Pathwise Functional Itô Calculus and Applications to Path-Dependent Derivatives

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Karsten Seeger

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Prof. Dr. Francesca Biagini

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Declaration of Authenticity

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Karsten Seeger

Date

Abstract

The thesis studies pathwise counterparts to classical results within Itô Calculus and is based on Ananova and Cont [1]. It disentangles any probabilistic notions from the underlying process, reducing the mathematics therein to an exercise of *pure* analysis.

The first chapter provides a brief introduction to the inherent drawbacks of defining a stochastic integral in the Lebesgue-Stieltjes sense. A historical account of contributions and developments within Mathematical Finance as well as a literature review on the modern field of Pathwise Functional Itô Calculus are also rendered. Crucially, we introduce and prove the seminal contribution in this field, the existence of the Föllmer integral and, consequently, a *pathwise* change of variable formula for C^2 functions with respect to càdlàg paths is obtained [9].

Initially, the second chapter recapitulates key probabilistic notions that underpin Stochastic Calculus. From Dupire [6], a framework is obtained to appropriately undertake analysis on functionals with respect to the space Λ_T^d. Key notions of regularity for càdlàg paths are then proven. From Fournie [10], the result of the Föllmer integral and change of variable formula obtained in the first chapter is adapted to functionals in $\mathbb{C}_b^{1,2}(\Lambda_T^d)$. In particular, a pathwise integral of a gradient-type functional $\nabla_\omega F(\cdot, \omega)$ is established.

Using the change of variable formula in the second chapter, a pathwise isometry formula, analogous to the probabilistic Itô isometry, is obtained for multiple classes of partitions. Finally, an injective isometry $I_{\overline{\omega}}$ between $(\mathbb{V}_a(\overline{\omega}), \|\cdot\|_{L^2([0,T],a)})$ and $(\mathcal{H}_a(\overline{\omega}), \|\cdot\|_\pi)$ is established.

The fourth chapter proves that the integral in [10],

$$\int_0^T \nabla_\omega F(t, \omega_{t-}) d^\pi \omega,$$

is indeed well-defined as a *pathwise* integral.

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Chapter 1

The Pathwise Approach

> ”For banks, the only way to avoid a repetition of the current crisis is to measure and control all their risks, including the risk that their models give incorrect results. On the other hand, the surest way to repeat this disaster is to trust the models blindly while taking large-scale advantage of situations where they seem to provide trading strategies that would yield results too good to be true.”
>
> – *Steven Shreve, 2008, on the risks of classical model-dependent finance.*[1]

The main results and commentary from this chapter rely on Föllmer [9], Shreve [20] and Sondermann [21].

1.1 Introduction

A partition in $[0, T]$ is a sequence $(\pi_i)_{i=0,\dots,m}$ in $[0, T]$ such that

$$0 = \pi_0 < \dots < \pi_m = T.$$

Further, we associate a *mesh size* $|\pi| := \sup_{i=1,\dots,m} |\pi_i - \pi_{i-1}|$ to a given partition π as a measure of regularity. Now denote $\boldsymbol{\pi} := (\pi^n)_{n\in\mathbb{N}}$ as a sequence of partitions in $[0, T]$, i.e. as a countable collection of partitions. Throughout this paper, we assume that $\boldsymbol{\pi}$ is of vanishing mesh size, in the sense

$$|\pi^n| \xrightarrow{n\to\infty} 0.$$

Continuously differentiable paths exhibit a number of properties, including the Mean Value Theorem, that render the analysis of such functions, efficient and the results therein, far-reaching. This suitability to deliver satisfying analytical results, however, comes at the expense of adequately modelling financial market dynamics. Notwithstanding the allure of continuous differentiability, a simple glance at the price

[1] S.E. Shreve, *Don’t Blame the Quants*, Forbes, 2008.

chart of any traded financial instrument with substantial amounts of liquidity renders integration along a continuously differentiable path obsolete, e.g. Figure 1.1. Since the Brownian motion is a fundamental building block of price dynamics, we seek to define an integral of the form:

$$Z_t := \int_0^t X_s dB_s, \tag{1.1}$$

where X is adapted to the filtration generated by the Brownian motion B. It has been proven that the Brownian motion is not even differentiable at any point almost surely [7]. Note that Z_t is indeed a random variable on the probability space, hence the *path-by-path* construction of the Riemann-Stieltjes integral, see Definition A.1, may initially seem appropriate, in the sense:

$$Z(t) = \lim_{n\to\infty} \sum_{t_i^n \in \pi^n} X_{t\wedge t_i^n} \cdot (B_{t\wedge t_{i+1}^n} - B_{t\wedge t_i^n}). \tag{1.2}$$

We thus evaluate whether

$$\lim_{n\to\infty} \sum_{t_i^n \in \pi^n} |B(t_{i+1}^n) - B(t_i^n)| \tag{1.3}$$

exists almost surely. Recalling from Theorem A.2 that a Brownian motion is of non-trivial quadratic variation along any partition in $[0, T]$ almost surely, we assume that the limit above exists. By the almost sure continuity of Brownian motion paths, we obtain

$$\sum_{t_i^n \in \pi^n} |B(t_{i+1}^n) - B(t_i^n)|^2 \leq \max_{t_i^n \in \pi^n} |B(t_{i+1}^n) - B(t_i^n)| \sum_{t_i^n \in \pi^n} |B(t_{i+1}^n) - B(t_i^n)| \text{ a.s.}, \tag{1.4}$$

and further that

$$\max_{t_i^n \in \pi^n} |B(t_{i+1}^n) - B(t_i^n)| \xrightarrow{n\to\infty} 0 \text{ a.s.}$$

Since $|\pi^n| \xrightarrow{n\to\infty} 0$,

$$\lim_{n\to\infty} \sum_{t_i^n \in \pi^n} |B(t_{i+1}^n) - B(t_i^n)|^2 = 0 \text{ a.s. } .$$

This clearly contradicts the fact that a Brownian motion is of non-trivial quadratic variation. The claim immediately follows.

This result implies that Lebesgue-Stieltjes integration yields an unsatisfactory expression for integrators of a particular irregularity. Indeed, further generalizations of Brownian motion only amplify the problem of infinite variation. The field of *Stochastic Calculus* provides a solid theoretical framework under which integration with respect to a stochastic process of particular irregularity is well-defined, and the reader is invited to consult Karatzas and Shreve [15] for greater insight.

Considering Figure 1.1, asset paths can also be approximated by paths that contain *jumps*, i.e. points of discontinuity. Crucially, paths exhibiting jumps of a particular class are ideal to model price evolutions. In this case, a jump time t on a path ω can be approximated precisely from the right of t but not from the left. This construction underpins a fundamental idea in Mathematical Finance concerning the evolution of information: a market participant is only aware of an asset price realization at time t, at, or shortly after, time t. The reason underlying these jumps can be a myriad of exogeneous shocks, e.g. a liquidity squeeze. Paths exhibiting such jumps lend themselves naturally to model asset paths. We formalize this class of functions by introducing càdlàg paths.

Definition 1.1 (Càdlàg – *continue à droite, limite à gauche* – paths)**.** An arbitrary path $\omega : [0, T] \to \mathbb{R}^d$ is càdlàg if:

1. ω is right-continuous, i.e., $\omega(t+) := \lim_{s \downarrow t} \omega(s) = \omega(t)$.

2. ω has left limits, i.e., $\omega(t-) := \lim_{s \uparrow t} \omega(s)$ exists.

We denote the space of $\mathbb{R}^d$-valued càdlàg paths on $[0, T]$ as $D([0, T], \mathbb{R}^d)$. In Chapter 2, properties of càdlàg paths are discussed in detail.

1.2 Historical Account & Literature Review

Jarrow, Protter, et al. [14] and Oblój [17] form the basis of the historical account, while Dupire [6] aids the literature review.

Brief historical account of stochastic integration in mathematical finance.
At its inception in 1827, Brownian motion was first observed by Robert Brown in the passage of pollen while suspended in liquid. As a continuous-time process, Brownian motion was viewed as the fundamental dynamic underpinning the modelling concerning prices of risky assets. While Louis Bachelier, widely recognized as the founding father of Mathematical Finance, reconstructed dynamic price behaviour on the Paris Bourse as a Brownian motion from as early as 1900, Thorvald N. Thiele of Copenhagen actually preceded Bachelier's use of Brownian motion by some 20 years. Thiele, however, conceptualized Brownian motion on the basis of time series. Independent of Bachelier and Thiele, Einstein introduced Brownian motion in 1905 as the underlying dynamic of a particle when suspended in liquid: a novel notion that still persists as the introduction of Brownian motion to students in Probability Theory lectures. Wiener cemented the above notions by constructing a measure space, the *Wiener measure*, on the space of continuous functions such that the canonical projection process on C^0 follows the distribution of a Brownian motion under this measure. As mathematical techniques, particularly those in measure theory, advanced, coupled with the advent of Itô calculus, new emphasis was placed on analysing further generalizations of the

process. We recall the key properties of Brownian motion paths that exemplify the need for stochastic integration:

- non-trivial quadratic variation, and by extension,
- infinite variation on compact intervals.

Kiyosi Itô, the pioneer of stochastic integration, published his first paper, *Stochastic Integration*, in the field during 1944 [12], while the omnipresent *Itô's formula* of the form:

$$f(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d[X,X]_t, \tag{1.5}$$

was introduced in 1951 for C^2 functions in the paper *On a Formula Concerning Stochastic Differentials* [13]. An extended overview of stochastic integration clearly falls beyond the scope of this paper. Itô limited the space of integrands to those processes H that are adapted to the filtration generated by B to establish the L^2 isometry

$$\mathbb{E}\left[\left(\int_0^t H_s dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t H_s^2 ds\right].$$

In 1953, Doob [5] extended Itô's initial L^2 isometry,

$$\mathbb{E}\left[\left(\int_0^t H_s dM_s\right)^2\right] = \mathbb{E}\left[\int_0^t H_s^2 dF(s)\right],$$

where F is non-decreasing and deterministic and $M^2 - F$ is a martingale. In a Mathematical Finance setting, we may intuitively view the stochastic integral as a random variable, representing the value of a portfolio based on the asset trajectories of the securities held and traded. These essential underpinnings in stochastic integration were coupled with the groundbreaking paper from Black and Scholes [2] that developed the Black and Scholes formula for option pricing. Shortly thereafter followed the development of the Fundamental Theorem of Asset Pricing, and by extension the notion of no-arbitrage. Subsequently, Mathematical Finance has vastly broadened its scope of research, particularly as computational possibilities expand exponentially and gain increasing relevance in the industry, e.g. Buehler, Gonon, Teichmann and Wood [3].

The classical financial model, which enjoys great application in the industry, is based on a filtered probability space $(\Omega, \mathbb{F}_T, \mathcal{F}_T, P)$. Thereafter an underlying price dynamic of, for notational purposes, one risky asset S, e.g.:

$$dS(t) = \mu(t, S(t))dt + \sigma(t, S(t))dB(t),$$

is defined where B is the standard brownian motion, while μ and σ are locally deterministic functions. Obłój [17] notes three drawbacks of the classical framework:

- In practice, models are regularly re-calibrated to reflect current market prices, and in the process, contradict the initial idea of a theoretical model.
- We may assume some *unique* probabilistic dynamics on the trajectories of risky asset S. This approach inherently poses model risk.
- The model is assumed *frictionless* in the sense that transaction costs and counterparty risk, amongst other factors, are neglected.

The pathwise approach tends to be *robust* in the sense that it imposes no probabilistic dynamics. This modern field of Mathematical Finance receives a great deal of attention from academia and industry.

Literature review on pathwise integration.
Föllmer's seminal paper *Calcul d'Ito sans probabilités* [9] during the *Séminaire de probabilités de Strasbourg* in 1981 introduced a *trajectoire par trajectoire* Itô's formula for C^2 functions with respect to càdlàg paths of finite quadratic variation. This, consequently, conceived the *Föllmer integral*

$$\int_0^T F'(\omega_{s-})d^\pi\omega,$$

see Theorem 1.1. In 2009, Dupire [6] developed a framework to enable Functional Itô Calculus, including the introduction of the vertical and horizontal derivative, see Chapter 2. His work laid the foundation to extend regular Itô Calculus to functionals of Itô processes. Cont then removed implicit probabilistic assumptions inherent in Dupire's framework to deliver *pathwise* or *purely analytical* results. Over recent years, Pathwise Functional Itô Calculus has delivered pathwise analogues to classical results in model-dependent finance, e.g. [19] and [10].

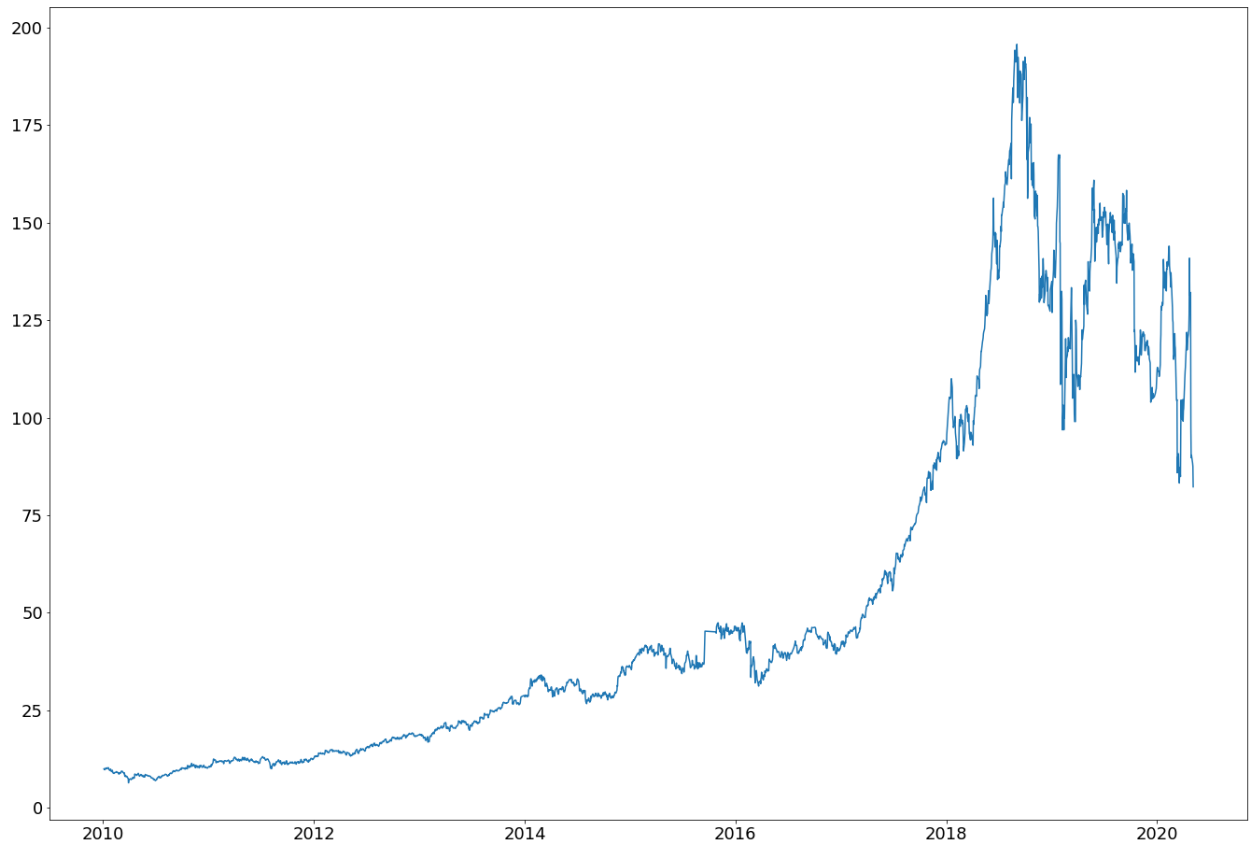

Figure 1.1: An exemplary stock price path of Wirecard AG, a German financial services provider, based on closing prices from Quandl.

Definition 1.2 (Radon measure [16])**.** Consider a topological space (E, τ) with the Borel σ-algebra $\mathcal{E} = \mathcal{B}(E) := \sigma(\tau)$ generated by some complete metric d. Any σ-finite measure μ on $(E, \mathcal{E})$ is said to be:

- inner regular if for $A \in \mathcal{E}$, $\mu(A) = \sup\{\mu(K) : K \subseteq A \text{ and } K \text{ compact}\}$.
- locally finite if for $x \in E$, there exists $U_x \in \tau$ with $x \in U_x$ and $\mu(U_x) < \infty$.

Crucially, a σ-finite measure μ that is inner regular and locally finite is defined as a *Radon measure*.

Remark. In the setting of the definition above, assume additionally that E is compact. Since μ is locally finite, we consider the collection of open sets $(U_e)_{e\in E}$ such that $e \in U_e$ and U_e is of finite measure for all $e \in E$. Therefore,

$$E \subseteq \bigcup_{e\in E} U_e,$$

and by the Heine-Borel property of compact sets, we obtain a finite subcover: $\exists N \in \mathbb{N}$ such that

$$E \subseteq \bigcup_{i=1}^{N} U_{e,i}.$$

By the monotonicity and subadditivity of measures,

$$\mu(E) \leq \mu\left(\bigcup_{i=1}^{N} U_{e,i}\right) \leq \sum_{i=1}^{N} \mu(U_{e,i}) < \infty, \tag{1.6}$$

thus μ is finite on $(E, \mathcal{E})$.

Definition 1.3 (Quadratic variation of a path along $\boldsymbol{\pi}$ [10])**.** A càdlàg path

$$\omega : [0,T] \to \mathbb{R}^d,$$

is said to be of *finite quadratic variation along* $\boldsymbol{\pi}$ if the sequence of measures

$$\mu_n := \sum_{t_i^n \in \pi^n} \left(\omega(t_{i+1}^n) - \omega(t_i^n)\right)^2 \delta_{t_i^n}, \quad n \in \mathbb{N} \tag{1.7}$$

converges weakly to a Radon measure μ on $[0,T]$ whose atomic part is null. The distribution function of μ is denoted as $[\omega]_\pi$ such that

$$\mu([0,t]) = [\omega]_\pi(t) = \lim_{n\to\infty} \sum_{\substack{t_i^n \in \pi^n \\ t_i^n \leq t}} (\omega(t_{i+1}^n) - \omega(t_i^n))^2 = [\omega]_\pi^c(t) + \sum_{0<s\leq t} |\Delta\omega(s)|^2 \tag{1.8}$$

where $[\omega]_\pi^c$ is a continuous non-decreasing function on $[0,T]$ and $|\Delta\omega(s)|$ denotes the absolute difference of the càdlàg function at a point and its left limit, i.e.: $|\omega(s) - \omega(s-)|$. In this case, we define $[\omega]_\pi$ as the *quadratic variation of* ω *along* $\boldsymbol{\pi}$ and $Q_\pi([0,T], \mathbb{R})$ as the set of càdlàg paths of *finite quadratic variation along* $\boldsymbol{\pi}$.

Remark. Since μ is locally finite by definition and $[0,T]$ is compact, μ is indeed a finite measure such that

$$[\omega]_\pi(T) = \lim_{n\to\infty} \sum_{t_i^n \in \pi^n} \left(\omega(t_{i+1}^n) - \omega(t_i^n)\right)^2 < \infty. \tag{1.9}$$

Recall $\mathcal{S}^+(d)$ as the set of positively definite, symmetric $d \times d$ matrices. We extend the definition of *quadratic variation along* $\boldsymbol{\pi}$ to its multidimensional counterpart.

Definition 1.4 (Quadratic variation of a multidimensional path along $\boldsymbol{\pi}$ [10])**.** Let ω be an $\mathbb{R}^d$-valued càdlàg path, $d \in \mathbb{N}$. ω is said to be of *finite quadratic variation along* $\boldsymbol{\pi}$ if, for any i and j with $1 \leq i, j \leq d$, both ω^i as well as $\omega^i + \omega^j$ have *finite quadratic variation along* $\boldsymbol{\pi}$ in terms of Definition 1.3. The *quadratic variation of* ω *along* $\boldsymbol{\pi}$, i.e. $[\omega]_\pi$, thus maps to $S^+(d)$ such that

$$t \mapsto [\omega]_\pi(t) := \lim_{n\to\infty} \sum_{\substack{t_i^n \in \pi^n \\ t_i^n \leq t}} \left((\omega(t_{i+1}^n) - \omega(t_i^n) \right)^{\,t} \left(\omega(t_{i+1}^n) - \omega(t_i^n) \right),$$

where each coordinate is defined:

$$[\omega]_\pi^{i,j}(t) = \begin{cases} [\omega^i]_\pi(t) & , i = j \\ \frac{1}{2}([\omega^i + \omega^j]_\pi(t) - [\omega^i]_\pi(t) - [\omega^j]_\pi(t)) & , i \neq j \end{cases}$$

Proposition 1.1 (Quadratic variation along $\boldsymbol{\pi}$ of a continuously differentiable path [20])**.** *A continuously differentiable function ω on $[0,T]$ has finite quadratic variation along $\boldsymbol{\pi}$, more precisely it has quadratic variation* $\mathbf{0}$.

