Pipe Flow
(Denn Chapter 3)

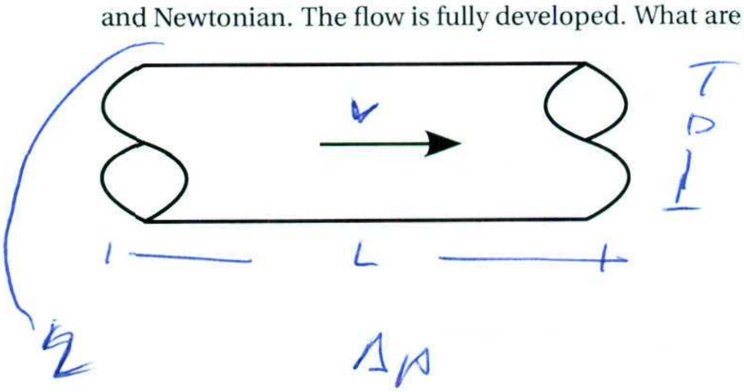
LEARNING OBJECTIVES

1. Apply dimensional analysis with physical insight and experimental data to find design equations for pipe flow.
2. Describe the laminar and turbulent regimes of fluid pipe flow both qualitatively and quantitatively.
3. Interpret the numerator and denominator of Reynold's number physically and in relation to laminar and turbulent flow.
4. Derive Poiseuille's law using dimensional analysis and physical insight.
5. Evaluate quantitatively power consumption and cost tradeoffs when designing pipe flow parameters.

DIMENSIONAL ANALYSIS

PIPE FLOW

We wish to measure the pressure drop for a fluid through a pipe. The fluid is incompressible and Newtonian. The flow is fully developed. What are potential variables involved?



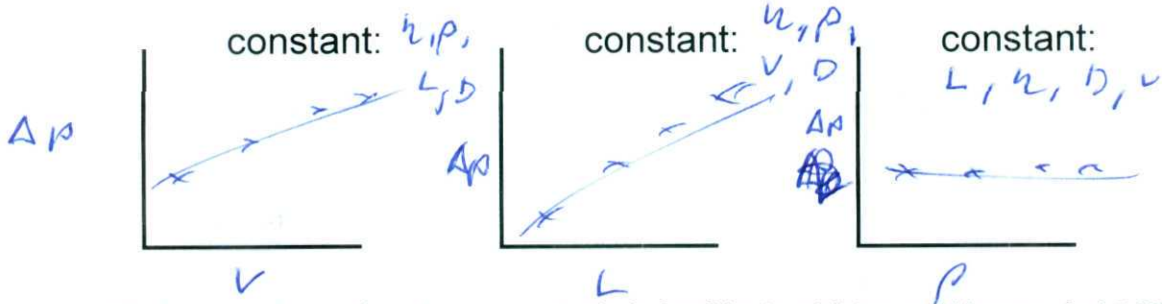
ρ

$$V = \langle v \rangle = Q/A = \frac{Q}{\pi D^2/4}$$

How do we figure out how these variables fit into a pipe flow model?

$$\Delta p = f(D, L, \eta, \rho, v)$$

Traditional Answer:



The last experiment doesn't seem particularly feasible. In addition, we'd have to do A LOT of experiments. With six variables where we hold four variables constant for each experiment, we'd have to do . . .

5 for Δp

$\binom{6}{2} = 15$ experiments ← for all combinations

→ need something ~~easier~~ to make easier

INTRODUCING DIMENSIONAL ANALYSIS

Dimensional analysis, along with Buckingham π theorem, allows us combine variables into dimensionless groups that characterize the process. These dimensionless groups serve as the main parameters describing our model.

Buckingham π theorem: $G = V - D$

V - # of variables characterizing process

D - # of fundamental dimensions (e.g. mass, length, time)

G - minimum # of dimensionless groups characterizing process

Why?

- reduce # of parameters to characterize our process
  → reduce # of experiments

Applying π theorem to our above pipe flow example, we have

$$G = V - D = 6 - 3 = \boxed{3}$$

So if we don't have a model for pipe flow, how do we find our 3 dimensionless groups?

VARIABLES INVOLVED?

