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CHEM-ENG 321: FLUID MECHANICS

Accelerating Flow (Denn Chapter 9)

LEARNING OBJECTIVES

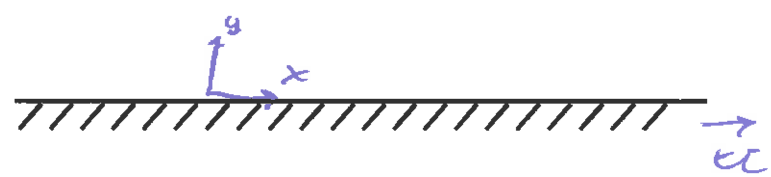
1. Derive and analyze detailed microscopic flow model for accelerating flow using Dimensional Analysis and the Similarity Solution
2. Derive and analyze detailed microscopic flow model for accelerating flow using the Integral Approximation method

(DVD - from pp. 612-621)
"impulsively started flow"

SCENARIO

Consider an infinite plate with fluid on top:

initially at rest, fluid on top
• plate set in motion at $t=0$, U



infinite plate

We seek a velocity field. $v_x(y,t)$; $v_y = v_z = 0$

DERIVING OUR MODEL

We know that $v_x(y,t)$, $v_y = 0 = v_z = 0$ just from our intuition and experience from before. (continuity equation satisfied)

We turn to the x component of our Navier-Stokes equations:

inertia → $$\rho\frac{\partial v_x}{\partial t} = \eta\frac{\partial^2 v_x}{\partial y^2}$$ ← viscous forces

Or:

$$\frac{\partial v_x}{\partial t} = \nu\frac{\partial^2 v_x}{\partial y^2}$$

(ν = kinematic viscosity = η/ρ [=] L^2/T
"momentum diffusivity")

same eq. governs transient 1-D diffusion for other transport

$\partial c/\partial t = D\,\partial^2 c/\partial x^2$

Fick's 2nd law

$\partial T/\partial t = \frac{k}{\rho c_p}\,\partial^2 T/\partial x^2$ ($\frac{k}{\rho c_p} = \alpha$)

Heat eq.

INITIAL CONDITIONS

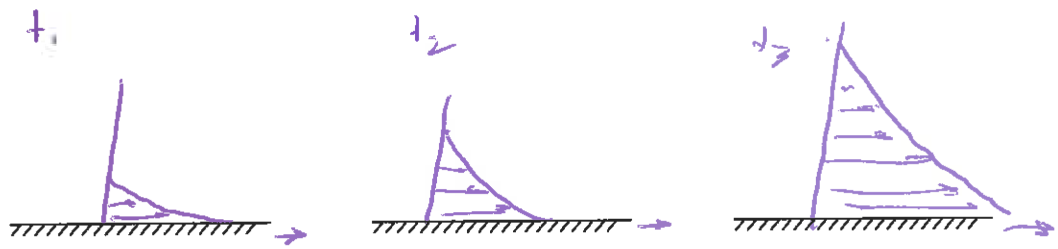
$v_x = 0$, $t = 0$ for all y

BOUNDARY CONDITIONS

$v_x = U$, $y = 0$, $t > 0$

$v_x = 0$, $y \to \infty$, $t > 0$

EXPECTED BEHAVIOR



How would we solve for $v_x(y, t)$? partial diff eq

→ can't solve by integration

DIMENSIONAL ANALYSIS AND SIMILARITY SOLUTION METHOD

Define $u(y, t) = v_x(y, t)/U$

$u(y, 0) = 0$

$u(0, t) = 1$

$u(\infty, 0) = 0$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

- u is dimensionless
- dimensional analysis tells us we u as function of other dimensionless groups

var	dimension
u	—
y	L
t	T
ν	L^2/T

only one way combine these

$u = f\left(\frac{y^2}{\nu t} \right)$ → solution must depend on a variable which combines y & t

Interpreting our dimensionless group we will see that

- $\sqrt{\nu t}$ = "diffusion distance" in time t
- $\frac{y}{\sqrt{\nu t}}$ - relative distance, distance scaled by diffusion distance

SIMILARITY SOLUTION

Velocity profile $u(y,t)$ looks similar at different times when plotted vs the relative distance $\frac{y}{\sqrt{\nu t}}$.