Proof. By the continuity of the derivative ω', the Mean Value Theorem delivers for any $n \in \mathbb{N}$ and $t_i^n \in \pi^n$:

$$\omega(t_{i+1}^n) - \omega(t_i^n) = \omega'(t_{i,i+1}^n)(t_{i+1}^n - t_i^n),$$

where $t_{i,i+1}^n \in [t_i^n, t_{i+1}^n]$. Since $[0,T]$ is compact, we may bound ω':

$$\exists C > 0, \ \sup_{s\in[0,T]} |\omega'(s)| \leq C.$$

Then using the above,

$$\begin{aligned} \sum_{t_i^n \in \pi^n} \left(\omega(t_{i+1}^n) - \omega(t_i^n) \right)^2 &\leq \sum_{t_i^n \in \pi^n} \left(|\omega'(t_{i,i+1}^n)| \cdot |t_{i+1}^n - t_i^n| \right)^2 \leq C^2 \sum_{t_i^n \in \pi^n} |t_{i+1}^n - t_i^n|^2 \\ &\leq C^2 |\pi^n| \sum_{t_i^n \in \pi^n} |t_{i+1}^n - t_i^n| = C^2 |\pi^n| \sum_{t_i^n \in \pi^n} \left(t_{i+1}^n - t_i^n \right) = C^2 T |\pi^n| \end{aligned}$$

for all $n \in \mathbb{N}$. As $\boldsymbol{\pi}$ is of vanishing *mesh size*, it must follow

$$[\omega]_\pi(T) = \lim_{n\to\infty} \sum_{t_i^n \in \pi^n} \left(\omega(t_{i+1}^n) - \omega(t_i^n) \right)^2 = 0.$$

Since $[\omega]_\pi$ is a monotonically increasing, positive function, it follows that $[\omega]_\pi \equiv 0$. $\square$

1.3 The Föllmer Integral

Theorem 1.1 (Change of variable formula for C^2 functions: Föllmer integral [9], [21])**.** *For a path $\omega \in Q_\pi([0,T])$ and F as a function of class C^2, it follows that the Föllmer* *integral*

$$\int_0^t F'(\omega_{s-})d^\pi\omega := \lim_{n\to\infty} \sum_{\substack{t_i^n \le t \\ t_i^n \in \pi^n}} F'(\omega_{t_i^n})\left(\omega(t_{i+1}^n) - \omega(t_i^n)\right) \tag{1.10}$$

exists as the series converges absolutely, while the pathwise Itô formula holds:

$$\begin{aligned} F(\omega_t) - F(\omega_0) &= \int_0^t F'(\omega_{s-})d^\pi\omega + \frac{1}{2}\int_{]0,t]} F''(\omega_{s-})d[\omega]_\pi \\ &+ \sum_{0\le s\le t}\left(F(\omega_s) - F(\omega_{s-}) - F^{'}(\omega_{s-})\Delta\omega(s) - \frac{1}{2}F''(\omega_{s-})\Delta\omega_s^2\right). \end{aligned} \tag{1.11}$$

Proof. Let $t \in (0, T]$. From the right-continuity of path ω and the fact that $\boldsymbol{\pi}$ is of vanishing mesh size, the following holds

$$F(\omega_t) - F(\omega_0) = \lim_{n\to\infty} \sum_{\substack{t_i^n \le t \\ t_i^n \in \pi^n}} \left(F(\omega_{t_{i+1}^n}) - F(\omega_{t_i^n})\right). \tag{1.12}$$

For the purposes of this paper, we shall only prove the elementary case where ω is a continuous path. The continuity of ω annihilates the third term on the right-hand side in (1.11), i.e.:

$$\sum_{0\le s\le t}\left(F(\omega_s) - F(\omega_{s-}) - F'(\omega_{s-})\Delta\omega_s - \frac{1}{2}F''(\omega_{s-})\Delta\omega_s^2\right) = 0.$$

Since F is indeed of class C^2, we obtain for any $n \in \mathbb{N}$ fixed and $t_i^n \in \pi^n$ that the second order Taylor expansion of F at $\omega_{t_i^n}$ yields:

$$F(\omega_{t_{i+1}^n}) - F(\omega_{t_i^n}) = F'(\omega_{t_i^n})(\omega_{t_{i+1}^n} - \omega_{t_i^n}) + \frac{1}{2}F''(\omega_{t_i^n})(\omega_{t_{i+1}^n} - \omega_{t_i^n})^2 + r(\omega_{t_{i+1}^n}, \omega_{t_i^n}),$$

where r is defined by the bound

$$|r(a,b)| \le \varphi(|b-a|)(b-a)^2, \tag{1.13}$$

such that φ is an increasing function on $[0,\infty)$ and $\varphi(c) \xrightarrow{c\to 0} 0$.

A simple summation over the partition π^n with indices less than or equal to t yields:

$$\begin{aligned} \sum_{\substack{t_i^n \le t \\ t_i^n \in \pi^n}} \left(F(\omega_{t_{i+1}^n}) - F(\omega_{t_i^n})\right) &= \sum_{\substack{t_i^n \le t \\ t_i^n \in \pi^n}} F'(\omega_{t_i^n})(\omega_{t_{i+1}^n} - \omega_{t_i^n}) + \frac{1}{2}\sum_{\substack{t_i^n \le t \\ t_i^n \in \pi^n}} F''(\omega_{t_i^n})(\omega_{t_{i+1}^n} - \omega_{t_i^n})^2 \\ &+ \sum_{\substack{t_i^n \le t \\ t_i^n \in \pi^n}} r(\omega_{t_{i+1}^n}, \omega_{t_i^n}). \end{aligned} \tag{1.14}$$

Since ω is continuous and $[\omega]_\pi$ inherits its continuity from ω

$$\int_{[0,t]} F''(\omega_s)d[\omega]_\pi = \int_{(0,t]} F''(\omega_{s-})d[\omega]_\pi.$$

In (1.14), the second term on the right-hand side can be represented as an integral with respect to the discrete measure μ_n, namely

$$\frac{1}{2}\int_{[0,t]} F''(\omega_s)d\mu_n := \frac{1}{2}\sum_{\substack{t_i^n \le t \\ t_i^n \in \pi^n}} F''(\omega_{t_i^n})(\omega_{t_{i+1}^n} - \omega_{t_i^n})^2. \tag{1.15}$$

Since the sequence of measures μ_n converges weakly to μ

$$\mu_n := \sum_{t_i^n \in \pi^n} (\omega_{t_{i+1}^n} - \omega_{t_i^n})^2 \delta_{\omega_{t_i^n}} \xrightarrow{w} \mu,$$

it follows

$$\int_{[0,t]} F''(\omega_s)d\mu_n \xrightarrow{n\to\infty} \int_{[0,t]} F''(\omega_s)d\mu. \tag{1.16}$$

Once again note that $[\omega]_\pi(t) = \mu([0,t])$ and hence

$$\int_{[0,t]} F''(\omega_s)d\mu = \int_{[0,t]} F''(\omega_s)d[\omega]_\pi = \int_{(0,t]} F''(\omega_{s-})d[\omega]_\pi$$

in the Lebesgue-Stieltjes sense, see Definition A.1.
For the third term on the right-hand side in (1.14), the following bound is obtained by (1.13):

$$\sum_{\substack{t_i^n \le t \\ t_i^n \in \pi^n}} r(\omega_{t_i^n}, \omega_{t_{i+1}^n}) \le \varphi(\max_{\substack{t_i^n \le t \\ t_i^n \in \pi^n}} |\omega_{t_{i+1}^n} - \omega_{t_i^n}|) \sum_{\substack{t_i^n \le t \\ t_i^n \in \pi^n}} (\omega_{t_{i+1}^n} - \omega_{t_i^n})^2. \tag{1.17}$$

Furthermore, the right-hand side in (1.17) tends to 0 as $n \to \infty$ since $\boldsymbol{\pi}$ is of vanishing mesh, ω is continuous, ω is of finite quadratic variation along $\boldsymbol{\pi}$ and φ vanishes near 0, by assumption.

Finally, evaluating all results above, the limit

$$\lim_{n\to\infty} \sum_{t_i^n \le t} F'(\omega_{t_i^n})(\omega_{t_{i+1}^n} - \omega_{t_i^n}) \quad \text{exists.}$$

We define the *Föllmer Integral* as

$$\int_0^t F'(x_{s-})d[\omega]_\pi := \lim_{n\to\infty} \sum_{t_i^n \le t} F'(\omega_{t_i^n})(\omega_{t_{i+1}^n} - \omega_{t_i^n}).$$

Consequently, the pathwise Itô formula as claimed in (1.11) is satisfied. $\square$

Chapter 2

Pathwise Integration of Gradient Functionals

The main results from this section can be found in Fournie [10] as well as Cont, Bally and Caramellino [18].

2.1 Non-Anticipative Functionals

In Probability Theory, the mathematical analogue to *information* or *causality* is the notion of filtration. A natural model choice to render particular random settings *fair* in the sense that no additional information on the future evolution of the process is divulged, is to adopt the filtration based on the evolution of a stochastic process. An observer of the stochastic process at time t is thus unable to anticipate the trajectory of the process beyond t. At time t, all paths that are equivalent on $[0,t]$ deliver identical information. In essence, this describes the *non-anticipative* nature of the phenomena we shall investigate. Clearly, the canonical process X on $[0,T] \times D([0,T] \times \mathbb{R}^d)$

$$X(t,\omega) = \omega(t)$$

is non-anticipative with respect to the filtration

$$\mathbb{F} := (\mathcal{F}_t^X)_{t\in[0,T]} = (\sigma(X_s : 0 \leq s \leq t))_{t\in[0,T]} \,.$$

General functionals that only depend on the *stopped* path are deemed *non-anticipative*. In order to develop rigorous arguments on the underlying structure of processes that follow this notion, Dupire [6] constructed an analytical framework that enables sensible calculus in the space of non-anticipative functionals. Non-Anticipative Functional Calculus underpins the modelling of numerous path-dependent stochastic processes inherent in Mathematical Finance and other random phenomena. Some examples include:

- Erratic weather patterns, e.g. leading to poor crop harvest quality.

- Path-dependent options pricing.
- Credit risk associated with leveraged borrowers.

As an introduction, we examine the simple random walk as the discrete-time counterpart to Brownian motion, a fundamental process that underpins continuous-time finance outlined in Chapter 1.

Example. Consider the trajectory of a particle p on path ω_p as p it is suspended in some liquid from time 0 to T. We may *stop* the path at some time $t \in [0, T]$. Intuitively, at time t, an observer without any additional information is only informed on the trajectory of p on ω_p until t. In this sense, if any additional path ω_p' tracks ω_p until t, both paths render identical information for the observer at time t. Therefore, a non-anticipative functional on the paths ω_p and ω_p' until time t will be identical as, at time t, the functional only depends on the stopped paths $\omega_p(t \wedge \cdot)$ and $\omega_p'(t \wedge \cdot)$ that by assumption satisfy $\omega_p(t \wedge \cdot) \equiv \omega_p'(t \wedge \cdot)$, see Figure 2.1 below.

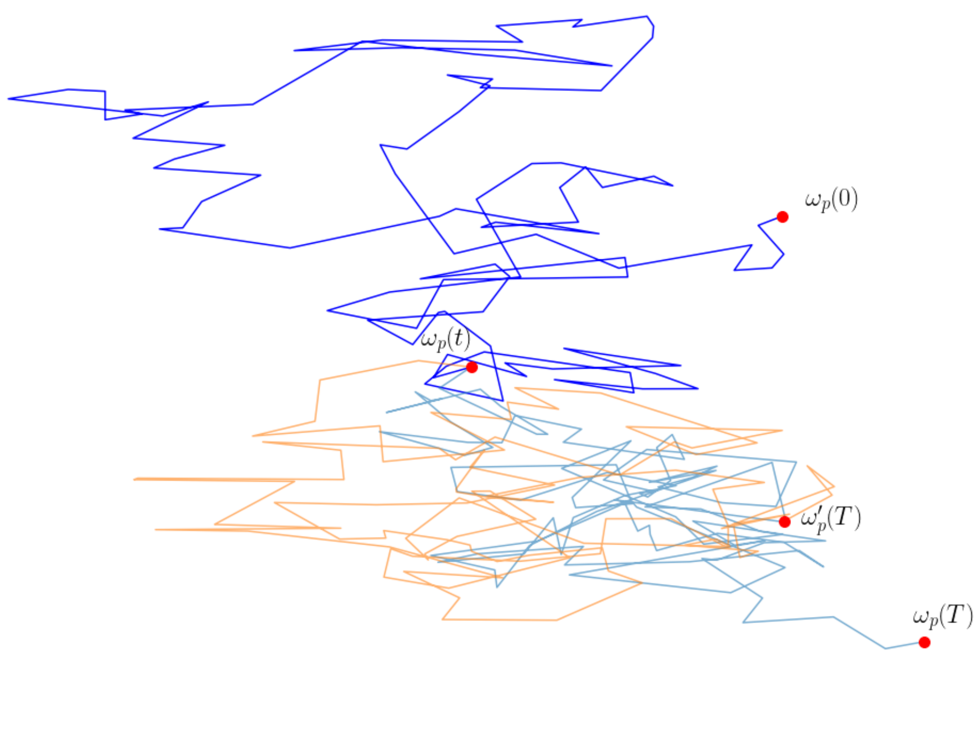

Figure 2.1: The paths of the particle p, ω_p and ω_p'.

2.2 The Mathematical Setting for Non-Anticipative Functionals

First and foremost, we need to appropriately define the space of paths that renders paths ω_p and ω_p', as above, identical at time t. We thus define the space of *stopped paths*, denoted Λ_T^d, as the set of equivalence classes

$$\Lambda_T^d := [0,T] \times D([0,T],\mathbb{R}^d) := \big([0,T] \times D([0,T],\mathbb{R}^d)\big) \,/ \sim$$

with respect to the equivalence relation

$$(t,\omega) \sim (t',\omega') \iff t = t' \text{ and } \omega(t \wedge \cdot) \equiv \omega'(t \wedge \cdot). \tag{2.1}$$

$\mathcal{W}_T$ denotes the subset of Λ_T^d containing only the continuous stopped paths:

$$\mathcal{W}_T := \{(t,\omega) \in \Lambda_T^d : \omega \in C^0([0,T],\mathbb{R}^d)\}.$$

To underscore the significance and utility in defining the space of *stopped paths* Λ_T^d as above, we plot multiple paths that at time t only represent one path or equivalence class in Λ_T^d.

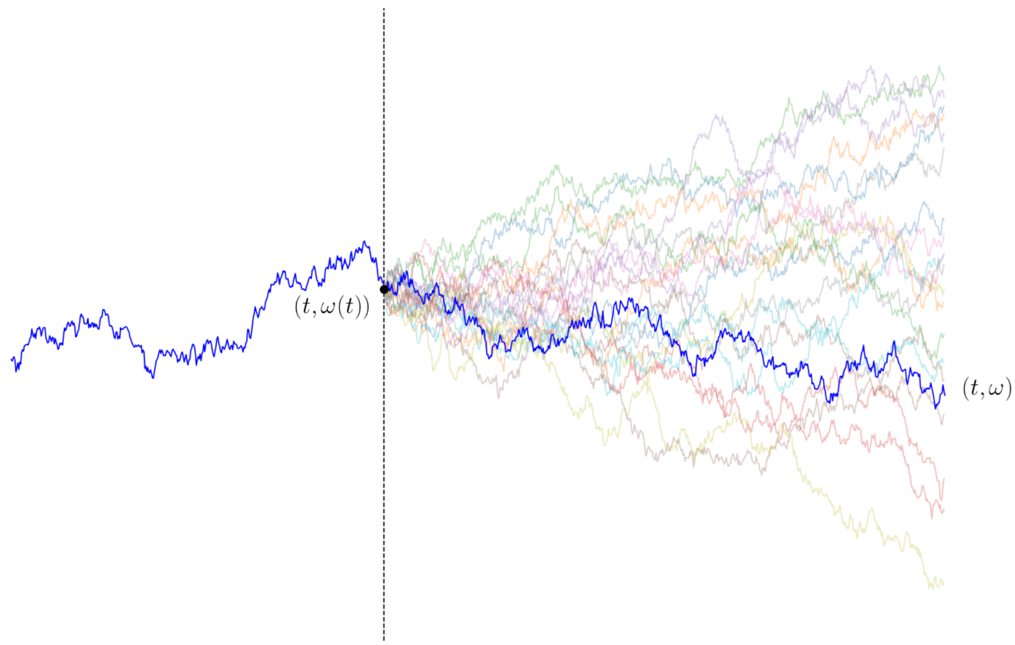

Figure 2.2: The equivalence class $(t, \omega) \in \Lambda_T^d$.

To separate distinct elements in Λ_T^d appropriately, we endow the space of stopped paths with the metric d_∞, defined as:

$$d_\infty((t, \omega), (t', \omega')) := \sup_{u \in [0,T]} |\omega(t \wedge u) - \omega'(t' \wedge u)| + |t - t'|.$$

In the context of this paper, we view the path (t, ω) as some underlying asset trajectory stopped at time t. We disentangle any probabilistic assumptions from the path, thus delivering purely *analytical* results. As previously stated, functionals of stopped paths are of particular interest, so functionals that for a given path $(t, \omega) \in [0, T] \times D([0, T], \mathbb{R}^d)$ depend only on the evolution of ω until t. Without defining an equivalence relation on $[0, T] \times D([0, T], \mathbb{R}^d)$, we formalize the idea of the *non-anticipative functional* as a map

$$F : [0, T] \times D([0, T], \mathbb{R}^d) \to \mathbb{R}$$

such that

$$\forall \omega \in \Omega, \quad F(t, \omega) = F(t, \omega_t), \tag{2.2}$$

where we use the notation

$$\omega_t(\cdot) := \omega(t \wedge \cdot). \tag{2.3}$$

Intuitively, a non-anticipative functional as in (2.2) lends itself to model the payoffs of an array of financial instruments.

Example. Consider a discrete-time market model $\{S_t^0, S_t^1\}_{t=0..T}$ on a discrete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with one risk-free asset S^0 and one risky asset S^1. Suppose we introduce some American contingent claim S^2 with payoff structure

$$S_t^2 = \begin{cases} 0, & t = 0 \\ \sum_{k=0}^{t-1} (S_k^1 - S_t^1)^+, & t \in \{1, ..., T\}. \end{cases}$$

It is clear that the payoff and subsequent pricing of S^2 at t is contingent on the evolution of S^1 *only* until t. We identify $S_t^2(\omega) = F(t, \omega)$, and note that for any other trajectory $\tilde{\omega}$ that follows ω until t, i.e.,

$$\tilde{\omega}(s) = \omega(s) \text{ for } s \leq t,$$

S^2 will attain the same payoff at t on $\tilde{\omega}$ as on ω. In particular, the stopped path ω_t is identical to ω until t and hence $S_t^2(\omega) = F(t, \omega_t)$. Therefore S^2 is amongst the class of non-anticipative functionals on $[0, T] \times D([0, T], \mathbb{R}^d)$.

The following analytical notions on the space of stopped paths will form the basis of our results.

Definition 2.1 (Horizontal extension of a path)**.** The horizontal extension of a path ω at t to h is a transformation $\omega_{t,h}$ where

$$\begin{aligned} \omega_{t,h} : [0, T] &\to \mathbb{R}^d \\ s &\mapsto \omega_{t,h}(s), \end{aligned}$$

such that

$$\omega_{t,h}(s) = \begin{cases} \omega(t), & \text{if } s \in (t, t+h] \\ \omega(s), & \text{otherwise} \end{cases}$$

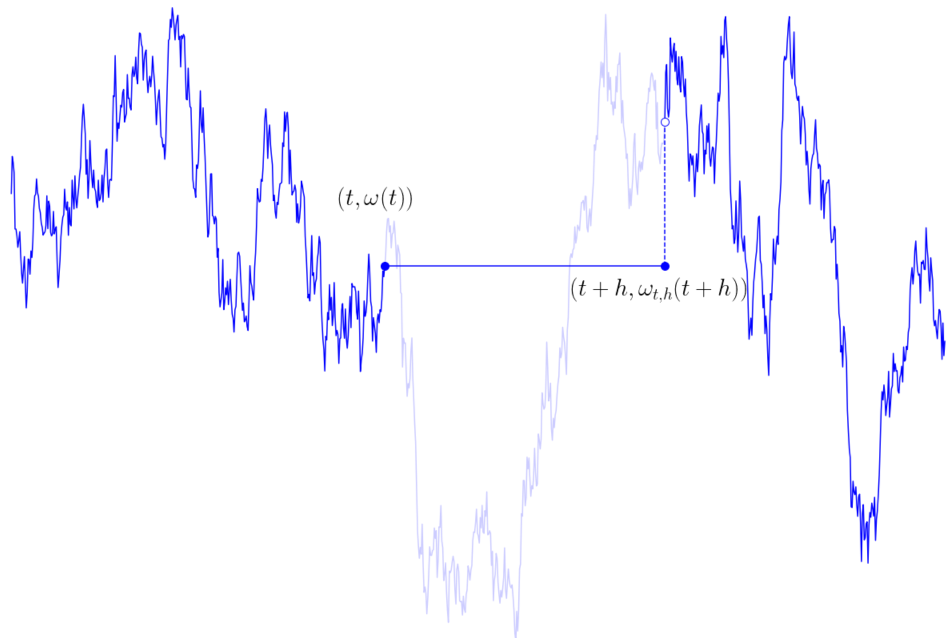

Figure 2.3: The horizontal extension of ω at t to h.

Definition 2.2 (Vertical perturbation of a path)**.** The *vertical perturbation* of ω by e at t is denoted by ω_t^e and is obtained by shifting the path ω by e at time t:

$$\omega_t^e(s) = \begin{cases} \omega(s), & s \in [0, t) \\ \omega(s) + e, & s \in [t, T] \end{cases}$$

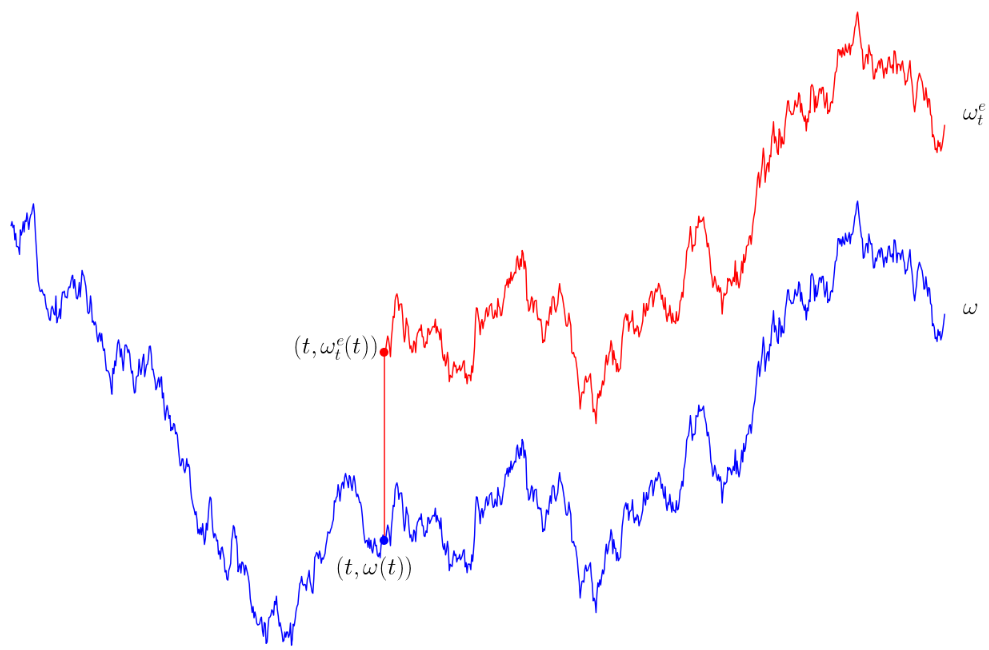

Figure 2.4: The path ω is shifted vertically by e at t to obtain ω_t^e.

Definition 2.3. A non-anticipative functional F is said to be:

- continuous *at fixed times* if for any $t \in [0, T]$:

$$F(t, \cdot) : (D([0, T], \mathbb{R}), \|\cdot\|_\infty) \to \mathbb{R} \text{ is continuous.}$$

  In other words, for a fixed $t \in [0, T]$ and an arbitrary path $\omega \in D([0, T], \|\cdot\|_\infty)$, $\forall \varepsilon > 0,\ \exists \eta > 0$ such that $\forall \omega' \in D([0, T], \mathbb{R})$ the following implication holds:

$$\|\omega_t - \omega'_t\|_\infty < \eta \Rightarrow |F(t, \omega) - F(t, \omega')| < \varepsilon,$$

  where

$$\|\omega_t - \omega'_t\|_\infty := \sup_{u \in [0,T]} |\omega_t(u) - \omega'_t(u)|.$$

- left-continuous if $\forall (t, \omega) \in \Lambda_T^d,\ \forall \varepsilon > 0,\ \exists \eta > 0$ so that $\forall (t', \omega') \in \Lambda_T^d$ and $t' < t$:

$$d_\infty((t, \omega), (t', \omega')) < \eta \Rightarrow |F(t, \omega) - F(t', \omega')| < \varepsilon.$$

Naturally denote $\mathbb{C}_l^{0,0}(\Lambda_T^d)$ as the space of left-continuous and $\mathbb{C}_r^{0,0}(\Lambda_T^d)$ as the space of right-continuous functionals on Λ_T^d.