Variable	Dimensions (Length, Time, Mass)
Δp	$\frac{M}{LT^2}$
ρ	M/L^3
μ	M/LT
D	L
L	L
V	L/T

$$\frac{\partial u}{\partial y}\mu = \tau_{yx}$$

$$[1/T]\ ? = [mL/T^2][1/L^2]$$

$$? = \left[\frac{m}{LT}\right]$$

Messing around:

L/D is easy $[=]$ L/L

We need something with Δp

$$\frac{\Delta p}{\rho V^2} \; [=] \; \frac{m}{LT^2}\frac{L^3}{m}\frac{T^2}{L^2}$$

$$\frac{D\,\Delta p}{\mu V} \; [=] \; \frac{m}{LT^2}\cdot\frac{LT}{m}\frac{T}{L}L \; ; \text{ could also do } \frac{L\,\Delta p}{\mu V}$$

eliminate mass btw ρ + μ

$$\frac{\rho V D}{\mu} \; [=] \; \frac{M}{L^3}\frac{LT}{M}\frac{L}{T}\cdot L \; ; \; \frac{\rho V L}{\mu}$$

Lots of groups; many more than 3! What's going on?

independence!

$$L/D \times \frac{Dv\rho}{\mu} = \frac{Lv\rho}{\mu} \leftarrow \text{not independent}$$

Which 3 should we pick?

1. $\boxed{L/D}$ - simple, involves geometry only

2. We wish to see how Δp depends on the other variables, so let's select only <u>one</u> group with that.

$$\boxed{\frac{\Delta p}{\rho v^2}} \text{; why vs } \frac{D \Delta p}{\mu v} \text{? (convention)}$$

3. need μ in one other groups

$$\boxed{\frac{Dv\rho}{\mu}} \text{; why vs } \frac{Lv\rho}{\mu} \text{? - convention - more ~~useful~~ informative}$$

Dimensional Analysis tells us that we can go from

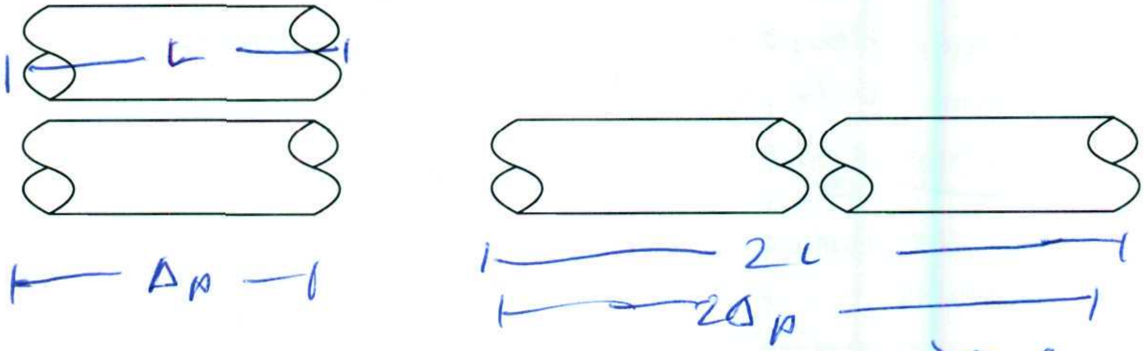
$$\Delta p = f(\rho, v, \mu, L, D)$$

$$\Downarrow$$

$$\frac{\Delta p}{\rho v^2} = f\left(L/D, \frac{Dv\rho}{\mu}\right) \leftarrow \text{3 parameters instead of 6}$$

USING PHYSICAL INSIGHT

We found appropriate dimensionless groups through dimensional analysis, now using physical insight, we can simplify further. Consider two sections of pipes.



→ $\Delta p \propto L$

This insight allows us to characterize our model further

$$\frac{\Delta p}{\rho v^2} = \frac{L}{D} f\left(\frac{D v \rho}{\mu}\right)$$

And now we can define the Fanning friction factor (f) and Reynold's number (Re).

$$f = \frac{\Delta p\, D}{2 \rho v^2}$$

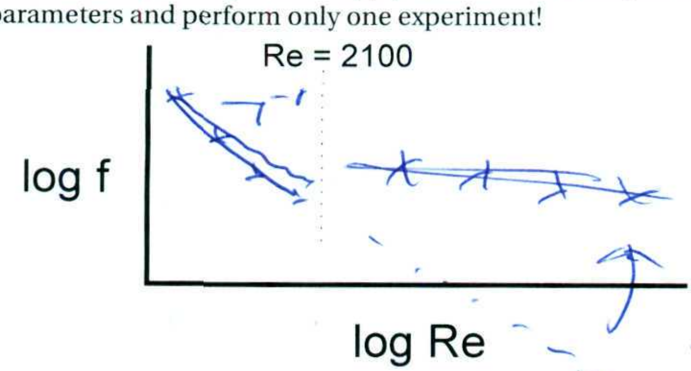
$$Re = D v \rho / \mu$$

$$\boxed{f = f(Re)}$$ ← one experiment

2 parameters!

