CURVATURE FLOWS IN CONFORMAL PARAMETRIZATION

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Abstract. This article is a survey on spherical conformal parametrization. Our ultimate goal is to find an efficient algorithm to measure and compare any two closed surfaces of genus-zero. Given a pair of such surfaces, one measure proposed by Hass and Koehl is an energy functional measuring mutual stretching between them. The difficulty of computing this energy is to an efficient and robust conformal parametrization onto a round unit sphere. In this article, we described a modified version of mean curvature flow, proposed by Kahzdan, Solomon, Ben Chen, to construct such a conformal parametrization.

1. Introduction

Shape comparison has a wide range of applications in biology and image processing. For example, the human brains and proteins exhibit complex geometric structures that play an important role in their biological functions. Neurobiologists study variations in the shapes of brain structure in neurodegenerative diseases such as Alzheimer. Structural biologists can infer the functionality of proteins through representing their structures in shape analysis. Therefore it remains a central problem in numerous disciplines to compare images and shapes.

The idea proposed by Hass and Koehl in [6], [7] is to compute a distance between two closed surfaces of genus zero (represented by triangular meshes embedded in $\mathbb{R}^3$). They first construct a conformal map between two genus zero surfaces that is as close to an isometry as possible. The unformization theorem ensures the existence of this conformal map but such map does not need to be unqiue. So they select the "optimal" conformal map that minimizes the local area distortion through an energy minimization process. They used this method to analyze and compare brain cortices, protein surfaces, tooth and brain surfaces in [3], [4], [5].

One key step in their method is to construct conformal maps to warp the surfaces onto a unit sphere. We look for an efficient and robust algorithm to compute such a spherical conformal parametrization. This algorithm should be

(i) applicable on a wide range of genus zero surfaces, not only with globular shapes, but also with protrusions or spikes.
(ii) fast enough to process triangular meshes of over millions of vertices, which are not uncommon in real-world application.
(iii) robust in the sense that any moderate distortion in the input mesh does not affect the algorithm's efficiency.

Key words and phrases. Conformalized Mean Curvature Flow, Discrete Conformal Map, Conformal Parametrization, Shape Analysis.

Here we describe two methods that were implemented in our project previously. A method proposed by Bobenko, Pinkall, Schröder ,and Springborn is to flatten a given mesh onto a plane by minimizing a convex energy functional which describes a precise notion of discrete conformal equivalence capturing the conformal equivalence of surfaces in the continuous case [1], [11]. However, if the input mesh has a region full of "flat" triangle, i.e. with one angle close to π, triangle inequality might fail during the energy minimization and the output set of edge length could not be embedded in $\mathbb{R}^3$. Edge flipping or subdivision of triangles can be used to fix this problem. Another method proposed by Chow, Luo and Gu is to evolve genus zero surface to a sphere [2], [8], [12]. The idea is to distribute evenly the total curvature (4π by Gauss-Bonnet Theorem) over the discretized surface through assigning a discrete conformal factor on each vertex, which can be obtained by solving a first-order nonlinear ODE systems. Their algorithm might lead to mesh degeneracy in the output mesh, i.e. overlapping of triangles. Indeed both methods described above are derived from the Ricci flow which is an intrinsic flow but with different notions of discrete conformal equivalence. Ricci flow evolves the metric of the surface but does not provide its embedding.

Here we descirbe a modified mean curvature flow proposed by Kazhdan, Solomon, and Ben-Chen [10]. Note that Kahzdan has made his code available [9]. We are interested in using their algorithm because it is

(i) easy to be implemented.
(ii) fast and robust to evolve meshes with complex geometry to a sphere. And the resulting mesh is discrete conformal to the input mesh.

The mean curvature flow (MCF) is an extrinsic geometric flow which provides the embedding of the mesh at every step and hence demonstrates how a shape can be morphed onto another. We test this flow on different meshes and discuss its limitation below.

2. Conformalized Mean Curvature Flow

All the mathematical derivations shown below are work by Kahzdan, Solomon, and Ben-Chen in [10].
They start with finite element discretization of MCF on a 2 dimensional surface M. Let $\Phi_t\colon M \to \mathbb{R}^3$ be a smooth family of immersions with parameter time $t \geqslant 0$ and g_t the induced metric at time t,