- horizontally locally Lipschitz if

$$\forall \omega \in D([0,T],\mathbb{R}),\ \exists C > 0,\ \eta > 0,\ \forall s < t \leq T,\ \forall \omega' \in D([0,T],\mathbb{R}) :$$
$$\|\omega_t - \omega'_t\|_\infty < \eta \Rightarrow |F(s,\omega'_t) - F(t,\omega'_t)| < C|t-s|$$

Non-anticipative functionals of particular interest are those that preserve boundedness. The property of *boundedness preservation* imposes a notion of local regularity on the functional F. We formally define this notion:

Definition 2.4. A non-anticipative functional F is said to be *boundedness-preserving* if for every compact subset K of $\mathbb{R}^d$, $\forall t_0 \in [0,T],\ \exists C_{K,t_0} > 0$ such that,

$$\forall t \in [0,t_0],\ (t,\omega) \in \Lambda_T^d,\ \omega([0,t]) \subseteq K \Rightarrow |F(t,\omega)| < C_{K,t_0},$$

and subsequently denote $\mathbb{B}(\Lambda_T^d)$ as the space of *boundedness-preserving* functionals.

Remark. A boundedness-preserving functional is clearly locally bounded in terms of paths:

$$\forall \omega \in D([0,T],\mathbb{R}),\ \exists C > 0,\ \eta > 0,\ \forall \omega' \in D([0,T],\mathbb{R}) :$$

$$\|\omega' - \omega\|_\infty < \eta \Rightarrow \forall t \in [0,T] :\ |F(t,\omega) - F(t,\omega')| < C.$$

Changes in the space of stopped paths Λ_T^d can be decomposed into a *horizontal* or *time* component and a *vertical* or *path* component. We thus construct a differential calculus on this space by introducing the *horizontal derivative* $\mathcal{D}F$ as well as the *vertical derivative* $\nabla_\omega F$ of F.

Definition 2.5. A non-anticipative functional F is said to be:

- horizontally differentiable at $(t,\omega) \in \Lambda_T^d$ if

$$\mathcal{D}F(t,\omega) = \lim_{h \downarrow 0} \frac{F(t+h,\omega_t) - F(t,\omega_t)}{h} \tag{2.4}$$

  exists. If $\mathcal{D}F(t,\omega)$ exists for every $(t,\omega) \in \Lambda_T^d$, then (2.4) defines a non-anticipative functional $\mathcal{D}F$ known as the *horizontal derivative* of F.

- vertically differentiable at $(t,\omega) \in \Lambda_T^d$ if the map:

$$\begin{aligned} g_{(t,\omega)} :\ & \mathbb{R}^d \to \mathbb{R} \\ & e \mapsto F(t,\omega_t + e\mathbf{1}_{[\mathbf{t},\mathbf{T}]}) \end{aligned}$$

is differentiable at 0. In the case of vertical differentiability, the gradient at 0 is the *Dupire derivative* or *vertical derivative* of F at (t, ω) and is denoted $\nabla_\omega F(t, \omega)$ where

$$\nabla_\omega F(t, \omega) := \nabla g_{(t,\omega)}(0) \in \mathbb{R}^d.$$

Intuitively, this leads to the equivalent expression

$$\nabla_\omega F(t, \omega) = \begin{pmatrix} \partial_1 F(t, \omega) \\ \vdots \\ \partial_d F(t, \omega) \end{pmatrix}$$

such that

$$\partial_i F(t, \omega) := \lim_{h \to 0} \frac{F(t, \omega_t + h e_i \mathbf{1}_{[t,T]}) - F(t, \omega_t)}{h}$$

where $\{e_i, i = 1, ..., d\}$ is the canonical basis in $\mathbb{R}^d$. If F is vertically differentiable at all $(t, \omega) \in \Lambda_T^d$, then $\nabla_\omega F : \Lambda_T^d \to \mathbb{R}^d$ defines a non-anticipative functional, the *vertical derivative* of F. We thus proceed inductively to define $\nabla_\omega^n F$ for any $n \in \mathbb{N}$.

In an application to a financial markets setting, $\nabla_\omega F(\cdot, \omega)$ evaluates sensitivities on an underlying asset path ω with respect to a price functional F along $[0, T]$. Therefore, the vertical derivative, as defined above, plays a key role in evaluating hedging strategies against an adverse price evolution.

Remark. If F is horizontally differentiable, then in general:

$$\lim_{h \to 0} \frac{F(t + h, \omega_t) - F(t, \omega_t)}{h} = \mathcal{D}F(t, \omega) \neq \partial_t F(t, \omega) = \lim_{h \to 0} \frac{F(t + h, \omega) - F(t, \omega)}{h}.$$

since $\partial_t F(t, \omega)$, as the partial derivative in t, describes the change in F from an infinitesimal increment in t with respect to **path** ω, while $\mathcal{D}F(t, \omega)$ observes the change in F from an increment in t with respect to the **stopped path** ω_t.

Definition 2.6. We define $\mathbb{C}_b^{1,k}(\Lambda_T^d)$ as the set of non-anticipative functionals of the form $F : (\Lambda_T^d, d_\infty) \to \mathbb{R}$ which are:

- horizontally differentiable and $\mathcal{D}F$ continuous at fixed times.
- k-times vertically differentiable and $\nabla_\omega^j F \in \mathbb{C}_l^{0,0}(\Lambda_T^d)$ for $j = 0, ..., k$.
- $\mathcal{D}F, \nabla_\omega F, ..., \nabla_\omega^k F \in \mathbb{B}(\Lambda_T^d)$.

Definition 2.7. $F \in \mathbb{C}_b^{0,0}(\Lambda_T^d)$ is locally regular if an increasing sequence of stopping times $(\tau_k)_{k \in \mathbb{N}}$ exists such that $\tau_0 = 0$ and $\tau_k \uparrow \infty$ while $F^k \in \mathbb{C}_b^{1,2}(\Lambda_T^d)$ so that

$$F(t, \omega) = \sum_{k \geq 0} F^k(t, \omega) \mathbf{1}_{[\tau_k(\omega), \tau_{k+1}(\omega))}(t).$$

Denote the space of such functionals as $\mathbb{C}_{\text{loc}}^{1,2}(\Lambda_T^d)$.

It is clear that $\mathbb{C}^{1,2}_{\text{loc}}(\Lambda^d_T)$ is a generalization of $\mathbb{C}^{1,2}_b(\Lambda^d_T)$, i.e.,

$$\mathbb{C}^{1,2}_b(\Lambda^d_T) \subset \mathbb{C}^{1,2}_{\text{loc}}(\Lambda^d_T)$$

as it allows discontinuities at times described by $(\tau_k)_{k\in\mathbb{N}}$.

2.3 Some Results on Càdlàg Paths

Lemma 2.1 (Jumps of a càdlàg path [11])**.** *An $\mathbb{R}$-valued càdlàg path ω has at most countably many jumps.*

Proof. Define

$$J_{\omega,\frac{1}{n}} := \{t \in [0,T] : |\Delta\omega(t)| > \frac{1}{n}\} \tag{2.5}$$

for any fixed $n \in \mathbb{N}$. We assume some $n \in \mathbb{N}$ exists so that $J_{\omega,\frac{1}{n}}$ contains infinitely many jump times. By the Axiom of Choice, we may construct a sequence of distinct jump times $(t_l)_{l\in\mathbb{N}} \subseteq J_{\omega,\frac{1}{n}}$, i.e.:

$$|\Delta\omega(t_l)| > \frac{1}{n} \;\; \forall l \in \mathbb{N} \text{ and } t_j \neq t_k \;\; \forall j \neq k \tag{2.6}$$

By compactness of $[0,T]$ and the Bolzano-Weierstraß Theorem, it follows that a convergent subsequence $(t_l)_{l\in\mathbb{N}}$ exists with its limit denoted

$$t := \lim_{l\to\infty} t_l$$

and $(t_{l_k})_{k\in\mathbb{N}}$ as the monotone subsequence converging to t.

Without loss of generality, let $(t_{l_k})_{k\in\mathbb{N}}$ be a monotonically increasing sequence to t. Since ω is càdlàg,

$$\forall k \in \mathbb{N} : \omega(t_{l_k}-) \text{ exists.} \tag{2.7}$$

For any particular k, consider the interval $(t_{l_{k-1}}, t_{l_k})$ that is non-empty by choice of sequence in (2.6). From (2.7), there exists some u^l_k in every interval $(t_{l_{k-1}}, t_{l_k})$ such that

$$|\omega(u^l_k) - \omega(t_{l_k})| \geq \frac{1}{n}. \tag{2.8}$$

The sequence $(u^l_k)_{k\in\mathbb{N}}$ also converges to t, and given the existence of the left-limit of ω at t, we obtain

$$\omega(u^l_k) \xrightarrow{k\to\infty} \omega(t-) \xleftarrow{k\to\infty} \omega(t_{l_k}).$$

Using assumption (2.8), clearly $\omega(t-)$ does not exist which is a contradiction to fact that $\omega \in D([0,T],\mathbb{R})$. We may thus assume $J_{\omega,\frac{1}{n}}$ contains only finitely many elements. The set of jump times J_ω can be constructed in the following sense:

$$J_\omega = \bigcup_{n\in\mathbb{N}} J_{\omega,\frac{1}{n}}$$

so that J_ω remains countable as a countable union of sets of finite cardinality. □

By virtue of Lemma 2.1, we may assume that the sequence of partitions exhausts jump times on the interval $[0,T]$, i.e.:

$$\sup_{t\in[0,T]-\pi^n} |\omega(t) - \omega(t-)| \xrightarrow{n\to\infty} 0 \tag{2.9}$$

where $[0,T]-\pi^n := [0,T]\backslash\pi^n$. In other words, we may construct a sequence of partitions $\boldsymbol{\pi}$ *of vanishing mesh size* that will tend to contain all discontinuity points of ω on $[0,T]$. For example, consider the sequence of partitions $\tilde{\boldsymbol{\pi}}$ of vanishing mesh size, we define the finite set $J_{\omega,\frac{1}{n}}$ as in (2.5) and realize that the sequence of partitions $\boldsymbol{\pi}$, defined as

$$\pi^n := \tilde{\pi}^n \cup J_{\omega,\frac{1}{n}},\ n \in \mathbb{N},$$

is of vanishing mesh size. Moreover, as $n\to\infty$, $\boldsymbol{\pi}$ tends to contain all jumps.

Lemma 2.2. [4] *For any càdlàg path* $\omega : [0,T] \to \mathbb{R}$

$$\forall\varepsilon > 0,\ \exists\eta > 0,\ |u-v| \leq \eta \Rightarrow |\omega(u) - \omega(v)| \leq \epsilon + \sup_{t\in[u,v]} |\Delta\omega(t)| \tag{2.10}$$

Proof. Assume that the implication does indeed not hold, i.e.: there exists some $\varepsilon > 0$ such that we can construct a sequence $(u_n, v_n)_{n\in\mathbb{N}} \subseteq [0,T]\times[0,T]$ satisfying

$$u_n \leq v_n \ \text{ and } \ v_n - u_n \xrightarrow{n\to\infty} 0, \tag{2.11}$$

while

$$|\omega(u_n) - \omega(v_n)| > \varepsilon + \sup_{t\in[u_n,v_n]} |\Delta\omega(t)|. \tag{2.12}$$

We can extract a sequence $(u_n)_{n\in\mathbb{N}}$ from the original sequence $(u_n,v_n)_{n\in\mathbb{N}}$, noting that $(u_n)_{n\in\mathbb{N}}$ is a sequence on the compact interval $[0,T]$. By the Bolzano-Weierstraß property, we construct a convergent subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$, denoting u as the limit of $(u_{n_k})_{k\in\mathbb{N}}$, i.e.

$$u := \lim_{k\to\infty} u_{n_k},$$

and thus by (2.11),

$$\lim_{k\to\infty} v_{n_k} = u.$$

Note that an infinite number of elements of the sequence $(u_{n_k}, v_{n_k})_{k\in\mathbb{N}}$ are either below or above u with respect to each component. We can further construct a monotone subsequence $(u_{n_{k_l}}, v_{n_{k_l}})_{l\in\mathbb{N}}$ that converges to u in each component that we shall denote as $(x_l, y_l)_{l\in\mathbb{N}}$ for notation purposes.

Case 1. Both $(x_l)_{l\in\mathbb{N}}$ and $(y_l)_{l\in\mathbb{N}}$ converge either from below or from above to u.

Since ω is a càdlàg path and x_l, y_l both either $\uparrow u$, or $\downarrow u$, as $l \to \infty$, it follows:

$$|\omega(x_l) - \omega(y_l)| \xrightarrow{l\to\infty} 0,$$

which contradicts (2.12).

Case 2. Without loss of generality $(x_l)_{l\in\mathbb{N}}$ converges from below and $(y_l)_{l\in\mathbb{N}}$ converges from above to u, i.e. $x_l \uparrow u$ and $y_l \downarrow u$ as $l \to \infty$.

First note that

$$\sup_{t\in[x_l,y_l]} |\Delta\omega(t)| \geq |\Delta\omega(u)|,$$

but

$$|\omega(x_l) - \omega(y_l)| \xrightarrow{l\to\infty} |\Delta\omega(u)|.$$

In (2.12), this leads to the inequality as $l \to \infty$:

$$|\Delta\omega(u)| > \varepsilon + |\Delta\omega(u)|,$$

yielding a contradiction.

Therefore (2.10) holds in both cases. $\square$

This proves to be a powerful lemma in establishing the regularity of convergence for piecewise constant approximations of càdlàg paths.

Proposition 2.1. *For $\omega \in D([0,T],\mathbb{R})$, the piecewise constant approximation*

$$\omega^n(t) = \sum_{i=0}^{k_n-1} \omega(t^n_{i+1}-)\mathbf{1}_{[t^n_i, t^n_{i+1})}(t) + \omega(T)\mathbf{1}_{\{T\}}(t) \tag{2.13}$$

converges uniformly to ω.

Proof. Let $\varepsilon > 0$. Choose $\eta > 0$ as in Lemma 2.2 such that for all $|u - v| < \eta$:

$$|\omega(u) - \omega(v)| < \frac{\varepsilon}{3} + \sup_{t\in[u,v]} |\Delta\omega(t)|. \tag{2.14}$$

Due to the vanishing mesh size of the partitions and (2.9), we select an $N \in \mathbb{N}$ large enough such that

$$|\pi^n| < \eta \text{ and } \sup_{t\in[0,T]-\pi^n} |\omega(t) - \omega(t-)| < \frac{\epsilon}{3} \text{ for every } n \geq N. \tag{2.15}$$

Now fix $n \geq N$ and $t \in [0,T]$ arbitrary, but more specifically $t \in [t_i^n, t_{i+1}^n]$. From the right-continuity of ω we select $s \in (t, t_{i+1}^n)$ close enough to t so that

$$|\omega(t) - \omega(s)| < \frac{\varepsilon}{3}. \tag{2.16}$$

Note that

$$\sup_{u\in[s,t_{i+1}^n)} |\Delta\omega(u)| \leq \sup_{u\in[0,T]-\pi^n} |\Delta\omega(u)| \tag{2.17}$$

Consequently,

$$\begin{aligned} |\omega^n(t) - \omega(t)| &= |\omega(t_{i+1}^n-) - \omega(t)| \leq |\omega(t_{i+1}^n-) - \omega(s)| + |\omega(s) - \omega(t)| \\ &< \frac{\varepsilon}{3} + \sup_{u\in[s,t_i^n)} \Delta\omega(u) + \frac{\varepsilon}{3} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This proves uniform convergence. □

2.4 Change of Variable Formula for Functionals on the Space of Stopped Paths

Lemma 2.3 (Pathwise regularity [10]). *If $F \in \mathbb{C}_l^{0,0}(\Lambda_T^d)$, for any $\omega \in D([0,T],\mathbb{R})$, the path $t \mapsto F(t,\omega_{t-})$ is left-continuous.*

Proof. We approximate (t,ω_{t-}) from below through some sequence $(t_s,\omega_{t_s})_{s\in\mathbb{N}} \subseteq \Lambda_T^d$. Trivially, since ω is càdlàg we know $\omega(t-)$ exists for any $t \in [0,T]$, and particularly

$$\lim_{s\to\infty} \omega(t_s) = \omega(t-) \Rightarrow \|\omega_{t_s} - \omega_{t-}\|_\infty \xrightarrow{s\to\infty} 0.$$

Since $F \in \mathbb{C}_l^{0,0}(\Lambda_T^d)$ and from the above, it follows

$$\lim_{s\to\infty} F(t_s,\omega_{t_s}) = F(t,\omega_{t-}).$$

Since $t \in [0,T]$ was arbitrary, it follows that the path $t \mapsto F(t,\omega_{t-})$ is left-continuous. □

Recalling the definition of the approximation in (2.13), it follows

$$\omega^n(t_i^n) = \omega(t_{i+1}^n-) \text{ and } \omega^n(t_i^n-) = \omega(t_i^n-).$$

We now seek to approximate the càdlàg path ω via horizontal and vertical increments along the partition π^n by defining

$$\omega_{t_i^n}^{n,\Delta\omega(t_i^n)} := \omega_{t_i^n} + \Delta\omega(t_i^n)\mathbf{1}_{[t_i^n,T]} \text{ such that } \omega_{t_i^n-}^{n,\Delta\omega(t_i^n)}(t_i^n) = \omega(t_i^n) \tag{2.18}$$

Theorem 2.1 (Pathwise change of variable formula for $\mathbb{C}_b^{1,2}$ functionals [10])**.** *Let $\omega \in Q_\pi([0,T],\mathbb{R}^d)$, then for any $F \in \mathbb{C}_b^{1,2}(\Lambda_T^d)$, the limit*

$$\int_0^T \nabla_\omega F(t,\omega_t) d^\pi\omega := \lim_{n\to\infty} \sum_{i=0}^{k(n)-1} F\left(t_i^n, \omega_{t_i^n-}^{n,\Delta\omega(t_i^n)}\right) \cdot (\omega(t_{i+1}^n) - \omega(t_i^n)), \tag{2.19}$$

exists and we obtain the pathwise Itô formula

$$\begin{aligned} F(T,\omega_T) - F(0,\omega_0) &= \int_0^T \mathcal{D}F(t,\omega_t)dt + \int_0^T \frac{1}{2}\operatorname{tr}({}^t\nabla_\omega^2 F(t,\omega_{t-})d[\omega]_\pi^c(t)) \\ &+ \int_0^T \nabla_\omega F(t,\omega_{t-})d^\pi\omega + \sum_{t\in(0,T]} (F(t,\omega_t) - F(t,\omega_{t-}) - \nabla_\omega F(t,\omega_{t-})\cdot\Delta\omega(t)). \end{aligned}$$

Proof. For the purposes of this paper, we restrict our efforts to the case where ω is a continuous path and further assume that $\omega \in C^0([0,T],U)$ where $U \subset \mathbb{R}$ open. Denote $\delta\omega_i^n := \omega(t_{i+1}^n) - \omega(t_i^n)$ and define

$$\eta_n := \sup\{|\omega(u) - \omega(t_i^n)| + |t_{i+1}^n - t_i^n| \ : \ 0 \le i \le k(n)-1, \ u \in [t_i^n, t_{i+1}^n)\}.$$

Lemma 2.2 implies that $\eta_n \xrightarrow{n\to\infty} 0$ such that for n sufficiently large, $\exists C > 0$, $\forall \omega' \in D([0,T],\mathbb{R})$:

$$\sup_{u\in[0,T]} \|\omega - \omega'\|_\infty < \eta_n \Rightarrow \ \forall t \in [0,T] : \ \max\{|\mathcal{D}F(\omega',t)|, |\nabla_\omega^2 F(\omega',t)|\} \le C, \tag{2.20}$$

by the local boundedness of $\mathcal{D}F$ and $\nabla_\omega^2 F$. Next define $K = \overline{\{\omega(u) : 0 \le u \le T\}}$ as a compact subset of U, thus $U^C = \mathbb{R} - U$. By compactness of K in U, we may find ε sufficiently small and n sufficiently large such that $\operatorname{dist}(K, U^C) > \varepsilon + \eta_n$, see Proposition A.1. It thus follows $B_{\eta_n}(\omega(u)) \subset U$ for all $u \in [0,T]$. Note that in the space of stopped paths Λ_T^d, we have

$$(t_{i+1}^n, \omega_{t_{i+1}^n-}^n) \sim (t_{i+1}^n, \omega_{t_i^n, h_i^n}^n)$$

with $h_i^n := t_{i+1}^n - t_i^n$. For any $i \le k_n - 1$, we decompose:

$$F(t_{i+1}^n, \omega_{t_{i+1}^n-}^n) - F(t_i^n, \omega_{t_i^n-}^n) = \overbrace{F(t_{i+1}^n, \omega_{t_{i+1}^n-}^n) - F(t_i^n, \omega_{t_i^n}^n)}^{(1)} + \underbrace{F(t_i^n, \omega_{t_i^n}^n) - F(t_i^n, \omega_{t_i^n-}^n)}_{(2)}. \tag{2.21}$$

Considering (1), define a function ψ on $[0,T]$ as

$$\psi(u) = F(t_i^n + u, \omega_{t_i^n,u}^n).$$

Since $F \in \mathbb{C}_b^{1,2}([0,T],\mathbb{R})$, F is indeed horizontally differentiable such that ψ must be right-differentiable. Furthermore from Lemma 2.3 and (2.20) we know that for $t > t_i^n$:

$$t \mapsto F(t, \omega_{t-}^n) = F(t_i^n + (t - t_i^n), \omega_{t_i^n, t-t_i^n}^n) = \psi(t - t_i^n) \tag{2.22}$$

is left-continuous. The Fundamental Theorem of Calculus yields:

$$F(t_{i+1}^n, \omega_{t_{i+1}^n-}^n) - F(t_i^n, \omega_{t_i^n}^n) = F(t_{i+1}^n, \omega_{t_i^n, h_i^n}^n) - F(t_i^n, \omega_{t_i^n}^n) = \psi(h_i^n) - \psi(0)$$

$$= \int_0^{t_{i+1}^n - t_i^n} \mathcal{D}F(t_i^n + u, \omega_{t_i^n,u}^n) du.$$

The term (2) of (2.21) can be expressed as $\phi(\delta\omega_i^n) - \phi(0)$ where

$$\phi(u) = F(t_i^n, \omega_{t_i^n-}^{n,u}).$$

As $F \in \mathbb{C}_b^{1,2}(\Lambda_T^d)$, ϕ is well-defined and C^2 on the convex set $B(\omega(t_i^n), \eta_n) \subset U$, such that:

$$\phi'(u) = \nabla_\omega F(t_i^n, \omega_{t_i^n-}^{n,u}) \tag{2.23}$$

$$\phi''(u) = \nabla_\omega^2 F(t_i^n, \omega_{t_i^n-}^{n,u}). \tag{2.24}$$

As in the proof of Theorem 1.1, we consider the second order Taylor expansion of ϕ about $u = 0$. First note that

$$F(t_i^n, \omega_{t_i^n}^n) = F(t_i^n, \omega_{t_i^n-}^{n,\delta\omega_i^n}), \tag{2.25}$$

since

$$(t_i^n, \omega_{t_i^n}^n) \sim (t_i^n, \omega_{t_i^n-}^{n,\delta\omega_i^n}),$$

and thus

$$\begin{aligned} F(t_i^n, \omega_{t_i^n}^n) - F(t_i^n, \omega_{t_i^n-}^n) &= \nabla_\omega F(t_i^n, \omega_{t_i^n-}^n)\delta\omega_i^n \\ &+ \frac{1}{2}\operatorname{tr}(\nabla_\omega^2 F(t_i^n, \omega_{t_i^n-}^n)\ {}^t\delta\omega_i^n \delta\omega_i^n) + r_i^n, \end{aligned} \tag{2.26}$$

where r_i^n is bounded by

$$L|\delta\omega_i^n| \sup_{x\in B(\omega(t_i^n),\eta_n)} |\nabla_\omega^2 F(t_i^n, \omega_{t_i^n-}^{n,x-\omega(t_i^n)}) - \nabla_\omega^2 F(t_i^n, \omega_{t_i^n-}^n)|, \tag{2.27}$$

for some $L > 0$. Denote the index $i^n(t)$ such that $t \in [t_{i^n(t)}^n, t_{i^n(t)+1}^n)$. Summing up all of the terms in (2.26) over $i = 0, ..., k_n - 1$:

- The left-hand side of (2.21) yields $F(T, \omega_{T-}^n) - F(0, \omega_0)$. Since $F \in \mathbb{C}_l^{0,0}(\Lambda_T^d)$ and ω^n converges uniformly to ω from Proposition 2.1, we obtain:

$$|F(T, \omega_{T-}^n) - F(T, \omega_{T-})| \xrightarrow{n\to\infty} 0, \tag{2.28}$$

such that

$$F(T, \omega_{T-}^n) - F(0, \omega_0) \xrightarrow{n\to\infty} F(T, \omega_{T-}) - F(0, \omega_0). \tag{2.29}$$

Moreover, since ω is continuous, $F(T, \omega_{T-}) = F(T, \omega_T)$.