EXPERIMENTS

We were successfully able to apply dimensional analysis and physical insight to relate two parameters and perform only one experiment!



$f_{turb} >> f_{lam}$; extrapolate

Data

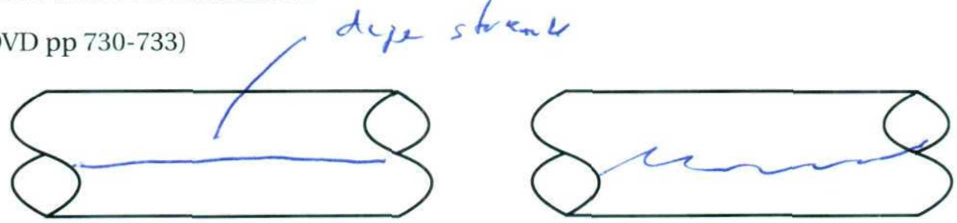
- Different viscosities
- Different diameters
- Different densities
- Different velocities

lumped into one parameter (Re)

"Dynamic Similarity" (DVD pp 534-540; 568-601)

REYNOLDS EXPERIMENT

(DVD pp 730-733)



Re < 2100	Re > 2100
<u>Laminar Flow</u>	<u>Turbulent Flow</u>
$v_z = v_z(r)$	$v_z = v_z(r, \theta, z, t)$
$v_r = 0$	$v_r = v_r(r, \theta, z, t)$
$v_\theta = 0$	$v_\theta = v_\theta(r, \theta, z, t)$

- poor mixing
- steady

- vigorous mixing
- good for heat/mass transport
- time-dependent
- greater dissipation of energy
- $f_{turbulent} \gg f_{laminar}$, extrapolate

ANOTHER DIMENSIONAL ANALYSIS EXAMPLE

Liquid is slowly dripping out of a faucet of diameter D under the influence of gravity g. The liquid has density ρ and surface tension σ (dimensions force/length). Using dimensional analysis, determine how fluid mass M will relate to the other variables in the problem.

M	m	
D	L	
g	L/T^2	
ρ	m/L^3	
σ	$F/L =$	m/T^2
5	3	$= 2$

$$\frac{m}{\rho D^3}$$

$$\frac{\rho g D^2}{\sigma}$$

$$\frac{m}{\rho D^3} = f\left(\frac{\rho g D^2}{\sigma}\right)$$

10/10/13 practice

PIPE FLOW DATA CORRELATIONS

RE < 2100

Laminar flow → poiseuille's ~~flow~~ law

derived earlier

$$Q = \frac{\pi \Delta p R^4}{8\eta L} \Rightarrow V = \frac{Q}{A} = \frac{Q}{\pi R^2} = \frac{\Delta p R^2}{8\eta L} = \frac{\Delta p D^2}{32\eta L}$$

Now rearrange to $\frac{\Delta p D}{L} = \frac{32 V\eta}{D}$

Bringing in the Fanning friction factor

multiply by $\frac{1}{2\rho v^2}$

$$f = \frac{\Delta p D}{2\rho v^2 L} = \frac{16\eta}{Dv\rho} \Rightarrow f = 16/Re$$

agrees w/ data well

RE > 2100

Turbulent flow → no flow model (yet)

