Linear Algebra Exercise 4

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■ **1.** Determine whether the transformation $T(x,y,z)=(4x,\frac{y^2}{z^2})$ linear or not ?

■ **2.** Find the standard matrix A for a linear transformation on $\mathbb{R}^2$; that is an reflection about the x-axis, followed by an clockwise rotation of $60°$ about the origin. Also, find the inverse matrix of A.

■ **3.** Determine a 2D transformation matrix (A) that is a concatenation of reflection to y-axis, followed by a rotation of 30 degree. Also determine A^{-1}.

■ **4.** There exists a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(5,13)=(31,-53,-2)$ and $T(11,7)=(25,13,-26)$. Find $T(2,-1)$.

■ **5.** Let T be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that

$$T\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix},\ T\begin{bmatrix}-2\\1\\1\end{bmatrix}=\begin{bmatrix}-2\\1\\1\end{bmatrix},\ T\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}4\\4\\4\end{bmatrix}$$

Find the standard matrix representation of T.

■ **6.** Find the coordinate vectors of $u=(4,5)$ relative to the following basis B of $\mathbb{R}^2$: $B=\{(2,1),(-1,1)\}$.

■ **7.** Find the coordinate vector of u relative to the given basis B in $\mathbb{R}^3$, where $u=(0,5,-3,-4)$ and $B=\{(1,2,3),(1,-1,0),(0,1,-2)\}$.

■ **8.** Let $v_1=\begin{bmatrix}4\\6\\7\end{bmatrix}$, $v_2=\begin{bmatrix}0\\1\\1\end{bmatrix}$ and $v_3=\begin{bmatrix}0\\1\\2\end{bmatrix}$, and let $u_1=\begin{bmatrix}1\\1\\1\end{bmatrix}$, $u_2=\begin{bmatrix}1\\2\\2\end{bmatrix}$ and $u_3=\begin{bmatrix}2\\3\\4\end{bmatrix}$. If $x=2v_1+3v_2-4v_3$, determine the coordinates of x with respect tp $\{u_1,u_2,u_3\}$.

■ **9.** In the vector space $\mathbb{R}^3$, there are two sets of basis $A=\left\{\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}1\\-2\\0\end{pmatrix}\right\}$ and $B=\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$. Let vector $[x]_B=\begin{pmatrix}1\\-2\\-1\end{pmatrix}$. Find

a) coordinate transitions matrix from B to A.

b) $[x]_A$.

■ **10.** Suppose $B=([2,3,1],[1,2,0],[2,0,3])$ and $B'=([1,0,0],[0,1,0],[0,0,1])$ are ordered bases for $\mathbb{R}^3$. Find the change of coordinates from B to B'.

■ **11.** Consider the bases $B = \{p_1, p_2\}$ and $B' = \{q_1, q_2\}$ for P_1, where $p_1 = 1 + 2x,\ p_2 = 3 - 3x,\ q_1 = x,\ q_2 = 4 + x$.

a) Find the transition matrix from B' to B.

b) Find the transition matrix from B to B'.

c) Compute the coordinate vector $[p]_B$, where $p = 1 + x$.

d) Use your answer to parts (b) and (c) to compute $[p]_{B'}$.

■ **12.** Let L be a transformation mapping P_2 into $\mathbb{R}^2$ defined by $L(p(x)) = \begin{bmatrix} \int_0^1 p(x)dx \\ p'(x) \end{bmatrix}$.

a) Is L a linear transformation ?

b) Find a matrix A such that $L(\alpha_1 x + \alpha_2) = A \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$.

■ **13.** Let $D = \frac{d}{dx}$ be the operation of taking the derivative. Find the matrix of D with respect to the basis $\{x^2, x, 1\}$ of P_2.

■ **14.** Let $T: P_2 \to \mathbb{R}^{2\times 2}$ be a linear transformation such that $T(a + bx + cx^2) = \begin{bmatrix} a & b+c \\ a+b & c \end{bmatrix}$. Find the matrix representation of T with respect to the standard bases.

■ **15.** Define the transformation T from P_2 to P_1 by $T(v) = \frac{v(1)-v(0)}{t}$. Let the ordered basis for P_2 be $B = \{1 + t, t + t^2, 1 + t^2\}$ and the ordered basis for P_1 be $C = \{1 + t, 1 - t\}$. Find the matrix representation of T.

■ **16.** Let the ordered basis for P_3 be $B = \{x^3, x^2, x, 1\}$ and let $T: P_3 \to P_3$ be defined by $T(p(x)) = \frac{d}{dx} p(x)$

a) Find the matrix representation A of T.

b) Use A to find $T(4x^3 - 5x^2 + 10x - 13)$.

■ **17.** Let $f: \mathbb{R}^2 \to \mathbb{R}^3$ be the linear map given on the standard basis by $f(e_1) = 2e_2 + e_3$ and $f(e_2) = e_1 + 3e_2 + 5e_3$. Determine the 3×2 matrix of this map with respect to the bases $3e_1 - e_2,\ 2e_1 + 3e_2$ of $\mathbb{R}^2$ and $e_1,\ e_1 + e_2,\ e_1 - e_2 - e_3$ of $\mathbb{R}^3$.