- For (1) on the right-hand side in (2.21),

$$\sum_{i=0}^{k_n-1} \Big(F(t_{i+1}^n, \omega_{t_{i+1}^n-}^n) - F(t_i^n, \omega_{t_i^n}^n) \Big) = \int_0^T \mathcal{D}F(u, \omega_{t_{i^n(u)}^n, u-t_{i^n(u)}^n}^n) du.$$

For any $u \in [0, T]$ and for $v \in [0, u]$, we have:

$$|\omega_{t_{i^n(u)}^n, u-t_{i^n(u)}^n}^n(v) - \omega_u(v)| = \begin{cases} |\omega^n(t_{i^n(u)}^n) - \omega_u(v)| & \text{if } v \in [t_{i^n(u)}^n, u] \\ |\omega^n(v) - \omega_u(v)| & \text{if } v \in [0, t_{i^n(u)}^n) \end{cases} \tag{2.30}$$

In the case $v \in [0, t_{i^n(u)}^n)$, it follows from Proposition 2.1 that

$$\sup_{v\in[0,t_{i^n(u)})} |\omega^n(v) - \omega_u(v)| \xrightarrow{n\to\infty} 0.$$

If $v \in [t_{i^n(u)}, u]$, since ω is uniformly continuous on $[t_{i^n(u)}, u]$ and $\boldsymbol{\pi}$ is of vanishing mesh size, we obtain

$$\sup_{v\in[t_{i^n(u)},u]} |\omega^n(t_{i^n(u)}^n) - \omega_u(v)| = \sup_{v\in[t_{i^n(u)},u]} |\omega(t_{i^n(u)+1}^n) - \omega_u(v)| \xrightarrow{n\to\infty} 0.$$

Summarizing both cases above,

$$\sup_{v\in[0,u]} |\omega_{t_{i^n(u)}^n, u-t_{i^n(u)}^n}^n(v) - \omega_u(v)| \xrightarrow{n\to\infty} 0. \tag{2.31}$$

By assumption, $\mathcal{D}F$ is continuous at fixed times so that the uniform convergence of $\omega^n_{t^n_{i^n(u)}}$ to ω_u on $[0,u]$ for every $u \in [0,T]$ delivers

$$\mathcal{D}F(u, \omega^n_{t^n_{i^n(u)}, u - t^n_{i^n(u)}}) \xrightarrow{n\to\infty} \mathcal{D}F(u, \omega_u). \tag{2.32}$$

Moreover, we choose $N \in \mathbb{N}$ large enough such that

$$\|\omega^m_{t^m_{i^m(u)}, u - t^m_{i^m(u)}} - \omega\|_\infty \leq \eta_n, \tag{2.33}$$

for all $m \geq N$. Thus by local boundedness described in (2.20), $\exists \tilde{C} > 0$

$$\mathcal{D}F(u, \omega^n_{t^n_{i^n(u)}, u - t^n_{i^n(u)}}) \leq \tilde{C} \text{ for all } n \in \mathbb{N}.$$

Using the Dominated Convergence Theorem, see Theorem A.1, yields:

$$\lim_{n\to\infty} \int_0^T \mathcal{D}F(u, \omega^n_{t^n_{i^n(u)}, u - t^n_{i^n(u)}}) du = \int_0^T \mathcal{D}F(u, \omega_u) du.$$

- The second line can be written as:

$$\begin{aligned} &\sum_{i=0}^{k_n-1} \nabla_\omega F(t^n_i, \omega^n_{t^n_i -})(\omega(t^n_{i+1}) - \omega(t^n_i)) \\ &+ \sum_{i=0}^{k_n-1} \frac{1}{2} \operatorname{tr}(\nabla^2_\omega F(t^n_i, \omega^n_{t^n_i -})\ {}^t\delta\omega^n_i \delta\omega^n_i) + \sum_{i=0}^{k_n-1} r^n_i. \end{aligned} \tag{2.34}$$

Furthermore, by (2.31), for any $u \in [0,T]$

$$\left(\nabla^2_\omega F(t^n_{i^n(u)}, \omega^n_{t^n_{i^n(u)}-})\right) \mathbf{1}_{(t^n_{i^n(u)}, t^n_{i^n(u)+1}]}$$

is bounded by some $\overline{C} > 0$, and

$$\nabla^2_\omega F(t^n_{i^n(u)}, \omega^n_{t^n_{i^n(u)}-}) \xrightarrow{n\to\infty} \nabla^2_\omega F(u, \omega_u)$$

by the left-continuity of $\nabla^2_\omega F$. From Lemma 2.3, the paths

$$t \mapsto \nabla^2_\omega F(t, \omega_t)$$

as well as

$$t \mapsto \sum_{i=0}^{k_n-1} \nabla^2_\omega F(t^n_i, \omega^n_{t^n_i -}) \mathbf{1}_{(t^n_i, t^n_{i+1}]}(t)$$

are left-continuous. We may thus apply Lemma B.2 to obtain the limit

$$\int_0^T \frac{1}{2}\,\mathrm{tr}({}^t\nabla_\omega^2 F(u,\omega_u)d[\omega]_\pi(u)).$$

Note that by definition of the remainder term r_i^n in (2.27),

$$|r_i^n| \le \varepsilon_i^n |\delta\omega_i^n|^2 \tag{2.35}$$

where $\varepsilon_i^n \xrightarrow{n\to\infty} 0$ and $\varepsilon_i^n \le 2C$ from the local boundedness of $\nabla_\omega^2 F$. Since $\omega \in Q_\pi([0,T],\mathbb{R})$ and applying Lemma B.2 once again,

$$\sum_{i=0}^{k_n-1} \varepsilon_i^n |\delta\omega_i^n|^2 \xrightarrow{n\to\infty} 0,$$

then using (2.35)

$$\sum_{i=i^n(0)+1}^{i^n(T)-1} r_i^n \xrightarrow{n\to\infty} 0.$$

In summary, all other terms converge such that the limit

$$\int_0^T \nabla_\omega F(t,\omega)d^\pi\omega := \lim_{n\to\infty} \sum_{t_i^n\in\pi^n} \nabla_\omega F(t_i^n,\omega_{t_i^n-}^n)(\omega(t_{i+1}^n)-\omega(t_i^n))$$

exists. This proves the theorem. $\square$

Chapter 3

Isometry Property of the Pathwise Integral

3.1 Pathwise Isometry Formula

The main results and definitions are developed in Ananova and Cont [1].

Our aim is to provide sufficient conditions under which the identity

$$[F(\cdot,\omega.)]_\pi(t) = \int_0^t \langle \nabla_\omega F(s,\omega_{s-})\ {}^t\Delta_\omega F(s,\omega_{s-}), d[\omega]_\pi(s)\rangle \tag{3.1}$$

holds, where the scalar product $\langle \cdot,\cdot\rangle$ denotes $\langle A,B\rangle := \mathrm{tr}(AB)$ for square matrices A, B. To guide the regularity of the functional, we impose the following assumptions:

Assumption 1. *Uniform Lipschitz continuity.*

$$\exists K>0,\ \ \forall \omega,\omega' \in D([0,T],\mathbb{R}^d),\ \forall t\in[0,T],\ \ |F(t,\omega)-F(t,\omega')| \le K\|\omega_t-\omega'_t\|_\infty.$$

We denote $\mathrm{Lip}(\Lambda^d_T, \|\cdot\|_\infty)$ as the space of all functionals that satisfy this property. Consider a nested sequence of partitions $\pi^n := \{t^n_i :\ i=0,...,m(n)\}$ of $[0,T],\ n\in\mathbb{N}$. We define the oscillation

$$\mathrm{osc}(f,\pi^n) := \max_{j=0..m(n)-1} \sup_{t\in(t^n_j,t^n_{j+1}]} |f(t)-f(t^n_j)|.$$

We assume ω is of vanishing oscillation along $\boldsymbol{\pi}$:

Assumption 2.

$$\mathrm{osc}(\omega,\pi^n) \xrightarrow{n\to\infty} 0$$

Assumption 3.

$$\max_{i=0..m(n)-1} |F(t^n_{i+1},\omega)-F(t^n_i,\omega)| \xrightarrow{n\to\infty} 0$$

Lemma 3.1. [1] *A Lipschitz-continuous functional $F : (\Lambda_T^d, d_\infty) \to \mathbb{R}$ satisfies Assumption 2 and Assumption 3 for continuous paths ω along a sequence of partitions of vanishing mesh.*

Proof. Note that ω is uniformly continuous on $[0,T]$. Let $\boldsymbol{\pi}$ be the usual sequence of partitions of vanishing mesh and $\varepsilon > 0$ arbitrary. There exists $\delta_\varepsilon > 0$ so that

$$|t-s| < \delta_\varepsilon \implies \|\omega(t) - \omega(s)\| < \varepsilon.$$

Since the mesh of $\boldsymbol{\pi}$ vanishes, there exists some $N \in \mathbb{N}$ so that for $n \geq N$: $|\pi^n| \leq \min\{\varepsilon, \delta_\varepsilon\}$ and uniform continuity delivers

$$\operatorname{osc}(\omega, \pi^n) = \max_{j=0..m(n)-1} \sup_{t \in (t_j^n, t_{j+1}^n]} |\omega(t) - \omega(t_j^n)| < \varepsilon,$$

and thus Assumption 2 holds. In addition,

$$\begin{aligned} \forall n \geq N: \ d_\infty((t_{i+1}^n, \omega), (t_i^n, \omega)) &= |t_{i+1}^n - t_i^n| + \|\omega_{t_{i+1}^n} - \omega_{t_i^n}\|_\infty \\ &\leq |\pi^n| + \sup_{t \in [t_i^n, t_{i+1}^n]} |\omega(t) - \omega(t_i^n)| \leq 2\varepsilon. \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, it is clear that $d_\infty((t_{i+1}^n, \omega), (t_i^n, \omega)) \xrightarrow{n\to\infty} 0$. The assumed Lipschitz-continuity of F then implies

$$\max_{i=0..m(n)-1} |F(t_{i+1}^n, \omega) - F(t_i^n, \omega)| \xrightarrow{n\to\infty} 0,$$

so Assumption 3 also holds. □

For $0 < \nu < 1$, denote the space of ν-Hölder continuous functions as $C^\nu([0,T], \mathbb{R}^d)$ such that

$$C^\nu([0,T], \mathbb{R}^d) := \left\{ f \in C^0([0,T], \mathbb{R}^d) \;\middle|\; \begin{array}{l} \exists \gamma > 0, \forall (t,s) \in [0,T]^2 : \\ |f(t) - f(s)| \leq C|t-s|^\nu \end{array} \right\} \tag{3.2}$$

and the left limit of ν-Hölder continuous functions as

$$C^{\nu-}([0,T], \mathbb{R}^d) = \bigcap_{0 \leq \alpha < \nu} C^\alpha([0,T], \mathbb{R}^d)$$

as the space of functions that are α-Hölder for any $\alpha < \nu$. Note that the Hölder condition as described in (3.2) can be equivalently stated as

$$\sup_{\substack{(t,s) \in [0,T]^2 \\ t \neq s}} \frac{\|f(t) - f(s)\|}{|t-s|^\nu} < \infty.$$

Furthermore, the α-Hölder condition on $[0,T]$ implies uniform continuity but not necessarily Lipschitz continuity in the following sense:

$$\operatorname{Lip}([0,T], \mathbb{R}^d) \subset C^\alpha([0,T], \mathbb{R}^d) \subset C^0([0,T], \mathbb{R}^d).$$

Proposition 3.1. *If $\omega \in C^\nu([0,T],\mathbb{R}^d)$, the piecewise constant approximation:*

$$\omega^n := \sum_{i=0}^{m(n)-1} \omega(t_{i+1}^n -)\mathbf{1}_{[t_i^n, t_{i+1}^n)} + \omega(T)\mathbf{1}_T$$

implies

$$\|\omega - \omega^n\|_\infty \le C|\pi^n|^\nu$$

where

$$C := \sup_{\substack{(t,s)\in[0,T]^2 \\ t\neq s}} \frac{\|\omega(t)-\omega(s)\|}{|t-s|^\nu} < \infty.$$

Proof. Fix $n \in \mathbb{N}$, and for some arbitrary $t_i^n \in \pi^n$, consider the interval $[t_i^n, t_{i+1}^n) \subset [0,T]$. We evaluate:

$$\sup_{t\in[t_i^n,t_{i+1}^n)} \|\omega(t) - \omega^n(t)\| = \sup_{t\in[t_i^n,t_{i+1}^n)} \|\omega(t) - \omega(t_{i+1}^n -)\|$$
$$\le \sup_{(s,t)\in[t_i^n,t_{i+1}^n)^2} \|\omega(t)-\omega(s)\| \le C|t_{i+1}^n - t_i^n|^\nu \le C|\pi^n|^\nu.$$

Given that the interval $[t_i^n, t_{i+1}^n)$ was arbitrary,

$$\|\omega - \omega^n\|_\infty \le C|\pi^n|^\nu.$$

□

We introduce the main result of this chapter:

Theorem 3.1 (Pathwise isometry formula [1])**.** *Let $\boldsymbol{\pi}$ be the usual sequence of partitions of $[0,T]$, and $\omega \in Q_\pi([0,T],\mathbb{R}^d) \cap C^\nu([0,T],\mathbb{R}^d)$ such that*

$$\nu > \frac{\sqrt{3}-1}{2}$$

to satisfy Assumption 2.

Let $F \in \mathbb{C}_b^{1,2}(\Lambda_T^d) \cap \mathrm{Lip}(\Lambda_T^d, \|\cdot\|_\infty)$ with $\nabla_\omega F \in \mathbb{C}_b^{1,1}(\Lambda_T^d)$ such that Assumption 3 holds. Then

$$[F(\cdot,\omega.)]_\pi(T) = \left[\int_0^{\cdot} \nabla_\omega F(s,\omega_s) d^\pi \omega\right]_\pi (T) = \int_0^T \langle \nabla_\omega F(s,\omega_s)\, {}^t\nabla_\omega F(s,\omega_s), d[\omega]_\pi\rangle. \tag{3.3}$$

Some preparations need to be made in order to prove the above Theorem.

Lemma 3.2. [1] *Let $\omega \in C^{\nu}([0,T],\mathbb{R}^d)$ for some $\nu \in (\frac{1}{3},\frac{1}{2}]$ and $F \in \mathbb{C}^{1,2}(\Lambda_T^d,\mathbb{R}^n)$ so that $\nabla_\omega F \in C_b^{1,1}(\Lambda_T^d,\mathbb{R}^{n\times d})$ and $F \in \mathrm{Lip}(\Lambda_T^d,\|\cdot\|_\infty)$. Define*

$$R^F_{t,s}(\omega) := F(s,\omega_s) - F(t,\omega_t) - \nabla_\omega F(t,\omega_t)(\omega(s)-\omega(t)), \tag{3.4}$$

then there exists a constant $C_{F,T,\|\omega\|_\nu} > 0$ increasing in T and $\|\omega\|_\nu$, such that

$$|R^F_{t,s}(\omega)| \leq C_{F,T,\|\omega\|_\nu}|s-t|^{\nu(1+\nu)}.$$

Proof. We will only treat the case of a real-valued F, i.e. $n=1$, but the extension to the general case is merely an exercise of notation. Note that by [18], Proposition 5.26, it follows for any continuous path $\lambda : [t,s] \to \mathbb{R}$ of bounded variation and $G \in \mathbb{C}_b^{1,1}(\Lambda_T^d) \subset \mathbb{C}_{\mathrm{loc}}^{1,1}(\Lambda_T^d)$ that

$$G(s,\lambda_s) - G(t,\lambda_t) = \int_t^s \mathcal{D}G(u,\lambda_u)du + \int_t^s \nabla_\omega G(u,\lambda_u)d\lambda(u) \tag{3.5}$$

where the integrals are defined in the Riemann-Stieltjes sense. Therefore fix a Lipschitz-continuous path $\lambda : [0,T] \to \mathbb{R}^d$, and using (3.5) repeatedly for $G = F, \nabla_\omega F, \cdots$, we obtain an integral formula for $R^F_{t,s}(\lambda)$ in terms of the derivatives of F. For the sake of convenience, we denote $\partial_i F$ and λ^i as the respective i-th coordinates of $\nabla_\omega F$ and λ. For brevity of notation, we also use Einstein's convention of summation in repeated indices and the additional notation:

$$\delta\lambda_{t,s} := \lambda(s) - \lambda(t).$$

By (3.5) for $G = F$, it follows

$$R^F_{t,s}(\lambda) = \int_t^s \mathcal{D}F(u,\lambda_u)du + \int_t^s \left(\partial_i F(u,\lambda_u) - \partial_i F(t,\lambda_t)\right) d\lambda^i(u). \tag{3.6}$$

Note that the functionals F, $\partial_i F$, $\mathcal{D}F$ and $\mathcal{D}\partial_i F$ are boundedness-preserving and measurable such that Fubini's Theorem may be applied below. For the second term on the right-hand side of the identity, we use (3.5) where $G = \partial_i F$ and then Fubini's theorem to deliver

$$
\begin{aligned}
&\int_t^s \left(\partial_i F(u,\lambda_u) - \partial_i F(t,\lambda_t)\right) d\lambda^i(u) \\
&= \int_t^s \left(\int_t^u \mathcal{D}\partial_i F(r,\lambda_r) dr + \int_t^u \partial^2_{ij} F(r,\lambda_r) d\lambda^j(r) \right) d\lambda^i(u) \\
&= \int_t^s \int_t^u \mathcal{D}\partial_i F(r,\lambda_r) dr d\lambda^i(u) + \int_t^s \int_t^u \partial^2_{ij} F(r,\lambda_r) d\lambda^j(r) d\lambda^i(u) \qquad (3.7)\\
&= \int_t^s \int_r^s \mathcal{D}\partial_i (r,\lambda_r) d\lambda^i(u) dr + \int_t^s \int_r^s \partial^2_{ij} F(r,\lambda_r) d\lambda^i(u) d\lambda^j(r) \\
&= \int_t^s \mathcal{D}\partial_i F(r,\lambda_r) \left(\lambda^i(s) - \lambda^i(r)\right) dr + \int_t^s \partial^2_{ij} F(r,\lambda_r) \lambda^j(r) \left(\lambda^i(s) - \lambda^i(r)\right).
\end{aligned}
$$

In summary, we obtain

$$
R^F_{t,s}(\lambda) = \int_t^s \left(\mathcal{D}F(u,\lambda_u) + \mathcal{D}\partial_i F(u,\lambda_u)\delta\lambda^i_{u,s}\right) du + \int_t^s \partial^2_{ij} F(r,\lambda_r)\delta\lambda^i_{r,s} d\lambda^j(r). \qquad (3.8)
$$

To use the above formula to estimate the error term $R^F_{t,s}(\omega)$ for Hölder continuous paths, we will use piece linear approximations. So consider $\omega \in C^\nu([0,T],\mathbb{R}^d)$, then ω^N is a piecewise linear approximation of ω on $[t,s]$ in the following form:

$$
\begin{aligned}
\omega^{N,i}(u) := \omega^i(u)\mathbf{1}_{[0,t)}(u) + \sum_{k=0}^{N} \left(\omega^i(\tau^N_k) + \frac{\delta\omega^i_{\tau^N_k,\tau^N_{k+1}}}{\tau^N_{k+1} - \tau^N_k}(u - \tau^N_k) \right) \mathbf{1}_{[\tau^N_k,\tau^N_{k+1})}(u) \\
+ \omega^i(u)\mathbf{1}_{[s,T]}(u)
\end{aligned} \qquad (3.9)
$$

where

$$
\tau^N_k = t + k\frac{s-t}{N} \text{ for every } k \in \{0,...,N\} \text{ and } i \in \{1,...,d\}.
$$

By construction, we obtain

- $\omega^N(r) = \omega(r),\ \forall r \in [0,t]$,
- $\omega^N(\tau^N_k) = \omega(\tau^N_k)$,
- ω^N is linear on every interval $[\tau^N_k, \tau^N_{k+1}]$.

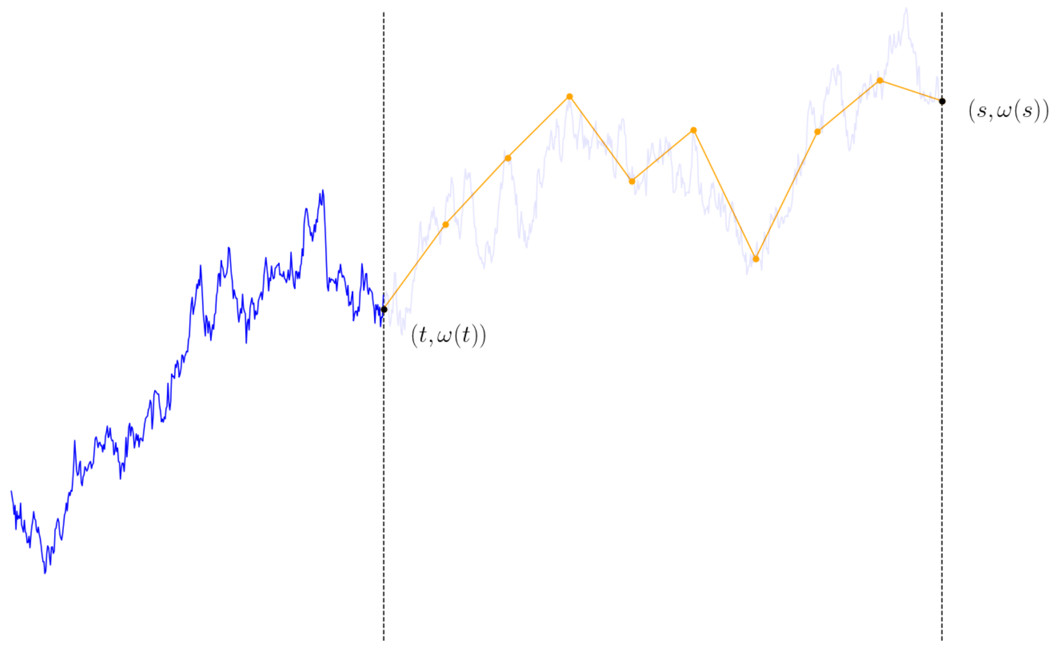

Figure 3.1: The piecewise linear approximation ω^N of ω.