EMPIRICAL EQUATIONS

Blasius Equation

$$f = 0.079\, Re^{-1/4} \quad (4000 < Re < 10^5)$$

von Karman-Nikuradse correlation

$$\frac{1}{\sqrt{f}} = 4.0 \log_{10}(Re\sqrt{f}) - 0.4 \quad (Re > 4000)$$

implicit for f

theoretical basis but parameters are experimentally determined

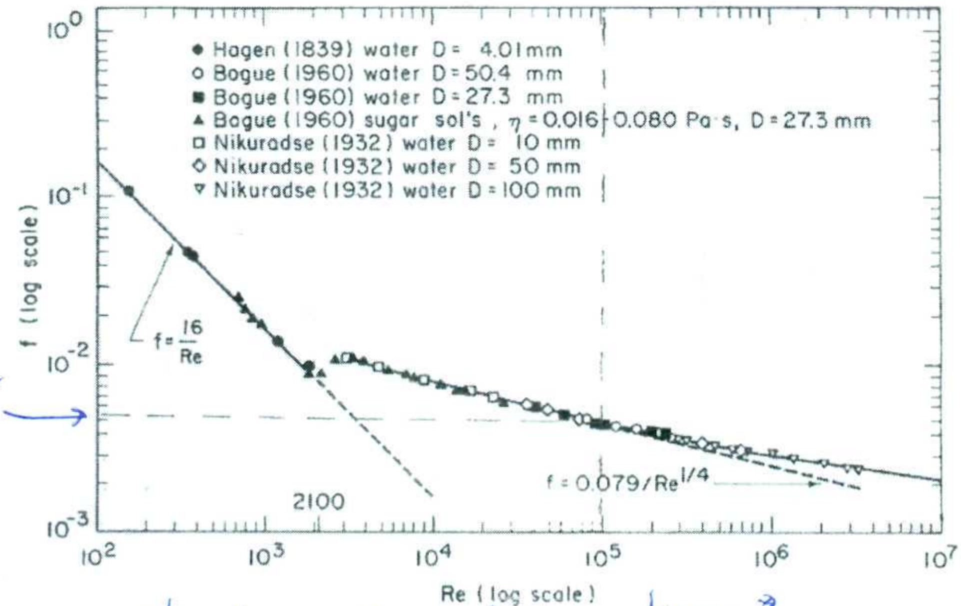
2100 < Re <4000

difficult "transition region"

Example Problem

~~What~~ water is pumped through 50m of a smooth pipe with an inside diameter of 5cm at a volumetric flow rate of 4 liters/second. What is the pressure drop? (Figure 3-4 information below)

change



What steps should we take here?

1) Re
2) use correlation or chart to get f →

$Q = 4\ L/s = \dfrac{0.004\ m^3}{s}$

$A = \dfrac{\pi D^2}{4} = \dfrac{\pi (0.05)^2}{4} = 0.00196\ m^2$

$V = Q/A = 2.04\ m/s$

$Re = \dfrac{D v \rho}{\eta} = \dfrac{(0.05\ m)(2.04\ m/s)(1000\ kg/m^3)}{0.001\ kg/m\cdot s} = 1.02\times10^5$ (turbulent)

Three options:

i) Chart: $f = 0.005$

ii) Blasius: $f = 0.079\ Re^{-1/4} = 0.0044$ (note at edge of validity)

iii) von Karman/Nikuradse:

$1/\sqrt{f} = 4.0 \log_{10}(Re\sqrt{f}) - 0.4$

$f = 0.0045$

$\Delta p = \dfrac{2\rho v^2 L f}{D} = 37{,}500\ Pa$

other aspect of DA

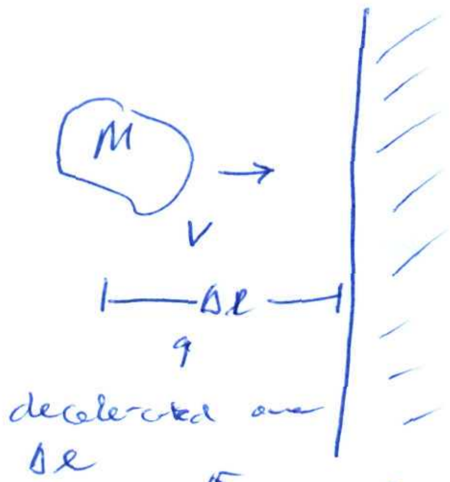
PHYSICAL INTERPRETATION OF REYNOLD'S NUMBER

Reynold's number can be thought of the ratio between inertial forces and viscous forces. (DVD pp. 496-508).

$$\mathrm{Re} = \frac{\text{Inertial Forces}}{\text{Viscous Forces}}$$

INERTIAL FORCES

Inertia is what must be overcome to change speed/direction of a fluid. Consider a blob of fluid impinging on a wall with mass M, moving at velocity V, with a distance Δl to the wall.



