Section: Chapter.5 Diagonalization

5.1 Eigenvalues and Eigenvectors

Def: Let A, B be square matrices. If $\exists$ invertible matrix P such that ________________. We say that A is similar (相似) to B. Denoted by $A \sim B$.

Thm: Suppose $A \sim B$, where A, B be square matrices. Then

(1) $tr(A) = tr(B)$.

(2) $\det(A) = \det(B)$.

(3) $rank(A) = rank(B)$.

(4) $nullity(A) = nullity(B)$.

(5) $A^T \sim B^T$.【105 中興資料、109 中央資工】

(6) $A^k \sim B^k$ for any $k \in \mathbb{N}$.【104.109 中央資工 105 中興資料】

(7) $cA \sim cB$ for any scalar c.

(8) $A + cI \sim B + cI$ for some scalar c.

(9) $f(A) \sim f(B)$ for any polynomial f.【94 彰師資工、105 中興資料】

(10) $A^{-1} \sim B^{-1}$, where A, B both invertible.【104.109 中央資工】

$Proof$:

Def: $T \in \mathcal{L}(V)$ is diagonalizable (可對角化) if $\exists$ some basis β such that $[T]_\beta = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$ is diagonal matrix.

Def: Let $T \in \mathcal{L}(V)$ and $v \in V$. $v \neq 0_v$ is an eigenvector (特徵向量) of T if $\exists\, \lambda \in \mathbb{F}$ such that $T(v) = \lambda v$. λ is called the eigenvalue (特徵值) of T with corresponding to V.

$Remark$: 上述定義的矩陣版本為$Av = \lambda v$.

Def: Let $T \in \mathcal{L}(V)$ and $v \neq 0_v$. Then (λ, v) is called an eigenpair (特徵對) of T.

$Remark$: eigenvalue 可為 0, 但 eigenvector 不可為0_v.

$Remark$: 求解A的 eigenvalue 及 eigenvector 可以計算$N(A - \lambda I_n) = \{0_v\}$.

$Proof$: λ is an eigenvalue of T $\Leftrightarrow$ $Av = \lambda v$ $\Leftrightarrow$ $Av - \lambda v = 0_v$ $\Leftrightarrow$ $(A - \lambda I_n)v = 0_v$ $\Leftrightarrow$ Solve $N(A - \lambda I_n)$

Thm: Suppose A and B is similar. Then

(1) AB and BA have the same eigenvalue.

(2) A and A^T have the same eigenvalue.

(3) If A is invertible, then λ^{-1} is an eigenvalue of A^{-1}.

(4) λ^k is an eigenvalue of A^k for some $k \in \mathbb{N}$.

(5) Let f is an polynomial, then $f(\lambda)$ is an eigenvalue of $f(A)$.

$Proof$:

Thm: Let $A \in M_n(\mathbb{F})$. Then $\lambda \in \mathbb{F}$ is an eigenvalue of $A \iff N(A-\lambda I_n) \neq \{0_v\}$

Thm: Let V be a finite dimensional vector space. Then $T \in \mathcal{L}(V)$ is invertible $\iff$ 0 is not an eigenvalue of T.

$Proof$:

$Remark$: (1) 特徵值相同，也有可能不相似.【97 彰師數學】

e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$及$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(2) 相似矩陣的特徵向量不一定相同.【105 中山通訊、107 台大資工】

e.g. $\begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix}$及$\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$.

Def: Let $A \in M_n(\mathbb{F})$. Defined $\det(A - xI_n)$ is called the characteristic polynomial (特徵多項式) of A, denoted by $C_A(x) = \det(A - xI_n)$.

Thm: Let $A, B \in M_n(\mathbb{F})$.

(1) If $A \sim B$, then A and B have the same characteristic polynomial.

(2) AB and BA have the same characteristic polynomial.

【89 中央數學、90 彰師數學、92 台大數學】

$Proof$:

Thm: Suppose $\lambda_1, \dots, \lambda_n$ be eigenvalues of A. Let

$$C_A(x) = (\lambda_1 - x)(\lambda_2 - x)\cdots(\lambda_n - x) = (-x)^n + (\lambda_1 + \lambda_2 + \cdots + \lambda_n)(-x)^{n-1} + \cdots + (\lambda_1\lambda_2\cdots\lambda_n).$$

(1) $\det(A) = \lambda_1\lambda_2\cdots\lambda_n$.

