Linear Algebra Exercise 2

114.03.10

■ **1.** Determine which of the following sets are subspaces of $\mathbb{R}^n$?

(A) $\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n | x_1 = 0 \text{ or } x_n = 0\}$.

(B) $\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n | x_1 + 2x_n = 0\}$.

(C) $\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n | \sum_{i=1}^{n} x_i = 1\}$.

■ **2.** Determine which sets of vectors are linearly independent.

(A) The vectors $x+1, x^2-x+1, x^2+3x-2, 2x^2+8x+1$ in P_2.

(B) The vectors $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 7 & 3 \end{pmatrix}$ in $\mathbb{R}^{2\times 2}$.

(C) The vectors $e^x + e^{-x}, e^x - e^{-x}, e^{2x}$ in $\mathbb{C}[0,1]$.

■ **3.** Please explain whether the following are subspaces of $\mathbb{R}^{2\times 2}$ or not.

a) all 2×2 matrices with integer entries.

b) all 2×2 matrices A such that $\det(A) = 0$.

c) all matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$.

■ **4.** Let $W = \mathbb{R}^3$, $S_1 = \{(x,y,z) | x - y + z = 0\}$, $S_2 = \{(x,y,z) | 2x + y - z = 0\}$.

a) Is $S_1 \cap S_2$ a subspace of W ? Describe the reason.

b) Is $S_1 \cup S_2$ a subspace of W ? Describe the reason.

■ **5.** Are the following sets of vectors linearly independent ?

a) $u = (4,6,6,-4)$, $v = (4,5,6,-1)$, $w = (1,2,3,-1)$.

b) $u = (1,2,100)$, $v = (9,9,10)$, $w = (7,8,0)$, $x = (7,8,9)$.

c) $u = (1,2,3)$, $v = (4,5,6)$, $w = (7,8,9)$.

■ **6.** Determine the values t such that $\{v_1, v_2, v_3\}$ is linearly independent, where $v_1 = (t,1,1,1)^T$, $v_2 = (1,t,1,t)^T$, $v_3 = (1,1,t,1)^T$.

■ **7.** Find the largest possible number of independent vectors from the following vector set:

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, v_5 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, v_6 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

■ **8.** Let $f_1 = e^{-x}$, $f_2 = e^{-2x}$, $f_3 = e^{-3x}$ and f_1, f_2, f_3 are function in $\mathbb{C}^n[-1,1]$.

a) Compute the Wronskian of f_1, f_2, f_3.

b) Are these vectors f_1, f_2, f_3 linearky independent in $\mathbb{C}^n[-1,1]$?

■ **9.** Are the following functions linearly independent or dependent on the positive x-axis ?

a) $\cos x, \sin x, \sin 2x$.

b) $\ln(x), -\ln(x^2), \ln(x^3)$.

■ **10.** Find a matrix $A \in \mathbb{R}^{3\times3}$ such that for $x, y, z \in \mathbb{R}$, $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x - 2y + 3z = 0\right\}$.

■ **11.** Find a basis of $U = span\{(1,-1,3,2), (0,-1,2,1), (2,1,0,1), (3,5,-3,0)\}$.

■ **12.** Find the dimension of the vector space spanned by $B = \left\{\sin t, \cos t, 2\cos(t + \frac{\pi}{4})\right\}$.

■ **13.** Let $X = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_1 + x_2 + x_3 + x_4 = 0\}$ and $Y = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x, y \in \mathbb{R}\}$ be subspaces of $\mathbb{R}^4$. What is the dimension of $X \cap Y$?

■ **14.** Let V be the vector space of all 2×2 matrices over the field $\mathbb{F}$. Let W_1 be the set of matrices of the form $\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$ and let W_2 be the set of matrices of the form $\begin{bmatrix} a & b \\ -a & c \end{bmatrix}$. Find the dimensions of W_1, W_2, $W_1 + W_2$ and $W_1 \cap W_2$.

■ **15.** Let $V = \mathbb{F}^{2\times2}$, $W_1 = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c \in \mathbb{F}\right\}$ and $W_2 = \left\{\begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \middle| a, b \in \mathbb{F}\right\}$. Find the dimensions of W_1, W_2, $W_1 + W_2$ and $W_1 \cap W_2$.

