Calculus

■ **1**. Prove that for all $a, b \in \mathbb{R}$, $|\sin a - \sin b| \leq |a - b|$.

■ **2**. Find a such that the integral $\int_1^\infty \left(\frac{ax}{x^2+1} - \frac{1}{2x}\right) dx$ converges.

Linear Algebra

■ **1**. A matrix $N \in M_n(\mathbb{F})$ is called *nilpotent* if $N^k = O$ for some positive integer k where O is the zero matrix. Show that $\det(N) = 0$ if N is nilpotent.

■ **2**. Let $B_1 = \{(1,1), (1,-1)\}$ and $B_2 = \{(1,1,0), (0,1,1), (1,0,1)\}$ be bases for $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, and $A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \\ 1 & 1 \end{pmatrix}$ be the matrix of a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ with reapect to B_1 and B_2. Find the matrix of T with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

Calculus

■ **1**. Evaluate $\int \ln(x^2 + x)\, dx$.

■ **2**. Find the Taylor polynomial of $f(x) = \tan^{-1} x$ and determine the interval such that $f(x)$ converges.

Linear Algebra

■ **1**. Let V and W be vector spaces, and let $T: V \to W$ be linear and invertible. Show that $T^{-1}: W \to V$ is linear.

■ **2**. Let A and B be n by n matrices and $A + B = AB$. Prove that $AB = BA$.

Calculus

■ **1**. If f is continuous on $[0,\pi]$. Show that $\int_0^\pi xf(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$.

■ **2**. Find the height and radius of the largest right circular cylinder that can be put in a sphere of radius $\sqrt{3}$.

Linear Algebra

■ **1**. Suppose that $A \in M_n(\mathbb{R})$ and $A^T = -A$. Find all possible values for $\det(A)$.

■ **2**. Let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $T\big(f(x)\big) = f(x) + f'(x) + f''(x)$. Determine whether T is invertible. If T is invertible, find T^{-1}.

Calculus

■ **1**. Please answer the following questions.

(a) Please statement what is the Mean Value Theorem.【Please describe completely】

(b) Please explain the geometric meaning of the Mean Value Theorem.
【Drawings or text descriptions are acceptable】

(c) Please answer this question True or False：In Mean value Theorem, if the value of c is found, is there only one value ?【Do not show your reason】

(d) In (c). If there is only one value, please explain why. If there is more than one, please give an example and explain how to find these values.

■ **2**. Determine whether the integral $\int_0^\infty e^{-x^2}\,dx$ converges or not. Please show your reason.

Linear Algebra

■ **1**. Let A and B be n by n matrices. Show that $I-AB$ is invertible if and only if $I-BA$ is invertible.

■ **2**. Define the transformation $T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by $T\big(f(x)\big) = \begin{pmatrix} f(1)-f(2) & 0 \\ 0 & f(0) \end{pmatrix}$. Find a basis for the range space $R(T)$.

Calculus

■ **1**. Prove that if f is differentiable, then f is continuous.

■ **2**. Evaluate $\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+|t|}}-\frac{1}{t}\right)$.

Linear Algebra

■ **1**. Find all possible $a \in \mathbb{R}$ such that the vectors $(1,3,a),(a,4,3),(0,a,1) \in \mathbb{R}^3$ are linearly dependent.

■ **2**. Let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear operator defined by $T\big(p(x)\big) = p(2x+5)$, that is,

$T(a+bx+cx^2) = a + b(2x+5) + c(2x+5)^2$.

(a) Find the matrix T with respect to the basis $B = \{1, x, x^2\}$.

(b) Is T one-to-one ? If so, find the matrix for T^{-1} with respect to the basis B.

Calculus

■ **1**. Evaluate $\int \sec^3 x \, dx$.

■ **2**. Use the Riemann integral to evaluate $\int_{-1}^{0} x - x^2 \, dx$.