■ **18.** Let $u_1 = (2,1)^T$ and $u_2 = (5,3)^T$ and let L be the linear operator that rotates vector in $\mathbb{R}^2$ by $45°$ in the counterclockwise direction. Find the matrix representation of L with respect to the ordered basis $[u_1, u_2]$.

■ **19.** Define $T: P_2 \to P_2$ by the equation $T(p(x)) = p(x) + (1 + x)p'(x)$.

a) Find the matrix A representing T relative to the standard basis $\{1, x, x^2\}$ for P_2.

b) Find the matrix B representing T relative to the basis $\{1, 1 + x, 1 + x + x^2\}$.

c) Find a matrix C such that $B = C^{-1}AC$.

■ **20.** Let $T: P_2 \to P_2$ be defined by $T(p(x)) = p(x - 1)$. Consider the two ordered bases $B = \{x^2, x, 1\}$ and $B' = \{x, x + 1, x^2 - 1\}$. Find the matrix representation $[T]_B$ and $[T]_{B'}$ of T and a matrix C such that $C^{-1}[T]_B C = [T]_{B'}$.

■ **21.** $L(x) = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \end{bmatrix}$, $x \in \mathbb{R}^3$. Find $\ker(L)$.

■ **22.** Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $T\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 + a_3 \\ a_1 + a_2 \\ a_2 - a_3 \end{bmatrix}$.

a) Find a basis for the kernel space, $\ker(T)$.

b) Find a basis for the range space, $R(T)$.

■ **23.** Let $T: \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be the linear operator defined by $T(X) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} X + X \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$.

a) Find the dimension of the range of T.

b) Find a basis for the null space of all matrices X.

■ **24.** Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(a,b,c) = (a + 2b + c, -a - b + 2c, 2a + 3b - c)$. Determine the values of k such that $(k, 3, -2)$ is in the range of T.

■ **25.** Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(x,y,z) = (x - y + 3z, 5x + 6y - 4z, 7x + 4y + 2z)$. Find $\dim(T(\mathbb{R}^3))$ and $\dim(\ker(T))$.

■ **26.** Consider a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^4$ with $T\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 2a_1 \\ a_1 + 2a_2 \\ a_2 + 2a_3 \\ a_3 + 2a_4 \end{bmatrix}$. Is T one-to-one ? Does T map $\mathbb{R}^3$ onto $\mathbb{R}^4$?

■ **27.** Let $T: P_2 \to P_3$ be given by $T(f(x)) = xf(x) + f'(x)$.

a) What is the null space $N(T)$ of T ?

b) What is the range $R(T)$ of T ?

c) Is T one-to-one ? Is T onto ?

■ **28.** Find a basis for the solution space $\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \\ 2x_1 + 3x_2 + 4x_3 + 5x_4 = 0 \\ x_1 + 3x_2 + 5x_3 + 7x_4 = 0 \end{cases}$.

■ **29.** Find the bases of the row space, column space and null space of the following matrix A, respectively.

$$A = \begin{bmatrix} 1 & 3 & 0 & -1 & 2 \\ 0 & -2 & 4 & -2 & 0 \\ 3 & 11 & -4 & -1 & 6 \\ 2 & 5 & 3 & -4 & 0 \end{bmatrix}.$$

■ **30.** Let L be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ having the following matrix representation with respect to the standard basis:

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$

a) Find $L^3(v)$, where $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

b) Find the dimension of the kernel space, and find a basis of the range space of L.

■ **31.** Define $T: P_2 \to P_2$ by $T(a + bx + cx^2) = (a + 2c) + (a + b)x + (a + c)x^2$. Find T^{-1}, if it exists.

■ **32.** Let $T: P_2 \to P_2$ be defined by $T(f(x)) = f''(x) - 2f'(x) - f(x)$.

a) Determine whether T is invertible.

b) Compute T^{-1} if it exists.

■ **33.** Let $J\colon P_n \to P_{n+1}$ be the integration transformation defined by

$$J(P) = \int (a_0 + a_1 x + \cdots + a_n x^n)dx = a_0 x + \frac{a_1}{2}x^2 + \cdots + \frac{a_n}{n+1}x^{n+1}$$

where $P = a_0 + a_1 x + \cdots + a_n x^n$. Find the matrix for J with respect to the standard bases for P_n and P_{n+1}.

■ **34.** Assume $T\colon P_2 \to \mathbb{R}^2$ is a linear transformation with the matrix representation $[T]_\beta^\gamma = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix}$ with respect to the basis $\beta = \left\{\frac{1}{2} - \frac{1}{2}x, \frac{1}{2} + \frac{1}{2}x, x^2\right\}$ for P_2, and the basis $\gamma = \left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$ for $\mathbb{R}^2$. Find the function T.