For ease of notation in the section below, we will consider only the case $d = 1$, i.e. $\omega \in C([0,T],\mathbb{R})$. Let $(u,v) \in [\tau_k^N, \tau_{k+1}^N]^2$, $u \neq v$, where $[\tau_k^N, \tau_{k+1}^N]$ is an arbitrary interval, and D_k denote the Fréchét derivative of ω^N on the interval $[\tau_k^N, \tau_{k+1}^N]$. Intuitively, the behaviour of the quotient near 0, with a linear function f that vanishes at 0 in the numerator and an exponential function with exponent α, $0 < \alpha < 1$, in the denominator

$$\frac{f(x)}{x^\alpha}$$

remains well-defined near 0 since the rate of convergence to 0 of numerator dominates that of the denominator. To formalize this notion, we denote

$$D_k := \frac{\omega(\tau_{k+1}^N) - \omega(\tau_k^N)}{\tau_{k+1}^N - \tau_k^N},$$

then

$$\frac{|\omega^N(u)-\omega^N(v)|}{|u-v|^\nu} \leq \frac{|D_k|\cdot|u-v|}{|u-v|^\nu} = \left|\frac{\omega(\tau_{k+1}^N)-\omega(\tau_k^N)}{\tau_{k+1}^N-\tau_k^N}\right|\cdot|u-v|^{1-\nu}$$

$$\leq \left|\frac{\omega(\tau_{k+1}^N)-\omega(\tau_k^N)}{\tau_{k+1}^N-\tau_k^N}\right|\cdot|\tau_{k+1}^N-\tau_k^N|^{1-\nu} = \frac{|\omega(\tau_{k+1}^N)-\omega(\tau_k^N)|}{|\tau_{k+1}^N-\tau_k^N|^\nu}.$$

Since $u, v,\ u \neq v$ were arbitrary in the interval $[\tau_k^N, \tau_{k+1}^N]$, this yields

$$\sup_{\substack{(u,v)\in[\tau_k^N,\tau_{k+1}^N]^2\\ u\neq v}} \frac{|\omega^N(u)-\omega^N(v)|}{|u-v|^\nu} \leq \frac{|\omega(\tau_{k+1}^N)-\omega(\tau_k^N)|}{|\tau_{k+1}^N-\tau_k^N|^\nu}.$$

In fact this inequality is indeed an equality since $\omega^N(\tau_k^N)=\omega(\tau_k^N)$ for $k\in\{0,...,N\}$ by assumption. By the triangle inequality, we may extend this inequality to the entire interval $[t,s]$, as for $u\in[\tau_k^N,\tau_{k+1}^N]$ and $v\in[\tau_{k+n}^N,\tau_{k+n+1}^N]$ where $n\in\{0,...,N-k-1\}$:

$$|\omega^N(u)-\omega^N(v)| \leq |\omega^N(u)-\omega^N(\tau_{k+1}^N)| + ... + |\omega^N(\tau_{k+n}^N)-\omega^N(v)|$$

$$\leq |D_k|\cdot|u-\tau_{k+1}^N|^\nu + ... + |D_{n+k}|\cdot|\tau_{k+n}^N-v|^\nu \leq n\cdot\max_{i=k..n+k}|D_i|\cdot|u-v|^\nu.$$

Hence, there exists a constant $\tilde{C}>0$ such that:

$$\|\omega^N\|_\nu \leq \tilde{C}\|\omega\|_\nu, \tag{3.10}$$

Furthermore, for $u\in[t,s]$, assuming without loss of generality that $u\in[\tau_k^N,\tau_{k+1}^N]$:

$$|\omega^N(u)-\omega(u)| \leq |\omega^N(u)-\omega^N(\tau_k^N)| + |\omega(\tau_k^N)-\omega(u)|$$

$$\leq \|\omega^N\|_\nu\cdot|\tau_k^N-u|^\nu + \|\omega\|_\nu\cdot|\tau_k^N-u|^\nu \leq \tilde{C}\|\omega\|_\nu\cdot|\tau_k^N-u|^\nu + \|\omega\|_\nu\cdot|\tau_k^N-u|^\nu$$

$$\leq (\tilde{C}+1)\|\omega\|_\nu\cdot|\tau_{k+1}^N-\tau_k^N|^\nu.$$

As a consequence, some $C>0$ exists such that

$$\|\omega^N-\omega\|_\infty \leq C\|\omega\|_\nu\frac{|s-t|^\nu}{N^\nu}. \tag{3.11}$$

Furthermore, since $\omega\in C^\nu([0,T],\mathbb{R})$,

$$|\delta\omega_{a,b}| \leq \|\omega\|_\nu\cdot|b-a|^\nu,$$

it follows

$$|\dot{\omega}^N| = \sum_{i=0}^{N-1}\frac{|\delta\omega_{\tau_i,\tau_{i+1}}|}{|\tau_{i+1}-\tau_i|}\mathbf{1}_{(\tau_i,\tau_{i+1})} \leq N^{1-\nu}\|\omega\|_\nu|s-t|^{\nu-1}. \tag{3.12}$$

By (3.8), we set $\lambda = \omega^N$. The boundedness preservation of the functionals $\mathcal{D}F$, $\mathcal{D}\nabla_\omega F$ and $\nabla_\omega^2 F$ as well as the continuity of ω^N on $[t,s]$ yields,

$$\exists C_F > 0, \ \sup_{u\in[t,s]} \max\{\left|\mathcal{D}F(u,\omega^N)\right|, \left|\mathcal{D}\nabla_\omega F(u,\omega^N)\right|, \left|\nabla_\omega^2 F(u,\omega^N)\right|\} < C_F.$$

This delivers a bound for $|R^F_{t,s}(\omega^N)|$, namely

$$C_F|s-t| + C_F\|\omega^N\|_\nu|s-t|^{1+\nu} + C_F\|\omega^N\|_\nu^2 N^{1-\nu}|s-t|^{2\nu}. \tag{3.13}$$

By construction of ω^N,

$$\omega_t^N \equiv \omega_t \text{ but } \omega^N(s) = \omega(s).$$

Thus since F is a non-anticipative functional and also uniformly Lipschitz continuous it follows that

$$\begin{aligned}|R^F_{t,s}(\omega^N) - R^F_{t,s}(\omega)| =& |F(s,\omega_s^N) - F(t,\omega_t^N) - \nabla_\omega F(t,\omega_t^N)(\omega^N(s) - \omega^N(t)) \\ &- F(s,\omega_s) - F(t,\omega_t) - \nabla_\omega F(t,\omega_t)(\omega(s)-\omega(t))| \\ =& |F(s,\omega^N) - F(s,\omega)| \le C_F\|\omega^N - \omega\|_\infty \\ \le& C_F\|\omega\|_\nu|s-t|^\nu N^{-\nu},\end{aligned} \tag{3.14}$$

using the bound obtained from (3.11), where we simply increase $C_F > 0$ if needed. We exploit the estimates from (3.13) and (3.14), as well as the triangle inequality to obtain

$$\begin{aligned}|R^F_{t,s}(\omega)| \le |R^F_{t,s}(\omega) - R^F_{t,s}(\omega^N)| + |R^F_{t,s}(\omega^N)| \le C_F\|\omega^N - \omega\|_\infty \\ \le C_F\|\omega\|_\nu|s-t|^\nu N^{-\nu} + C_F|s-t| + C_F\|\omega^N\|_\nu|s-t|^{1+\nu} + C_F\|\omega^N\|_\nu^2 N^{1-\nu}|s-t|^{2\nu}.\end{aligned}$$

To optimize the above bound, we choose N such that

$$\|\omega\|_\nu N^{1-\nu}|s-t|^{2\nu} \approx N^{-\nu}|s-t|^\nu,$$

in other words,

$$N \approx \|\omega\|_\nu^{-1}|s-t|^{-\nu}. \tag{3.15}$$

Therefore

$$\begin{aligned}C_F\|\omega\|_\nu|s-t|^\nu N^{-\nu} + C_F|s-t| + C_F\|\omega^N\|_\nu|s-t|^{1+\nu} + C_F\|\omega^N\|_\nu^2 N^{1-\nu}|s-t|^{2\nu} \\ \le C_F|s-t| + C_F\|\omega\|_\nu|s-t|^{1+\nu} + C_F\|\omega\|_\nu^{1+\nu}|s-t|^{\nu^2+\nu} + C_F\|\omega\|_\nu^{1+\nu}|s-t|^{\nu^2+\nu}.\end{aligned}$$

In the second term, we bound $|s-t|$ by T to obtain a constant $C_{F,T} > 0$ and furthermore by choice of N in (3.15), we obtain the constant $C_{F,\nu} > 0$ such that the inequality

$$|R^F_{t,s}(\omega)| \le C_{F,T}(1 + \|\omega\|_\nu)|s-t| + C_{F,\nu}\|\omega\|_\nu^{1+\nu}|s-t|^{\nu+\nu^2} \tag{3.16}$$

holds. Based on (3.16), $\exists C_{F,T,\|\omega\|_\nu} > 0$ increasing in T and $\|\omega\|_\nu$ so that

$$|R^F_{t,s}(\omega)| \leq C_{F,T,\|\omega\|_\nu} |s-t|^{\nu^2+\nu}.$$

This proves the lemma. $\square$

We now progress to the proof of the isometry property:

Proof of Theorem 3.1. First recall that

$$F(t^n_{i+1}, \omega_{t^n_{i+1}}) - F(t^n_i, \omega_{t^n_i}) = R^F_{t^n_i, t^n_{i+1}}(\omega) + \nabla_\omega F(t^n_i, \omega)(\omega(t^n_{i+1}) - \omega(t^n_i)),$$

and thus

$$\begin{aligned}\big(F(t^n_{i+1}, \omega) - F(t^n_i, \omega)\big)^2 &= \Big(R^F_{t^n_i, t^n_{i+1}}(\omega)\Big)^2 - 2R^F_{t^n_i, t^n_{i+1}}(\omega)\nabla_\omega F(t^n_i, \omega)(\omega(t^n_{i+1}) - \omega(t^n_i)) \\ &\quad + \Big\langle \nabla_\omega F(t, \omega_{t^n_i})\ {}^t\nabla_\omega F(t, \omega_{t^n_i}), \delta\omega_{t^n_i, t^n_{i+1}}\ {}^t\delta\omega_{t^n_i, t^n_{i+1}} \Big\rangle .\end{aligned}$$

From above and the local boundedness of $\nabla_\omega F$:

$$\begin{aligned}&\left|\Big(F(t^n_{i+1}, \omega_{t^n_{i+1}}) - F(t^n_i, \omega_{t^n_i})\Big)^2 - \langle \nabla_\omega F(t, \omega_{t^n_i})\ {}^t\nabla_\omega F(t, \omega_{t^n_i}), \delta\omega_{t^n_i, t^n_{i+1}}\ {}^t\delta\omega_{t^n_i, t^n_{i+1}} \rangle\right| \\ &= \left|\Big(R^F_{t^n_i, t^n_{i+1}}(\omega)\Big)^2 - 2R^F_{t^n_i, t^n_{i+1}}(\omega)\nabla_\omega F(t^n_i, \omega)(\omega(t^n_{i+1}) - \omega(t^n_i))\right| \\ &\leq \left|R^F_{t^n_i, t^n_{i+1}}(\omega)\right|^2 + C_F\left|R^F_{t^n_i, t^n_{i+1}}(\omega)\right| |\delta\omega_{t^n_i, t^n_{i+1}}|.\end{aligned}$$

Assumption 2 and local boundedness implies that

$$M_n := \max_{t^n_i \in \pi^n} |R^F_{t^n_i, t^n_{i+1}}(\omega)| \xrightarrow{n\to\infty} 0, \tag{3.17}$$

so recall from Lemma 3.2

$$|R^F_{t^n_i, t^n_{i+1}}(\omega)| \leq C|t^n_{i+1} - t^n_i|^{\nu^2+\nu},$$

thus implying

$$|R^F_{t^n_i, t^n_{i+1}}(\omega)|^{\frac{1}{\nu^2+\nu}} \leq C^{\frac{1}{\nu^2+\nu}}|t^n_{i+1} - t^n_i|$$

since the terms in the inequality are positive and $\big(\nu > (\sqrt{3}-1)/2\big) \iff (\nu^2+\nu > 1/2)$. Hence with $D := C^{\frac{1}{\nu^2+\nu}}$, we obtain the bound

$$\begin{aligned}\sum_{t^n_i \in \pi^n} \left|R^F_{t^n_i, t^n_{i+1}}(\omega)\right|^2 &= \sum_{t^n_i \in \pi^n} \left(\left|R^F_{t^n_i, t^n_{i+1}}(\omega)\right|^{2-\frac{1}{\nu^2+\nu}} \cdot \left|R^F_{t^n_i, t^n_{i+1}}(\omega)\right|^{\frac{1}{\nu^2+\nu}}\right) \\ &\leq DM_n^{2-\frac{1}{\nu^2+\nu}} \sum_{t^n_i \in \pi^n} |t^n_{i+1} - t^n_i| \leq DTM_n^{2-\frac{1}{\nu^2+\nu}}.\end{aligned}$$

By (3.17),

$$DTM_n^{2-\frac{1}{\nu^2+\nu}} \xrightarrow{n\to\infty} 0. \tag{3.18}$$

As $\omega \in Q_\pi([0,T],\mathbb{R}^d)$,

$$\lim_{n\to\infty} \sum_{t_i^n\in\pi^n} |\delta\omega_{t_i^n,t_{i+1}^n}|^2 \to \mathrm{tr}([\omega]_\pi). \tag{3.19}$$

The Cauchy-Schwarz inequality, (3.18) and (3.19) implies

$$\sum_{t_i^n\in\pi^n} |R^F_{t_i^n,t_{i+1}^n}(\omega)||\delta\omega_{t_i^n,t_{i+1}^n}| \leq \sqrt{\sum_{t_i^n\in\pi^n} \left|R^F_{t_i^n,t_{i+1}^n}(\omega)\right|^2} \sqrt{\sum_{t_i^n\in\pi^n} \left|\delta\omega_{t_i^n,t_{i+1}^n}\right|^2} \xrightarrow{n\to\infty} 0.$$

From the above bounds, we conclude via summation:

$$\left|\sum_{t_i^n\in\pi^n} \left(F(t_{i+1}^n,\omega_{t_{i+1}^n}) - F(t_i^n,\omega_{t_i^n})\right)^2 - \sum_{t_i^n\in\pi^n} \left\langle \nabla_\omega F(t,\omega_{t_i^n})\,{}^t\nabla_\omega F(t,\omega_{t_i^n}), \delta\omega_{t_i^n,t_{i+1}^n}\,{}^t\delta\omega_{t_i^n,t_{i+1}^n}\right\rangle\right|$$

$$\leq \sum_{t_i^n\in\pi^n} \left|R^F_{t_i^n,t_{i+1}^n}(\omega)\right|^2 + C_F \sum_{t_i^n\in\pi^n} \left|R^F_{t_i^n,t_{i+1}^n}(\omega)\right| |\delta\omega_{t_i^n,t_{i+1}^n}| \xrightarrow{n\to\infty} 0,$$

and since

$$\sum_{t_i^n\in\pi^n} \left\langle \nabla_\omega F(t,\omega_{t_i^n})\,{}^t\nabla_\omega F(t,\omega_{t_i^n}), \delta\omega_{t_i^n,t_{i+1}^n}\,{}^t\delta\omega_{t_i^n,t_{i+1}^n}\right\rangle \xrightarrow{n\to\infty} \int_0^T \langle \nabla_\omega F(s,\omega_s)\,{}^t\nabla_\omega F(s,\omega_s), d[\omega]_\pi\rangle,$$

the triangle inequality indeed yields

$$[F(\cdot,\omega.)]_\pi(T) = \int_0^T \langle \nabla_\omega F(s,\omega_s)\,{}^t\nabla_\omega F(s,\omega_s), d[\omega]_\pi\rangle.$$

It is thus left to prove:

$$\left[F(\cdot,\omega.)\right]_\pi(T) = \left[\int_0^{\cdot} \nabla_\omega F(s,\omega_s) d^\pi\omega\right]_\pi (T).$$

Using the change of variable formula established in Theorem 2.1, it suffices to show that the paths

$$\int_0^{\cdot} \langle \nabla_\omega^2 F(t,\omega_t), d[\omega]_\pi(t)\rangle \text{ and } \int_0^{\cdot} \mathcal{D}F(t,\omega_t)dt \tag{3.20}$$

have zero quadratic variation along $\boldsymbol{\pi}$ since

$$
\begin{aligned}
\left[F(\cdot,\omega_\cdot)\right]_\pi(T) = \quad & \left[\int_0^{\cdot} \mathcal{D}F(t,\omega_t)dt + \frac{1}{2}\int_0^{\cdot} \langle \nabla_\omega^2 F(t,\omega_t), d[\omega]_\pi(t)\rangle\right]_\pi (T) \\
& + 2\left[\int_0^{\cdot} \mathcal{D}F(t,\omega_t)dt + \frac{1}{2}\int_0^{\cdot} \langle \nabla_\omega^2 F(t,\omega_t), d[\omega]_\pi(t)\rangle, \int_0^{\cdot} \nabla_\omega F(t,\omega_t) d^\pi\omega\right]_\pi (T) \\
& + \quad \left[\int_0^{\cdot} \nabla_\omega F(t,\omega_t) d^\pi\omega\right]_\pi (T).
\end{aligned}
\tag{3.21}
$$

For the first term on the right-hand side of (3.21), we have

$$
\begin{aligned}
& \left[\int_0^{\cdot} \mathcal{D}F(t,\omega_t)dt + \frac{1}{2}\int_0^{\cdot} \langle \nabla_\omega^2 F(t,\omega_t), d[\omega]_\pi(t)\rangle\right]_\pi (T) \\
= \quad & \left[\int_0^{\cdot} \mathcal{D}F(t,\omega_t)dt\right]_\pi (T) + \left[\frac{1}{2}\int_0^{\cdot} \langle \nabla_\omega^2 F(t,\omega_t), d[\omega]_\pi(t)\rangle\right]_\pi (T) \\
+2 & \left[\int_0^{\cdot} \mathcal{D}F(t,\omega_t)dt, \frac{1}{2}\int_0^{\cdot} \langle \nabla_\omega^2 F(t,\omega_t), d[\omega]_\pi(t)\rangle\right]_\pi (T),
\end{aligned}
$$

and the Cauchy-Schwarz inequality yields:

$$
\begin{aligned}
& \left[\int_0^{\cdot} \mathcal{D}F(t,\omega_t)dt, \frac{1}{2}\int_0^{\cdot} \langle \nabla_\omega^2 F(t,\omega_t), d[\omega]_\pi(t)\rangle\right]_\pi (T) \\
\leq & \sqrt{\left[\int_0^{\cdot} \mathcal{D}F(t,\omega_t)dt\right]_\pi (T)} \cdot \sqrt{\left[\frac{1}{2}\int_0^{\cdot} \langle \nabla_\omega^2 F(t,\omega_t), d[\omega]_\pi(t)\rangle\right]_\pi (T)}.
\end{aligned}
$$

Using the Cauchy-Schwarz inequality once again, the second term on the right-hand side of (3.21) is bounded:

$$
\begin{aligned}
& \left[\int_0^{\cdot} \mathcal{D}F(t,\omega_t)dt + \frac{1}{2}\int_0^{\cdot} \langle \nabla_\omega^2 F(t,\omega_t), d[\omega]_\pi(t)\rangle, \int_0^{\cdot} \nabla_\omega F(t,\omega_t) d^\pi\omega\right]_\pi (T) \\
\leq & \sqrt{\left[\int_0^{\cdot} \mathcal{D}F(t,\omega_t)dt + \frac{1}{2}\int_0^{\cdot} \langle \nabla_\omega^2 F(t,\omega_t), d[\omega]_\pi(t)\rangle\right]_\pi (T)} \cdot \sqrt{\left[\int_0^{\cdot} \nabla_\omega F(t,\omega_t) d^\pi\omega\right]_\pi (T)}.
\end{aligned}
$$

Hence, if the paths $\int_0^{\cdot} \mathcal{D}F(t,\omega_t)dt$ and $\int_0^{\cdot} \langle \nabla_\omega^2 F(t,\omega_t), d[\omega]_\pi(t)\rangle$ are of zero quadratic variation along $\boldsymbol{\pi}$, the first two terms on the right-hand side in (3.21) vanish. Assumption 2 naturally implies

$$\max_{t_i^n \in \pi^n} |[\omega]_\pi(t_{i+1}^n) - [\omega]_\pi(t_i^n)| \xrightarrow{n\to\infty} 0, \tag{3.22}$$

as

$$\max_{t_i^n \in \pi^n} |\omega(t_{i+1}^n) - \omega(t_i^n)|^2 \leq \left(\mathrm{osc}(\omega,\pi^n)\right)^2 \quad \forall n \in \mathbb{N}.$$

It follows from the local boundedness of $\nabla_\omega F$ that for arbitrary $n \in \mathbb{N}$,

$$\begin{aligned}
&\sum_{t_i^n \in \pi^n} \left(\int_{t_i^n}^{t_{i+1}^n} \langle \nabla_\omega F(t,\omega_t), d[\omega]_\pi(t)\rangle \right)^2 \\
\leq & C \sum_{t_i^n \in \pi^n} |[\omega]_\pi(t_{i+1}^n) - [\omega]_\pi(t_i^n)|^2 \\
\leq & C \max_{t_i^n \in \pi^n} |[\omega]_\pi(t_{i+1}^n) - [\omega]_\pi(t_i^n)| \cdot \sum_{t_i^n \in \pi^n} |[\omega]_\pi(t_{i+1}^n) - [\omega]_\pi(t_i^n)|
\end{aligned}$$

Furthermore, $[\omega]_\pi$ is by definition a monotone increasing function such that it is of finite variation, hence:

$$\lim_{n\to\infty} \sum_{t_i^n \in \mathbb{N}} |[\omega]_\pi(t_{i+1}^n) - [\omega]_\pi(t_i^n)| \text{ exists.}$$

