Linear Algebra Exercise 3

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■ **1.** The transformation $T(v_1, v_2) = (v_1, 0)$ is linear.

■ **2.** If T maps $\mathbb{R}^n$ into $\mathbb{R}^m$, and $T(0) = 0$, then T is linear.

■ **3.** $T_1T_2 = T_2T_1$ where $T_1: \mathbb{R}^3 \to \mathbb{R}^3$ is the rotation about the x-axis through an angle θ_1 and $T_2: \mathbb{R}^3 \to \mathbb{R}^3$ is the rotation about the z-axis through an angle θ_2.

■ **4.** Suppose $T(v_1) = (2,5)$, $T(v_2) = (1,3)$, $v_1 = (1,0)$, $v_2 = (0,1)$. Let $v_3 = (2,-1)$. Then $T(v_3) = (3,7)$.

■ **5.** $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ is the standard matrix for the linear operation $T: \mathbb{R}^2 \to \mathbb{R}^2$ that maps a point (x, y) into its reflection about the origin.

■ **6.** Given two bases for the same vector space. There is always a transition matrix from one basis to the other basis.

■ **7.** The transition matrix from B to B is always the identity matrix.

■ **8.** Any invertible $n \times n$ matrix is the transition matrix for some pair of bases for $\mathbb{R}^n$.

■ **9.** Let $[u_1, u_2, u_3]$ be a basis for $\mathbb{R}^3$, and let $U = (u_1, u_2, u_3)$ be the matrix with columns u_1, u_2, u_3. Then U^T is the transition matrix for the change of basis from the standard basis $[e_1, e_2, e_3]$ to $[u_1, u_2, u_3]$.

■ **10.** If $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$, then matrix $P = [c_1 \quad c_2]^{-1}[b_1 \quad b_2]$ satisfies $[x]_C = P[x]_B$.

■ **11.** If $m = \dim(V)$ and $n = \dim(W)$, β, γ are ordered basis of V and W, respectively, and T is a linear transformation, then $[T]_\beta^\gamma$ is an $m \times n$ matrix.

■ **12.** Suppose S and T are linear operators on a finite dimensional vector space V, α and β are ordered bases for V. If $[S\}_\alpha = [T]_\beta$, then $S = T$.

■ **13.** Let $L: V \to V$ be a linear transformation. If x is a vector in the kernal of L, then $L(v + x) = L(v)$ for all $v \in V$.

■ **14.** There does not exist linear transformation $T: V \to W$ such that $N(T) = R(T)$.

■ **15.** Let D be the differentation operator on P_3. Then $\ker(D) = \{0\}$.

■ **16.** Assume V and W are vector spaces and $L: V \to W$ is a linear transformation. If $\dim(V) = n$ and $\dim(W) = m$, then $\dim(\ker(L)) + \dim(L(V)) = m$.

■ **17.** Let $L: \mathbb{R}^6 \to \mathbb{R}^{10}$ be a linear transformation defined by $L(x) = Ax$ for x in $\mathbb{R}^6$. If $\dim(R(L)) = 3$, then $\dim(\ker(L)) = 7$.

■ **18.** The linear transformation $L: P_2 \to P_2$ defined by $L(ax^2 + bx + c) = 2ax + b$ is one-to-one.

■ **19.** If T is a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^4$ such that $N(T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_1 = 5x_2 \textit{ and } x_3 = 7x_4\}$. Then T is surjective.

■ **20.** There does not exist a linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^2$ whose null space equals

$$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 | x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.$$

■ **21.** If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a one-to-one linear transformation, then there are no distinct vectors u and v in $\mathbb{R}^n$ such that $T(u - v) = 0$.

■ **22.** If $n < m$ and $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then T is one-to-one.

■ **23.** There are linear transformations $L: \mathbb{R}^3 \to \mathbb{R}^5$ that are onto.

■ **24.** Let V and W be finite-dimensional vector spaces and T be a linear transformation from V into W, then T maps linearly independent subsets of V into linearly independent subsets of W.

■ **25.** Let α be a basis for a finite dimensional vector space V. If T is a linear operator on V then T is onto if and only if $T(\alpha)$ is a basis for V.

