# CODING THEORY

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# Contents

# Chapter 1

# Basics

## 1.1 Basic Notations and Definitions

**Definition 1.1.1: Code**

A *code* $C$ is a subset of $\Sigma^n$ where $\Sigma$ is an alphabet, where $n$ is the block length of $C$. We typically use $q$ to denote $|\Sigma|$.

Another way to view the definition of a code to be a map $C : [M] \to \Sigma^n$, where $M = |C|$.

**Definition 1.1.2: Dimension of a code**

*Dimension* of a code $C$, denoted as $k$, is defined as the following way,

$$k := \log_q |C|.$$

**Remark.** *Note that,*

1. *For any* $C \subseteq \Sigma^n$, $k \leq n$.

2. $k$ *can be non-integral.*

One way to quantify *Redundancy* in a code is via its rate.

**Definition 1.1.3: Rate of a Code**

*Rate* of a code $C$ of block length $n$ and dimension $k$, denoted as $R$, is defined as

$$R := \frac{k}{n}.$$

**Example 1.1.1.** *Define a code* $C \subseteq \{0, 1\}^5$ *that maps a binary string* $(x_1, x_2, x_3, x_4)$ *to* $(x_1, \ldots, x_4, x_1 \oplus \ldots \oplus x_4)$.

## 1.2 Formalizing Error Correction

**Definition 1.2.1: Encoding & Decoding Functions**

- Let $C \subseteq \Sigma^n$. An equivalent description of the code $C$ is an injective mapping $E : [|C|] \to \Sigma^n$ called the *encoding function*.
- Let $C \subseteq \Sigma^n$ be a code. A mapping $D : \Sigma^n \to [|C|]$ is called a *decoding function*.

We can refer to the following figure from now on.

### Quantifying Error

**Definition 1.2.2: Hamming Distance**

For any $a, b \in \Sigma^n$, *hamming distance* of $a, b$ is defined as

$$\Delta(a, b) := \#\{i \in [n] : a_i \neq b_i\}$$

where $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$. We also define *relative hamming distance* as

$$\delta(a, b) := \frac{1}{n}\Delta(a, b).$$

**Remark.** *We can verify easily that this $\Delta$ is actually a distance on $\Sigma^n$.*

**Definition 1.2.3: Hamming Weight**

We define *hamming weight* of an element $v \in \Sigma^n$ as, $wt(c) := \Delta(0, v)$.

Refering to the above figure, we define the error for that transmitted codeword to be $\Delta(c, y)$.

## 1.3 Performance of an Error Correcting Code

**Definition 1.3.1: t-Error Channel & t-Error Correcting Code**

- (t-Error Channel) : An $n$ symbol *t-Error Channel* is a relation $ch : \Sigma^n \to \Sigma$ such that $\forall c \in \Sigma^n$,
$$\Delta(c, ch(c)) \leq t.$$
- (t-Error Correcting Code): Let $C \in \Sigma^n$ is a *t-Error Correcting code* if $\forall$ t-Error Channel $ch$ and $\forall m \in [|C|]$ ,
$$D(ch(E(m))) = m.$$

**Example 1.3.1.** *$C_{3,rep}$ is a 1-error correcting code.*