Consequently, (3.22) implies that $\left[\int_0^{\cdot} \langle \nabla_\omega F(t,\omega_t), d[\omega]_\pi(t)\rangle\right]_\pi \equiv 0$. Note for the second path, $\int_0^{\cdot} \mathcal{D}F(t,\omega_t)dt$,

$$\sum_{t_i^n \in \pi^n} \left| \int_{t_i^n}^{t_{i+1}^n} \mathcal{D}F(t,\omega_t)dt \right| \leq \sum_{t_i^n \in \pi^n} \int_{t_i^n}^{t_{i+1}^n} |\mathcal{D}F(t,\omega_t)|\, dt = \int_0^T |\mathcal{D}F(t,\omega_t)| dt \quad \forall n \in \mathbb{N},$$

which is necessarily finite by the local boundedness of $\mathcal{D}F$ and thus the path $\int_0^{\cdot} \mathcal{D}F(t,\omega_t)dt$ is of finite variation.

$$\sum_{t_i^n \in \pi^n} \left(\int_{t_i^n}^{t_{i+1}^n} \mathcal{D}F(t,\omega_t)dt \right)^2 \leq \max_{t_i^n \in \pi^n} \left\{ \left| \int_{t_i^n}^{t_{i+1}^n} \mathcal{D}F(t,\omega_t)dt \right| \right\} \sum_{t_i^n \in \pi^n} \left| \int_{t_i^n}^{t_{i+1}^n} \mathcal{D}F(t,\omega_t)dt \right| \tag{3.23}$$

Now consider using the Fundamental Theorem of Calculus,

$$\begin{aligned}
\int_{t_i^n}^{t_{i+1}^n} \mathcal{D}F(t,\omega_t)dt &= \int_{t_i^n}^{t_{i+1}^n} \left(\mathcal{D}F(t,\omega_t) - \mathcal{D}F(t,\omega_{t_i^n})\right) dt + \int_{t_i^n}^{t_{i+1}^n} \mathcal{D}F(t,\omega_{t_i^n})dt \\
&= \int_{t_i^n}^{t_{i+1}^n} \left(\mathcal{D}F(t,\omega_t) - \mathcal{D}F(t,\omega_{t_i^n})\right) dt + \left(F(t_{i+1}^n,\omega_{t_i^n}) - F(t_i^n,\omega_{t_i^n})\right) \\
&= \int_{t_i^n}^{t_{i+1}^n} \left(\mathcal{D}F(t,\omega_t) - \mathcal{D}F(t,\omega_{t_i^n})\right) dt + \Big(F(t_{i+1}^n,\omega_{t_i^n}) - F(t_{i+1}^n,\omega_{t_{i+1}^n})\Big) \\
&\quad + \Big(F(t_{i+1}^n,\omega_{t_{i+1}^n}) - F(t_i^n,\omega_{t_i^n})\Big).
\end{aligned}$$

From the continuity of F and $\mathcal{D}F$ as well as the condition imposed on ω along $\boldsymbol{\pi}$ in Assumption 2, the first two summands converge *uniformly* to 0 as $n \to \infty$, so

$$\max_{t_i^n \in \pi^n} \left| \int_{t_i^n}^{t_{i+1}^n} \left(\mathcal{D}F(t,\omega_t) - \mathcal{D}F(t,\omega_{t_i^n})\right) dt \right| \xrightarrow{n\to\infty} 0$$

and

$$\max_{t_i^n \in \pi^n} \left| F(t_{i+1}^n,\omega_{t_i^n}) - F(t_{i+1}^n,\omega_{t_{i+1}^n}) \right| \xrightarrow{n\to\infty} 0.$$

The third summand converges *uniformly* to 0 as a consequence of Assumption 3 and the fact that F is non-anticipative:

$$\max_{t_i^n \in \pi^n} \left| F(t_{i+1}^n,\omega_{t_{i+1}^n}) - F(t_i^n,\omega_{t_i^n}) \right| \xrightarrow{n\to\infty} 0,$$

so therefore

$$\max_{t_i^n \in \pi^n} \left| \int_{t_i^n}^{t_{i+1}^n} \mathcal{D}F(t,\omega_t)dt \right| \xrightarrow{n\to\infty} 0.$$

By (3.22), the path $\int_0^{\cdot} \mathcal{D}F(t,\omega_t)dt$ is of finite quadratic variation along $\boldsymbol{\pi}$ and thus

$$\left[\int_0^{\cdot} \nabla_\omega F(t,\omega_t) d^\pi \omega \right]_\pi (T) = \left[F(\cdot,\omega_\cdot)\right]_\pi (T) = \int_0^T \langle \nabla_\omega F(s,\omega_s)\ {}^t\nabla_\omega F(s,\omega_s), d[\omega]_\pi \rangle.$$

This proves the theorem. □

To underscore the relevance of this result, we establish a linkage to the *classical Itô isometry*.

Remark. Theorem 3.1 develops a pathwise analogue to the Itô isometry formula in the following sense:

Let $\mathbb{P}$ be the Wiener measure on the space of continuous paths defined on the compact interval $[0,T]$, $C^0([0,T],\mathbb{R})$. Then the *stochastic process*:

$$\begin{aligned}\int_0^{\cdot} \nabla_\omega F(s,\cdot)d^\pi\omega \ : \ [0,T]\times C^0([0,T],\mathbb{R}) &\longrightarrow \mathbb{R} \\ (t,\omega) &\longmapsto \int_0^t \nabla_\omega F(s,\omega)d^\pi\omega\end{aligned} \tag{3.24}$$

is defined almost surely under the Wiener measure $\mathbb{P}$, and is a version of the classic Itô integral $\int_0^{\cdot} \nabla_\omega F(s,W)dW$, i.e.:

$$\mathbb{P}\left(\int_0^{\cdot} \nabla_\omega F(s,\cdot)d^\pi\omega = \int_0^{\cdot} \nabla_\omega F(s,W)dW,\ \forall s\in[0,T]\right) = 1.$$

Recalling the *classical* Itô isometry formula, the expectation of (3.3) under $\mathbb{P}$ yields:

$$\mathbb{E}_{\mathbb{P}}\left([\int_0^{\cdot} \nabla_\omega F(s,W)dW](T)\right) = \mathbb{E}_{\mathbb{P}}\left(|\int_0^{T} \nabla_\omega F(s,W)dW|^2\right) = \mathbb{E}_{\mathbb{P}}\left(\int_0^{T} |\nabla_\omega F(t,W)|^2 dt\right).$$

Note that in this interpretation, the underlying isometry still depends on the Wiener measure, thus delivering inherent model risk. In contrast, Theorem 3.1 delivers a purely analytic notion that is independent of the Wiener measure, proving that the pathwise isometry indeed underpins the classical Itô isometry.

3.2 Partitions based on Hitting Times

We may also analyze quadratic variation and Riemann sums under partitions of hitting times of uniformly spaced levels, also termed *Lebesgue partitions*. In contrast to the approaches of Föllmer and Theorem 3.1 that merely require a sequence partitions of vanishing mesh, partitions are based on, broadly speaking, variations of the particular path ω itself. We will observe the partitions of the form:

$$\tau_0^n(\omega) = 0, \quad \tau_{k+1}^n(\omega) = \inf\{t > \tau_k^n : |\omega(t) - \omega(\tau_k^n)| \geq 2^{-n}\} \wedge T. \tag{3.25}$$

One advantage inherent in this approach is the inherent *stationarity* of the quadratic variation: computations are invariant under time changes. For notation purposes, and uniformity with respect to the previous section, we define:

$$m(n) = \inf\{k \geq 1 : \tau_k^n(\omega) = T\} \text{ while } \tau^n(\omega) = \{\tau_k^n(\omega)\}_{k=0..m(n)},$$

so $\tau^n(\omega)$ is the Lebesgue partition analogue to π^n in the previous section. By the construction of the Lebesgue partition and the continuity of ω, note that Assumption 2:

$$\operatorname{osc}(\omega, \tau^n(\omega)) = \max_{i=0..m(n)-1} \sup_{t \in (\tau_k^n(\omega), \tau_{k+1}^n(\omega)]} |\omega(t) - \omega(t_i^n)| \leq 2^{-n}$$

still holds. A sequence of Lebesgue partitions $\tau(\omega)$ is not necessarily of vanishing mesh: the most illustrative example is a constant function. We shall prove that if a path ω is of strictly increasing quadratic variation along the Lebesgue partition, then

$$|\tau^n(\omega)| := \sup_{\tau_k^n(\omega) \in \tau^n(\omega)} |\tau_{k+1}^n(\omega) - \tau_k^n(\omega)| \xrightarrow{n \to \infty} 0,$$

i.e., the sequence of partitions τ if of vanishing mesh.

Lemma 3.3. [1] *Let $\omega \in C^0([0,T], \mathbb{R}^d)$ be of finite quadratic variation along its Lebesgue partition $\tau(\omega)$ as in* (3.25)*:*

$$\forall t \in (0,T], \lim_{n \to \infty} \sum_{\tau_k^n \leq t} |\omega(\tau_{k+1}^n) - \omega(\tau_k^n)|^2 = [\omega]_{\tau(\omega)}(t) > 0. \tag{3.26}$$

If $t \mapsto [\omega]_{\tau(\omega)}(t)$ is a strictly increasing function then $|\tau^n(\omega)| \xrightarrow{n \to \infty} 0$.

Proof. Let $h > 0$ and $t \in [0, T-h]$. Define a constant $m(t, t+h)$ as the number of partition points of $\tau^n(\omega)$ in $[t, t+h]$. Then:

$$\sum_{t \leq \tau_k^n \leq t+h} |\omega(\tau_{k+1}^n) - \omega(\tau_k^n)|^2 = 4^{-n}(m(n,t,t+h) - 1),$$

by definition of $\tau^n(\omega)$, and moreover $|\omega(\tau_{k+1}^n) - \omega(\tau_k^n)| = 2^{-n}$ $\forall k \in \{0, ..., m(n)-1\}$. From the assumption that $t \mapsto [\omega]_{\tau(\omega)}(t)$ is strictly increasing, it follows

$$\sum_{t \leq \tau_k^n \leq t+h} |\omega(\tau_{k+1}^n) - \omega(\tau_k^n)|^2 = 4^{-n}(m(n,t,t+h) - 1) \xrightarrow{n \to \infty} [\omega]_{\tau(\omega)}(t+h) - [\omega]_{\tau(\omega)}(t) > 0.$$

Consequently, $m(n,t,t+h) \sim 4^{-n} \xrightarrow{n \to \infty} 0$. $t \in [0, T-h]$ was arbitrary, hence for large enough n, $|\tau^n| < h$ for any $h > 0$, and thus

$$|\tau^n| \xrightarrow{n \to \infty} 0.$$

□

The *strict* monotonicity of $[\omega]_{\tau(\omega)}$ imposes an irregularity condition on ω in that its quadratic variation over any interval is non-zero. We extend Theorem 3.1 to paths of strictly increasing quadratic variation along the Lebesgue partitions $\tau(\omega)$.

Theorem 3.2 (Pathwise isometry formula along Lebesgue partitions [1]). *Let $\omega \in Q_{\tau(\omega)}([0,T],\mathbb{R}^d)\cap C^\nu([0,T],\mathbb{R}^d)$ where $\nu \in \left(\frac{\sqrt{3}-1}{2},\frac{1}{2}\right)$ and $F \in \mathbb{C}_b^{1,1}(\Lambda_T^d)\cap \mathrm{Lip}(\Lambda_T^d,\|\cdot\|_\infty)$ so that $\nabla_\omega F \in \mathbb{C}_b^{1,1}(\Lambda_T^d)$. If ω has strictly increasing quadratic variation $[\omega]_{\tau(\omega)}$ with respect to $\tau(\omega)$ then the isometry formula holds along the partition of hitting times defined by:*

$$[F(\cdot,\omega.)]_{\tau(\omega)}(t) = \left[\int_0^{\cdot} \nabla_\omega F(s,\omega_s)d^\pi\omega\right]_{\tau(\omega)}(t) = \int_0^t \langle \nabla_\omega F(s,\omega_s)\, {}^t\nabla_\omega F(s,\omega_s), d[\omega]_{\tau(\omega)}\rangle.$$

Proof. As noted above, the sequence of partitions $\tau(\omega)$ satisfies Assumption 2:

$$\mathrm{osc}(\omega,\tau^n(\omega)) \le 2^{-n}.$$

By Lemma 3.3, $|\tau^n(\omega)| \xrightarrow{n\to\infty} 0$. Let $\varepsilon > 0$, then since F is left-continuous, for any $(t,\omega)\in\Lambda_T^d$, there exists $\delta > 0$ so that for any $(t',\omega')\in\Lambda_T^d$ with $t' < t$ and

$$d_\infty((t,\omega),(t',\omega')) \le \delta : \quad |F(t,\omega)-F(t',\omega')| \le \varepsilon.$$

We choose $N \in \mathbb{N}$ so that $2^{-n} \le \frac{\delta}{2}$ and $|\tau^n(\omega)| \le \left(\frac{\delta}{2\|\omega\|_\nu}\right)^{\frac{1}{\nu}}$, $\forall n \ge N$. Since $\omega \in C^\nu([0,T],\mathbb{R}^d)$, it follows

$$\max_{k=0..m(n)-1} \sup_{(i,j)\in[\tau_k^n(\omega),\tau_{k+1}^n(\omega)]^2} |\omega(i)-\omega(j)| \le \|\omega\|_\nu |\tau^n(\omega)|^\nu \le \frac{\delta}{2}$$

. We thus obtain

$$d_\infty((\tau_{k+1}^n(\omega),\omega_{\tau_{k+1}^n}),(\tau_k^n(\omega),\omega_{\tau_k^n})) \le \delta \quad \forall k = 0,...,m(n)-1$$

and therefore by the left-continuity of F:

$$\max_{k=0..m(n)-1} |F(\tau_{k+1}^n(\omega),\omega_{\tau_{k+1}^n}) - F(\tau_k^n(\omega),\omega_{\tau_k^n})| \le \varepsilon.$$

Finally, F is *non-anticipative* so that $F(\tau_k^n(\omega),\omega) = F(\tau_k^n(\omega),\omega_{\tau_k^n})$ for all $k \in \{0,...,m(n)-1\}$ and this proves that Assumption 3 is satisfied. The claim then follows as a consequence of Theorem 3.1 since Assumption 2 and Assumption 3 are satisfied. □

3.3 Isometry Property

Proposition 3.2 (Preservation of irregularity [1]). *Let $\omega \in Q_\pi([0,T],\mathbb{R}^d)\cap C^\nu([0,T],\mathbb{R}^d)$ for some $\nu \in (\frac{\sqrt{3}-1}{2},\frac{1}{2}]$ such that $\frac{d[\omega]_\pi}{dt} := a(t) > 0$ is a right-continuous, positive definite function and $F \in \mathbb{C}_b^{1,2}(\Lambda_T^d)\cap \mathrm{Lip}(\Lambda_T^d,\|\cdot\|_\infty)$ is a non-anticipative functional with $\nabla_\omega F \in \mathbb{C}_b^{1,1}(\Lambda_T^d)$. If Assumption 2 and 3 hold, then the path $t\mapsto F(t,\omega)$ is of zero quadratic variation along $\boldsymbol{\pi}$ if and only if $\nabla_\omega F(t,\omega) = 0$, $\forall t\in[0,T]$.*

Proof. It follows immediately from Theorem 3.1

$$[F(\cdot,\omega.)]_\pi(T) = \int_0^T {}^t\nabla_\omega F(s,\omega_s)a(s)\nabla_\omega F(s,\omega_s)ds. \tag{3.27}$$

As $a(t)$ is positive definite, the integrand is necessarily non-negative, and vanishes at $s \in [0,T]$ if and only if, $\nabla_\omega F(s,\omega) = 0$. In the case that the integrand vanishes, the integral clearly vanishes too. For the proof of the converse, since the integrand is right-continuous, if some $s \in [0,T]$ exists such that $\nabla_\omega F(s,\omega) > 0$,

Case 1. $s \in [0,T)$.

This implies ${}^t\nabla_\omega F(s,\omega)a(s)\nabla_\omega F(s,\omega) > 0$, then there exists some lower bound $C > 0$ for the integral in (3.27). Consequently, if $F(\cdot,\omega.)$ is of zero quadratic, this implies $\nabla_\omega F(t,\omega) = 0,\ \forall t \in [0,T)$. Since $\nabla_\omega F$ is assumed continuous at T it follows that $\nabla_\omega F(t,\omega) = 0,\ \forall t \in [0,T]$.

Case 2. $s = T$.

The continuity of $\nabla_\omega F$ as well as the ν-Hölder continuous path ω imply the continuity of $\nabla_\omega F(\cdot,\omega)$. Therefore, in some neighbourhood of T, there exists some $s \in [0,T)$ such that $\nabla_\omega F(s,\omega) > 0$ and exploiting Case 1 delivers the desired result.

This proves the proposition. □

Let $F \in \mathbb{C}_b^{1,2}(\Lambda_T^d)$ be a functional that satisfies the assumptions embedded in Theorem 3.1. We denote the set of these functionals as $\mathcal{R}(\Lambda_T^d)$, i.e., the set of *regular functionals*. The claim of Proposition 3.2 is significant as it shows a regular functional transformation $F \in \mathbb{C}_b^{1,2}(\Lambda_T^d)$ of a path ω that is of *strictly* increasing quadratic variation inherits this 'irregularity' if and only if $\nabla_\omega F \neq 0$. In other words, we can view this proposition as a statement on the local preservation of irregularity of paths under functionals. As noted in Case 2 of the proof above for $\nabla_\omega F(\cdot,\omega)$ with $\nu \in (\frac{\sqrt{3}-1}{2}, \frac{1}{2}]$:

$$\omega \in C^\nu([0,T],\mathbb{R}^d) \Rightarrow F(\cdot,\omega) \in C^\nu([0,T],\mathbb{R}),$$

and therefore

$$\omega \in C^{\frac{1}{2}-}([0,T],\mathbb{R}^d) \Rightarrow F(\cdot,\omega) \in C^{\frac{1}{2}-}([0,T],\mathbb{R}).$$

Let $a : [0,T] \to S_+^d$ be a continuous functional taking on values in the positive-definite symmetric matrices.

Definition 3.1 (Harmonic functionals)**.** A regular functional $F \in \mathcal{R}(\Lambda_T^d)$ is called $a(\cdot)$-harmonic if

$$\forall (t,\omega) \in \Lambda_T^d,\ \ \mathcal{D}F(t,\omega_t) + \frac{1}{2}\langle \nabla_\omega^2 F(t,\omega_t), a(t)\rangle = 0.$$

Let $\mathcal{H}_a(\Lambda_T^d)$ denote the space of $a(\cdot)$-harmonic functionals.

Proposition 3.3. *For any $F \in \mathcal{R}(\Lambda_T^d)$, the functional defined as*

$$G(t,\omega) := F(t,\omega) - \int_0^t \mathcal{D}F(s,\omega)ds - \frac{1}{2}\int_0^t \langle \nabla_\omega^2 F(s,\omega), a(s)\rangle ds$$

is indeed $a(\cdot)$-harmonic.

Proof. First note that by the Fundamental Theorem of Calculus and linearity of the derivative

$$\begin{aligned}\mathcal{D}G(t,\omega) &= \mathcal{D}\left(F(t,\omega) - \int_0^t \mathcal{D}F(s,\omega)ds - \frac{1}{2}\int_0^t \langle \nabla_\omega^2 F(s,\omega), a(s)\rangle ds\right)\\ &= \mathcal{D}F(t,\omega) - \mathcal{D}F(t,\omega) - \frac{1}{2}\langle \nabla_\omega^2 F(t,\omega), a(t)\rangle\\ &= -\frac{1}{2}\langle \nabla_\omega^2 F(t,\omega), a(t)\rangle.\end{aligned}$$

Furthermore, in the proof of Theorem 3.1, we were also able to show that for $F \in \mathcal{R}(\Lambda_T^d)$, the paths $\int_0^\cdot \mathcal{D}F(s,\omega)ds$ and $\int_0^\cdot \langle \nabla_\omega^2 F(s,\omega), a(s)\rangle ds$ are of zero quadratic variation along $\boldsymbol{\pi}$, thus by Proposition 3.2

$$\nabla_\omega\left(\int_0^t \mathcal{D}F(s,\omega)ds\right) = 0 \text{ and } \nabla_\omega\left(\int_0^t \langle \nabla_\omega^2 F(s,\omega), a(s)\rangle ds\right) = 0,\ \forall t \in [0,T]. \quad (3.28)$$

Applying ∇_ω once more, it follows:

$$\nabla_\omega^2\left(\int_0^t \mathcal{D}F(s,\omega)ds\right) = \nabla_\omega^2\left(\int_0^t \langle \nabla_\omega^2 F(s,\omega), a(s)\rangle ds\right) = 0. \quad (3.29)$$

Now note that

$$\begin{aligned}\nabla_\omega^2 G(t,\omega) &= \nabla_\omega^2 F(t,\omega) - \nabla_\omega^2\left(\int_0^t \mathcal{D}F(s,\omega)ds\right) - \frac{1}{2}\nabla_\omega^2\left(\int_0^t \langle \nabla_\omega^2 F(s,\omega), a(s)\rangle ds\right)\\ &= \nabla_\omega^2 F(t,\omega).\end{aligned}$$

Finally,

$$\mathcal{D}G(t,\omega_t) + \frac{1}{2}\langle \nabla_\omega^2 G(t,\omega_t), a(t)\rangle = -\frac{1}{2}\nabla_\omega F(t,\omega) + \frac{1}{2}\nabla_\omega F(t,\omega) = 0.$$

We have shown that G is $a(\cdot)$-harmonic. $\square$

Harmonic functionals are a key class of functionals not only in Mathematical Finance, but also in broader Stochastic Analysis and contribute to probabilistic interpretations of Functional Itô Calculus.

