Section: Chapter.2 Linear transformations and Matrices

2.1 Linear Transformation, Null spaces and Ranges

Def: Let V and W be vector spaces over $\mathbb{F}$. A function $T: V \to W$ as a linear transformation (線性轉換)

if

1)

2)

for all $\alpha \in \mathbb{F}$ and $u, v \in V$

$Remark$: We define $\mathcal{L}(V, W) = \{\text{all linear transformation from } V \text{ into } W\}$.

Thm: $\mathcal{L}(V, W)$ is a subspace of V.

$Remark$: 1) $T \in \mathcal{L}(V, W) \Leftrightarrow$ ______________________________ (驗證線性的條件)

2) If $T \in \mathcal{L}(V, W)$, then $T\left(\sum_{i=1}^{k} \alpha_i v_i\right) = \sum_{i=1}^{k} \alpha_i T(v_i)$.

Def: 1) $T \in \mathcal{L}(V, W)$ is called the zero transformation (零轉換) if ________________, denoted by $0_{V \to W}$.

2) $T \in \mathcal{L}(V)$ is called the identity transformation (單位轉換) if ________________, denoted by id_V.

$Ex.$ The differentiable operator is a linearly mapping.【101 政大應數、105 中央統計】

$Proof$:

$Ex.$ Determine whether the following transformations are linear.

a) $T[x_1, x_2, x_3]^T = [x_1 + 1, x_2 - 1, x_3]^T$.【107 中央資工】

b) $T: \mathbb{R}^2 \to \mathbb{R}^2,\ T(x, y) = (x + y, x)$.【107 中央資工類題、107 台大電機】

c) $L: \mathbb{R}^3 \to \mathbb{R}^2,\ L(x) = (x_1 + 2x_2, 3x_2 + x_3)^T$.【108 台北資工】

d) $L: \mathbb{R}^2 \to \mathbb{R}^3,\ L(x) = [(x_1 + x_2)^2, x_1 + x_2, x_2]^T$.【97.99 成大資工、101 台大電信】

$Ex.$ Determine whether the following transformations are linear.

a) $T: M_{22} \to M_{22},\ T(A) = A^T$.【90 成大電信、95 中興資料、99 中正數學】

b) $T: M_{22} \to \mathbb{R},\ T(A) = \det(A)$.【107 高雄應數、109 成大電信】

c) L is defined on $\mathbb{R}^{n\times n}$, $L(A) = e^A$.【97.99 成大資工、107 中央資工】

$Ex.$ Determine whether the following transformations are linear.

a) $T(ax^2+bx+c)=(a+b)x+(b+c)$.【107 中央資工】

b) $T: P_2(\mathbb{R}) \to P_2(\mathbb{R}),\ T\big(f(x)\big)=xf'(x)$.【101 中正數學】

c) $T: P_2 \to P_3$ by $T\big(p(t)\big)=(t+3)p(t)$.【110 成大電機】

d) $T: P_3 \to P_3,\ L\big(p(t)\big)=p''(t)+p(0)$.【101 台北統計】

e) $T\big(f(x)\big)=f(x)\cos x : V \to V$, where V be the set of continuous real valued functions on $[0,2\pi]$.

【108 交大應數】

Thm: Let $T \in \mathcal{L}(V,V')$.

a) If $W \le V$, then $T(W) \le V'$.

b) If $W' \le V'$, then $T^{-1}(W') \le V$.

(線性映射保子空間)

◎ 一些特殊的線性映射

Thm: Let $T \in \mathcal{L}(\mathbb{R}^2,\mathbb{R}^2)$. Then $T\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ is a rotation mapping by θ.

(表示平面上的點(x,y)，經逆時針旋轉θ角後到達的位置)

$Proof$:

Cor: 繞$\mathbb{R}^3$上x, y, z軸的旋轉矩陣.

a) $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$表示繞著$z$軸，由$x$方向往$y$方向旋轉$\theta$角的旋轉矩陣.

b) $\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$表示繞著$y$軸，由$x$方向往$z$方向旋轉$\theta$角的旋轉矩陣.

c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$表示繞著$z$軸，由$x$方向往$y$方向旋轉$\theta$角的旋轉矩陣.

Thm: Let $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$. Then $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ is a reflection mapping about L with angel θ. (表示平面上的點(x, y)，對直線L做鏡射後到達的位置，其中直線L與x軸夾角為θ)

$Proof$:

Cor: 對$\mathbb{R}^3$上平面的鏡射矩陣.

a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$為$\mathbb{R}^3$上對$xy$平面做鏡射的鏡射矩陣.

b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$為$\mathbb{R}^3$上對$xz$平面做鏡射的鏡射矩陣.

c) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$為$\mathbb{R}^3$上對$yz$平面做鏡射的鏡射矩陣.

