Contents: Computer Science and Engineering Stream

LAB 1: Programme to compute area, volume and center of gravity

1.1 Objectives:

Use python

1. to evaluate double integration.
2. to compute area and volume.
3. to calculate center of gravity of 2D object.

Syntax for the commands used:

1. Data pretty printer in Python:

```
pprint()
```

2. integrate:

```
integrate(function,(variable, min_limit, max_limit))
```

1.2 Double and triple integration

Example 1:

Evaluate the integral $\int_0^1 \int_0^x (x^2 + y^2) dy dx$

```
from sympy import *
x,y,z=symbols('x y z')
w1=integrate(x**2+y**2,(y,0,x),(x,0,1))
print(w1)
```

```
1/3
```

Example 2:

Evaluate the integral $\int_0^3 \int_0^{3-x} \int_0^{3-x-y} (xyz) dz dy dx$

```
from sympy import *
x=Symbol('x')
y=Symbol('y')
z=Symbol('z')
w2=integrate((x*y*z),(z,0,3-x-y),(y,0,3-x),(x,0,3))
print(w2)
```

```
81/80
```

Example 3:

Prove that $\int\int(x^2+y^2)dydx = \int\int(x^2+y^2)dxdy$

```
from sympy import *
x=Symbol('x')
y=Symbol('y')
z=Symbol('z')
w3=integrate(x**2+y**2,y,x)
pprint(w3)
w4=integrate(x**2+y**2,x,y)
pprint(w4)
```

1.3 Area and Volume

Area of the region R in the cartesian form is $\int\limits_R\int dxdy$

Example 4:

Find the area of an ellipse by double integration. A=4 $\int\limits_0^a \int\limits_0^{(b/a)\sqrt{a^2-x^2}} dydx$

```
from sympy import *
x=Symbol('x')
y=Symbol('y')
#a=Symbol('a')
#b=Symbol('b')
a=4
b=6
w3=4*integrate(1,(y,0,(b/a)*sqrt(a**2-x**2)),(x,0,a))
print(w3)
```

```
24.0*pi
```

1.4 Area of the region R in the polar form is $\int\limits_R\int rdrd\theta$

Example 5:

Find the area of the cardioid $r = a(1 + cos\theta)$ by double integration

```
from sympy import *
r=Symbol('r')
t=Symbol('t')
a=Symbol('a')
#a=4

w3=2*integrate(r,(r,0,a*(1+cos(t))),(t,0,pi))
pprint(w3)
```

1.5 Volume of a solid is given by $\int\limits_V \int \int dxdydz$

Example 6:

Find the volume of the tetrahedron bounded by the planes x=0,y=0 and z=0, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

```
from sympy import *
x=Symbol('x')
y=Symbol('y')
z=Symbol('z')
a=Symbol('a')
b=Symbol('b')
c=Symbol('c')
w2=integrate(1,(z,0,c*(1-x/a-y/b)),(y,0,b*(1-x/a)),(x,0,a))
print(w2)
```