Let $\overline{\omega} \in Q_\pi([0,T],\mathbb{R}^d) \cap C^\nu([0,T],\mathbb{R}^d)$ so that $d[\overline{\omega}]/dt = a$. The functional change of variable formula then delivers

$$\forall F \in \mathcal{H}_a(\Lambda_T),\ \forall t \in [0,T],\ F(t,\overline{\omega}) = F(0,\overline{\omega}) + \int_0^t \nabla_\omega F(u,\overline{\omega}_u) d^\pi \omega. \tag{3.30}$$

Applying Theorem 3.1,

$$[F(\cdot,\overline{\omega}.)]_\pi(t) = \int_0^t {}^t\nabla_\omega F(u,\overline{\omega}) a(u) \nabla_\omega F(u,\overline{\omega}) du = \|\nabla_\omega F(\cdot,\overline{\omega}_\omega)\|^2_{L^2([0,T],a)} < \infty. \tag{3.31}$$

We define vector spaces, parameterized by a path $\overline{\omega}$, as images of $\overline{\omega}$ under harmonic functionals and corresponding vertical derivatives, respectively,

$$\mathcal{H}_a(\overline{\omega}) := \{F(\cdot,\overline{\omega}.) | F \in \mathcal{H}_a(\Lambda_T)\} \subset Q_\pi([0,T],\mathbb{R}),$$

$$\mathbb{V}_a(\overline{\omega}) := \{\nabla_\omega F(\cdot,\overline{\omega}.) | F \in \mathcal{H}_a(\Lambda_T)\} \subset L^2([0,T],a).$$

From Proposition 3.2, the map

$$\omega \in \mathcal{H}_a(\overline{\omega}) \to \|\omega\|_\pi = \sqrt{[\omega]_\pi(T)}$$

defines a norm on the space of images of $\overline{\omega}$ under a functional in $\mathcal{H}_a(\Lambda_T)$. Now consider a *linear* operator

$$I_{\overline{\omega}} : (\mathbb{V}_a(\overline{\omega}), \|\cdot\|_{L^2([0,T],a)}) \to (\mathcal{H}_a(\overline{\omega}), \|\cdot\|_\pi),$$

where

$$I_{\overline{\omega}}(\nabla_\omega F(\cdot,\overline{\omega}.)) = \int_0^{\cdot} \nabla_\omega F(t,\overline{\omega}) d^\pi \omega := \lim_{n\to\infty} \sum_{\substack{t_i^n \in \pi^n \\ t_i^n \leq \cdot}} \nabla_\omega F(t_i^n, \overline{\omega}^n_{t_i^n}) \delta\overline{\omega}_{t_i^n,t_{i+1}^n}.$$

Proposition 3.4. *$I_{\overline{\omega}}$, as defined above, is indeed an injective isometry.*

Proof. Let $\nabla_\omega F(\cdot,\overline{\omega}.) \in \mathbb{V}_a$ such that $I_{\overline{\omega}}(\nabla_\omega F(\cdot,\overline{\omega})) \equiv 0$. From (3.30), it follows $F(t,\overline{\omega}_t) = F(0,\overline{\omega}_0)$ for any $t \in [0,T]$. Therefore, the path $F(\cdot,\overline{\omega})$ is of zero quadratic variation along $\boldsymbol{\pi}$ so that Proposition 3.2 implies $\nabla_\omega F(\cdot,\overline{\omega}) \equiv 0$. Consequently, $I_{\overline{\omega}}$ is injective as $\ker(I_{\overline{\omega}}) = \{0\}$. In terms of the isometry, (3.31) yields:

$$\|I_{\overline{\omega}}(\nabla_\omega F(\cdot,\overline{\omega}.))\|_\pi = \|\int_0^{\cdot} \nabla_\omega F(\cdot,\overline{\omega}.) d^\pi \omega\|_\pi = \|F(\cdot,\omega.)\|_\pi = \|\nabla_\omega F(\cdot,\overline{\omega}.)\|_{L^2([0,T],a)},$$

where we used Theorem 3.1 in the second equality and (3.31) in the third. This proves the proposition, and by extension, the continuity of $I_{\overline{\omega}}$. □

Chapter 4

Pathwise Nature of the Integral

Results from this chapter can be found in Ananova and Cont [1].

The construction of the pathwise integral with respect to a continuous path ω in Theorem 2.1 clearly relies on piecewise constant approximations of a given path ω. In order to ensure that the integral in (2.19) is well-defined, we show that the integral indeed only depends on the path $\nabla_\omega F(\cdot, \omega_\cdot)$.

Property 4.1 (Pathwise nature of the integral [1])**.** *Let* $F_1,\ F_2 \in \mathbb{C}_b^{1,2}(\Lambda_T^d)$ *and* $\omega \in Q_\pi([0,T],\mathbb{R}) \cap C^\nu([0,T],\mathbb{R})$ *for some* $0 < \nu < 1/2$*. If*

$$\nabla_\omega F(t,\omega_t) = \nabla_\omega F_2(t,\omega_t),\ \forall t \in [0,T]$$

then

$$\int_0^T \nabla_\omega F_1 \cdot d^\pi \omega = \int_0^T \nabla_\omega F_2 \cdot d^\pi \omega.$$

We recall and assume the horizontal Lipschitz property imposed on a functional.

Assumption 4 (Horizontal Lipschitz property)**.** *A non-anticipative functional* $F : \Lambda_T^d \to V$ *such that* V *is finite dimensional real-valued vector space, satisfies the horizontal locally Lipschitz property if*

$$\begin{gathered}\forall \omega \in D([0,T],\mathbb{R}^d), \exists C > 0, \eta > 0, \forall h \geq 0, \forall t \leq T-h, \forall \omega' \in D([0,T],\mathbb{R}^d),\\ \|\omega_t - \omega'_t\|_\infty < \eta \implies |F(t+h,\omega'_t) - F(t,\omega'_t)| \leq Ch.\end{gathered}$$

This property is weaker than that of horizontal differentiability.

Lemma 4.1 (Expansion formula for regular functionals)**.** *Let* $\omega \in C^\nu([0,T]\mathbb{R}^d)$ *where* $\nu \in (1/3, 1/2]$ *and* $F \in \mathbb{C}_b^{1,2}(\Lambda_T^d, \mathbb{R}^n)$ *be a non-anticipative functional such that*

- $\nabla_\omega F \in \mathbb{C}_b^{1,1}(\Lambda_T^d, \mathbb{R}^{n\times d})$,
- $\nabla_\omega^2 F \in \mathbb{C}_b^{1,1}(\Lambda_T^d, \mathbb{R}^{n\times d\times d})$,

- $F,\ \mathcal{D}F,\ \nabla^3_\omega F \in \mathrm{Lip}(\Lambda^d_T, \|\cdot\|_\infty)$,
- $\nabla^3_\omega F$ *horizontally locally Lipschitz.*

Then

$$F(s,\omega_s) - F(t,\omega_t) = \nabla_\omega F(t,\omega_t)(\omega(s)-\omega(t)) + \int_t^s \mathcal{D}F(u,\omega_u)du$$
$$+\frac{1}{2}\langle \nabla^2_\omega F(t,\omega_t), (\omega(s)-\omega(t))^t(\omega(s)-\omega(t))\rangle + O(|s-t|^{3\nu^2+\nu}),$$

as $|s-t| \to 0$, *uniformly in* $t,\ s \in [0,T]$.

Proof. The second term of the third line in (3.7) can be expressed as

$$\begin{aligned}
&\int_t^s \int_t^u \partial^2_{ij} F(r,\lambda_r) d\lambda^j(r) d\lambda^i(u) \\
=&\partial^2_{ij}F(t,\lambda_t)\delta\lambda^i_{t,s}\delta\lambda^j_{t,u} - \partial^2_{ij}F(t,\lambda_t)\delta\lambda^i_{t,s}\delta\lambda^j_{t,u} + \int_t^s \int_t^u \partial^2_{ij} F(r,\lambda_r) d\lambda^j(r) d\lambda^i(u) \qquad (4.1)\\
=&\partial^2_{ij}F(t,\lambda_t)\int_t^s \int_t^u d\lambda^j(r)d\lambda^i(u) + \int_t^s \int_t^u \left(\partial^2_{ij}F(r,\lambda_r) - \partial^2_{ij}F(t,\lambda_t)\right) d\lambda^j(r)d\lambda^i(u).
\end{aligned}$$

Recall from (3.5) for $G = \partial^2_{ij}F$,

$$\partial^2_{ij}F(r,\lambda_r) - \partial^2_{ij}F(t,\lambda_t) = \int_t^r \mathcal{D}\partial^2_{ij}F(\tau,\lambda_\tau)d\tau + \int_t^r \partial^3_{ijk}F(\tau,\lambda_\tau)d\lambda(\tau),$$

so that Fubini's theorem delivers

$$\begin{aligned}
&\int_t^s \int_t^u \left(\partial^2_{ij}F(r,\lambda_r) - \partial^2_{ij}F(t,\lambda_t)\right) d\lambda^j(r)d\lambda^i(u) \\
=&\int_t^s \int_t^u \left(\int_t^r \mathcal{D}\partial^2_{ij}F(\tau,\lambda_\tau)d\tau + \int_t^r \partial^3_{ijk}F(\tau,\lambda_\tau)d\lambda(\tau)\right) d\lambda^j(r)d\lambda^i(u) \qquad (4.2)\\
=&\int_t^s \mathcal{D}\partial^2_{ij}F(\tau,\lambda_\tau)\Lambda^{ji}_{\tau,s}d\tau + \int_t^s \partial^3_{ijk}F(\tau,\lambda_\tau)\Lambda^{ji}_{\tau,s}\dot{\lambda}^k(\tau)d\tau,
\end{aligned}$$

where

$$\Lambda^{ji}_{a,b} = \int_a^b \int_a^u d\lambda^j(r)d\lambda^i(u) = \int_a^b (\lambda^j(u) - \lambda^j(a))\dot{\lambda}^i(u)du.$$

Recalling the definition of $R^F_{t,s}(\lambda)$ as in (3.6)

$$R^F_{t,s}(\lambda) = \int_t^s \mathcal{D}F(u,\lambda_u)du + \int_t^s (\partial_i F(u,\lambda_u) - \partial_i F(t,\lambda_t))d\lambda^i(u),$$

we further use (4.1) and (4.2) to obtain:

$$\begin{aligned} R^F_{t,s}(\lambda) = &\int_t^s \mathcal{D}F(u,\lambda_u)du + \int_t^s \mathcal{D}\partial_i F(r,\lambda_r)\delta\lambda^i_{r,s}dr + \partial^2_{ij}F(t,\lambda_t)\Lambda^{ji}_{t,s} \\ &+ \int_t^s \mathcal{D}\partial^2_{ij}F(\tau,\lambda_\tau)\Lambda^{ji}_{\tau,s}d\tau + \int_t^s \partial^3_{ijk}F(\tau,\lambda_\tau)\Lambda^{ji}_{\tau,s}\dot{\lambda}^k(\tau)d\tau. \end{aligned} \tag{4.3}$$

Note that:

$$\begin{aligned} 2R^F_{t,s}(\lambda) =& 2\int_t^s \mathcal{D}F(u,\lambda_u)du + 2\int_t^s \mathcal{D}\partial_i F(r,\lambda_r)\delta\lambda^i_{r,s}dr \\ &+ \partial^2_{ij}F(t,\lambda_t)\Lambda^{ji}_{t,s} + \partial^2_{ji}F(t,\lambda_t)\Lambda^{ij}_{t,s} \\ &+ \int_t^s \left(\mathcal{D}\partial^2_{ij}F(\tau,\lambda_\tau)\Lambda^{ji}_{\tau,s} + \mathcal{D}\partial^2_{ji}F(\tau,\lambda_\tau)\Lambda^{ij}_{\tau,s}\right)d\tau \\ &+ \int_t^s \left(\partial^3_{ijk}F(\tau,\lambda_\tau)\Lambda^{ji}_{\tau,s} + \partial^3_{jik}F(\tau,\lambda_\tau)\Lambda^{ij}_{\tau,s}\right)\dot{\lambda}^k(\tau)d\tau. \end{aligned} \tag{4.4}$$

By the symmetry of the second and third order derivatives, in the sense

$$\partial^2_{ij}F = \partial^2_{ji}F \text{ and } \partial^3_{ijk}F = \partial^3_{jik}F,$$

we may rewrite (4.4) as

$$\begin{aligned} R^F_{t,s}(\lambda) = &\int_t^s \mathcal{D}F(u,\lambda_u)du + \int_t^s \mathcal{D}\partial_i F(r,\lambda_r)\delta\lambda^i_{r,s}dr + \partial^2_{ij}F(t,\lambda_t)\tilde{\Lambda}^{ji}_{t,s} \\ &+ \int_t^s \mathcal{D}\partial^2_{ij}F(\tau,\lambda_\tau)\tilde{\Lambda}^{ji}_{\tau,s}d\tau + \int_t^s \partial^3_{ijk}F(\tau,\lambda_\tau)\tilde{\Lambda}^{ji}_{\tau,s}\dot{\lambda}^k(\tau)d\tau, \end{aligned}$$

where $\tilde{\Lambda}$ is the symmetric part of the matrix Λ:

$$\tilde{\Lambda}^{ji}_{t,s} := \frac{\Lambda^{ji}_{t,s} + \Lambda^{ij}_{t,s}}{2}.$$

Using partial integration, we evaluate

$$\Lambda^{ji}_{t,s} = \int_t^s (\lambda^j(u) - \lambda^j(t))\dot{\lambda}^i(u)du = (\lambda^j(u) - \lambda^j(t))(\lambda^i(u) - \lambda^i(t))\Big|_t^s$$
$$- \int_t^s (\lambda^i(u) - \lambda^i(t))\dot{\lambda}^j(u)du$$
$$= (\lambda^j(s) - \lambda^j(t))(\lambda^i(s) - \lambda^i(t)) - \Lambda^{ij}_{t,s}$$

and thus

$$\frac{\Lambda^{ji}_{t,s} + \Lambda^{ij}_{t,s}}{2} = \frac{1}{2}(\lambda^j(s) - \lambda^j(t))(\lambda^i(s) - \lambda^i(t)) = \frac{1}{2}\delta\lambda^j_{t,s}\delta\lambda^i_{t,s}.$$

Consequently,

$$\begin{aligned} R^F_{t,s}(\lambda) = &\int_t^s \mathcal{D}F(u,\lambda_u)du + \int_t^s \mathcal{D}\partial_i F(r,\lambda_r)\delta\lambda^i_{r,s}dr + \frac{1}{2}\partial^2_{ij}F(t,\lambda_t)\delta\lambda^j_{t,s}\delta\lambda^i_{t,s} \\ &+ \frac{1}{2}\int_t^s \mathcal{D}\partial^2_{ij}F(\tau,\lambda_\tau)\delta\lambda^j_{t,s}\delta\lambda^i_{t,s}d\tau + \frac{1}{2}\int_t^s \partial^3_{ijk}F(\tau,\lambda_\tau)\delta\lambda^j_{t,s}\delta\lambda^i_{t,s}\dot{\lambda}^k(\tau)d\tau. \end{aligned} \tag{4.5}$$

It is clear that we may decompose the last term in (4.5) as follows

$$\begin{aligned} \int_t^s \partial^3_{ijk}F(\tau,\lambda_\tau)\delta\lambda^i_{\tau,s}\delta\lambda^j_{\tau,s}\dot{\lambda}^k(\tau)d\tau = &\int_t^s \partial^3_{ijk}F(t,\lambda_t)\delta\lambda^i_{\tau,s}\delta\lambda^j_{\tau,s}\dot{\lambda}^k(\tau)d\tau \\ &+ \int_t^s \left(\partial^3_{ijk}F(\tau,\lambda_\tau) - \partial^3_{ijk}F(t,\lambda_t)\right)\delta\lambda^i_{\tau,s}\delta\lambda^j_{\tau,s}\dot{\lambda}^k(\tau)d\tau. \end{aligned} \tag{4.6}$$

We may rewrite the integrand of the first term, using the Fundamental Theorem of Calculus,

$$\begin{aligned} &\partial^3_{ijk}F(t,\lambda_t)\delta\lambda^i_{\tau,s}\delta\lambda^j_{\tau,s}\dot{\lambda}^k(\tau) = \partial^3_{ijk}F(t,\lambda_t)\delta\lambda^i_{\tau,s}\delta\lambda^j_{\tau,s}\frac{d}{d\tau}\left(\int_s^\tau \dot{\lambda}^k(v)dv\right) \\ =&\partial^3_{ijk}F(t,\lambda_t)\delta\lambda^i_{\tau,s}\delta\lambda^j_{\tau,s}\frac{d}{d\tau}\left(-\int_\tau^s \dot{\lambda}^k(v)dv\right) = -\partial^3_{ijk}F(t,\lambda_t)\delta\lambda^i_{\tau,s}\delta\lambda^j_{\tau,s}\frac{d}{d\tau}\lambda^k_{\tau,s} \end{aligned} \tag{4.7}$$

where $\lambda^k_{\tau,s} = \int_\tau^s \dot{\lambda}^k(v)dv$. Furthermore, we recall that by definition

$$\lambda^k_{\tau,s} = \delta\lambda^k_{\tau,s}.$$

As in (4.4), we once again exploit the symmetry of $\partial^3_{ijk}F$ in the indices, as well as the product rule for differentiation, to obtain

$$\begin{aligned}
&-\partial^3_{ijk}F(t,\lambda_t)\delta\lambda^i_{\tau,s}\delta\lambda^j_{\tau,s}\frac{d}{d\tau}\lambda^k_{\tau,s} \\
=&-\frac{1}{3}\partial^3_{ijk}F(t,\lambda_t)\left(\delta\lambda^i_{\tau,s}\delta\lambda^j_{\tau,s}\frac{d}{d\tau}\lambda^k_{\tau,s}+\delta\lambda^j_{\tau,s}\delta\lambda^k_{\tau,s}\frac{d}{d\tau}\lambda^i_{\tau,s}+\delta\lambda^k_{\tau,s}\delta\lambda^i_{\tau,s}\frac{d}{d\tau}\lambda^j_{\tau,s}\right) \qquad (4.8)\\
=&-\frac{1}{3}\partial^3_{ijk}F(t,\lambda_t)\frac{d}{d\tau}(\delta\lambda^i_{\tau,s}\delta\lambda^j_{\tau,s}\lambda^k_{\tau,s})
\end{aligned}$$

Therefore we write

$$\int_t^s \partial^3_{ijk}F(t,\lambda_t)\delta\lambda^i_{\tau,s}\delta\lambda^j_{\tau,s}\dot{\lambda}^k(\tau)d\tau=\frac{1}{3}\partial^3_{ijk}F(t,\lambda_t)\delta\lambda^i_{t,s}\delta\lambda^k_{t,s}\lambda^k_{t,s}.$$

Proceeding from (4.5),

$$\begin{aligned}
R^F_{t,s}(\lambda)=&\int_t^s \mathcal{D}F(u,\lambda_u)du+\int_t^s \mathcal{D}\partial_iF(r,\lambda_r)\delta\lambda^i_{r,s}dr+\frac{1}{2}\partial^2_{ij}F(t,\lambda_t)\delta\lambda^i_{t,s}\delta\lambda^j_{t,s}\\
&+\frac{1}{2}\int_t^s \mathcal{D}\partial^2_{ij}F(\tau,\lambda_\tau)\delta\lambda^i_{\tau,s}\delta\lambda^j_{\tau,s}d\tau+\frac{1}{6}\partial^3_{ijk}F(t,\lambda_t)\delta\lambda^i_{t,s}\delta\lambda^j_{t,s}\lambda^k_{t,s} \qquad (4.9)\\
&+\frac{1}{2}\int_t^s \left(\partial^3_{ijk}F(\tau,\lambda_\tau)-\partial^3_{ijk}F(t,\lambda_t)\right)\delta\lambda^i_{\tau,s}\delta\lambda^j_{\tau,s}\dot{\lambda}^k(\tau)d\tau.
\end{aligned}$$

Set $\lambda=\omega^N$ and as in Lemma 3.2, we estimate $R^F_{t,s}(\omega^N)$. In preparation, we define $\omega^N_{t,h}$ as the *horizontal extension* of ω^N on $[t,t+h]$ as in Chapter 2 and thus clearly for $\tau\in[t,s]$:

$$\omega^N_{t,\tau-t}(\tau\wedge u)=\omega^N_t(u),\ \forall u\in[0,T]. \qquad (4.10)$$

As already demonstrated above, we denote the stopped path of the horizontal extension of ω on $[t,t+h]$ at τ as $\omega^N_{t,\tau-t}(\tau\wedge\cdot)$ in order to avoid notation ambiguities. Nonetheless, in absence of a horizontal extension, the convention still holds to denote a stopped path of ω^N at t as ω^N_t. Since $\partial^3_{ijk}F$ is locally horizontally Lipschitz, we may exploit (4.10) as well as the the fact that $\partial^3_{ijk}F$ is a *non-anticipative* functional to obtain

$$\begin{aligned}
&|\partial^3_{ijk}F(\tau,\omega^N_{t,\tau-t})-\partial^3_{ijk}F(t,\omega^N_t)|=|\partial^3_{ijk}F(t+(\tau-t),\omega^N_{t,\tau-t})-\partial^3_{ijk}F(t,\omega^N_t)|\\
=&|\partial^3_{ijk}F(t+(\tau-t),\omega^N_{t,\tau-t}(\tau\wedge\cdot))-\partial^3_{ijk}F(t,\omega^N_t)| \qquad (4.11)\\
=&|F(t+(\tau-t),\omega^N_t)-F(t,\omega^N_t)|\le C_F|\tau-t|.
\end{aligned}$$

Since $\omega \in C^{\nu}([0,T],\mathbb{R}^d)$, $\partial^3_{ijk}F \in \mathrm{Lip}([0,T],\|\cdot\|_\infty)$, and using the bound in (4.11), we estimate with the triangle inequality:

$$\begin{aligned}
|\partial^3_{ijk}F(\tau,\omega^N_\tau) - \partial^3_{ijk}F(t,\omega^N_t)| \le & |\partial^3_{ijk}F(\tau,\omega^N_\tau) - \partial^3_{ijk}F(\tau,\omega^N_{t,\tau-t})| \\
& + |\partial^3_{ijk}F(\tau,\omega^N_{t,\tau-t}) - \partial^3_{ijk}F(t,\omega^N_t)| \\
\le & C_F\|\omega^N_\tau - \omega^N_{t,\tau-t}\|_\infty + C_F|\tau-t| \\
\le & C_{F,\|\omega\|_\nu}|\tau-t|^\nu + C_F|\tau-t| \\
\le & C_{F,\|\omega\|_\nu,T}|s-t|^\nu .
\end{aligned}$$

Returning to the expression (4.9), we exploit the local boundedness of

$$\mathcal{D}F,\ \mathcal{D}\partial_i F,\ \partial^2_{ij}F,\ \mathcal{D}\partial^2_{ij}F,\ \partial^3_{ijk}F$$

and the bounds in (3.10), (3.11) and (3.12) to obtain

$$\begin{aligned}
&\left| R^F_{t,s}(\omega^N) - \int_t^s \mathcal{D}F(u,\omega^N_u)du - \frac{1}{2}\partial^2_{ij}F(t,\omega_t)\delta\omega^i_{t,s}\delta\omega^j_{t,s} \right| \\
\le & \int_t^s |\mathcal{D}\partial_i F(r,\lambda_r)\delta\lambda^i_{r,s}|dr + \frac{1}{2}\int_t^s |\mathcal{D}\partial^2_{ij}F(\tau,\lambda_\tau)\delta\lambda^i_{\tau,s}\delta\lambda^j_{\tau,s}|d\tau \\
& + \frac{1}{6}|\partial^3_{ijk}F(t,\lambda_t)\delta\lambda^i_{t,s}\delta\lambda^j_{t,s}\lambda^k_{t,s}| \\
& + \frac{1}{2}\int_t^s \left| \left(\partial^3_{ijk}F(\tau,\lambda_\tau) - \partial^3_{ijk}F(t,\lambda_t)\right)\delta\lambda^i_{\tau,s}\delta\lambda^j_{\tau,s}\dot{\lambda}^k(\tau) \right| d\tau \\
\le & C_F|s-t|\cdot|\omega^N(r)-\omega^N(t)| + C_F|s-t|\cdot|\omega^N(\tau)-\omega^N(t)|^2 \\
& + C_F|\omega^N(s)-\omega^N(t)|^3 + C_{F,\|\omega\|_\nu}N^{1-\nu}|s-t|^{4\nu} \\
\le & C_{F,\|\omega\|_\nu}|s-t|^{1+\nu} + C_{F,\|\omega\|_\nu}|s-t|^{1+2\nu} + C_{F,\|\omega\|_\nu}|s-t|^{3\nu} + C_{F,\|\omega\|_\nu,T}N^{1-\nu}|s-t|^{4\nu} \\
\le & C_{F,\|\omega\|_\nu,T}|s-t|^{3\nu} + C_{F,\|\omega\|_\nu,T}N^{1-\nu}|s-t|^{4\nu},
\end{aligned} \tag{4.12}$$

where we use the definition ω^N as $\omega^N_t = \omega_t$ and $\omega^N(s) = \omega(s)$. Recall from (3.14) that

$$|R^F_{t,s}(\omega^N) - R^F_{t,s}(\omega)| \le C_F\|\omega\|_\nu|s-t|^\nu N^{-\nu}.$$