(2) $tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

(3) A is invertible if and only if all eigenvalues of A are not zero.

Ex. Find the eigenvalues of A^{25} for $A = \begin{pmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$.【102 中正資工】

Ex. Let A be a nilpotent matrix. Prove that $I+A$ is invertible.

【93 中央統計、96 政大統計、103 台大數學、103 清大數學、105 台大資工】

$Proof$:

Ex. Prove that if all eigenvalues of A are positive, then all eigenvalues of $A+A^{-1}$ must ≥ 2.

【110 台大流行】

$Proof$:

Ex. Find the trace of $I+A+A^2+\cdots+A^{28}$ for $A=\begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix}$.【93 交大應數】

Ex. Find the eigenvectors of $(I+A)^{100}$ given $A=\begin{pmatrix} -4 & -5 \\ 10 & 11 \end{pmatrix}$.【105 台大資工】

Ex. Find the eigenvalues and eigenvectors of :

$$T:\mathbb{C}^2 \to \mathbb{C}^2,\ T(a_1, a_2) = (a_1\cos\theta - a_2\sin\theta, a_1\sin\theta + a_2\cos\theta).$$

【89 師大數學、90 交大資料、94 交大電資、96 中正電機、108 台聯電機】

5.2 Diagonalizability

Thm: Let $T \in \mathcal{L}(V)$, where $\dim(V) < \infty$. Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be distinct eigenvalues of T. If $v_1, \dots, v_k$ are eigenvectors of T such that λ_i corresponding to v_i. Then $\{v_1, \dots, v_k\}$ is linearly independent.

Def: Let $T \in \mathcal{L}(V)$ and λ an eigenvalue of T. We called $\mathcal{E}_T(\lambda)$ the eigenspace (固有空間) of T.

Thm: Let V be a n-dimensional vector space and let $T \in \mathcal{L}(V)$. Suppose β is a basis for V, then $[T]_\beta$ is diagonal matrix $\Leftrightarrow$ all vectors of β are eigrnvectors of T.

$Proof$:

Thm: Let $A \in M_n(\mathbb{F})$. Let $P \in M_n(\mathbb{F})$ is invertible matrix. Then $P^{-1}AP$ is diagonal matrix $\Leftrightarrow$ all column vectors of P are eigenvectors of A.

$Proof$:

Thm: Suppose A is diagonalizable.

(1) A^T is diagonalizable.

(2) A is invertible $\Longrightarrow A^{-1}$ is diagonalizable.

(3) A^k is diagonalizable for $k \in \mathbb{N}$.

(4) $f(A)$ is diagonaliable.

$Proof$:

$Remark$: 方陣是否可對角化與方陣是否可逆不相關.

e.g. $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$可逆但不可對角化；$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$不可逆但可對角化；$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$可逆也可對角化；

$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$可逆但不可對角化.

Def: A polynomial $f(t)$ in $P(\mathbb{F})$ splits over $\mathbb{F}$ (在$\mathbb{F}$上完全分解) if there are scalars $c, a_1, \dots, a_n$ in $\mathbb{F}$ such that $f(t) = c(t-a_1)(t-a_2)\cdots(t-a_n)$.

Lem: T is diagonalizable $\Longrightarrow$ $C_T(x)$ splits over $\mathbb{F}$.

$Remark$: 上述$Lemma$的反方向是錯誤的，反例為$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Def: Let $T \in \mathcal{L}(V)$. Suppose $C_T(x) = (-1)^n(x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2}\cdots(x-\lambda_k)^{m_k}$ for $i = 1,2,\dots,k$.

(1) alg-mul(λ_i) =_____ is called a algebraic multiplicity (代數重數) of λ_i.

(2) geo-mul(λ_i) =__________ is called a geometric multiplicity (幾何重數) of λ_i.

Thm: Let λ be an eigenvalue of T. Then _________________________. (幾何重數與代數重數之關係)

Cor: alg-mul$(\lambda_i) = 1$ $\Longrightarrow$ geo-mul$(\lambda_i) = 1$

Thm: Let $T \in \mathcal{L}(V)$, where $n = \dim(V)$ such that $C_T(x)$ splits over $\mathbb{F}$. Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be distinct eigenvalues of T. Then

(1) T is diagonalizable $\Longleftrightarrow$ ____________________ for each $i = 1,2,\dots,k$. (對角化必要條件)

(2) If T is diagonalizable and β_i is a basis for $\mathcal{E}_T(\lambda_i)$ for each i, then $\beta = \beta_1 \cup \cdots \cup \beta_k$ is a basis for V. (找基底的方法)

Thm: Let $T \in \mathcal{L}(V)$ and $\dim(V) = n$. Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be distinct eigenvalues of T, TFAE

(1) T is diagonalizable.