■ **16.** Let W_1 be the set of 3×3 matrices of the form $\begin{bmatrix} x & 0 & z \\ 0 & -x & 0 \\ y & 0 & x \end{bmatrix}$, $x, y, z \in \mathbb{R}$, W_2 be the set of 3×3 matrices of the form $\begin{bmatrix} -a & 0 & b \\ 0 & a & 0 \\ c & 0 & a \end{bmatrix}$, $a, b, c \in \mathbb{R}$. Find the dimensions of W_1, W_2, $W_1 + W_2$ and $W_1 \cap W_2$.

■ **17.** Use the Lagrange interpolation formula to find a polynomial f with real coefficients such that f has degree ≤ 3 and $f(-1) = -6$, $f(0) = 2$, $f(1) = -2$, $f(2) = 6$.

■ **18.** Suppose the product of A and B is the zero matrix: $AB = O$, then the __(1)__ space of A contains the __(2)__ space of B. Also the __(3)__ space of B contains the __(4)__ space of A. Fill in those blank words (1) (2) (3) (4).

■ **19.** Suppose $\{x_1, x_2, x_3\}$ is a linearly independent set of vectors in $\mathbb{R}^3$. Find the values d such that $\{x_1 + 2x_2, x_1 + dx_2 + 3x_3, x_2 + dx_3\}$ is also linearly independent in $\mathbb{R}^3$.

■ **20.** If vectors $x_1, x_2, \dots, x_n$ are linearly independent for $n > 2$ and n can be an odd or even number. Determine $(x_1 + x_2), (x_2 + x_3), (x_3 + x_4), \dots, (x_n + x_1)$ to be linearly dependent or independent.

■ **21.** Let $A \in \mathbb{R}^{3\times3}$ and $\{v_1, v_2, v_3\}$ be a linearly independent subset of $\mathbb{R}^3$. If $Av_1 = 3v_1 + 2v_2 + v_3$, $Av_2 = v_1 + 2v_2 - v_3$ and $Av_3 = v_1 + v_2 + v_3$, find $\det(A)$.

■ **22.** Let $W_1 = span\{(1,2,3), (2,1,1)\}$ and $W_2 = span\{(1,0,1), (3,0,-1)\}$. Find a basis for $W_1 \cap W_2$.

■ **23.** Assume U is the space spanned by $\{(1,-2,0,3), (0,1,0,-1)\}$ and V is the space spanned by $\{(1,0,0,0), (0,1,0,0), (0,0,1,0)\}$. Find a basis for $U \cap V$.

■ **24.** Let the subspace V be generated by $v_1 = (2,0,0,8)$, $v_2 = (0,3,0,1)$, $v_3 = (2,0,1,2)$ and let the subspace W be generated by $w_1 = (1,2,3,4)$, $w_2 = (1,5,0,-3)$, $w_3 = (1,0,5,4)$. Find a basis for $V \cap W$.

■ **25.** Let V and W be vector spaces over a field $\mathbb{F}$ of dimensions m and n, respectively. Let Z be a vector space given by $Z = \{(v, w) | v \in V \text{ and } w \in W\}$. What is the dimension of Z ?

■ **26.** Let V be a vector space and $u, v \in V$. Prove that if $\{u, v\}$ is linearly independent, then $\{2u + v, u - 2v\}$ is linearly independent.

■ **27.** Show that the function $f_1 = 1$, $f_2 = e^x$, $f_3 = e^{2x}$ form a linearly independent set of vectors in $\mathbb{C}^2(-\infty, \infty)$.

■ **28.** Let $f(x) = x$ and $g(x) = |x|$.

a) Show that f and g are linearly independent in $\mathbb{C}[-1,1]$.

b) Show that f and g are linearly dependent in $\mathbb{C}[0,1]$.

■ **29.** Let $L = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2} \middle| a + d = 0 \right\}$.

a) Show that L is a subspace of $\mathbb{R}^{2\times 2}$.

b) Find the dimension of L.

■ **30.** Let M be the vector space of all 2×2 matrices. Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $U = \{X \in M | AX = XA\}$.

a) Show that U is a subspace of M.

b) Find the dimension of U.