In summary, we obtain the estimate

$$\left| R^F_{t,s}(\omega) - \int_t^s \mathcal{D}F(u,\omega_u)du - \frac{1}{2}\partial^2_{ij}F(t,\omega_t)\delta\omega^i_{t,s}\delta\omega^j_{t,s} \right|$$

$$\leq \; |R^F_{t,s}(\omega) - R^F_{t,s}(\omega^N)| + \left| R^F_{t,s}(\omega^N) - \int_t^s \mathcal{D}F(u,\omega^N_u)du - \frac{1}{2}\partial^2_{ij}F(t,\omega_t)\delta\omega^i_{t,s}\delta\omega^j_{t,s} \right|$$

$$+ \int_t^s |\mathcal{D}F(u,\omega^N_u) - \mathcal{D}F(u,\omega_u)|du$$

$$\leq \; C_{F,\|\omega\|_\nu}|s-t|^\nu N^{-\nu} + C_{F,\|\omega\|_\nu,T}|s-t|^{3\nu} + C_{F,\|\omega\|_\nu,T}N^{1-\nu}|s-t|^{4\nu}$$

$$+ \int_t^s |\mathcal{D}F(u,\omega^N_u) - \mathcal{D}F(u,\omega_u)|du.$$

Since $N > 1$ is arbitrary and we may choose $t-s$ arbitrarily close to 0 by assumption, we optimize $N \approx |s-t|^{-3\nu}$. Furthermore, since $\mathcal{D}F$ is uniformly Lipschitz continuous

$$|\mathcal{D}F(u,\omega^N_u) - \mathcal{D}F(u,\omega_u)| \leq C_F\|\omega^N_u - \omega_u\|_\infty \leq C_F\|\omega^N - \omega\|_\infty$$

$$\leq C_{F,\|\omega\|_\nu}\frac{|s-t|^\nu}{N^\nu} \approx C_{F,\|\omega\|_\nu}|s-t|^{\nu+3\nu^2}.$$

We therefore obtain the bound

$$\left| R^F_{t,s}(\omega) - \int_t^s \mathcal{D}F(u,\omega_u)du - \frac{1}{2}\partial^2_{ij}F(t,\omega_t)\delta\omega^i_{t,s}\delta\omega^j_{t,s} \right|$$

$$\leq \int_t^s |\mathcal{D}F(u,\omega^N_u) - \mathcal{D}F(u,\omega_u)|du + C_{F,\|\omega\|_\nu,T}|s-t|^{\nu+3\nu^2}$$

$$\leq C_{F,\|\omega\|_\nu,T}|s-t|^{\nu+3\nu^2}.$$

Finally, note that

$$\left| R^F_{t,s}(\omega) - \int_t^s \mathcal{D}F(u,\omega_u)du - \frac{1}{2}\partial^2_{ij}F(t,\omega_t)\delta\omega^i_{t,s}\delta\omega^j_{t,s} \right|$$

$$\begin{aligned} =&|F(s,\omega_s) - F(t,\omega_t) - \nabla_\omega F(t,\omega_t)(\omega(s)-\omega(t)) - \int_t^s \mathcal{D}F(u,\omega_u)du \\ &- \frac{1}{2}\langle \nabla^2_\omega F(t,\omega_t), (\omega(s)-\omega(t))\ {}^t(\omega(s)-\omega(t))\rangle| =: M_{t,s}, \end{aligned}$$

and therefore $M_{t,s} = O(|s-t|^{\nu+3\nu^2})$ for $|t-s| \to 0$ uniformly. This proves the lemma. □

The lemma above will prove instrumental in the following theorem that underpins the pathwise nature of the integral. We place sufficient conditions under which the integral

$$\int_t^s \nabla_\omega F(u,\omega_u)d^\pi\omega(u) = \lim_{n\to\infty} \sum_{[t,s]\in\pi^n} \nabla_\omega F(t,\omega)(\omega(s)-\omega(t)).$$

Theorem 4.1 (Pathwise nature of the integral [1])**.** *Let* $\omega \in Q_\pi([0,T],\mathbb{R}^d)\cap C^\nu([0,T],\mathbb{R}^d)$ *with* $\nu > \frac{\sqrt{13}-1}{6}$ *to satisfy Assumption 2. Furthermore, assume* $F \in \mathbb{C}^{1,2}_b(\Lambda^d_T)$ *such that* $\nabla_\omega F,\ \nabla^2_\omega F \in \mathbb{C}^{1,1}_b(\Lambda^d_T)$*,* $F,\ \mathcal{D}F, \nabla^3_\omega F \in \mathrm{Lip}(\Lambda^d_T, \|\cdot\|_\infty)$ *and* $\nabla^3_\omega F$ *is horizontally locally Lipschitz. Then the pathwise integral defined in* (2.19) *is a limit of non-anticipative Riemann sums, in the sense:*

$$\int_0^T \nabla_\omega F(u,\omega)d^\pi\omega(u) = \lim_{n\to\infty} \sum_{[t,s]\in\pi^n} \nabla_\omega F(t,\omega)(\omega(s)-\omega(t)). \tag{4.13}$$

In particular, the integral depends on the path $\nabla_\omega F(\cdot,\omega)$ *so that*

$$\nabla_\omega F(\cdot,\omega) \equiv 0 \Rightarrow \int_0^T \nabla_\omega F(u,\omega)d^\pi\omega(u) = 0.$$

Proof. We denote

$$\begin{aligned} A^n_i :=& F(t^n_{i+1},\omega_{t^n_{i+1}}) - F(t^n_i,\omega_{t^n_i}) - \nabla_\omega F(t,\omega_{t^n_i})(\omega(t^n_{i+1}) - \omega(t^n_i)) \\ &- \int_{t^n_i}^{t^n_{i+1}} \mathcal{D}F(u,\omega)du - \frac{1}{2}\langle \nabla^2_\omega F(t,\omega_{t^n_i}), \delta\omega_{t^n_i,t^n_{i+1}}\ {}^t\delta\omega_{t^n_i,t^n_{i+1}}\rangle. \end{aligned} \tag{4.14}$$

Lemma 4.1 implies that for a large enough $N \in \mathbb{N}$ with a sufficiently small mesh size $|\pi^N|$:

$$|A_i^n| \leq C|t_{i+1}^n - t_i^n|^{3\nu^2+\nu} \leq C|\pi^n|^{3\nu^2+\nu}, \ \forall n \geq N, \ \forall t_i^n \in \pi^n,$$

thus

$$\max_i |A_i^n| \xrightarrow{n\to\infty} 0.$$

The choice of $\nu > 0$ implies $3\nu^2 + \nu > 1$, such that

$$\sum_{t_i^n \in \pi^n} |A_i^n| \leq C(\max_i |A_i^n|)^{1-\frac{1}{3\nu^2+\nu}} \sum_{t_i^n \in \pi^n} |t_{i+1}^n - t_i^n| \leq CT \max_{t_i^n \in \pi^n} |A_i^n|)^{1-\frac{1}{3\nu^2+\nu}} \xrightarrow{n\to\infty} 0,$$

therefore

$$\sum_{t_i^n \in \pi^n} A_i^n \xrightarrow{n\to\infty} 0.$$

From the above as well as the change of variable formula in (2.19), the limit of Riemann sums must exist and satisfies

$$\begin{aligned}\lim_{n\to\infty} \sum_{t_i^n \in \pi^n} \nabla_\omega F(t, \omega_{t_i^n})(\omega(t_{i+1}^n) - \omega(t_i^n)) &= F(T, \omega) - F(0, \omega) - \int_0^T \mathcal{D}F(u, \omega) du \\ - \frac{1}{2} \int_0^T \langle \nabla_\omega^2 F(t, \omega), d[\omega]_\pi(t) \rangle &= \int_0^T \nabla_\omega F(t, \omega) d^\pi \omega(t).\end{aligned}$$

□

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Appendix A

Results in Real Analysis and Stochastic Calculus

Definition A.1 (Lebesgue-Stieltjes Integral)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a right-continuous and monotone increasing function and $\mathcal{I} := \{(a,b] : a < b\}$ the semiring of left half-open intervals. Then we uniquely extend the Lebesgue-Stieltjes premeasure $\tilde{\mu}_f$ on $\mathcal{I}$ where $\tilde{\mu}_f((a,b]) = f(b) - f(a)$ to the Lebesgue-Stieltjes measure μ_f on $\mathcal{B}(\mathbb{R})$ and define the *Lebesgue-Stieltjes* integral

$$\int_{\mathbb{R}} g(x)df(x) := \int_{\mathbb{R}} g d\mu_f$$

Theorem A.1 (Dominated Convergence Theorem [8])**.** *Consider the measure space $(\Omega, \mathcal{F}, \mu)$. Let f and a sequence $(f_n)_{n\in\mathbb{N}}$ be some arbitrary functions from Ω to $\mathbb{R}$ that are $\mathcal{F}$-measurable such that*

$$f_n \xrightarrow{n\to\infty} f \ \mu\text{-a.s.}$$

If there exists some $g \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu)$ such that $|f_n| \leq g$, μ-a.s. for all $n \in \mathbb{N}$, then

$$f_n,\ f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu)\ \ \forall n \in \mathbb{N}.$$

Crucially,

$$\lim_{n\to\infty} \int_{\Omega} f_n d\mu = \int_{\Omega} f d\mu,$$

and

$$\lim_{n\to\infty} \int_{\Omega} |f_n - f| d\mu = 0.$$

Proposition A.1. *Let (X, d) be a metric space. If $K \subseteq X$ is compact and $L \subseteq X$ is closed such that $K \cap L = \varnothing$, then* $\operatorname{dist}(K, L) := \inf\{d(x,y) : x \in K, y \in L\} > 0$.

Proof. Consider the map $d_L : K \to \mathbb{R}$, where $d_L(k) := \inf\{d(k,l) : l \in L\}$. For any $k, k' \in K$ and $l \in L$, we have from the triangle inequality in metric spaces:

$$d(k,l) \leq d(k,k') + d(k',l),$$

and

$$d(k',l) \leq d(k',k) + d(k,l).$$

By definition of d_L, we obtain

$$d_L(k) \leq d(k,k') + d(k',l),$$

and

$$d_L(k') \leq d(k',k) + d(k,l).$$

Since $l \in L$ is arbitrary, we obtain with the symmetry of metric spaces,

$$|d_L(k) - d_L(k')| \leq d(k,k').$$

d_L is thus Lipschitz-continuous on K, and in particular continuous. Since K is compact, d_L must take on a minimum, so assume that $\exists \tilde{k} \in K$ such that $d_L(\tilde{k}) = 0$. By definition of the infimum, there exists a sequence $(l_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} d(l_n, \tilde{k}) = 0.$$

Clearly $(l_n)_{n\in\mathbb{N}}$ is a convergent sequence in L to $\tilde{k}$. L is closed by assumption such that $\tilde{k} \in L$. This contradicts the assumption that $K \cap L = \varnothing$. $\square$

Theorem A.2 (Quadratic Variation of Brownian Motion [20])**.** *Let B be a Brownian motion, then $\langle B\rangle_T = T$ almost surely along a sequence of partitions $\boldsymbol{\pi}$ of vanishing mesh.*

Proof. Let $\pi := \{t_0, ..., t_{k_n}\}$ be some partition of $[0,T]$. We compute

$$\mathbb{E}[(B(t_{j+1}) - B(t_j))^2] = \mathrm{Var}\,(B(t_{j+1}) - B(t_j)) = t_{j+1} - t_j,$$

since $B_t - B_s \sim \mathcal{N}(0, t-s)$ for any $t > s$. This implies

$$\begin{aligned}\mathbb{E}[V_n(B)] = \mathbb{E}\left[\sum_{i=0}^{k_n-1}(B(t_{i+1}) - B(t_i))^2\right] &= \sum_{i=0}^{k_n-1}\mathbb{E}\left[(B(t_{i+1}) - B(t_i))^2\right] \\ &= \sum_{i=0}^{k_n-1}(t_{i+1} - t_i) = t_{k_n} - t_0 = T\end{aligned}$$

Moreover,

$$\mathrm{Var}(V_n(B)) = \mathbb{E}\left[(V_n(B) - \mathbb{E}[V_n(B)])^2\right] = \mathbb{E}\left[(V_n(B) - T)^2\right].$$

Since L^2-convergence implies almost sure convergence, it thus suffices to prove that

$$\mathrm{Var}(V_n(B)) \xrightarrow{n\to\infty} 0.$$

By the independence of increments, we obtain

$$\mathrm{Var}(V_n(B)) = \sum_{i=0}^{k_n-1} \mathrm{Var}\left((B(t_{i+1}) - B(t_i))^2\right),$$

and for any i:

$$\begin{aligned}\mathrm{Var}\left((B(t_{i+1}) - B(t_i))^2\right) &= \mathbb{E}\left[\left((B(t_{i+1}) - B(t_i))^2 - (t_{i+1} - t_i)\right)^2\right] \\ =&\mathbb{E}\left[(B(t_{i+1}) - B(t_i))^4\right] - 2(t_{i+1} - t_i)\mathbb{E}[(B(t_{i+1}) - B(t_i))^2] + (t_{i+1} - t_i)^2\end{aligned}$$

Since $B(t_{i+1}) - B(t_i) \sim \mathcal{N}(0, t_{i+1} - t_i)$:

$$\mathbb{E}\left[(B(t_{i+1}) - B(t_i))^4\right] = 3(t_{i+1} - t_i)^2,$$

and thus

$$\mathrm{Var}\left((B(t_{i+1}) - B(t_i))^2\right) = 3(t_{i+1} - t_i)^2 - 2(t_{i+1} - t_i)^2 + (t_{i+1} - t_i) = 2(t_{i+1} - t_i)^2.$$

In summary,

$$\mathrm{Var}(V_n(B)) = \sum_{i=0}^{k_n-1} 2(t_{i+1} - t_i) \leq 2|\pi^n| \sum_{i=0}^{k_n-1} (t_{i+1} - t_i) = 2|\pi^n|T.$$

Since $\boldsymbol{\pi}$ is assumed to be of vanishing mesh, this proves the theorem. $\square$

Appendix B

Results for the Change of Variable Formula

Lemma B.1. [10] *Let f be a bounded left-continuous function defined on $[0,T]$ and μ_n a sequence of Radon measures on $[0,T]$ such that μ_n converges weakly to a Radon measure μ with no atoms. Then for all $0 \leq s < t \leq T$, with $\mathcal{I}$ being $[t,s]$, $(s,t]$, $[s,t)$ or (s,t):*

$$\lim_{n\to\infty}\int_{\mathcal{I}} f(u)d\mu_n(u) = \int\limits_{\mathcal{I}} f(u)d\mu(u) \tag{B.1}$$

Proof. Let M be an upper bound for $|f|$, $F_n(t) = \mu_n([0,t])$ and $F(t) = \mu([0,t])$ the cumulative distribution functions with respect to μ_n and μ, respectively. For $\varepsilon > 0$ and $u \in (s,t]$, define:

$$\eta(u) = \inf\{h > 0 : |f(u-h) - f(u)| \geq \varepsilon\} \wedge u$$

and we have $\eta(u) > 0$ through the right-continuity of f. In this sense, we define $\theta(u)$:

$$\theta(u) = \inf\left\{h > 0 : |f(u-h) - f(u)| \geq \frac{\varepsilon}{2}\right\} \wedge u.$$

The uniform continuity of F on $[0,T]$ implies, there exists $\zeta(u)$ so that $\forall v \in [T - \zeta(u), T]$, $F(v + \zeta(u)) - F(v) < \varepsilon\nu(u)$. We obtain the finite covering

$$[t,s] \subset \bigcup_{i=0}^{N}(u_i - \theta(u_i), u_i + \zeta(u_i)),$$

where the u_i's are increasing order. We select $u_0 = s$ and $u_N = t$. Define the decreasing v_j as follows: $v_0 = t$, and when v_j has been constructed, choose the minimum index $i(j)$ such that $v_j \in (u_{i(j)}, u_{i(j)+1}]$, then either $u_{i(j)} \leq v_j - \eta(v_j)$ and in this case $v_{j+1} = u_{i(j)}$, else $u_{i(j)} > v_j - \eta(v_j)$, and in this case $v_{j+1} = \max\{v_j - \eta(v_j), s\}$. Stop the procedure when s is reached, and denote M the maximum index of the v_j. Define the following piecewise constant approximation of f on $[s,t]$:

$$g(u) = \sum_{j=0}^{M-1} f(v_j)\mathbf{1}_{(v_{j+1},v_j]}(u) \tag{B.2}$$

Denote J_1 the set of indices j where v_{j+1} has been constructed as in the first case, and J_2 as its complement. If $j \in J_1$, $|f(u) - g(u)| < \varepsilon$ on the interval $[v_j - \eta(v_j), v_j]$ and $v_j - \eta(u_{i(j)}) - v_{j+1} < \zeta(u_{i(j)+1}) = \zeta(v_{j+1})$, because of the remark that $v_j - \eta_{v_j} < u_{i(j)} - \theta(u_{i(j)})$. Therefore:

$$\int_{(v_j,v_{j+1}]} |f(u) - g(u)| d\mu(u) \le \varepsilon[F(v_{j+1}) - F(v_j)] + 2M\varepsilon\eta(v_{j+1}). \tag{B.3}$$

If $j \in J_2$, $|f(u) - g(u)| < \varepsilon$ on $[v_{j+1}, v_j]$. Summing up all the terms,

$$\int_{[s,t]} |f(u) - g(u)| d\mu(u) \le \varepsilon(F(t) - F(s) + 2M(t-s)), \tag{B.4}$$

since $\eta(v_j) \le v_j - v_{j+1}$ for $j < M$. Exploiting the same argument with respect to μ_n yields

$$\begin{aligned} \int_{[s,t]} |f(u) - g(u)| d\mu_n(u) \le \varepsilon[F_n(t) - F_n(s-)] \\ + 2M \sum_{j=0}^{M-1} F_n(v_{j+1}) - F_n(v_{j+1} - \zeta(v_{j+1})). \end{aligned} \tag{B.5}$$

Since $F_n(u)$ converges to $F(u)$ for every u, the lim sup of (B.5) satisfies (B.4). On the other hand, it is immediately observed that

$$\lim_{n\to\infty} \int_{\mathcal{I}} g(u) d\mu_n(u) = \int_{\mathcal{I}} g(u) d\mu(u), \tag{B.6}$$

since $F_n(u)$ and $F_n(u-)$ both converge to $F(u)$ as μ has no atoms (g is a linear combination of indicators of intervals). This establishes the lemma. □

Lemma B.2. [10] *Let $(f_n)_{n\in\mathbb{N}}$, with f left-continuous on $[0,T]$, satisfying:*

$$\forall t \in [0,T],\ \lim_{n\to\infty} f_n(t) = f(t) \ \ \forall t \in [0,T],\ f_n(t) \le K. \tag{B.7}$$

Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of Radon measures on $[0,T]$ such that μ_n converges weakly to a Radon measure μ with no atoms. Then $\forall 0 \le s < t \le R$, where $\mathcal{I}$ is $[s,t]$, $(s,t]$, $[s,t)$, or (s,t):

$$\int_{\mathcal{I}} f_n(u) d\mu_n(u) \xrightarrow{n\to\infty} \int_s^t f(u) d\mu(u) \tag{B.8}$$

Proof. Let $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that $\mu(\{\sup_{m\geq n_0}|f_m - f| > \varepsilon\}) < \varepsilon$. Note that

$$\{\sup_{m\geq n_0}|f_m - f| > \varepsilon\} = \bigcup_{m\geq n_0}\{|f_m - f| > \varepsilon\}$$

and that for any $m \in \mathbb{N}$, we have $|f_m - f|$ left-continuous. In particular,

$$\{|f_m - f| > \varepsilon\}$$

is a countable union of disjoint intervals for any $m \in \mathbb{N}$, and therefore

$$\{\sup_{m\geq n_0}|f_m - f| > \varepsilon\}$$

is countable union of disjoint intervals. Since μ is assumed to have no atoms, $\{\sup_{m\geq n_0}|f_m - f| > \varepsilon\}$ is a continuity set such that the Portmanteau Lemma implies $\lim_{n\to\infty}\mu_n(\{\sup_{m\geq n_0}|f_m - f| > \varepsilon\}) = \mu(\{\sup_{m\geq n_0}|f_m - f| > \varepsilon\}) < \varepsilon$. Therefore, for $n \geq n_0$, we have

$$\begin{aligned}
&\int_{\mathcal{I}}|f_n(u) - f(u)|d\mu_n(u)\\
=&\int_{\mathcal{I}}|f_n(u) - f(u)|\mathbf{1}_{\{\sup_{n\geq n_0}|f_n - f|>\varepsilon\}}d\mu_n(u) + \int_{\mathcal{I}}|f_n(u) - f(u)|\mathbf{1}_{\{\sup_{m\geq n_0}|f_n - f|\leq\varepsilon\}}d\mu_n(u)\\
\leq&2K\mu_n\left(\{\sup_{n\geq n_0}|f_n - f| > \varepsilon\}\right) + \varepsilon\mu_n(\mathcal{I})
\end{aligned}$$

The $\limsup$ of the last term above is bounded by

$$2K\mu(\{\sup_{m\geq n_0}|f_m - f| > \varepsilon\}) + \varepsilon\mu(\mathcal{I}) \leq (2K + \mu(\mathcal{I}))\varepsilon. \tag{B.9}$$

As $\varepsilon > 0$ is arbitrary, we obtain

$$\limsup_{n\to\infty}\int_{\mathcal{I}}|f_n(u) - f(u)|d\mu_n(u) = 0. \tag{B.10}$$

Furthermore since $|f_n(u) - f(u)| \geq 0$, we have

$$\liminf_{n\to\infty}\int_{\mathcal{I}}|f_n(u) - f(u)|d\mu_n(u) \geq 0,$$

thus

$$\lim_{n\to\infty}\int_{\mathcal{I}}|f_n(u) - f(u)|d\mu_n(u) = 0.$$

Recalling from Lemma B.1 that

$$\lim_{n\to\infty} \int_{\mathcal{I}} f(u)d\mu_n(u) = \int_{\mathcal{I}} f(u)d\mu(u),$$

we evaluate

$$\begin{aligned}
&\int_{\mathcal{I}} f_n(u)d\mu_n(u) - \int_s^t f(u)d\mu(u) \\
&= \int_{\mathcal{I}} f_n(u)d\mu_n(u) - \int_{\mathcal{I}} f(u)d\mu_n(u) + \int_{\mathcal{I}} f(u)d\mu_n(u) - \int_s^t f(u)d\mu(u) \\
&\leq \int_{\mathcal{I}} |f_n(u) - f(u)|d\mu_n(u) + \int_{\mathcal{I}} f(u)d\mu_n(u) - \int_s^t f(u)d\mu(u).
\end{aligned}$$

Therefore

$$\int_{\mathcal{I}} f_n(u)d\mu_n(u) \xrightarrow{n\to\infty} \int_s^t f(u)d\mu(u),$$

completing the proof. $\square$