Linear Algebra Exercise 6

114.05.19

一、計算與問答題

■ **1.** Find all eigenvalues and the corresponding eigenvectors of the matrix A below.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

■ **2.** Find the eigenvalues and the corresponding eigenspaces of $A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$.

■ **3.** Let $T: P_3 \to P_3$ be defined by $T(f) = \frac{f(x)+f(-x)}{2}$. Find all the eigenvalues and eigenvectors of T.

■ **4.** Let $\theta \in [0,2\pi)$ be any real number and let A be the following 2×2 matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

a) Find all the eigenvalues of A.

b) For each eigenvalue, find the corresponding eigenspace of A.

■ **5.** Suppose that the marix $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$ is diagonnalizable. Please find the matrix P such that $P^{-1}AP$ is diagonal.

■ **6.** Suppose that T is linear operator on the vector space $V = \mathbb{R}^2$ defined by

$$T(a,b) = (-2a + 3b, -10a + 9b).$$

Find the eigenvalues of T and an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

■ **7.** Suppose that A is $n \times n$ matrix that has two distinct eigenvalues λ_1 and λ_2. If $\dim\big(V(\lambda_1)\big) = n-1$, is A diagonalizable ?

■ **8.** Compute e^A for the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 3 \\ 1 & 1 & -1 \end{bmatrix}$.

■ **9.** Let $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$, find e^A.

■ **10.** Let A be a real $n \times n$ symmetric matrix, P be an $n \times n$ invertible matrix. Let v be the eigenvector of A corresponding to the eigenvalue λ. Find the eigenvector of $(P^{-1}AP)^T$ corresponding to the eigenvalue λ.

■ **11.** Find $A^{103} - 4A^{102} + 5A^{101} - 2A^{100}$ where $A = \begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix}$.

■ **12.** Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$. Find $A^{50} - 2A^{48} + A$.

二、證明題

■ **1.** Let T be a linear operator on a finite-dimensional vector space V, and let $W_1, W_2, \dots, W_k$ be T-invariant subspaces of V such that $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$. Prove that

$$\det(T) = \det(T_{W_1})\det(T_{W_2})\cdots\det(T_{W_k}).$$

■ **2.** Let A be a non-singular matrrix. Show that if $\lambda > 0$ is an eigenvalue of A^2, then either $\sqrt{\lambda}$ or $-\sqrt{\lambda}$ is an eigenvalue of A.

■ **3.** Let A, B be two $n \times n$ complex matrices such that $AB = BA$. Suppose A has n distinct eigenvalues. Show that B is diagonalizable.

■ **4.** Let $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$, show that $A^{3n} - 3A^{2n} + 3A^n - I = O$ for any positive integer n.

■ **5.** Let T be a linear operator on a two-dimensional vector space V. Prove that either V is a T-cyclic subspace of itself or $T = cI$ for some scalar c.