By convention, we use the similarity variable (ζ):

$$\zeta = \frac{y}{\sqrt{4\nu t}}$$

$du/d\zeta$ instead of du/dt, du/dy

We seek a solution to $u(\zeta)$ instead of $u(y,t)$. First we need to rewrite our differential equations and initial/boundary conditions in terms of ζ using chain rule:

First with respect to time t:

$$\frac{\partial u}{\partial t} = \frac{du}{d\zeta}\frac{\partial \zeta}{\partial t}, \qquad \frac{\partial \zeta}{\partial t} = -\frac{1}{2}\frac{y}{(4\nu t)^{3/2}} \cdot 4\nu$$

$$\Rightarrow \frac{\partial u}{\partial t} = \left[-\frac{2\nu y}{(4\nu t)^{3/2}}\right]\frac{du}{d\zeta}$$

Next with respect to distance from the plate y:

$$\frac{\partial u}{\partial y} = \frac{du}{d\zeta}\frac{\partial \zeta}{\partial y}, \qquad \frac{\partial \zeta}{\partial y} = \frac{1}{\sqrt{4\nu t}}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \left[\frac{1}{\sqrt{4\nu t}}\right]\frac{du}{d\zeta}$$

Now let's go back to our Navier-Stokes term:

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left[\frac{1}{\sqrt{4\nu t}}\frac{du}{d\zeta}\right] = \frac{1}{4\nu t}\,\partial/\partial y\left(\partial u/d\zeta\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\sqrt{4\nu t}}\,\frac{\partial}{\partial}\left(\frac{\partial u}{d\zeta}\right)\frac{\partial \zeta}{\partial y} = \frac{1}{4\nu t}\frac{\partial^2 u}{d\zeta^2}$$

Plug back into our Navier-Stokes Equation:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$\left[\frac{-2\nu y}{(4\nu t)^{3/2}} \right] \frac{du}{d\zeta} = \left[\frac{\cancel{\nu}}{4\nu t} \right] \frac{d^2 u}{d\zeta^2}$$

$$\left[\frac{-2\zeta}{\cancel{4\nu t}} \right] \frac{du}{d\zeta} = \left[\frac{1}{\cancel{4\nu t}} \right] \frac{d^2 u}{d\zeta^2}$$

$$\boxed{\frac{d^2 u}{d\zeta^2} = -2\zeta \frac{du}{d\zeta}}$$

ordinary differential equation

Now let's consider the boundary/initial conditions in terms of our similarity variable and relative velocity

- $t = 0 \Rightarrow \zeta = \infty$, $u = 0$
- $y = 0 \Rightarrow \zeta = 0$, $u = 1$
- $y = \infty \Rightarrow \zeta = \infty$, $u = 0$

I.C. and B.C. @ $y = \infty$ collaspe

At this stage, we have

$$\frac{d^2 u}{d\zeta^2} = -2\zeta \, du/d\zeta$$

$u(0) = 1$

$u(\infty) = 0$

Now let's try to solve it. Let $v = \frac{du}{d\zeta}$

$$\frac{dv}{d\zeta} = -2\zeta v$$

We can separate variables

$$\frac{dv}{v} = -2\zeta \, d\zeta$$

Then integrate

$$\ln v = -\zeta^2 + K$$

And exponentiate

$$v = C_1 e^{-\zeta^2}$$

Plug back in for v:

$$\frac{du}{d\zeta} = C_1 e^{-\zeta^2}$$

Separate variables; integrate using $u = 1$ at $\zeta = 0$

$$\int_1^{u(\zeta)} du = C_1 \int_0^{\zeta} e^{-s^2} ds$$

dummy variable

$$u(\zeta) - 1 = C_1 \int_0^{\zeta} e^{-s^2} ds$$

To determine C_1, use our other boundary condition $u(\zeta) = 0$ at $\zeta = \infty$

$$0 - 1 = C_1 \underbrace{\int_0^{\infty} e^{-s^2} ds}_{=\sqrt{\pi}/2} \quad \Rightarrow C_1 = -2/\sqrt{\pi}$$