Inertial Force = $F_I = \dfrac{\Delta\,\text{momentum}}{\Delta\,\text{time}} = \dfrac{mv - 0}{\Delta t}$

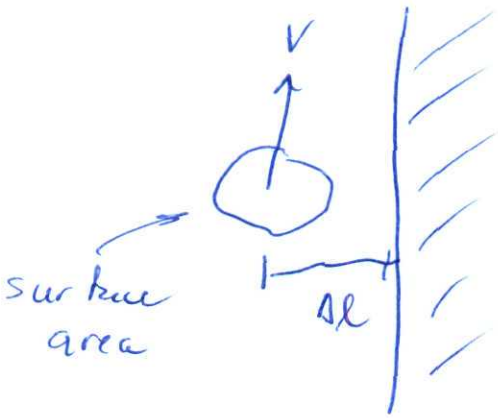
$\Delta t \approx \Delta l / v$

$\therefore$ inertial force $F_I \approx \dfrac{mv^2}{\Delta l}$

overcome inertial force

VISCOUS FORCES

Viscous forces are associated with deformation of a fluid. Consider a blog of fluid moving parallel to the wall.



Shear rate $\Gamma_s \approx V/\Delta l$

Shear stress $\tau_s \approx \dfrac{\eta V}{\Delta l}$

Viscous Force $F_V \approx \dfrac{\eta V A}{\Delta l}$

RATIO

$$\frac{F_I}{F_V} = \frac{m v^{\cancel{2}}}{\cancel{d}} \cdot \frac{\cancel{d}}{\eta v A} = \frac{m v}{\eta A}$$

$D \sim$ size of blob

$m \sim \rho D^3$

$A \sim D^2$

$$\frac{F_I}{F_V} = \frac{\rho D^{\cancel{3}} v}{\eta \cancel{D^2}} = \frac{\rho D v}{\eta} \quad \text{(Reynold's number)}$$

REVISITING DIMENSIONAL ANALYSIS FOR LAMINAR PIPE FLOW

Streamlines are straight, there is constant velocity (no acceleration), so therefore ...

no inertial effects

→ density shouldn't matter

Δp	M/LT^2
η	M/LT
V	L/T
D	L
L	L

$5 - 3 = 2$

$$\frac{\Delta p D}{\eta v}, \ L/D \Rightarrow \frac{\Delta p D}{\eta v} = f(L/D)$$

physical intuition suggests

$$\Delta p \propto L$$

$$\frac{\Delta p D}{\eta v} \propto L/D \Rightarrow \Delta p \propto \frac{\eta v L}{D^2}$$

$V \propto Q/D^2$

$$\boxed{Q \propto \frac{\Delta p\, D^4}{\eta L}}$$

We obtained a mathematical form of Poiseuile's law without a detailed solution using only dimensional analysis and physical insight!

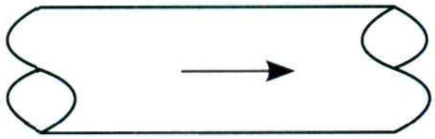
nothing about geometry here!

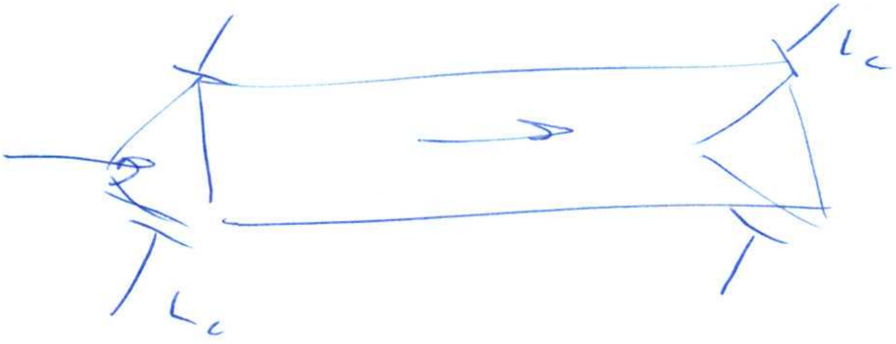
COROLLARY

f = c/Re

For laminar flow in any conduit w/ char. length scale L_c

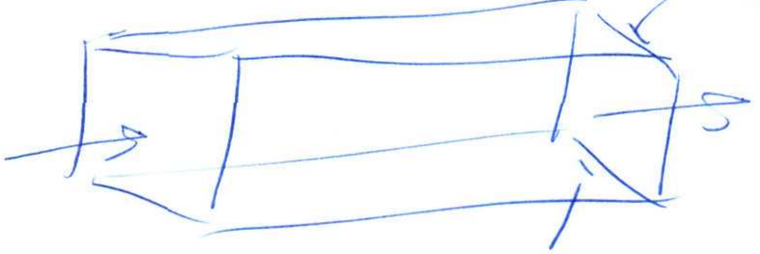
$\left(Re = \frac{L_c v \rho}{\eta}\right)$





How do we get c?

