Section: Chapter.1 Vector Spaces

1.2 Vector Spaces

Def: Let $\mathbb{F}$ be a field. A vector space (向量空間) V over $\mathbb{F}$ is a ________ set V together with two operations.

$$+: V + V \to V \quad (\text{向量加法})$$

$$\times: \mathbb{F} \times V \to V \quad (\text{純量積、純量乘法})$$

such that the following conditions hold.

(V1)【加法交換性】

(V2)【加法結合性】

(V3)【加法單位元】

(V4)【加法反元素】

(SM1)【純量積與純量乘法結合性】

(SM2)【純量乘法單位元】

(SM3)【純量積對向量加法分配性】

(SM4)【純量積對純量加法分配性】

Remark: 向量空間中必定存在零向量，且零向量是______的.

Ex. 常見的向量空間.

1. (n-tuple Euclifean space：n維歐式空間) 記為 $\mathbb{F}^n$.
2. (Matrix space：矩陣空間) 記為 $M_{m\times n}(\mathbb{F})$、$\mathbb{F}^{m\times n}$.
3. (Function space 函數空間)
   (1) $P(\mathbb{F}) = \mathbb{F}[x] =$
   (2) $P_n(\mathbb{F}) = \mathbb{F}_n[x] =$
   (3) (space of continuous function：連續函數空間)

1.3 Subpsaces

Def: $\emptyset \neq W \subseteq V$ is a subspace (子空間) of V if W is itself a vector space over $\mathbb{F}$ under the same operations on V.

1) If W is a subspace of V, then we denoted by __________.
2) If W is a subspace of V and $W \neq V$, then we called W is a ____________________ of V.

Thm: Subspaces are vector spaces.

(因此有時候要驗證一個集合V是否為向量空間，只需驗證W為V的一個子空間即可)

Remark: 1) V is a subspace of V, and $\{0\}$ is a subspace of any vector space. We call these subspace is a ____________________(顯然子空間).

2) $\mathbb{R}^n$ is not a subspace of $\mathbb{R}^{n+1}$.

3) $\mathbb{R}^2$上的子空間包含______、__________________、____________ 三種類型.

4) $\mathbb{R}^3$上的子空間包含______、__________________，__________________、___四種類型.

Ex. 1) 為什麼$\{0\}$是任何 vector space 的 subspace？

2) 空集合是任何 vector space 的 subspace 嗎 ？

Thm: (Theorem 1.3：子空間判斷定理)

Let V be a vector and W is a subspace of V. Then W is a subspace of V if and only if

1)

2)

3)

Cor: If __________, then W is a subspace of V if and only if ____________________. (Exercise 18)

$Ex.$ 判斷下列集合哪些是$\mathbb{R}^{n\times n}$的子空間.

1) $W_1 = \{A|A^T = A\}$. 【100 中央資工、103 政大應數、105 成大資工、110 台大電信】
2) $W_2 = \{A|A^T = -A\}$. 【109 台聯電信、110 台大電信】
3) $W_3 = \{A|A$ 為對角方陣$\}$.
4) $W_4 = \{A|A$ 為上三角矩陣$\}$. 【96 暨南資工、105 交大資工、105 成大資工】
5) $W_5 = \{A|tr(A) = 0\}$.
6) $W_6 = \{A|\det(A) = 0\}$. 【96 暨南資工、105 成大資工】
7) $W_7 = \{A|A$ 為可逆$\}$. 【96 台科資工、98 清大統計】
8) $W_8 = \{X|AX = XA\}$, where A is a fixed matrix.

$Ex.$ Which of the following is a subspace of P_4 ? 【101 交大資工】

1) W: The set of polynomials in P_4 of even degree.
2) W: The set of polynomials in P_4 of degree 3.
3) W: The set of polynomials in P_4 such that $P(0) = 0$.
4) W: The set of polynomials in P_4 having at least one real root.

Thm: Let W_1 and W_2 be a subspace of V. Then

1) $W_1 \cap W_2$ is a subspace of V.
2) $W_1 \cup W_2$ is not necessarily a subspace of V.

Cor: If $W_1, \dots, W_n$ is a subspace of V, then $\cap W_i$ is a subspace of V.

Cor: The intersection of the subspace must not be an empty. (原因：__________________________)

Thm: $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

【100 彰師數學，101.103 中興應數、103 高雄應數、109 中正數學】

$Proof$:

Def: If $W_1, \dots, W_k$ is a subspace of V. Define $W_1 + \cdots + W_k =$ ______________________________

is the sum space (和空間) of $W_1, \dots, W_k$.