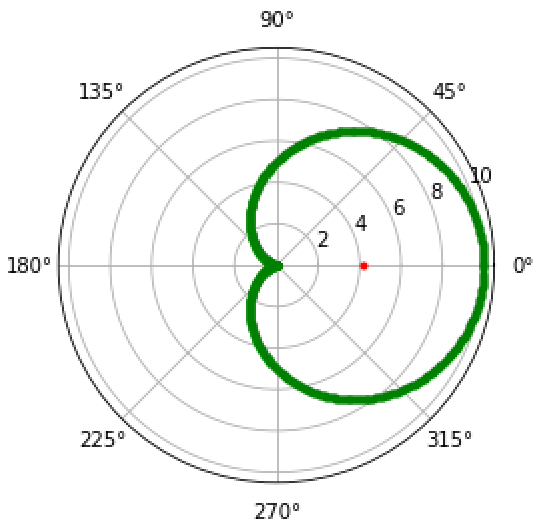
```
a*b*c/6
```

1.6 Center of Gravity

Find the center of gravity of cardioid . Plot the graph of cardioid and mark the center of gravity.

```
import numpy as np
import matplotlib.pyplot as plt
import math
from sympy import *
r=Symbol('r')
t=Symbol('t')
a=Symbol('a')
I1=integrate(cos(t)*r**2,(r,0,a*(1+cos(t))),(t,-pi,pi))
I2=integrate(r,(r,0,a*(1+cos(t))),(t,-pi,pi))
I=I1/I2
print(I)
I=I.subs(a,5)
plt.axes(projection = 'polar')
a=5


rad = np.arange(0, (2 * np.pi), 0.01)

# plotting the cardioid
for i in rad:
    r = a + (a*np.cos(i))
    plt.polar(i,r,'g.')

plt.polar(0,I,'r.')
plt.show()
```

```
5*a/6
```



1.7 Exercise:

1. Evaluate $\int_0^1 \int_0^x (x+y)dydx$
   Ans: 0.5

2. Find the $\int_0^{log(2)} \int_0^x \int_0^{x+log(y)} (e^{x+y+z})dzdydx$
   Ans: -0.2627

3. Find the area of positive quadrant of the circle $x^2 + y^2 = 16$
   Ans: 4π

4. Find the volume of the tetrahedron bounded by the planes x=0,y=0 and z=0, $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$
   Ans: 4

LAB 2: Evaluation of improper integrals, Beta and Gamma functions

2.1 Objectives:

Use python

1. to evaluate improper integrals using Beta function.
2. to evaluate improper integrals using Gamma function.

Syntax for the commands used:

1. `gamma`
   ```
   math.gamma(x)
   ```
   Parameters :

   `x` : The number whose gamma value needs to be computed.
2. `beta`
   ```
   math.beta(x,y)
   ```
   Parameters :

   `x` ,y: The numbers whose beta value needs to be computed.
3. **Note:** We can evaluate improper integral involving infinity by using `inf`.

Example 1:

Evaluate $\int\limits_0^\infty e^{-x}dx$.

```
from sympy import *
x=symbols('x')
w1=integrate(exp(-x),(x,0,float('inf')))
print(simplify(w1))
```

1

Gamma function is $x(n) = \int_0^\infty e^{-x}x^{n-1}dx$

Example 2:

Evaluate $\Gamma(5)$ by using definition

```
from sympy import *
x=symbols('x')
w1=integrate(exp(-x)*x**4,(x,0,float('inf')))
print(simplify(w1))
```

24

Example 3:

Evaluate $\int_0^\infty e^{-st}\cos(4t)dt$. That is Laplace transform of $\cos(4t)$

```
from sympy import *
t,s=symbols('t,s')
# for infinity in sympy we use oo
w1=integrate(exp(-s*t)*cos(4*t),(t,0,oo))
display(simplify(w1))
```

$$\begin{cases} \frac{s}{s^2+16} & \text{for } 2\left|\arg(s)\right| < \pi \\ \int_0^\infty e^{-st}\cos(4t)\,dt & \text{otherwise} \end{cases}$$

Example 4:

Find Beta(3,5), Gamma(5)

```
#beta and gamma functions
from sympy import beta, gamma
m=input('m :');
n=input('n :');
m=float(m);
n=float(n);
s=beta(m,n);
t=gamma(n)
print('gamma (',n,') is %3.3f'%t)
print('Beta (',m,n,') is  %3.3f'%s)
```

```
m :3
n :5
gamma ( 5.0 ) is 24.000
Beta ( 3.0 5.0 ) is  0.010
```

Example 5:

Calculate Beta(5/2,7/2) and Gamma(5/2).

```
#beta and gamma functions
# If the number is a fraction give it in decimals. Eg 5/2=2.5
from sympy import beta, gamma
m=float(input('m : '));
n=float(input('n :'));

s=beta(m,n);
t=gamma(n)
print('gamma (',n,') is %3.3f'%t)
print('Beta (',m,n,') is %3.3f '%s)
```

```
m : 2.5
n :3.5
gamma ( 3.5 ) is 3.323
Beta ( 2.5 3.5 ) is 0.037
```

Example 6:

Verify that $Beta(m, n) = Gamma(m)Gamma(n)/Gamma(m + n)$ for m=5 and n=7

```
from sympy import beta, gamma
m=5;
n=7;
m=float(m);
n=float(n);
s=beta(m,n);
t=(gamma(m)*gamma(n))/gamma(m+n);
print(s,t)
if (abs(s-t)<=0.00001):
    print('beta and gamma are related')
else:
    print('given values are wrong')
```

```
0.0004329004329004329004329 0.0004329004329004329004329
beta and gamma are related
```

2.2 Exercise:

1. Evaluate $\int\limits_0^\infty e^{-t}cos(2t)dt$
   Ans: 1/5

2. Find the value of Beta(5/2,9/2)
   Ans: 0.0214

3. Find the value of Gamma(13)
   Ans: 479001600

4. Verify that $Beta(m, n) = Gamma(m)Gamma(n)/Gamma(m + n)$ for m=7/2 and n=11/2
   Ans: True

LAB 3: Finding gradient, divergent, curl and their geometrical interpretation

1.1 Objectives:

Use python

1. to find the gradient of a given scalar function.
2. to find find divergence and curl of a vector function.

1.2 Method I:

To find gradient of $\phi = x^2y + 2xz - 4$.

```
#To find gradient of scalar point function.
from sympy.vector import *
from sympy import symbols
N=CoordSys3D('N') #Setting the coordinate system
x,y,z=symbols('x y z')
A=N.x**2*N.y+2*N.x*N.z-4 #Variables x,y,z to be used with coordinate
                                    system N
delop=Del() #Del operator
display(delop(A)) #Del operator applied to A
gradA=gradient(A) #Gradient function is used
print(f"\n Gradient of {A} is \n")
display(gradA)
```

$$\left(\frac{\partial}{\partial \mathbf{x_N}}(\mathbf{x_N}^2\mathbf{y_N} + 2\mathbf{x_N}\mathbf{z_N} - 4)\right)\hat{\mathbf{i}}_\mathbf{N} + \left(\frac{\partial}{\partial \mathbf{y_N}}(\mathbf{x_N}^2\mathbf{y_N} + 2\mathbf{x_N}\mathbf{z_N} - 4)\right)\hat{\mathbf{j}}_\mathbf{N} + \left(\frac{\partial}{\partial \mathbf{z_N}}(\mathbf{x_N}^2\mathbf{y_N} + 2\mathbf{x_N}\mathbf{z_N} - 4)\right)\hat{\mathbf{k}}_\mathbf{N}$$

```
Gradient of N.x**2*N.y + 2*N.x*N.z - 4 is
```

$$(2\mathbf{x_N}\mathbf{y_N} + 2\mathbf{z_N})\hat{\mathbf{i}}_\mathbf{N} + (\mathbf{x_N}^2)\hat{\mathbf{j}}_\mathbf{N} + (2\mathbf{x_N})\hat{\mathbf{k}}_\mathbf{N}$$