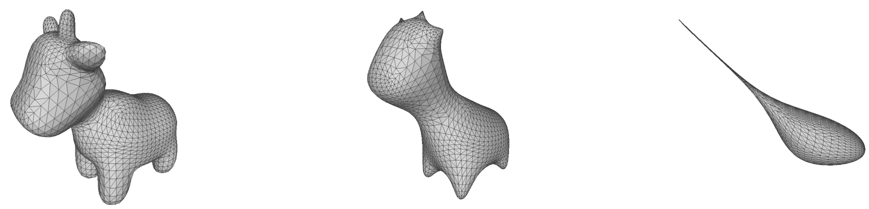
$$\frac{\partial \Phi_t}{\partial t} = \Delta_{g_t}\Phi_t = 2H_t\hat{N}_t \tag{2.1}$$

where $H_t(p)$ is the scalar mean curvature and $\hat{N}_t(p)$ is the inward unit surface normal. They approximate $\Phi_t = \sum_{i=1}^{N} x_i(t)B_i(p)$ with a finite set of function basis $B_i\colon M \to \mathbb{R}$ for $1 \leqslant i \leqslant n$. and a set of coefficient vectors at time t denoted by $\vec{X}(t) \coloneqq \{x_1(t), \cdots, x_N(t)\} \subset \mathbb{R}^3$. Using the weak formulation and applying backward Euler to discretize the time derivative of $\vec{X}(t)$, they solve following the linear system for $\vec{X}(t+\tau)$ in $x, y, z-$ directions

$$(D^t - \tau L^t)\vec{X}(t+\tau) = D^t\vec{X}(t) \tag{2.2}$$

where the mass matrix entries $D^t_{ij} \coloneqq \int_M B_i \cdot B_j \, dA_t$ is the inner product of the basis on M and the stiffness matrix entries $L^t_{ij} \coloneqq -\int_M g_t(\nabla_t B_i, \nabla_t B_j) dA_t$ is inner product of the gradients of basis. dA_t denotes the volume form on M at time t. Formulas for computing entries in the mass and stiffness matrices using hat basis can be found in [10].

Since singularity emerges (formation of spike), hence MCF cannot provide a spherical parametrization. Below is the mesh “spot” with 2930 vertices and 5856 faces. *Surface area is kept constant at every step*, with step size = 0.05. Spot’s head evolves into a spike where singularity forms and the flow stops within 4 steps.



Kazhdan, Solomon, and Ben-Chen proposed a “conformalized” mean curvature flow (cMCF) by replacing the metric g_t with a metric $\tilde{g}_t = \sqrt{|g_t||g_0|^{-1}}g_0$ that is conformal to the initial metric g_0 at time $t = 0$, the formulation becomes

$$\frac{\partial \Phi_t}{\partial t} = \sqrt{|g_0||g_t|^{-1}}\Delta_{g_0}\Phi_t \tag{2.3}$$

$$(D^t - \tau L^0)\vec{X}(t+\tau) = D^t\vec{X}(t) \tag{2.4}$$

While the mass matrix is updated at every step, they use the initial stiffness matrix and do not update it. Refer to [10] for the detailed derivation. The cMCF provides a conformal parametrization if the flow converges to a sphere:

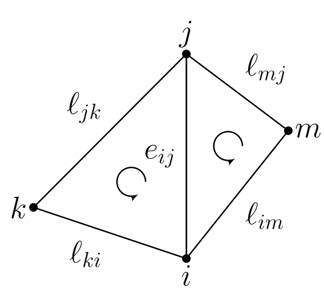
Theorem 2.1. *[10] If cMCF converges, then it converges to a map onto the sphere if and only if the limit map is conformal*

3. Results

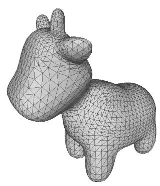
Here we use the following two criteria to measure the conformality of our maps:

(I). Angular distortion associated to each triangle, i.e. $\max\limits_{\theta=\alpha,\beta,\gamma}\left(\frac{|\theta-\theta'|}{\theta}\right)$ where α, β, γ are three angles in a triangle and α', β', γ' are the transformed angles.

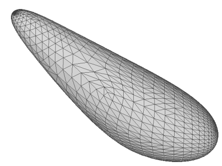
(II). Ratio of cross length ratio (clr.) associated to each edge, i.e. $\frac{\mathfrak{c}'_{ij}}{\mathfrak{c}_{ij}}$ where the clr. at an edge e_{ij} is defined by $\mathfrak{c}_{ij} := \frac{\ell_{im}\cdot\ell_{jk}}{\ell_{mj}\cdot\ell_{ki}}$ whereas $\mathfrak{c}'_{ij}$ denotes the clr. at the transformed edge. Two meshes are discretely conformal equivalent if and only if $\mathfrak{c}_{ij} = \mathfrak{c}'_{ij}$ for all edges e_{ij} [11].



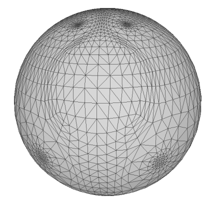
In addition, we use a dimensionless parameter $s := \frac{\left(36\pi V^2\right)^{\frac{1}{3}}}{A}$ to measure the sphericity of the mesh, where V and A denote the volume and the total surface area of the mesh respectively. Clearly $0 < s < 1$ and $s = 1$ implies We applied the cMCF on "spot" and two meshes of human brain (the first is a full brain is with $65,538$ vertices and $131,072$ faces and the second one is half of a brain with $189,906$ vertices and $379,808$ faces). It took around 0.1 sec., 8 sec., and 32 sec to obtain conformal parametrization of spot, full brain, half brain onto a sphere.