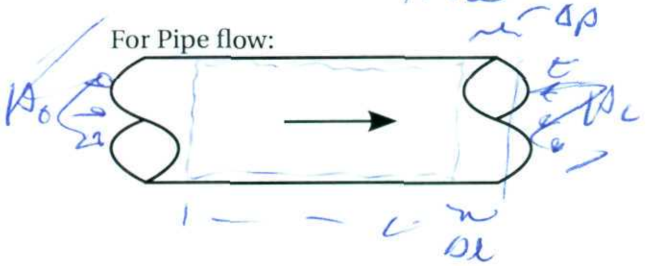
- can use detailed laminar solution



POWER REQUIREMENT FOR PUMPING

Power = $\dfrac{\text{work}}{\text{time}}$ work = force · distance

For Pipe flow:



Over time Δt, the fluid moves distance Δl

Work = $\dfrac{\Delta p \, \pi D^2}{4} \Delta l$

Power = $\dfrac{\Delta p \, \pi D^2}{4} \dfrac{\Delta l}{\Delta t} = \Delta p \dfrac{\pi D^2}{4} v$ Q (volumetric flow rate)

Power = $\Delta p \cdot Q$ where does this energy go? (dissipated in fluid

OPTIMAL PIPE DIAMETER

We want to pump fluid distance L with volumetric flow rate Q. What pipe diameter do we choose? How do we choose?

→ minimize work

Total Cost = Operating Cost + Capital Costs

how will D affect these?

CAPITAL COSTS : pipes, fittings, construction, etc.

$\sim D^n L$ (fig 3-4, $n = 1.37$)

cost per year

Capital Costs = $\Lambda D^n L$

OPERATING COSTS

power $\sim \Delta p Q$

$\Delta p = \dfrac{2 f \rho v^2 L}{D}$; $v = \dfrac{4Q}{\pi D^2}$

use Blasius eq: $f = 0.079 \left(\dfrac{D v \rho}{\eta}\right)^{-0.25}$

gamma $\rightarrow \gamma$

costs $= \dfrac{HK}{E} \Delta p Q$ H - hours operated/year

$\lambda = 0.16 (4/\pi)^{1.75}$

Operating Costs = $\lambda \dfrac{L \rho^{0.75} \eta^{0.25} Q^{2.75}}{D^{4.75}}$ K - cost per energy unit

E - efficiency

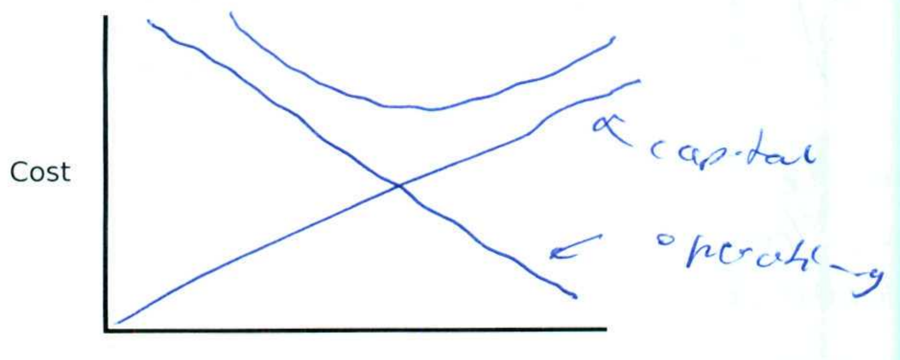
TOTAL COSTS

Total Cost =

$$\Lambda D^n L + \lambda \frac{L \rho^{0.75} \eta^{0.25} Q^{2.75}}{D^{4.75}}$$

goes up w/ D

goes down with D



$$\partial C / \partial D = 0 = n \Lambda D^{n-1} L - 4.75 \lambda \frac{L \rho^{0.75} \eta^{0.25} Q^{2.75}}{D^{5.75}}$$

$$D = \left(\frac{4.75 \lambda}{n \Lambda} \rho^{0.75} \eta^{0.25} Q^{2.75} \right)^{1/(4.75+n)}$$