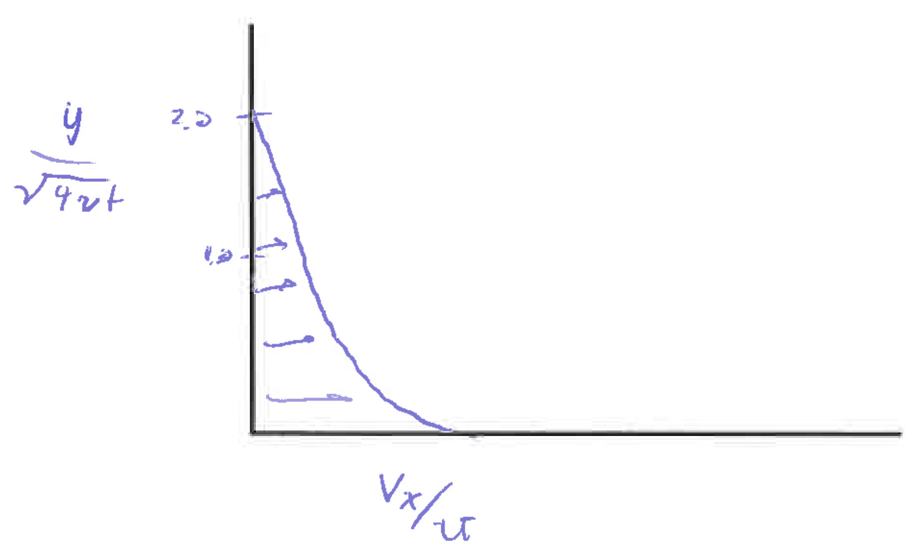
$$\boxed{u(\zeta) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\zeta} e^{-s^2} ds}$$

Define the <u>error function</u> (tabulated function like sine, cosine, tangent but more exotic)

$$\text{erf}(\zeta) = \frac{2}{\pi} \int_0^{\zeta} e^{-s^2} ds$$

$$u(\zeta) = 1 - \text{erf}\,\zeta$$

$$\boxed{v_x(y, t) = U\left[1 - \text{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right]}$$ (Denn 9.4)



Note that for $\frac{y}{\sqrt{4\nu t}} \gtrsim 2$, flow remains undisturbed

<u>Boundary Layer</u> – region close to solid surface where influence of viscosity is felt

Here, the boundary layer thickness, $\delta \approx 4\sqrt{\nu t}$

see $\sqrt{\nu t}$ dependence

→ characteristic of diffusive process
(see slot in 322, 323)

EXAMPLE NUMBERS

Boundary layer thickness after one minute

water $\nu = 0.01$ cm^2/s ⇒ $\delta = 4\sqrt{0.01 \cdot 60} = 3.1$ cm

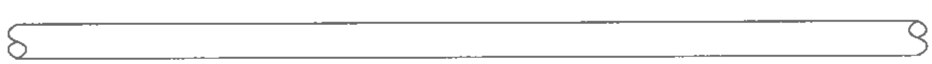
air $\nu = 0.15$ cm^2/s $\delta = 12$ cm