■ **35.** Let $T\colon P_2 \to P_2$ be defined by $T\big(p(x)\big) = 2p(x) + 2p(1)x - p''(1)x^2$. And let $B = \{1, x, x^2\}$.

a) Find $[T]_B$.

b) Find $N(T)$.

■ **36.** Let V be the linear space of all functions in two variables of the form $q(x_1, x_2) = a{x_1}^2 + b x_1 x_2 + c{x_2}^2$. Consider the linear transformation $T\colon V \to V$

$$T(f) = \frac{\partial f}{\partial x_1}x_2 - \frac{\partial f}{\partial x_2}x_1.$$

a) Find the matrix of T with respect to the basis ${x_1}^2,\ x_1 x_2,\ {x_2}^2$.

b) Find bases of kernel and image of T.

■ **37.** Let the linear transformation be $L\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be defined by $L(A) = \frac{1}{2}(A + A^T)$ for all $A \in \mathbb{R}^{2\times 2}$.

a) Find the matrix representation of L with respect to the standard (ordered) basis

$$\left\{E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}.$$

b) Determine the kernel and range of L. What are their dimensions ?

■ **38.** Let A be an $n \times n$ matrix, and $v_1, \ldots, v_n$ be linearly independent vectors in $\mathbb{R}^n$.

a) What must be true about A for $Av_1 \ldots, Av_n$ to be linearly independent ?

b) Justify your answer for (a) using a formal proof that is based on the definition of linear independence for $v_1, \ldots, v_n$.

■ **39.** Consider the system of linear equations:

$$x + (\lambda - 1)y + (\lambda - 2)z = 2$$

$$\lambda x + (2\lambda - 2)y + (\lambda - 2)z = \lambda + 1$$

$$(\lambda^2 - 2\lambda)x + \lambda z = -2\lambda$$

Find the real values of λ such that this system of equations satisfies the following condition.

a) The system has no solution.

b) The system has a unique solution.

c) The set of the solutions of the equations is a line.

d) The set of the solutions of the equations is a plane.

■ **40.** Let L be the operator on P_2 defined by $L\big(p(x)\big) = xp'(x) - p(x)$.

a) Find the matrix A representing L with respect to the standard basis $\{1, x, x^2\}$ of P_2.

b) Find the matrix B representing L with respect to the basis $\{1, 1+x, 1+2x+x^2\}$.

c) Find the matrix S such that $B = S^{-1}AS$.

d) If $p(x) = a_0 + a_1(1+x) + a_2(1+2x+x^2)$, calculate $L''(p(x))$ with respect to the basis $\{1, 1+x, 1+2x+x^2\}$.

■ **41.** Show that the following mapping F are linear:

a) $F: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (x + y, x)$.

b) $F: \mathbb{R}^3 \to \mathbb{R}$ defined by $F(x, y, z) = 2x - 3y + 4z$.

■ **42.** Define the transformation $T: P_2 \to P_2$ by $T(a_2x^2 + a_1x + a_0) = (a_2 - 1)x^2$. Prove or disprove that T is linear.

■ **43.** Let B be an $n \times n$ invertible matrix. Define $\varphi(A) = B^{-1}AB$. Prove that φ is an isomorphism.

■ **44.** Let $X = C[a, b]$ and $\phi: X \to X$ be defined by $\big(\phi(f)\big)(t) = \int_a^t f(x)dx$.

a) Show that ϕ is a linear transformation.

b) Find $\ker(\phi)$.

■ **45.** Let V be an n-dimensional vector space over $\mathbb{R}$ and let $T: V \to V$ be a linear transformation such that $N(T) = R(T)$.

a) Prove that n must be even.

b) Give an example of such a linear transformation T for $V = \mathbb{R}^2$.

■ **46.** Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Prove that if $T\big(T(x)\big) = T(x) + T(x) + 3x$ for all $x \in \mathbb{R}^n$, then T is a one-to-one mapping of $\mathbb{R}^n$ into $\mathbb{R}^n$.

■ **47.** Let V be the vector space of all polynomials of degree less or equal than 20.

a) Show that the map $T: V \to \mathbb{R}$ given by $T\big(p(t)\big) = \int_0^1 p(t)dt$ is a linear transformation.

b) Show that T is onto.

c) Find the dimension of the kernel of T.

■ **48.** If A and B are $n \times n$ matrices satisfying $AB = O$, prove that $rank(A) + rank(B) \leq n$.

■ **49.** Let $L: V \to W$ be a one-to-one and onto, hence invertible, linear transformation between vector space V and W. Let $\{z_1, \dots, z_r\}$ be a basis for W. Show that $\{L^{-1}(z_1), \dots, L^{-1}(z_r)\}$ is a basis for V.

■ **50.** Consider $\mathbb{C}$ as a vector space over $\mathbb{R}$. Let A be a linear map of $\mathbb{C}$ into itself given by $Az = az + b\bar{z}$, where $a, b \in \mathbb{C}$. Prove that this map is not invertible if and only if $|a| = |b|$. (4-222)