To find divergence of $\vec{F} = x^2yz\hat{i} + y^2zx\hat{j} + z^2xy\hat{k}$

```
#To find divergence of a vector point function
from sympy.vector import *
from sympy import symbols
N=CoordSys3D('N')
x,y,z=symbols('x y z')
A=N.x**2*N.y*N.z*N.i+N.y**2*N.z*N.x*N.j+N.z**2*N.x*N.y*N.k
delop=Del()
divA=delop.dot(A)
display(divA)

print(f"\n Divergence of {A} is \n")
display(divergence(A))
```

$$\frac{\partial}{\partial \mathbf{z_N}} \mathbf{x_N y_N z_N}^2 + \frac{\partial}{\partial \mathbf{y_N}} \mathbf{x_N y_N}^2 \mathbf{z_N} + \frac{\partial}{\partial \mathbf{x_N}} \mathbf{x_N}^2 \mathbf{y_N z_N}$$

```
Divergence of N.x**2*N.y*N.z*N.i + N.x*N.y**2*N.z*N.j + N.x*N.y*N.z**2*N.k is
```

$$6\mathbf{x_N y_N z_N}$$

To find curl of $\vec{F} = x^2yz\hat{i} + y^2zx\hat{j} + z^2xy\hat{k}$

```
#To find curl of a vector point function
from sympy.vector import *
from sympy import symbols
N=CoordSys3D('N')
x,y,z=symbols('x y z')
A=N.x**2*N.y*N.z*N.i+N.y**2*N.z*N.x*N.j+N.z**2*N.x*N.y*N.k
delop=Del()
curlA=delop.cross(A)
display(curlA)

print(f"\n Curl of {A} is \n")
display(curl(A))
```

$$\left(\frac{\partial}{\partial \mathbf{y_N}} \mathbf{x_N y_N z_N}^2 - \frac{\partial}{\partial \mathbf{z_N}} \mathbf{x_N y_N}^2 \mathbf{z_N}\right) \hat{\mathbf{i}}_\mathbf{N} + \left(-\frac{\partial}{\partial \mathbf{x_N}} \mathbf{x_N y_N z_N}^2 + \frac{\partial}{\partial \mathbf{z_N}} \mathbf{x_N}^2 \mathbf{y_N z_N}\right) \hat{\mathbf{j}}_\mathbf{N} + \left(\frac{\partial}{\partial \mathbf{x_N}} \mathbf{x_N y_N}^2 \mathbf{z_N} - \frac{\partial}{\partial \mathbf{y_N}} \mathbf{x_N}^2 \mathbf{y_N z_N}\right) \hat{\mathbf{k}}_\mathbf{N}$$

```
Curl of N.x**2*N.y*N.z*N.i + N.x*N.y**2*N.z*N.j + N.x*N.y*N.z**2*N.k is
```

$$\left(-\mathbf{x_N y_N}^2 + \mathbf{x_N z_N}^2\right) \hat{\mathbf{i}}_\mathbf{N} + \left(\mathbf{x_N}^2 \mathbf{y_N} - \mathbf{y_N z_N}^2\right) \hat{\mathbf{j}}_\mathbf{N} + \left(-\mathbf{x_N}^2 \mathbf{z_N} + \mathbf{y_N}^2 \mathbf{z_N}\right) \hat{\mathbf{k}}_\mathbf{N}$$

1.3 Method II:

To find gradient of $\phi = x^2yz$.

```
#To find gradient of a scalar point function x^2yz
from sympy.physics.vector import *
from sympy import var,pprint
var('x,y,z')
v=ReferenceFrame('v')
F=v[0]**2*v[1]*v[2]
G=gradient(F,v)
F=F.subs([(v[0],x),(v[1],y),(v[2],z)])
print("Given scalar function F=")
display(F)
G=G.subs([(v[0],x),(v[1],y),(v[2],z)])
print("\n Gradient of F=")
display(G)
```

Given scalar function F=

$$x^2yz$$

Gradient of F=

$$2xyz\hat{\mathbf{v}}_{\mathbf{x}} + x^2z\hat{\mathbf{v}}_{\mathbf{y}} + x^2y\hat{\mathbf{v}}_{\mathbf{z}}$$