■ **26.** The rank of any upper triangular matrix is the number of nonzero entries on its diagonal.

■ **27.** The rank of a matrix is equal to either the number of its nonzero rows or the number of its nonzero columns, whichever is smaller.

■ **28.** The product of two matrices has a rank equal to the smaller rank of the two matrices.

■ **29.** For two square $n \times n$ matrices A and B, $rank(AB) = rank(A)$ if and only if B is nonsingular.

■ **30.** Elementary row operations preserves matrix rank but elementary column operations do not.

■ **31.** Let $A \in \mathbb{R}^{m \times n}$, where $m < n$. Suppose $a_{ij} < 0$ if $j < i$ and $a_{ij} \neq 0$ if $i \leq j$. Let $B = E_k E_{k-1} \cdots E_2 E_1 A$, where $E_1, \dots, E_k$ denote a sequence of $m \times m$ elementary matrices, then the row vectors of B are linearly independent.

■ **32.** Let A be $n \times n$. Then $rank(A) = rank(A^2)$.

■ **33.** Let A and B be $n \times n$ real matrices, if $rank(A) = rank(B)$, then $rank(A^2) = rank(B^2)$.

■ **34.** If v is not a linear combination of $\{u_1, u_2, \dots, u_k\}$, then $rank([u_1 \quad u_2 \quad \cdots \quad u_k \quad v]) = 1 + rank([u_1 \quad u_2 \quad \cdots \quad u_k])$.

■ **35.** $rank(A) \leq rank(A + B)$, where A and B are both $m \times n$ matrices.

■ **36.** If A is an $m \times n$ matrix, then the nullity of A^T plus the rank of A equals m.

■ **37.** For an $m \times n$ matrix A, the nullity of A equals the nullity of its transpose A^T.

■ **38.** If A is an $n \times m$ matrix, then $\dim\big(N(A)\big) + \dim\big(RS(A)\big) = m$.

■ **39.** If an $m \times n$ matrix A is row equivalent to an echelon matrix U and if U has k nonzero rows, then the dimension of the solution space of $Ax = O$ is $m - k$.

■ **40.** Let A be an $m \times n$ matrix. Then $nullity(A) \geq n - m$.

■ **41.** A is a 4×7 matrix and has four pivot columns. Thus, $R(A) = \mathbb{R}^4$ and $N(A) = \mathbb{R}^3$.

■ **42.** If the matrix A is 8×5 and $rank(A) = 3$, then the dimension of the row space of A is 5.

■ **43.** If the row vectors of A are linearly independent, then null space $N(A) = \{0\}$.

■ **44.** If the columns of a matrix are dependent, so are the rows of the matrix.

■ **45.** The column of a 5×8 matrix A whose rank is 5 form a linearly dependent set.

■ **46.** A is an $m \times n$ matrix $(m \neq n)$. If A has a right inverse C such that $AC = I_m$, then the column vectors of A span $\mathbb{R}^m$.

■ **47.** A is an $m \times n$ matrix ($m \neq n$). If A has a left inverse B such that $BA = I_m$, then the column vector of A are linearly independent.

■ **48.** For an $n \times n$ matrix A, the columns of A are linearly independent if and only if the rows of A are linearly independent.

■ **49.** If a system $Ax = b$ has more than one solution, then so does the system $Ax = O$.

■ **50.** If A is a 3×5 matrix such that $\dim\big(N(A)\big) = 2$, then the equation $Ax = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$ has infinite many solutions.

■ **51.** If the rows of an $m \times n$ matrix A are linearly independent, then $Ax = b$ is always solvable.

■ **52.** If the rows of an $m \times n$ matrix A are linearly independent, then the solution $Ax = b$, if it exists, is always unique.

■ **53.** If the columns of A are linearly independent, then $Ax = b$ has exactly one solution for every b.

■ **54.** If the columns of an $m \times n$ matrix A are linearly independent, then $Ax = b$ is always solvable.

■ **55.** Let A be an $m \times n$ matrix. If $Au = Av$ implies $u = v$, then $rank(A) = n$.