Thm: If W_1 and W_2 be a subspace of V, then $W_1 + W_2$ is a subspace of V.

$Proof$:

Thm: $W_1 \cup W_2 \subseteq W_1 + W_2$.

$Proof$:

$Remark$: $W_1 + W_2$是包含$W_1 \cup W_2$的最小子空間.

$Ex.$ 常見的基本子空間

1) (Row space：列空間) $RS(A) =$

2) (Column space：行空間) $CS(A) =$

3) (Kernel：核空間) $\ker(A) =$

Remark: 1) $CS(A)$ 也被記為______________、__________、__________.

2) $\ker(A)$ 也被記為_____.

Thm: $Ax = b$ has solution if and only if $b \in CS(A)$.

Proof:

Def: Let V be a vector space and W_i is a subspace of V for $i = 1,2,\dots k$. If

1)【滿足和空間】$V = W_1 + \cdots + W_k$

2)【彼此無交集】

Then $V = W_1 \oplus \cdots \oplus W_k$ is called the direct sum (直和).

Thm: Let $V = W_1 \oplus \cdots \oplus W_k$.

1) If β_i is a basis for W_i, where $i = 1,2,\dots,k$, then $\beta = \beta_1 \cup \cdots \cup \beta_k$ is a basis for V.

2) $\forall v \in V$, $\exists w_i \in W_i$ for $i = 1,2,\dots,k$ such that $v = w_1 + w_2 + \cdots + w_k$ is uniquely.

◆ Application of direct sum.

Ex. 任何一個方陣均可表示成一個對稱矩陣與一個斜對稱矩陣之和.

【94 中央數學、95 交大應數、96 成大資工、95.108 政大應數】

Proof:

$Ex.$ 任一函數必可表示成一個奇函數與一個偶函數的和.

【86.97 成大應數、92 交大資料、94 暨南資工、99 台大數學】

$Proof$:

1.4 Linear Combinations and Systems of Linear Equations

Def: Let S is a subset of V. If $V = span(S)$, then S is called the spanning set (生成集) of V.

(我們會說___是___的生成集)

Thm: (Theorem 1.5)

Let $\emptyset \neq S \subseteq V$, then $span(S) =$____.

Cor: $span(S)$ is a subspace of V.

【103 台大資工】

$Proof$:

$Remark$: $span(\{0\}) = \{0\}$. 定義$span(\emptyset) =$__________為零空間.

$Queestion$: 為什麼$span(\{0\}) = \{0\}$?

$Remark$: 1) $RS(A) = span\{A_{(1)}, A_{(2)}, \dots, A_{(m)}\}$.

2) $CS(A) = span\{A^{(1)}, A^{(2)}, \dots, A^{(n)}\}$.

Thm: (一些生成集的性質)

Let V be a vector space and let S, S_1, S_2 is a subset of V.

1) $S \subseteq span(S)$. 【101 中央資工、109 台北統計】

2) If $S_1 \subseteq S_2 \subseteq S$, then $span(S_1) \subseteq span(S_2)$.

3) $span(S_1) \cup span(S_2) \subseteq span(S_1 \cup S_2)$.

4) $span(S_1 \cap S_2) \subseteq span(S_1) \cap span(S_2)$.

5) For any subspace W, if $S \subseteq W$, then $span(S) \subseteq W$. 【103.108 台大資工，109 台北統計】

($span(S)$為包含S的最小子空間)

6) $span(S_1) + span(S_2) = span(S_1 \cup S_2)$. (Exercise 14)

$Proof$:

1.5 Linearly Dependent and Linearly Independent

Def: Let V be a vecctor space. Let $v_1, v_2, \dots, v_n$ be distinct vectors on V. We say that $\{v_1, v_2, \dots, v_n\}$ is linearly dependent (線性相依) set if $\exists$ scalars $c_1, c_2, \dots, c_n$, ________________, such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0.$$

(直觀來看：存在向量可表為其他向量的 linear combination，所以寫成矩陣時行列式為 0)

$Remark$: 1) Linearly dependent set are ___________.

2) $\{0\}$ is a ________________ set. (因為________________________________)

Def: $S \subseteq V$ is linearly independent (線性獨立) if S is not linearly dependent. i.e., $S \subseteq V$ is linearly independent if and only if $\forall v_1, \dots, v_n \in S$ and $c_1, \dots, c_n \in \mathbb{F}$, if $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$, then ___________________________.

(直觀來看：不存在向量可表為其他向量的 linear combination，所以寫成矩陣時行列式不為 0)

$Remark$: 1) $\emptyset$ is a ____________________ set.

2) Let $v \in V$ with $v \neq 0_v$, then $\{v\}$ is _____________________.