Thm: Let $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$. Then

1) $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ is a projection matrix about x-axis.

(表示平面上的點(x, y)，投影在x軸上的位置)

2) $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ is a projection matrix about y-axis.

(表示平面上的點(x, y)，投影在y軸上的位置)

$Proof$:

Cor: $\mathbb{R}^3$上的投影矩陣

a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$為投影在$xy$平面上的投影矩陣.

b) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$為投影在$yz$平面上的投影矩陣.

c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$為投影在$xz$平面上的投影矩陣.

d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$為投影在$x$軸上的投影矩陣.

e) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$為投影在$y$軸上的投影矩陣.

f) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$為投影在$z$軸上的投影矩陣.

Def: If a linear operator T that satisfies $T^2 = T$. We called that T is an idempotent operator (冪等算子).

$Ex.$ Show that $T: \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ defined by $T(A) = \frac{A+A^T}{2}$ is an idempotent.

$Proof$:

Def: Let $T \in \mathcal{L}(V,W)$. We define

1) $N(T) = \ker(T) = \{v \in V | T(v) = 0_W\}$ is called the null space (核空間) of T.

2) $R(T) = \{T(v) | v \in V\}$ is called the range (值域) of T.

$Remark$: 1) $N(id_V) =$________ and $R(id_V) =$________.

2) $N(0_{V\to W})$=________ and $R(0_{V\to W}) =$________.

Def: Define $nullity(T) = \dim(N(T))$ and $rank(T) = \dim(R(T))$.

Thm: 1) $N(T)$ is a subspace of V.【101 成大應數、105 台大資工】

2) $R(T)$ is a subspace of W.

$Proof$:

Thm: (Theorem 2.3: Dimension Theorem: 維度定理)

Let $T \in \mathcal{L}(V,W)$. If $\dim(V) < \infty$, then ________________________________.

Ex. For the linear transformation T defined by $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 - 2x_2 + x_3 + x_4 \\ 2x_1 - 5x_2 + x_3 + 3x_4 \\ x_1 - 3x_2 + 2x_4 \end{pmatrix}$.

a) Find a basis for the range of T.

b) Find a basis for the null space of T.【109 師大資工】

Ex. Let $T: P_2 \to P_2$ be given by $T\big(p(x)\big) = p(x-1)$. Consider the two ordered bases $\beta = \{x^2, x, 1\}$ and $\gamma = \{x, x+1, x^2 - 1\}$. If the basis is γ, find the dimension of $\ker(T)$ and the basis of $\ker(T)$.

【110 政大資料】

Ex. Let the mapping $L: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $L(x) = (x_1 + 2x_2, 3x_2 + x_3)^T$. Determine the kernel of L.【108 台北資工】

$Ex.$ Define $T(A)=\frac{A+A^T}{2}$, where A is a $n\times n$ matrix. Then

a) $\ker(T)=$____________________.

b) $\left(nullity(T), rank(T)\right)=$____________________.

【101 清大數學、103 政大應數、103 中興應數、106 高雄應數、102 台大資工】

$Remark$: 1) 任何 linear transformation 均保 linearly dependent.

2) T為一對一函數 if and only if T保 linearly independent.

3) T為映成函數 if and only if T保 spanning set.

2.2 The matrix representation of a linear transformation

Def: Let $T \in \mathcal{L}(V,W)$. Let $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_n\}$ be ordered basis for V and W, respectively. Then $[T]_\beta^\gamma = \left([T(v_1)]_\gamma \middle| [T(v_2)]_\gamma \middle| \cdots \middle| [T(v_n)]_\gamma\right)$____ is called the matrix representation (表現矩陣) of T in the ordered basis β and γ.

$Remark$: If $V = W$ and $\beta = \gamma$, then $[T]_\beta^\gamma = [T]_\beta$.

$Ex.$ Let $T: P_3 \to P_3$ be the linear transformation is defined by $T(p(x)) = 2p(x) - p'(x)$. Let $B = \{3 + 2x + x^2, -x + 2x^2, 1 + x^2\}$. Find the matrix T with respect to the basis B.【109 台科資工】

$Ex.$ Let $V = \{p(t) | p(t) = a_0 + a_1 t$ defined on $[-1,1]$ for some $a_0, a_1 \in \mathbb{R}\}$ and $W = \{q(t) | q(t) = b_0 + b_1 t + b_2 t^2$ defined on $[-1,1]$ for some $b_0, b_1, b_2 \in \mathbb{R}\}$. Note V and W are vector spaces and having natural bases $S = \{p_1(t) = 1, p_2(t) = t\}$ and $T = \{q_1(t) = 1, q_2(t) = t, q_3(t) = t^2\}$, respectively. Let the linear mapping $L: V \to W$ be defined by $L(p(t))(t) = \int_{-1}^{1} p(x)\, dx$.

a) Please find $[L(p_1(t))]_T$ and $[L(p_2(t))]_T$, the coordinaate vectors of $L(p_1(t))$ and $L(p_2(t))$ with respect ot the ordered basis T. Show your work.

b) Please find the matrix representing L with respect to the bases S and T.【107 政大統計】

2.3 Composition of the Linear Transformation and Matrix Multiplication

Thm: (Theorem 2.9)

Let $T \in \mathcal{L}(V,W)$ and $W \in \mathcal{L}(W,Z)$. Then $(U \circ T)(x) = U(T(x)) \in \mathcal{L}(V,Z)$.