(2) $C_T(x)$ is splits over $\mathbb{F}$ and alg-mul(λ_i) =geo-mul(λ_i) for $i = 1,2,\dots,k$.

(3) $V = \mathcal{E}_T(\lambda_1) \cup \mathcal{E}_T(\lambda_2) \cup \cdots \cup \mathcal{E}_T(\lambda_k)$.

Def: If A is not diagonalizable, then A is called the defective (缺陷的).

Ex. Diagonalize $A = \begin{pmatrix} 1 & -3 & 2 \\ 1 & -2 & 1 \\ 2 & -3 & 1 \end{pmatrix}$ to be a diagonak marix D.

【98.99 中正統計、100 中正資工、97.98 成大資工】

Ex. Let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $T(a_0 + a_1 x + a_2 x^2) = 2(a_1 - a_2) + (2a_0 + 3a_2)x + 3a_2 x^2$.

(1) Let $B = \{1, x, x^2\}$ be a basis for $P_2(\mathbb{R})$. Give the matrix T with respect to B.

(2) Find the eigenvectors and the associated eigenvalues for T.

(3) Let C denote the basis of $P_2(\mathbb{R})$ that consists of the eigenvectors for T. Give the matrix of T with respect to C.

【94.98.103 師大資工】

◆ 對角化的應用

一、指數矩陣

Def: $e^{A}=\sum_{i=0}^{\infty}\frac{A^{i}}{i!}=I+\frac{A}{1!}+\frac{A^{2}}{2!}+\frac{A^{3}}{3!}+\cdots$

Thm: (1) $A\sim B \implies e^{A}\sim e^{B}$.【93 交大應數、101 台大數學】

(2) λ is an eigenvalue of $A \implies e^{\lambda}$ is an eigenvalue of e^{A}.【93 成大統計】

(3) $\det(e^{A})=e^{tr(A)}$.【93 成大統計】

(4) If A can be diagonalize to D, then $e^{A}=Pe^{D}P^{-1}$ is invertible matrix.【95 彰師統計】

$Proof$:

Ex. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Evaluate A^n, where $n \in \mathbb{N}$, e^A, $\sin A$, $A^{\frac{1}{2}}$.

二、遞迴關係式

Ex. The Fibonacci sequence can be recursively defined by $\begin{cases} x_n = x_{n-1} + x_{n-2},\ n \geq 3 \\ x_1 = x_2 = 1 \end{cases}$.

(1) Determine the matrix A that can recursively generate the Fibonacci sequence by

$\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = A \begin{pmatrix} x_{n-1} \\ x_{n-2} \end{pmatrix}$.【99 交大資工、100 中興電機、100 成大電信】

(2) Starting with $\begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Show that $\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = A^{n-2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(3) Find a matrix P that diagonalizes A.

(4) Derive an explicit formula for the n-th term of the Fibonacci sequences.

【85 成大資工、88 中山應數、89 師大數學、89.102 成大應數、103 中央統計】

5.4 Invariant Subspaces and the Cayley-Hamilton Theorem

Def: Let $T \in \mathcal{L}(V)$ and $W \leq V$. If $T(W) \subseteq W$, then W is called the T-invariant (T的不動子空間).

Thm: $\{0_V\}$, V, $N(T)$, $R(T)$, $\mathcal{E}_T(\lambda)$ are T-invariant.

Thm: Let $T \in \mathcal{L}(V)$ and $v \neq 0_v$. Then $span\{v\}$ is T-invariant $\Leftrightarrow$ v is an eigenvector of T.

Thm: If $W_1, \dots, W_k$ are T-invariant, then

1) $W_1 + \cdots + W_k$ is T-invariant.【99 交大應數】

2) $W_1 \cap \cdots \cap W_n$ is T-invariant.【99 交大應數】

Cor: Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \dots, \lambda_k$ are eigenvalues of T. Then $\mathcal{E}_T(\lambda_1) \oplus \cdots \oplus \mathcal{E}_T(\lambda_k)$ is T-invariant.

