to \mathcal{H}_E$ and $\hat{A}_\ell^+ : \mathcal{H}_E \to \mathcal{H}_S \otimes \mathcal{H}_E$, which makes sense from an operator perspective. This is technically correct, but Gould & Albert use the Kraus operators differently. Specifically, Gould uses $\hat{\rho}_E(t) = \mathrm{Tr}_S\{\hat{U}(\hat{\rho}_{in,S} \otimes \hat{\rho}_{in,E})\hat{U}^+\}$ as a machine to get $\hat{\rho}_E(t)$ from $\hat{\rho}_{in,S} \otimes \hat{\rho}_{in,E}$; so for him we have $\hat{A}_\ell, \hat{A}_\ell^+ : \mathcal{H}_S \otimes \mathcal{H}_E \to \mathcal{H}_E$. Similarly, Albert uses $\hat{\rho}_S(t) = \sum_e{}' \hat{E}^e(t)\,\hat{\rho}_{in,S}\,\hat{E}^{e+}(t)$ as a machine to get $\hat{\rho}_S(t)$ from $\hat{\rho}_{in,S}$. Since the use is different, the type of mapping is different too.
- Also called Kraus maps, quantum channels, & completely positive trace preserving (CPTP) maps. Properties:
  * Completely positive: $\hat{\rho}_S(t) \geq \hat{\emptyset}$ for all t. (Preserves positivity.)
  * Preserves positivity when acting on a larger system: $\hat{\rho}_S(t) \otimes \hat{\rho}_A(t) \geq \emptyset$ for all t.
  * Trace preserving: $\mathrm{Tr}\{\hat{\rho}_S(t)\} = \mathrm{Tr}_S\{\hat{\rho}(t)\} = 1$ for all t.

Unitarity of $e^{-i\hat{H}_{SE}t}$ and completeness of $\{|e\rangle_E\}$ give $\sum'_e \hat{E}^{e\dagger}\hat{E}^e = \mathbb{1}_S$:

$$\sum_e{}' \hat{E}^{e\dagger}\hat{E}^e = \sum_e{}' \langle\emptyset| e^{i\hat{H}_{SE}t} |e\rangle_E \langle e| e^{-i\hat{H}_{SE}t} |\emptyset\rangle_E = \sum_e{}' (\mathbb{1}_S \otimes \langle\emptyset|_E)\, e^{i\hat{H}_{SE}t} (\mathbb{1}_S \otimes |e\rangle_E)(\mathbb{1}_S \otimes \langle e|_E)\, e^{-i\hat{H}_{SE}t} (\mathbb{1}_S \otimes |\emptyset\rangle_E)$$

$$\sum_e{}' \hat{E}^{e\dagger}\hat{E}^e = (\mathbb{1}_S \otimes \langle\emptyset|_E)\, e^{i\hat{H}_{SE}t} \sum_e{}' (\mathbb{1}_S \otimes |e\rangle_E)(\mathbb{1}_S \otimes \langle e|_E)\, e^{-i\hat{H}_{SE}t} (\mathbb{1}_S \otimes |\emptyset\rangle_E) = (\mathbb{1}_S \otimes \langle\emptyset|_E)\, e^{i\hat{H}_{SE}t} (\mathbb{1}_S \otimes \mathbb{1}_E)\, e^{-i\hat{H}_{SE}t} (\mathbb{1}_S \otimes |\emptyset\rangle_E)$$

$$\sum_e{}' \hat{E}^{e\dagger}\hat{E}^e = (\mathbb{1}_S \otimes \langle\emptyset|_E)\, e^{i\hat{H}_{SE}t} e^{-i\hat{H}_{SE}t} (\mathbb{1}_S \otimes |\emptyset\rangle_E) = (\mathbb{1}_S \otimes \langle\emptyset|_E)\, \mathbb{1}_{SE}\, (\mathbb{1}_S \otimes |\emptyset\rangle_E) = \mathbb{1}_S \otimes \langle\emptyset|\emptyset\rangle_E = \mathbb{1}_S \otimes 1 = \mathbb{1}_S.$$

<u>Stinespring's dilation theorem</u>: examine a superoperator $\hat{\mathcal{A}} \in \mathcal{L}_1(O_p(\mathcal{H}_A))$ applied to $\hat{\rho} \in \mathcal{D}(\mathcal{H}_A)$ with a Kraus decomposition given by:

$$\hat{\mathcal{A}}[\hat{\rho}] = \sum_e{}' \hat{E}^e \hat{\rho}\, \hat{E}^{e\dagger}$$

We can expand $\hat{\mathcal{A}}$ to a larger space $\mathcal{H}_A \otimes \mathcal{H}_B$ using a pure initial state $|s\rangle\langle s| \in \mathcal{D}(\mathcal{H}_B)$ over $\mathcal{H}_B$ and a joint unitary operator $\hat{U} \in O_p(\mathcal{H}_A \otimes \mathcal{H}_B)$:

$$\hat{\mathcal{A}}[\hat{\rho}] = \mathrm{Tr}_B\{\hat{U}(\hat{\rho} \otimes |s\rangle\langle s|)\hat{U}^\dagger\} = \sum_b{}' (\mathbb{1}_A \otimes \langle b|_B)\, \hat{U}(\hat{\rho}_A \otimes |s\rangle\langle s|_B)\, \hat{U}^\dagger (\mathbb{1}_A \otimes |b\rangle_B)$$

<u>Proof</u>: as Kraus representation theorem in Caves. Include in typeset notes.

→ Expressing quantum channels $\hat{\hat{C}}: \hat{\rho} \mapsto \hat{\sigma}$ as $\hat{\sigma} = \sum_e' \hat{E}^e \hat{\rho} \hat{E}^{e\dagger}$ with Kraus operators $\{\hat{E}^e\}$ is called Kraus decomposition / operator-sum representation.

→ For a set $\{\hat{\mathcal{A}}_\alpha\}$ of superoperators, have set of projection operators $\hat{P}_\alpha \in \mathcal{O}_p(\mathcal{H}_B)$. Now, Kraus representation theorem reads:

$$\hat{\mathcal{A}}[\hat{\rho}] = \mathrm{Tr}_B\{\hat{P}_\alpha \hat{U}(\hat{\rho} \otimes |s\rangle\langle s|)\hat{U}^\dagger\} = \sum_b' (\mathbb{1}_A \otimes \langle b|_B)(\mathbb{1}_A \otimes \hat{P}_{\alpha_B})\hat{U}(\hat{\rho}_A \otimes |s\rangle\langle s|_B)\hat{U}^\dagger(\mathbb{1}_A \otimes |b\rangle_B)$$