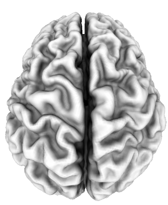
$s = 0.679$



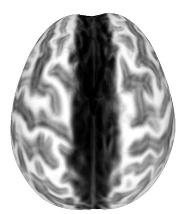
$s = 0.839$



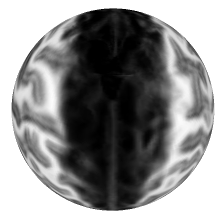
$s = 0.999$



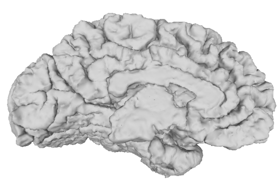
$s = 0.5877$



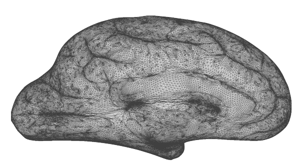
$s = 0.9290$



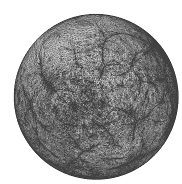
$s = 0.9999$



$s = 0.2664$



$s = 0.8133$



$s = 0.9999$

Here are the distribution of angular distortion (ang.) over all triangles and that of the ratio of cross length ratio (clr.) over all edges. An ideal conformal parametrization should give 0 for ang. and 1 for clr.. Their standard deviations (std) are also provided.

	mean μ	std σ
ang.	0.0910	0.0767
clr.	1.0010	0.0441

	mean μ	std σ
ang.	0.0326	0.0207
clr.	1.0004	0.0294

	mean μ	std σ
ang.	0.1270	0.2268
clr.	1.0112	0.9959

4. Implementation

We programmed on `C++` with a open source library `OpenMesh`. All meshes are stored in *.off* or *.obj* formats. Snapshots are taken using the software `MeshLab`. The implementation of the algorithm and all the resulting images are our own work.

5. Ongoing Work

The conformalized MCF evolves meshes with sharp and thin protrusions to a sphere but with mesh degeneracy, i.e. some triangles shrink to a point in which their edge length is less than the machine epsilon. We would like to address this problem in our future work.
Moreover, the underlying idea of conformalized MCF described above is that the two eigenvalues of the conformalized metric are equal to each other and they are equal to the geometric mean of the two eigenvalues from the traditional MCF. An interesting question would be whether we can choose another conformal metric to derive another conformalized mean curvature flow.

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References

[1] Bobenko, A., Pinkall, U., and Springborn, B. Discrete conformal maps and ideal hyperbolic polyhedra. *Geometry & Topology 19*, 4 (2015), 2155 – 2215. doi: 10.2140/gt.2015.19.2155.

[2] Chow, B., and Luo, F. Combinatorial ricci flows on surfaces. *J. Differential Geom. 63*, 1 (2003), 97–129. doi: 10.4310/jdg/1080835659.

[3] Hass, J., and Koehl, P. Automatic alignment of genus-zero surfaces. *IEEE Trans. Pattern Anal. Mach. Intell. 36* (2014), 466 – 478. doi:10.1109/TPAMI.2013.139.

[4] Hass, J., and Koehl, P. How round is a protein? exploring protein structures for globularity using conformal mapping. *Front. Mol. Biosci. 1* (2014), 26. doi: 10.3389/fmolb.2014.00026.

[5] Hass, J., and Koehl, P. Landmark-free geometric methods in biological shape analysis. *J. R. Soc. Interface 12* (2015). doi: 10.1098/rsif.2015.0795.

[6] Hass, J., and Koehl, P. A metric for genus-zero surfaces. http://arxiv.org/abs/1507.00789.

[7] Hass, J., and Koehl, P. Comparing shapes of genus-zero surfaces. *J. Appl. and Comput. Topology 1* (2017), 57 – 87. doi: 10.1007/s41468-017-0004-y.

[8] Jin, M., Kim, J., Luo, F., and Gu, X. Discrete surface ricci flow. *IEEE Transactions on Visualization and Computer Graphics 14*, 5 (2008), 1030–1043. doi:10.1109/TVCG.2008.57.

[9] Kazhdan, M. Source code for conformalized mean curvature flow. http://www.cs.jhu.edu/~misha/Code/ConformalizedMCF/Version2/.

[10] Kazhdan, M., Solomon, J., and Ben-Chen, M. Can mean curvature flow be modified to be non-singular? *Comput. Graph. Forum 31* (2012), 5. doi: 10.1111/j.1467-8659.2012.03179.x.

[11] Springborn, B., Schröder, P., and Pinkall, U. Conformal equivalence of triangle meshes. *ACM Trans. Graph. 27*, 3 (2008), 77:1–77:11. doi:10.1145/1360612.1360676.

[12] Zhang, M., Zeng, W., Guo, R., Luo, F., and Gu, X. Survey on discrete surface ricci flow. *Journal of Computer Science and Technology 30*, 3 (2015), 598–613. doi:10.1007/s11390-015-1548-8.

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