Def: Let $T \in \mathcal{L}(V)$ and $v \in V$. Define

$$\begin{aligned} \mathcal{Z}(v;T) &= span\{v, T(v), T^2(v), \dots, T^n(v), \dots\} \\ &= span\{a_0 v + a_1 T(v) + \cdots + a_n T^n(v) + \cdots | a_i \in \mathbb{F}, n \in \mathbb{N}^0\} \\ &= span\{f(T)(v) | f(x) \in P(\mathbb{F})\} \end{aligned}$$

$\mathcal{Z}(v;T)$ is called the T-cyclic subspace (T的循環子空間) generated by v.

$Remark$: $\mathcal{Z}(v;T)$ is the smallest T-invariant subspace containing v.

Lem: (1) $\mathcal{Z}(v;T) = \{0_v\} \Leftrightarrow v = 0_v$.

(2) $\dim \mathcal{Z}(v;T) = 1 \Leftrightarrow v$ is an eigenvector of T.

Thm: Let $T \in \mathcal{L}(V)$. Let $v \neq 0_v$ and $W = \mathcal{Z}(v;T)$. Let $k = \dim(W)$.

(1) $\beta = \{v, T(v), T^2(v), \dots, T^{k-1}(v)\}$ is a basis (此基底為循環基底) for W.【108 政大應數】

(2) If $a_0 v + a_1 T(v) + \cdots + a_{k-1} T^{k-1}(v) + T^k(v) = 0_v$, then

$$[T_W]_\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix}$$

① $[T_W]_\beta$ is called an compansion matrix (友矩陣).【107 台大資工】

② $C_{T_W}(x) = (-1)^k (a_0 + a_1 x + \cdots + a_{k-1} x^{k-1} + x^k)$.【107 台大資工】

※ 此定理最重要是在告訴你求解循環子空間的方法，請務必熟記.

Ex. Let $T: \mathbb{R}^4 \to \mathbb{R}^4$ be defined by $T(x, y, u, v) = (x + y, y - u, x + u, x + v)$. Let W be the T-cyclic subspace of $\mathbb{R}^4$ generated by $e_1 = (1,0,0,0)$. Find $tr(T_W)$ and $\det(T_W)$.

【100 中興應數、88 清大應數】

Thm: (Cayley-Hamilton Theorem)

Let $T \in \mathcal{L}(V)$ and $\dim(V) < \infty$. Then $C_T(T) = 0_{V \to V}$.

$Ex.$ $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Sol: $C_A(x) = (x-1)^2 \implies C_A(A) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 0_{2\times 2}$.

◆ Cayley-Hamilton Theorem 的應用

一、求解反矩陣

Thm: Let $A \in M_n(\mathbb{F})$. Suppose that $C_A(x) = (-1)^n x^n + a_{n-1}x^{n-1} + \cdots + a_1 A + a_0 I_n = 0_n$. If A is invertible, then

(1) $a_0 = \det(A) \neq 0$.

(2) $A^{-1} = -\frac{1}{a_0}((-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n)$.

【91 成大應數、93 政大應數、93 中正應數、98 中興統計】

$Proof$:

$Ex.$ $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $ad - bc \neq 0$. Find A^{-1}.

二、高次方矩陣對角化問題

Thm: Let $A \in M_n(\mathbb{F})$. Suppose $f(x) \in P(\mathbb{F})$ and $C_A(x) = g(x)$. If $f(x) = g(x)q(x) + r(x)$, where $der(r(x)) < \deg(g(x))$ or $r(x) = 0$. Then $f(A) = r(A)$.

$Proof$:

Ex. Let $A=\begin{pmatrix}14 & 9\\ -16 & -10\end{pmatrix}$. Compute A^{100}.【100 交大應數】

Ex. Evaluate the following matrix by Cayley-Hamilton Theorem.

(1) $A=\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}$, find A^n.【92 台大資工、98 清大統計、110 政大應數】

(2) $A=\begin{pmatrix}0 & 1\\ -8 & 6\end{pmatrix}$, find e^{At}.【107.109 台聯電機、109 成大資訊聯招】

(3) $A=\begin{pmatrix}0 & 4\\ -1 & 4\end{pmatrix}$, find A^n.