■ **31.** Let $v_1 = (1,0,0)$, $v_2 = (-1,1,1)$ and $v_3 = (0,4,2)$ be three vectors.

a) Prove that $\beta = \{v_1, v_2, v_3\}$ is a basis for three-dimensional Euclidean space $\mathbb{R}^3$.

b) Let $x = (x_1, x_2, x_3)$ be a vector and $x = c_1 v_1 + c_2 v_2 + c_3 v_3$. Derive the values of c_1, c_2 and c_3 in terms of x_1, x_2 and x_3.

■ **32.** Prove that $\{1, x - 1, x^2 - 2x + 1\}$ is a basis for P_2.

■ **33.** Let $S = \{v_1, v_2, \dots, v_k\}$ be the set of vectors in a vector space V. If $v_j \in S$ and v_j can be expressed as a linear combination of other vectors in S, prove that $S - \{v_j\}$ spans the same space as S.

■ **34.** Let V be a vector space over a field $\mathbb{F}$ and $n \geq 2$ be a positive integer. Show that n distinct vectors $u_1, u_2, \dots, u_n$ in V are linearly dependent if and only if some vector of them is a linear combination of the other $n - 1$ vectors.

■ **35.** Prove that any nonempty subset of a linearly independent set of vectors $x_1, \dots, x_n$ is also linearly independent.

■ **36.** Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be distinct numbers $\neq 0$. Show that $e^{\alpha_1 t}, e^{\alpha_2 t}, \dots, e^{\alpha_n t}$ are linearly independent over the complex numbers.

■ **37.** Let v and w be two linearly independent column vectors in $\mathbb{R}^3$, and let $A \in \mathbb{R}^{3\times 3}$ be an invertible 3×3 matrix. Prove that the vectors Av and Aw are linearly independent.

■ **38.** Let A be an $n \times n$ matrix and assume $v, w \in \mathbb{R}^n$ are nonzero vectors with $Av = v$ and $Aw = 2w$. Prove that $\{v, w\}$ is linearly independent.

■ **39.** Prove that if the vector $\{u, v, w, z\}$ are linearly independent, then $\{u+v+w, u+v+z, u+w+z, v+w+z\}$ are also linearly independent.

■ **40.** If $A \neq O$ is a symmetric matrix and $B \neq O$ is a skew-symmetric matrix in the space of $n \times n$ matrices, show that A and B are linearly independent.

■ **41.** Let W be the set of all vectors $b = (b_1, b_2, b_3)$ in $\mathbb{R}^3$ such that the system of linear equations

$$\begin{cases} x_1 - 2x_2 - x_3 = b_1 \\ 2x_1 - 3x_2 + x_3 = b_2 \\ 5x_1 - 8x_2 + x_3 = b_3 \end{cases}$$

has a solution.

a) Show that W is a subspace of $\mathbb{R}^3$.

b) Find the dimension of W.

■ **42.** Let $V = \left\{\theta \in \mathbb{R} \middle| -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right\}$. Define the following two operations $\oplus$ and $\cdot$:

vector addition: $\alpha \oplus \beta = \tan^{-1}(\tan\alpha + \tan\beta), \forall \alpha, \beta \in V$

scalar multiplication: $r \cdot \alpha = \tan^{-1}(r\tan\alpha), \forall \alpha \in V, r \in \mathbb{R}$.

Show that V is a vector space.

■ **43.** Suppose that $S = \{u_1, u_2, \dots, u_n\}$ is a set of vectors from $\mathbb{R}^m$. Prove that S is linearly independent if and only if the set $S' = \{u_1, \sum_{i=1}^{2} u_i, \sum_{i=1}^{3} u_i, \dots, \sum_{i=1}^{n} u_i\}$ is linearly independent.

■ **44.** Let S be a set. For each $s \in S$, the characteristic function of s is the function $\chi_s: S \longrightarrow \mathbb{R}$ defined by $\chi_s(t) = \begin{cases} 1 & \text{if } t = s \\ 0 & \text{if } t \neq s \end{cases}$.

Prove that if $s_1, s_2, \dots, s_k \in S$ are distinct, then their characteristic function $\chi_{s_1}, \chi_{s_2}, \dots, \chi_{s_k}$ are linearly independent over $\mathbb{R}$.