Linear Algebra Exercise 1

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一、是非題

■ **1.** Let $V = \{(a_1, a_2) | a_1, a_2 \in \mathbb{R}\}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$

$$c \cdot (a_1, a_2) = \begin{cases} 0 & \text{if } c = 0 \\ (a_1, a_2) & \text{if } c \neq 0 \end{cases}$$

Then $(V, +, \cdot)$ is a vector space.

■ **2.** A set V is a vector space if V satisfies the following properties:

(i) V has a zero vector;

(ii) whenever u and v benong to V, then $u + v$ belongs to V; and

(iii) whenever v belongs to V and c is a scalar,

then cv belongs to v.

■ **3.** The set of vectors (x, y) in $\mathbb{R}^2$ with $y = -3x + 1$ is a vector space.

■ **4.** The number of vectors in each vector space is infinite.

■ **5.** Every vector space has at least two distinct subspaces.

■ **6.** The empty set is a vector space over any field.

■ **7.** The condition that a subset of a vector space contains the zero vector is a necessary and sufficient condition for the subset to be a subspace.

■ **8.** The subset of vectors in $\mathbb{R}^3$ with $b_1 b_2 b_3 = 0$ forms a subspace.

■ **9.** The subset of vectors in $\mathbb{R}^3$ with $b_1 + b_2 + b_3 = 0$ forms a subspace.

■ **10.** All vectors $v = (v_1, v_2, v_3)$ with $v_1 > v_2 > v_3$ form a subspace.

■ **11.** The set $\{(x, \cos x) | x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$.

■ **12.** The line passing through $(0,1)$ and $(1,0)$ is a subspace of $\mathbb{R}^2$.

■ **13.** The set of x satisfying $Ax = b$, where A is a real $m \times n$ matrix and b is a real $m \times 1$ matrix with $b \neq 0$ is a subspace of $\mathbb{R}^n$.

■ **14.** The only vector space that contains a finite number of vectors is the zero vector space $Z = \{0\}$.

■ **15.** If V is a vector space other than the zero vector space $\{0\}$, then V contains a subspace H such that $W \neq V$.

■ **16.** Let $\mathbb{R}$ denote the set of all real numbers. If S is a closed and bounded interval in $\mathbb{R}$ and contains 0, then S is not a subspace of $\mathbb{R}$.

■ **17.** If A is a subspace, then its completement be a subspace.

■ **18.** For any $m \times n$ matrix A and $n \times p$ matrix B, the null space of B is contained in the null space of AB.

■ **19.** Any three nonzero vectors span $\mathbb{R}^3$.

■ **20.** If w_1, w_2, w_3 are indepensent vectors, the differences $v_1 = w_2 - w_3$ and $v_2 = w_1 - w_3$ and $v_3 = w_1 - w_2$ are independent.

■ **21.** In a vector space V, if v_i and v_j are linearly independent for $i, j = 1,2,3$, $i \neq j$, then v_1, v_2, v_3 are linearly independent.

■ **22.** If none of the vectors in the set $S = \{v_1, v_2, v_3\}$ in $\mathbb{R}^3$ is a multiple of one of the other vectors, then S is linearly independent.

■ **23.** If both $\{v_1, v_2, v_3\}$ and $\{v_2, v_3, v_4\}$ are linearly independent sets, then $\{v_1, v_2, v_3, v_4\}$ is linearly independent, where vectors v_1, v_2, v_3 and v_4 are in $\mathbb{R}^4$.

■ **24.** If $S = \{v_1, \dots, v_n\}$ is linearly dependent in a vector space V, where $n \geq 2$, then every vector in S can be expressed as a linear combination of the others.

■ **25.** Let $W = \{v_1, v_2, \dots, v_k\} \subseteq \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$. If W is a linearly independent set, then $\{Av_1, Av_2, \dots, Av_k\}$ is a linearly independent set.

