Scaling/Approximation (Denn Chapter 11 and 14)

LEARNING OBJECTIVES

1. Scale conservation equations. Analyze scaled equation.
2. Solve and analyze inviscid flow problems using stream functions and potential flows.

MOTIVATION

Throughout this course we've encountered situations where an approximation has simplified our problem solving and analysis. Examples?

As a corollary, we should be able to rewrite our conservation equations (e.g. Navier-Stokes) in a form approximate to the situation. Hopefully, a solution to the approximate equations is an approximate solution.

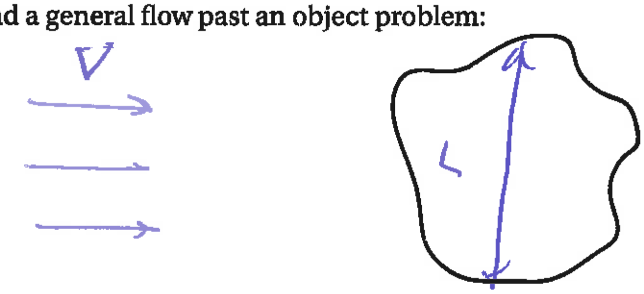
REWRITING NAVIER-STOKES EQUATIONS (CHAPTER 11)

Consider Steady Navier-Stokes Equation:

Fancy P bundle in gravity "modified"

$$\rho \underline{\mathbf{v}}\cdot\underline{\nabla}\underline{\mathbf{v}} = -\underline{\nabla}\mathscr{P} + \eta\nabla^2\underline{\mathbf{v}}$$

And a general flow past an object problem:



V = characteristic velocity

L = characteristic length

Re-write the Navier Stokes problems in terms of the "Scaled Variables"

- dimensionless
- scales are chosen such that new variables are the order of magnitude 1 $O(1)$

SCALING OUR VARIABLES

We can rescale our variables appropriately:

Scaled velocity $\tilde{\mathbf{v}} = \mathbf{v}/V$ e.g. $\tilde{v}_x = v_x/V$, $\tilde{v}_y = v_y/V$, etc.

$\mathbf{v}/V$

Scaled coordinate $\tilde{\mathbf{x}} = \mathbf{x}/L$ e.g. $\tilde{x} = x/L$, $\tilde{y} = y/L$, etc

$\mathbf{x}/L$

What about pressure? $\tilde{\mathcal{P}} = \mathcal{P}/\Pi$ ← characteristic pressure (will determine later)

$P/\Pi L$

Typical Inertial Term

$$\rho v_x \frac{\partial v_y}{\partial x} \qquad v_x = \tilde{v}_x V \qquad v_y = \tilde{v}_y V \qquad x = \tilde{x} L$$

Plug in...

$$\rho(\tilde{v}_x V)\frac{\partial(\tilde{v}_y V)}{\partial(\tilde{x} L)} = \frac{\rho V^2}{L}\tilde{v}_x \frac{\partial \tilde{v}_y}{\partial \tilde{x}}$$

As a result we can rewrite the ~~right~~ left side of N-S as

inertial term

$$\rho \mathbf{v}\cdot\nabla\mathbf{v} \Rightarrow \frac{\rho V^2}{L}\tilde{\mathbf{v}}\cdot\tilde{\nabla}\tilde{\mathbf{v}}$$

inertial term

$$\left(\tilde{\nabla} = i\,\partial/\partial\tilde{x} + j\,\partial/\partial\tilde{y} + k\,\partial/\partial\tilde{z}\right)$$

Now for the viscous term:

$$\eta\frac{\partial^2 v_x}{\partial y^2} \qquad v_x = \tilde{v}_x V \qquad y = \tilde{y} L \qquad \eta\nabla^2$$

Plug in...

$$= \eta\frac{\partial^2(\tilde{v}_x V)}{\partial(\tilde{y} L)^2} = \frac{\eta V}{L^2}\frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2}$$

And pressure:

$$\frac{\partial \mathcal{P}}{\partial x} \qquad \mathcal{P} = \tilde{\mathcal{P}}\Pi \qquad x = \tilde{x} L$$

Plug in...

$$\cancel{\frac{\rho V^2}{L}\tilde{\mathbf{v}}\cdot\tilde{\nabla}\tilde{\mathbf{v}} = -\frac{\Pi}{L}\tilde{\nabla}\tilde{P}} +$$

$$\partial P/\partial x = \Pi/L\ \partial\tilde{P}/\partial\tilde{x} \qquad \nabla P \to \frac{\Pi}{L}\tilde{\nabla}\tilde{P}$$

Rewrite our Navier-Stokes Equation:

$$\frac{\rho V^2}{L}\,\underline{\tilde{V}}\cdot\underline{\tilde{\nabla}}\,\underline{\tilde{V}} = -\frac{\Pi}{L}\,\underline{\tilde{\nabla}}\tilde{p} + \frac{\mu V}{L^2}\,\tilde{\nabla}^2\underline{\tilde{V}}$$

If variables are properly scaled, then all dimensionless terms are $O(1)$ quantities.

