Scaling/Approximation (Denn Chapter 11 and 14)

LEARNING OBJECTIVES

1. Scale conservation equations. Analyze scaled equation.
2. Solve and analyze inviscid flow problems using stream functions and potential flows.

MOTIVATION

Throughout this course we've encountered situations where an approximation has simplified our problem solving and analysis. Examples?

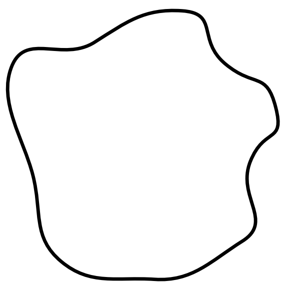
As a corollary, we should be able to rewrite our conservation equations (e.g. Navier-Stokes) in a form approximate to the situation. Hopefully, a solution to the approximate equations is an approximate solution.

REWRITING NAVIER-STOKES EQUATIONS (CHAPTER 11)

Consider Steady Navier-Stokes Equation:

$$\rho \underline{\mathbf{v}} \cdot \underline{\nabla} \underline{\mathbf{v}} = -\underline{\nabla} \mathscr{P} + \eta \nabla^2 \underline{\mathbf{v}}$$

And a general flow past an object problem:



Re-write the Navier Stokes problems in terms of the "Scaled Variables"

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Scaling Our Variables

We can rescale our variables appropriately:

Scaled velocity $\underline{\tilde{\mathbf{v}}} = \tilde{\mathbf{v}}/V$

Scaled coordinate $\underline{\tilde{\mathbf{x}}} = \tilde{\mathbf{x}}/L$

What about pressure? $\underline{\widetilde{\mathscr{P}}} = \widetilde{\mathscr{P}}/\Pi$

Typical Inertial Term

$$\rho v_x \frac{\partial v_y}{\partial x} \qquad v_x = \widetilde{v_x} V \qquad v_y = \widetilde{v_y} V \qquad x = \tilde{x} L$$

Plug in...

As a result we can rewrite the left side of N-S as

$$\rho \underline{\mathbf{v}} \cdot \underline{\nabla \mathbf{v}} \Rightarrow$$

Now for the viscous term:

$$\eta \frac{\partial^2 v_x}{\partial y^2} \qquad v_x = \widetilde{v_x} V \qquad y = \tilde{y} L$$

Plug in...

Viscous terms in general: $\eta \nabla^2 \underline{\mathbf{v}} \Rightarrow$

And pressure:

$$\frac{\partial \mathscr{P}}{\partial x} \qquad \mathscr{P} = \widetilde{\mathscr{P}} \Pi \qquad x = \tilde{x} L$$

Plug in...

Rewrite our Navier-Stokes Equation:

If variables are properly scaled, then all dimensionless terms are $O(1)$ quantities.

$\Rightarrow$

In particular,

This provides basis for approximations.

When Re >> 1, expect

$\Rightarrow$ Approximate by neglecting

$\Rightarrow$ Inviscid flow

When Re << 1, expect

$\Rightarrow$ Approximate by neglecting

$\Rightarrow$ Creeping flow

What about pressure?

When inertial terms dominate (high Re), pressure balances with

$$\Rightarrow \Pi =$$

When viscous terms dominate (low Re), pressure balances with

$$\Rightarrow \Pi =$$

Thus, for High Reynold's Number (inertial dominated) flow, we can write:

$$\underline{\tilde{\mathbf{v}}} \cdot \underline{\tilde{\nabla}\tilde{\mathbf{v}}} = -\underline{\tilde{\nabla}}\widetilde{\mathscr{P}} + \frac{1}{\mathrm{Re}}\tilde{\nabla}^2\underline{\tilde{\mathbf{v}}} \Rightarrow$$

Thus, for Low Reynold's Number (viscous dominated) flow, we can write:

$$\mathrm{Re}(\underline{\tilde{\mathbf{v}}} \cdot \underline{\tilde{\nabla}\tilde{\mathbf{v}}}) = -\underline{\tilde{\nabla}}\widetilde{\mathscr{P}} + \tilde{\nabla}^2\underline{\tilde{\mathbf{v}}} \Rightarrow$$

We hope that solutions of the approximation equations is an approximate solution to the problem. (Denn 11.6)

High Re

Low Re

High Reynold's Number (Inviscid Flow)

$$\text{Re} = \frac{Dv\rho}{\eta} \Rightarrow$$

However, we face some challenges with this equation. Need to define some other concepts before solving inviscid flow solutions. In particular, we seek Potential Flows (Denn Chp. 14).

Stream Function (Chapter 14)

Useful in 2-D flow (e.g. $v_x(x,y), v_y(x,y)$, and $v_z = 0$)

Case without a Stream function:

$$v_x(x,y)$$
$$v_y(x,y)$$
$$\mathscr{P}(x,y)$$

Define Stream Function, $\Psi(x,y)$:

$$v_x(x,y) =$$
$$v_y(x,y) =$$

Plug into continuity:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} =$$

Now we have

$$\Psi(x,y)$$
$$\mathscr{P}(x,y)$$

In steady flow, the stream function doesn't change along the streamline. We can prove this by taking the substantial derivative:

$$\frac{D\Psi}{Dt} = v_x \frac{\partial \Psi}{\partial x} + v_y \frac{\partial \Psi}{\partial y} =$$

For cylindrical coordinates:

$$v_r = \qquad\qquad v_\theta =$$