glycerine $\nu = 10$ cm^2/s → $\delta \approx 98$ cm

EXAMPLE PROBLEM

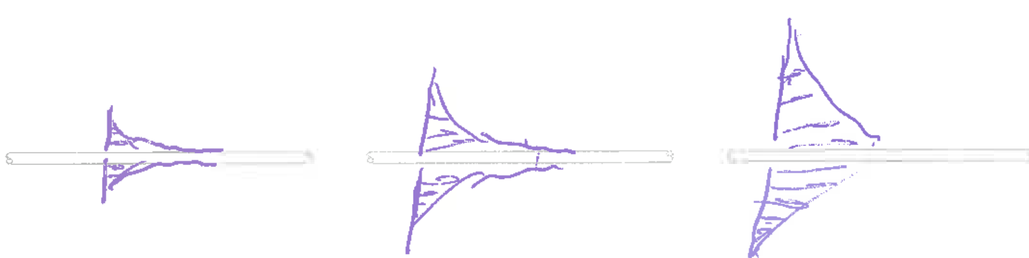
A long thin wire is surrounded by a viscous fluid that extends to infinity. At $t = 0$, the wire is set in motion along its own axis with velocity = V.

fluid extends to infinity



→ U

A. What would we expect the velocity profile to look like over time?



time →

B. Simplify the Navier-Stokes equations to obtain a partial differential equation governing the time-dependent velocity profile $v_z(r,t)$. Also, what are the initial and boundary conditions on the velocity field?

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial P}{\partial t}+\rho g_z+\eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]$$

$$\partial v_z/\partial t = \nu \, 1/r \, \partial/\partial r\,(r\, \partial v_z/\partial r)$$

$t=0$, $v_z=0$ at all r

$t>0$, $v_z(0,t) = U$

$t>0$, $v_z(\infty,t) = 0$

C. Introduce the appropriate similarity variable (ζ) that allows this problem to be turned into an ordinary differential equation with appropriate boundary conditions. Rewrite the partial differential equations in terms of this variable. Do not solve.

From DA argument

$$\zeta = r/\sqrt{\nu t} \qquad u = v_z/U$$

$$\partial u/\partial t = \nu \, 1/r \, \partial/\partial r\,(r\, \partial u/\partial r)$$

$$\frac{\partial\zeta}{\partial t} = -\tfrac{1}{2}\frac{r}{(\nu t)^{3/2}}\,\nu = -\frac{\nu r}{3(\nu t)^{3/2}}$$

$$\frac{\partial\zeta}{\partial r} = 1/\sqrt{\nu t}$$

Re-write the partial derivatives

$$\partial u/\partial t = \nu \, 1/r \, \partial/\partial r\,(r\, \partial u/\partial r)$$

$$du/\partial t) = \nu \, 1/r\left[\partial u/dr + r\, d^2u/dr\right]$$

$$= \nu \, 1/r\,(\partial u/dr) + \nu\,(d^2u/dr)$$

$$\frac{\partial u}{\partial t} = \partial u/\partial \xi \; \partial \xi/\partial t = du/\partial \xi \left(-1/2 \frac{r}{(\nu t)^{3/2}}\right)$$

$$\frac{\partial u}{\partial r} = \partial u/\partial \xi \; \partial \xi/\partial r = \frac{1}{\sqrt{\nu t}} \; \partial u/\partial \xi$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{1}{\nu t} \; \partial u/\partial \xi$$

plug back in

$$\left(-1/2 \frac{\nu r}{(\nu t)^{3/2}}\right) du/d\xi = \nu \left[1/r \; \partial u/\partial r + \partial^2 u/\partial r^2\right]$$

$$= \frac{1}{r\sqrt{\nu t}} \, du/d\xi + \frac{1}{\nu t} \, d^2u/d\xi^2$$

$$-1/2 \frac{\xi}{(\nu t)} \, du/\partial \xi = \frac{1}{r\sqrt{\nu t}} \, \partial u/d\xi + \frac{1}{\nu t} \, \partial^2 u/d\xi^2$$

$$\boxed{d^2u/d\xi^2 + \left(\frac{\xi}{2} + 1/\xi\right) du/d\xi = 0}$$

$$u(0) = 1$$

$$u(\infty) = 0$$