Remark: TU is not defined.

2.4 Invertibility and Isomorphism

Def: Let $T \in \mathcal{L}(V,W)$. If $T \circ U = id_W$ and $U \circ T = id_V$, then we called $U \in \mathcal{L}(W,V)$ is inverse (可逆元/反元素). If T has an inverse, then we called T is invertible (可逆的). Finally, if T is invertible, then the inverse of T is ________ and we denoted by T^{-1}.

Thm: T is invertible if and only if T is bijective.

Thm: Let V and W are finite-dimensional vector space and let $T \in \mathcal{L}(V,W)$.

a) T is injective if and only if $N(T) = \{0_V\}$.【97 輔大資工、102.108 台北統計】

b) T is surjective if and only if $R(T) = W$.

c) If $\dim(V) = \dim(W)$, then T is injective if and only if T is onto.

【97 政大統計、101 中興應數】

Proof:

Thm: 1) If T is injective, then $\dim(V) \leq \dim(W)$.

2) If T is surjective, then $\dim(W) \leq \dim(V)$.

$Proof$:

Def: Let V and W be vector spaces over $\mathbb{F}$. Let $T \in \mathcal{L}(V, W)$. T is an isomorphism (同構函數) if T is bijection. If $T \in \mathcal{L}(V, W)$ is an isomorphism, we say that V and W are isomorphic (同構：此指向量空間), denoted by $V \cong W$.

Thm: Let V and W are finite-dimensional vector space over $\mathbb{F}$.

1) $V \cong W$ if and only if $\dim(V) = \dim(W)$. (同維即同構) 【100 交大資工、103 師大數學】

2) $\mathcal{L}(V, W)$ is a vector space over $\mathbb{F}$.

$Proof$:

Def: Let $T \in \mathcal{L}(V)$ and $W \leq V$. If $T(W) \subseteq W$, then W is called the T-invariant (T的不動子空間).

Thm: $\{0\}$, V, $N(T)$, $R(T)$ are T-invariant.

Thm: If $W_1, \dots, W_k$ are T-invariant, then

1) $W_1 + \cdots + W_k$ is T-invariant.【99 交大應數】

2) $W_1 \cap \cdots \cap W_n$ is T-invariant.【99 交大應數】

Thm: (Theorem 2.18)

Let $T \in \mathcal{L}(V, W)$ and let β and γ be the orderd bases for V and W. Then T is isomorphism if and only if $[T]_\beta^\gamma$ is invertible and $[T^{-1}]_\beta^\gamma = \left([T]_\beta^\gamma\right)^{-1}$.

※ 求 inverse 時經常會用到這條定理，請務必熟記.

Thm: If T and U are invertible, then TU is invertible and $(TU)^{-1} =$__________.

2.5 The Change of Coordinate matrix.

Thm: Let β and β' be ordered bases for V. Let $Q = [I_V]_{\beta}^{\beta'}$.

1) Q is invertible.

2) $[v]_{\beta'} = [I_v]_{\beta}^{\beta'}[v]_{\beta}$. (一般化的座標變換公式)

Thm: (Change of variable：座標變換)

Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V)$. Let β and γ be ordered bases over V.

Then

$$\boxed{\phantom{[T]_\gamma = Q^{-1}[T]_\beta Q \qquad\qquad\qquad\qquad}}$$

where $Q = [I_V]_{\gamma}^{\beta}$.

※ 任何向量空間的基底變換都依照這個定理，請務必記下，且要學會追蹤座標~~~

$Ex.$ Let $u_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, and let L be the linear operator that rotates vectors in $\mathbb{R}^2$ by $45°$ in the counterclockwise direction. Find the matrix representation of L with respect to the ordered basis $\{u_1, u_2\}$.【96 台科電機類題、102.104 成大資工】

$Ex.$ Let $T: P_2 \to P_2$ be given by $T\big(p(x)\big) = p(x-1)$. Consider the two ordered bases $\beta = \{x^2, x, 1\}$ and $\gamma = \{x, x+1, x^2-1\}$.

a) Find $[T]_\beta$ and $[T]_\gamma$.

b) Find the matrix S such that $[T]_\gamma = S[T]_\beta S^{-1}$.【110 政大資料】