⇒ relative importance of terms is revealed by what's out in front!

In particular, $\frac{\text{inertial terms}}{\text{viscous terms}} = \frac{\rho V^2/L}{\mu V/L^2} = \frac{\rho V L}{\mu}$ ← reynold's number

This provides basis for approximations.

When Re >> 1, expect inertia >> viscous forces

⇒ Approximate by neglecting viscosity

⇒ Inviscid flow ←

When Re << 1, expect viscous >> inertial forces

⇒ Approximate by neglecting inertia

⇒ Creeping flow

What about pressure? must keep it in equation to maintain eq/unknown balance

When inertial terms dominate (high Re), pressure balances inertial term

$$\frac{\rho V^2}{L} \sim \frac{\Pi}{L}$$ ⇒ Π = char. pressure = ρV^2 ← Bernoulli's eq

When viscous terms dominate (low Re), pressure balances viscous term

$$\frac{\mu V}{L^2} \sim \frac{\Pi}{L}$$ ⇒ Π = $\Pi = \mu V/L$

Thus, for High Reynold's Number (inertial dominated) flow, we can write:

$$\underset{\sim}{\tilde{v}} \cdot \tilde{\nabla}\underset{\sim}{\tilde{v}} = -\tilde{\nabla}\tilde{\mathscr{P}} + \underbrace{\frac{1}{\text{Re}}\tilde{\nabla}^2\underset{\sim}{\tilde{v}}}_{\to 0} \Rightarrow \quad \rho \frac{Dv}{Dt} = -\nabla p \quad (\text{euler's eq})$$

Thus, for Low Reynold's Number (viscous dominated) flow, we can write:

$$\underbrace{\text{Re}(\underset{\sim}{\tilde{v}} \cdot \tilde{\nabla}\underset{\sim}{\tilde{v}})}_{0} = -\tilde{\nabla}\tilde{\mathscr{P}} + \tilde{\nabla}^2\underset{\sim}{\tilde{v}} \Rightarrow \quad -\underset{\sim}{\nabla} p + \eta \nabla^2 \underset{\sim}{v}$$

We hope that solutions of the approximation equations is an approximate solution to the problem. (Denn 11.6)

<u>High Re</u>	<u>Low Re</u>
"large" V, L	- small V, L
"small" ν	- large ν
- every day fluid mech	- highly viscous materials (polymers, foods)
- process → pipes, pumps, flow meters	- microscopic phenomena (e.g. settling of particles)
- planes, cars, trains ...	
- sports (golf, tennis, baseball)	

HIGH REYNOLD'S NUMBER (INVISCID FLOW)

ignore viscosity

$$\text{Re} = \frac{Dv\rho}{\eta} \Rightarrow \rho \, Dv/Dt = -\nabla p$$ (eulers eq of motion)

However, we face some challenges with this equation. Need to define some other concepts before solving inviscid flow solutions. In particular, we seek Potential Flows (Denn Chp. 14).

STREAM FUNCTION (CHAPTER 14)

Useful in 2-D flow (e.g. $v_x(x, y), v_y(x, y)$, and $v_z = 0$)

Case without a Stream function:

$$\left.\begin{array}{l} v_x(x,y) \\ v_y(x,y) \\ \mathscr{P}(x,y) \end{array}\right\}$$ 3 unknowns ; 3 eqns — continuity $\partial v_x/\partial x + \partial v_y/\partial y = 0$ (1) x, y component of N-S

Define Stream Function, $\Psi(x, y)$:

$$\boxed{\begin{array}{l} v_x(x,y) = -\partial\Psi/\partial y \\ v_y(x,y) = +\partial\Psi/\partial x \end{array}}$$

Plug into continuity:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = \partial/\partial x\left(-\partial\Psi/\partial y\right) + \partial/\partial y\left(\partial\Psi/\partial x\right) = 0$$ — continuity automatically satisfied

Now we have

$$\left.\begin{array}{l} \Psi(x,y) \\ \mathscr{P}(x,y) \end{array}\right\}$$ 2 unknowns ; 2 eqns (N-S x & y)

In steady flow, the stream function doesn't change along the streamline. We can prove this by taking the substantial derivative:

$$\underbrace{\frac{D\Psi}{Dt}}_{\text{on 2-D, steady flow}} = v_x\frac{\partial \Psi}{\partial x} + v_y\frac{\partial \Psi}{\partial y} = -\frac{\partial\Psi}{\partial y}\frac{\partial\Psi}{\partial x} + \frac{\partial\Psi}{\partial x}\frac{\partial\Psi}{\partial y} = 0$$

For cylindrical coordinates:

$$v_r = -\frac{1}{r}\frac{\partial\Psi}{\partial\theta} \qquad v_\theta = \frac{\partial\Psi}{\partial r}$$

$$(v_z = 0)$$