■ **26.** Let $W = \{v_1, v_2, \dots, v_k\} \subseteq \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$. If $\{Av_1, Av_2, \dots, Av_k\}$ is a linearly independent set, then W is a linearly independent set.

■ **27.** The vector space $\mathbb{R}^3$ has a basis containing the vector $(1,2,3)$.

■ **28.** Two vectors in $\mathbb{R}^3$ always span a two dimension subspaces.

■ **29.** Any two nonzero vectors in $\mathbb{R}^2$ that do not form a basis are collinear.

■ **30.** If $W_1 = \{A \in \mathbb{F}^{n \times n} | tr(A) = 0\}$, then $\dim(W_1) = n^2 - 1$.

■ **31.** Every subset of $\mathbb{R}^n$ with more than n elements is a spanning set for $\mathbb{R}^n$.

■ **32.** Every vector space is spanned by a linearly dependent set.

■ **33.** If V and W are subspaces of $\mathbb{R}^n$ having the same dimension, then $V = W$.

■ **34.** There are three linearly independent vectors in the $\mathbb{R}^2$.

■ **35.** A subset of a vector space V is a basis if and only if the subset is linearly independent and finite.

■ **36.** If x_1, $x_2, \dots$, x_n span $\mathbb{R}^n$, then $\{x_1, x_2, \dots, x_n\}$ is a basis for $\mathbb{R}^n$.

■ **37.** If $V = span\{v_1, \dots, v_n\}$, then $\dim(V) \leq n$.

■ **38.** If an n-element subset of a finite-dimensional vector space V is linearly independent, then the dimension of V is greater than n.

■ **39.** Let $\{v_1, v_2, \dots, v_n\}$ be a spanning set for the vector space V and let v be any other vector in V, then $v, v_1, v_2, \dots, v_n$ are linear independent.

■ **40.** A set of three vectors in $\mathbb{R}^2$ can be linearly independent.

■ **41.** If V is a nonzero finite-dimensional vector spaces, and there exists a linearly dependent set $\{v_1, \dots, v_p\}$ in V, then $\dim(V) \leq p$.

■ **42.** If V is a nonzero finite-dimensional vector spaces, and if every set of p elements in V fails to span V, then $\dim(V) > p$.

■ **43.** If the column of a matrix are dependent, so are the rows.

■ **44.** If a square matrix A has independent columns, so does A^2.

■ **45.** Let V be a vector space of finite dimension. Let S, T and U be vector subspaces of V. Then

$$\dim(S+T+U) = \dim(S) + \dim(T) + \dim(U) - \dim(S \cap T) - \dim(T \cap U) - \dim(U \cap S) + \dim(S \cap T \cap U).$$

■ **46.** Let V be an n-dimensional vector space and W_1, $W_2, \dots,$ W_k be subspaces of V. Then $V = W_1 \oplus W_2$ if and only if $V = W_1 + W_2 + \cdots + W_k$ and $W_i \cap W_j = \{0\}$ for $i \neq j$, $1 \leq i, j \leq k$.

二、選擇題

■ **1.** Consider the following sets of vectors:

(A) $\{(a, b, a+3) | a, b \in \mathbb{R}\}$.

(B) $\{(a, 4a, -3a | a \in \mathbb{R}\}$.

(C) $\{(a, b, 2) | a, b \in \mathbb{R}\}$.

(D) $\{(0, b, a+3b) | a, b \in \mathbb{R}\}$.

(E) $\{(a, b, c) \in \mathbb{R}^3 | a+b+c=0\}$.

(F) $\{(a, b, c) \in \mathbb{R}^3 | ab=0\}$

(G) $\{(a, b, c) \in \mathbb{R}^3 | a+b+c=1\}$.

Determine which of the sets are subspaces of $\mathbb{R}^3$.

■ **2.** Which of the following subsets of $\mathbb{R}^3$ are also subspaces of $\mathbb{R}^3$?