Linear Algebra

■ **1**. Let $W_1, W_2, \ldots$ be subspaces of a vector space V for which $W_1 \subseteq W_2 \subseteq \cdots$. Let

$$W = \bigcup_{i=1}^{\infty} W_i = W_1 \cup W_2 \cup \ldots$$

Prove that W is a subapace of V.

■ **2**. Let $T: \mathbb{R}_2[x] \to M_2(\mathbb{R})$ be the linear transformation satisfying

$$T(1+x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ T(x+x^2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ T(1+x^2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Find $T(a + bx + cx^2)$.

Calculus

■ **1**. Evaluate $\int \tan^{-1} x \, dx$.

■ **2**. Suppose $f(x) = \int_0^x \frac{\sin^{-1} t}{\sqrt{1-t^2}} dt - \int_0^x \frac{t}{\sqrt{1-t^2}} dt + \int_0^{\frac{\pi}{2}} \cos t \, dt$. Prove that f is constant function.

Linear Algebra

■ **1**. Suppose $f(x) = x$ and $g(x) = |x|$.

(a) Prove that f and g are linearly independent in $C[-1,1]$.

(b) Prove that f and g are linearly dependent in $C[0,1]$.

■ **2**. An operator T is *idempotent* if $T^2 = T$. Suppose that A and B are symmetric matrices. If AB is idempotent. Show that BA is idempotent.

Calculus

■ **1**. Let f be a conntinuous real function satisfying the identity $f(2x) = 3f(x)$ for all x. If $\int_0^1 f(x)\,dx = 1$, find $\int_1^2 f(x)\,dx$.

■ **2**. Prove that $\sum_{n=1}^{\infty}\left(\frac{a_n}{1+a_n}\right)$ converges if $a_n > 0$ for all n and $\sum_{n=1}^{\infty} a_n$ converges.

Linear Algebra

■ **1**. In $\mathbb{R}^2$, let L be the line $y = mx$ where $m \neq 0$. Find an expression for $T(x, y)$, where T is the reflection of $\mathbb{R}^2$ about L.

■ **2**. Let A be a nonsingular matrix and $A - I$ is nonsingular. If $A^r = I$ for integer $r \geq 2$, prove that $A + A^2 + A^3 + \cdots + A^{r-1} = -I$.

Calculus

■ **1**. Suppose that $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$.

(a) Is $\lim_{x\to 0^+} f(x)$ exist ?

(b) Is $\lim_{x\to 0^-} f(x)$ exist ?

(c) Is $\lim_{x\to 0} f(x)$ exist ?

■ **2**. Show that $\lim_{x\to a} f(x)$ is unique.【Hint：Use $\varepsilon - \delta$ definition】

Linear Algebra

■ **1**. Let A and B be $n \times n$ matrices. Assume that A is invertible and $B^3 = O$. If $AB = BA$, prove that $A + B$ is also invertible.

■ **2**. Prove that $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are colinear if and only if $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$.

Calculus

■ **1**. Evaluate $\int \left(\frac{1}{2}+x^2\right)e^{x^2}\,dx.$

■ **2**. A resistor consists of two parts of the resistors R_1 and R_2 connected in parallel, and the total resistor R satisfies

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$$

If R_1 and R_2 increase by $0.3\,\Omega/\mathrm{sec}$ and $0.2\,\Omega/\mathrm{sec}$, respectively. What is the rate of increase in R when $R_1 = 80\,\Omega/\mathrm{sec}$ and $R_2 = 120\,\Omega/\mathrm{sec}$?

Linear Algebra

■ **1**. Let V be a vector space over the real number $\mathbb{R}$. Let u, v, w be distinct vectors in V. Show that $\{u, v, w\}$ is linearly independent if and only if $\{u+v, u+w, v+w\}$ is linearly independent.

■ **2**. Let a、b、c be the lengths of the sides of △ABC. If the determinant $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$, then what must be the shape of △ABC ?

(1) Equilateral triangle

(2) Isosceles triangle

(3) Right triangle

(4) Acute triangle

(5) Obtuse triangle