■ **56.** If A is $m \times n$ and $rank(A) = m$, then the linear transformation $x \mapsto Ax$ is one-to-one.

■ **57.** If A is $m \times n$ and the linear transformation $x \mapsto Ax$ is onto, then $rank(A) = m$.

■ **58.** The non-pivot columns of a matrix are always linearly dependent.

■ **59.** Row operations on a matrix A can change the linear dependence relations among the rows of A.

■ **60.** If A is an $m \times n$ matrix whose rows span $\mathbb{R}^n$, then the set of columns in A is linearly dependent.

■ **61.** A and A^T have the same null space.

■ **62.** If A is an $n \times n$ matrix, then the row space of A equals the column space.

■ **63.** If the row space equals the column space then $A^T = A$.

■ **64.** If E is an $m \times m$ elementary matrix, then A and EA must have the same null space.

■ **65.** If E is an $m \times m$ elementary matrix, then A and EA must have the same row space.

■ **66.** If E is an $n \times n$ elementary matrix, then A and AE must have the same column space.

■ **67.** Let A be an $m \times n$ matrix. If B is a nonsingular $n \times n$ matrix, then AB and A have the same null space.

■ **68.** If matrices A and B have the same reduced row-echelon form, then they have the same row space.

■ **69.** If $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{1 \times n}$, then the null space of $\begin{bmatrix} A \\ B \end{bmatrix}$ is the intersection of the null spaces of A and B.

■ **70.** The row space and null space of an invertible matrix A are the same.

■ **71.** If H is a subspace of $\mathbb{R}^3$, then there is a 3×3 matrix A such that H equals the column space of A.

■ **72.** If A and B share the same four subspaces then A is a multiple of B.

■ **73.** Let $T, U : V \to W$ be linear transformations. If T and U are both one-to-one, then $T + U$ is also one-to-one

二、選擇題

■ **1.** $T: P_3 \to P_3$ is defined by $T(a_3x^3 + a_2x^2 + a_1x + a_0) = (a_3 + a_2 - a_1)x^3 + (a_2 + a_1 - 2a_0)x$. Determine which of the following polynomials are in the kernel of T.

(A) $x^2 + x + 1$ (B) $2x^3 + 2x + 1$ (C) $2x^3 + x^2 + x + 2$ (D) $x^2 + x + 2$ (E) $-4x^3 + 5x^2 + x + 3$.

■ **2.** Suppose that A is a 4×6 matrix. Determine which of the following statements are true ?

(A) The rank of A is at most 4.

(B) The rank of A^T is at most 4

(C) The number of general solution of $Ax = O$ is at most 6.

(D) The nullity of A^T is at most 6.

■ **3.** Let H be an arbitrary $n \times n$ matrix. Then

(A) The row space of H is contained in the column space.

(B) The row space of H equals the column space of H.

(C) The row space of H has the same dimensions as the column space of H.

(D) The row space of H equals the null space of H.

(E) None of the preceding statements is true.

■ **4.** Given the linear operator T with standard matrix $[T]_E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ and B-matrix $[T]_B =$

$\begin{bmatrix} 1 & 9 & -6 \\ 0 & 7 & -4 \\ 2 & 11 & -8 \end{bmatrix}$, which can be the correct basis for B ?

(A) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 9 \end{bmatrix} \right\}$ (B) $\left\{ \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \right\}$ (C) $\left\{ \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \right\}$ (D) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ -4 \end{bmatrix} \right\}$ (E) None of the above.

■ **5.** An affine transformation of $\mathbb{R}^2$ is a function $T: \mathbb{R}^2 \to \mathbb{R}^2$ of the form $T(x) = Ax + b$, where A is an invertible 2×2 matrix and $b \in \mathbb{R}^2$. Which of the following statements are correct ?

(A) $T^{-1}(x) = A^{-1}x - A^{-1}b$.

(B) Affine transformations map straight lines to straight lines.

(C) There is no affine transformation that can map a straight line to a circle.

(D) Affine transformations maps parallel straight lines to parallel straight lines.

(E) There exists an affine transformation that maps parallel straight lines to intersecting straight lines.