3) Let $v_1 \neq v_2$ be in V. Then $\{v_1, v_2\}$ is linearly independent $\Leftrightarrow$ ________________________.

(彼此沒有_____________) (Exercise 9)

$Remark$: $A_{m \times n}$行獨立$\Leftrightarrow Ax = O$只有零解. (當$m = n$時，代表方陣A行獨立$\Leftrightarrow A$可逆)

Thm: If $\{v_1, \dots, v_n\}$ is a linearly independent set of $\mathbb{R}^n$ and A is invertible, then $\{Av_1, \dots, Av_n\}$ is also a linearly independent set of $\mathbb{R}^n$. 【99 交大資工、107 中山應數】

$Proof$:

Cor: If $\{v_1, \dots, v_n\}$ and $\{Av_1, \dots, Av_n\}$ are linearly independent set of $\mathbb{R}^n$, then A is invertible.

【102.107 中山應數】

$Proof$:

$Ex.$ 已知$\{u, v, w\}$為線性獨立集，試判斷下列是否為線性獨立集，請證明之。

1) $\{u+v, u-v\}$.
2) $\{u, u+v, u+v+w\}$.
3) $\{u+v, v+w, u+w\}$. 【105 交大資工、106 中山應數】
4) $\{3u, 2u-v, u+w\}$. 【95 中山電機類題、98 清大統計】

Def: 令$C^{(n-1)}[a,b]$為所有在閉區間上$n-1$次可微函數所形成的向量空間. 假設$f_1, f_2, \dots, f_n \in C^{(n-1)}[a,b]$. 定義

$$W[f_1, f_2, \dots, f_n](x) = \det \begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1{}'(x) & f_2{}'(x) & \cdots & f_n{}'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1{}^{(n-1)}(x) & f_2{}^{(n-1)}(x) & \cdots & f_n{}^{(n-1)}(x) \end{bmatrix}$$

為$f_1, f_2, \dots, f_n$的 Wronskian. (課本沒有，但微方有教喔)

1.6 Bases and Dimension

Def: Let V be a vector space. Let $\beta \subseteq V$, then β is a basis (基底) of V if

1) ________________ (最小的__________).
2) ________________________________ (最大的_______________).

Def: Let V be a finite dimensional vector space. Let β be a basis for V. We define $|\beta|$ to be dimension (維度) of V, denoted by $\dim_{\mathbb{F}}(V)$ or $\dim(V)$.

Thm: β is a basis for V if and only if $\forall v \in V$ can be expressed uniquely as a linearr combination of β.
【101 成大應數】

$Proof$:

$Ex.$ 常見的標準基底與維度

1) $V = \{0_v\}$, standard basis: ________________, dimension: ___________.
2) $V = \mathbb{F}^n$, standard basis: ________________, dimension: ___________.
3) $V = P(\mathbb{F})$, standard basis: ________________, dimension: ___________.
4) $V = P_n(\mathbb{F})$, standard basis: ________________, dimension: ___________.
5) $V = M_{m\times n}(\mathbb{F})$, standard basis: ________________, dimension: ___________.

$Remark$: 1) 基底選法不維一.

2) 基底有無限多種，但每種選擇的基底元素各數均相同. (Theroem1.10, Corollary 1)

Thm: (Theorem 1.10：Steinitz Replacement Theorem：替代定理)

Let V be an $\mathbb{F}$-vector space. Suppose $S = \{w_1, \dots, w_n\}$ is a spanning set of V. If $L = \{v_1, \dots, v_m\}$ is a linearly independent subset of V, then $m \le n$ and their exists a subset H of S such that $|H| = n - m$ and $V = span(L \cup H)$.

Cor: Suppose S is a finite spanning set of V. If $\beta \subseteq V$ is linearly indpendent, then $|\beta| \leq |S|$.

(線性獨立集小於生成集)

Thm: (Basis Extension Theorem：基底擴張定理)

Let W be a subspace of a finite dimensional vector space V. Then all linearly subsets of W are finite and each can be extended to a basis for W. (可參考 Section 3.4)

$Ex.$ $S = \{(1,0,0), (0,1,0)\}$為$\mathbb{R}^3$一組獨立集，可用向量$u = (1,1,1)$將S擴張為基底.

Thm: (Dimension Theorem：維度定理)

If W_1 and W_2 are finite-dimensional subspace of a vector space V, then the subspace $W_1 + W_2$ is a finite dimensional and ____. (Exercise 29)

$Question$: 將 Dimension Theorem 推廣至三個向量空間會成立嗎？【95 交大應數、99 台大電機】