(A) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 + 2x_2 + 3x_3 = 0\}$.

(B) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 + 2x_2 + 3x_3 = 4\}$.

(C) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 x_2 x_3 = 0\}$.

(D) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 = 5x_3\}$.

(E) None of the above are subspaces of $\mathbb{R}^3$.

■ **3.** Which of the following statement is true ?

(A) The set of all invertible matrices is a vector space.

(B) The set of all diagonal matrices is a vector space.

(C) The set of all symmetric matrices is a vector space.

(D) The set of a line passing through $(1, -1)$ and $(-1, 1)$ is a vector space.

■ **4.** Let $\{u, v, w, z\}$ be linearly independent vectors. Which of the following are linearly independent ?

(A) $\{u - v, v - w, u - w\}$.

(B) $\{u + v, v + w, w + u\}$.

(C) $\{u - v, v - w, w - z, z - u\}$.

(D) $\{u + v, v + w, w + z, z + u\}$.

■ **5.** Which of the following sets of functions are linearly independent ?

(A) $\{1, \sin x, \cos x\}$.

(B) $\{1, \sin^2 x, \cos^2 x\}$.

(C) $\{e^x, e^{-x}\}$.

(D) $\{1, \ln(2x), \ln(x^2)\}$.

(E) $\{\sin x, \sin 2x, \sin 3x\}$.

■ **6.** Which of the following is correct ?

(A) $x^2 - 2x + 1$ and $|x - 1|$ are linearly dependent in the vector space $\mathbb{C}[0,2]$.

(B) $\cos x, 1, \sin x$ are linearly independent in $\mathbb{C}[-\pi, \pi]$.

(C) $A = \begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & \sin\left(\frac{\pi}{4}\right) \\ -\sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{pmatrix}$, $B = \begin{pmatrix} e & 1 \\ 1 & e^{-1} \end{pmatrix}$, $C = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$, $E = \begin{pmatrix} 0 & 8 \\ 4 & 0 \end{pmatrix}$ are linearly independent in $\mathbb{R}^{2\times 2}$.

(D) $v_1 = (1 \quad 8 \quad 9 \quad 5)^T$, $v_2 = (1 \quad 7 \quad 8 \quad 9)^T$, and $v_3 = (3 \quad 0 \quad 0 \quad 1)^T$ form a spanning set for $\mathbb{R}^4$.

(E) None of the above.

■ **7.** Consider the vector space $S = \{(a, a + b, a + b, -b) | a, b \in \mathbb{R}\}$. Determine which of the following sets of vectors are spanning sets of S.

(A) $\{(1,0,0,1), (1,2,2,-1)\}$.

(B) $\{(1,1,0,0), (0,0,1,-1)\}$.

(C) $\{(2,1,1,1), (3,1,1,2), (3,2,2,1)\}$.

(D) $\{(1,0,0,0), (0,1,1,0), (0,0,0,1)\}$.

■ **8.** Let V be a vector space with dimension n. Then in the following, pick up the correct statements.

(A) Any linearly independent subset for V containing exactly n vectors is a basis for V.

(B) Any finite generating set for V contains at most n vectors.

(C) Any two bases for V have the same number of vectors.

(D) If $\{v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$ is a basis for V, then $\{v_1, v_1 + 2v_2, v_1 + 2v_2 + 3v_3, v_1 + 2v_2 + 3v_3 + 4v_4, \ldots, v_1 + 2v_2 + 3v_3 + \cdots + nv_n\}$ is also a basis for V.

■ **9.** For any vector space V.

(A) If W is finite-dimensional, then W is a subspace of $\mathbb{R}^n$ for some positive integer n.

(B) If W is finite-dimensional, then no infinite subset of W is linearly independent.

(C) If W is a function space, then W must be infinite-dimensional.

(D) If W is a infinite-dimensional, then every infinite subset of W is linearly independent.

(E) None of the preceding statements are true.