avg roughness

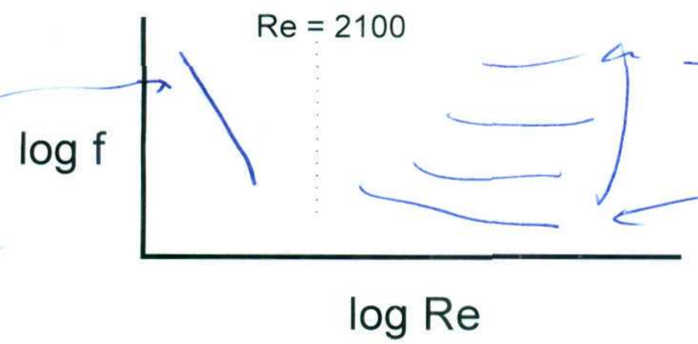
ROUGH PIPE

Dimensional Analysis... add a new variable (k, dimensions of length)

$f = f(Re, k/D)$ (usually $k/D << 1$)

Experiments, Nikuradse (glue sand to pipes)

no change in laminar



– increasing k/D

smooth

for $k/D = 0.002$
$Re = 10^7$
→ get triple the Δp!
→ Big effect!

Estimating f in rough pipes: Colebrook formula

(plotted in Fig 3.7 "Moody chart")

$$\frac{1}{\sqrt{f}} = -4.0 \log_{10}\left(\frac{k}{D} + \frac{4.67}{Re\sqrt{f}}\right) + 2.28$$

1. Calculate Re
2. Calculate k/D
3. Use the formula or chart to determine f

COMMERCIAL PIPES

nominal size (Table 3.3, appendix)

Table 3.2
↓
different materials

"2 inch, schedule 80 pipe"

ID = 1.939" = 49.25 mm

OD = 2.375" = 60.33 mm

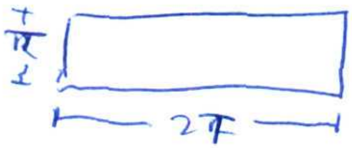
NON-CIRCULAR CROSS SECTIONS

For turbulent flow...

"hydraulic diameter"

1. Define the hydraulic diameter: $D_H = \dfrac{4 \cdot \text{cross sectional area}}{\text{wetted perimeter}}$

circular pipes



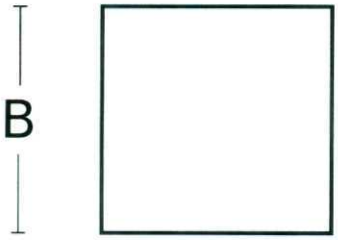
$$\frac{4 \cdot 4T^2}{6T} = 8/3\ T$$

2. Use D_H in place of D in calculating Re, f, k

3. Use correlations for circular tubes

EXAMPLE PROBLEM

Water is pumped through 20 m of a channel with a square cross section (see below) with B=50 mm. The channel is made from commercial steel with a surface roughness of $k = 0.05$ mm. What will be pressure drop assuming the average fluid velocity is 4 m/s. Assume $\rho = 1000\frac{\text{kg}}{\text{m}^3}$ and $\eta = 0.001$ Pa s



$$D_H = \frac{4B^2}{4B} = B = 0.05\ m$$

$$Re = \frac{D_H V \rho}{\eta} = \frac{(0.05\ m)(4\ m/s)(1000\ kg/m^3)}{0.001\ Pa\text{-}s}$$

$$= 2 \times 10^5$$

$$k/D = \frac{0.05}{50} = 0.001$$

$$f = 0.0053$$

CHAPTER SUMMARY

Aim: To develop a fluid model for pipe flow as a function of known, involved parameters without a detailed flow solution (macroscopically).

Challenge:

Solution: Apply Dimensional Analysis and

Additionally, dimensional analysis allows us to

- Simplify the our model by reducing number of parameters (and thus experiments)
-
-

For simple pipe flow, we found two dimensionless groups to characterize flow through a pipe, Reynold's Number (Re) and Fanning Friction factor (f).

Reynold's Number reveals the following about fluid flow:

- The balance of viscous and inertial forces (laminar vs turbulent regime).
- The correlation/equation to be used with friction factor.

From experiments and derivations we can relate friction factor to Reynold's number. Using pipe flow models we can

- Calculate the power requirement
- Find optimal sizing for our process
- Account for pipe roughness
- Non-circular cross sections