To find divergence of $\vec{F} = x^2y\hat{i} + yz^2\hat{j} + x^2z\hat{k}$.

```
#To find divergence of F=x^2yi+yz^2j+x^2zk
from sympy.physics.vector import *
from sympy import var
var('x,y,z')
v=ReferenceFrame('v')
F=v[0]**2*v[1]*v.x+v[1]*v[2]**2*v.y+v[0]**2*v[2]*v.z
G=divergence(F,v)
F=F.subs([(v[0],x),(v[1],y),(v[2],z)])
print("Given vector point function is ")
display(F)

G=G.subs([(v[0],x),(v[1],y),(v[2],z)])
print("Divergence of F=")
display(G)
```

Given vector point function is

$$x^2y\hat{\mathbf{v}}_{\mathbf{x}} + yz^2\hat{\mathbf{v}}_{\mathbf{y}} + x^2z\hat{\mathbf{v}}_{\mathbf{z}}$$

Divergence of F=

$$x^2 + 2xy + z^2$$

To find curl of $\vec{F} = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$

```
#To find curl of F=xy^2i+2x^2yzj-3yz^2k
from sympy.physics.vector import *
from sympy import var
var('x,y,z')
v=ReferenceFrame('v')
F=v[0]*v[1]**2*v.x+2*v[0]**2*v[1]*v[2]*v.y-3*v[1]*v[2]**2*v.z
G=curl(F,v)
F=F.subs([(v[0],x),(v[1],y),(v[2],z)])
print("Given vector point function is ")
display(F)

G=G.subs([(v[0],x),(v[1],y),(v[2],z)])
print("curl of F=")
display(G)
```

Given vector point function is

$$xy^2\,\hat{\mathbf{v}}_{\mathbf{x}} + 2x^2yz\hat{\mathbf{v}}_{\mathbf{y}} - 3yz^2\,\hat{\mathbf{v}}_{\mathbf{z}}$$

curl of F=

$$(-2x^2y - 3z^2)\hat{\mathbf{v}}_{\mathbf{x}} + (4xyz - 2xy)\hat{\mathbf{v}}_{\mathbf{z}}$$

1.4 Exercise:

1. If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$, find grad$u$, grad$v$ and grad$w$.
   Ans: $\hat{\imath} + \hat{\jmath} + \hat{k}$, $2(x\hat{\imath} + y\hat{\jmath} + z\hat{k})$, $(y + z)\hat{\imath} + (z + x)\hat{\jmath} + (z + x)\hat{k}$.

2. Evaluate div$F$ and curl$F$ at the point (1,2,3), given that $\vec{F} = x^2yz\hat{\imath} + xy^2z\hat{\jmath} + xyz^2\hat{k}$.
   Ans: $6xyz$, $x(z^2 - y^2)\hat{\imath} + y(x^2 - z^2)\hat{\jmath} + z(y^2 - x^2)\hat{k}$.

3. Prove that the vector $(yz - x^2)\hat{\imath} + (4y - z^2x)\hat{\jmath} + (2xz - 4z)\hat{k}$ is solenoidal.

4. Find the vector normal to the surface $xy^3z^2 = 4$ at the point $(-1, -1, 2)$.
   Ans: $-4\hat{\imath} - 12\hat{\jmath} + 4\hat{k}$.

5. If $\vec{R} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, show that (i) $\nabla \cdot \vec{R} = 3$, (ii) $\nabla \times \vec{R} = 0$.

LAB 4: Computation of basis and dimension for a vector space and graphical representation of linear transformation

4.1 Objectives:

Use python

1. to verify the Rank nullity theorem of given linear transformation
2. to compute the dimension of vector space
3. to represent linear transformations graphically

4.2 Rank Nullity Theorem

Verify the rank-nullity theorem for the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x + 4y + 7z, 2x + 5y + 8z, 3x + 6y + 9z)$.

```python
import numpy as np
from scipy.linalg import null_space

# Define a linear transformation interms of matrix
A = np.array([[1, 2, 3], [4, 5, 6], [7, 8, 9]])

# Find the rank of the matrix A
rank = np.linalg.matrix_rank(A)
print("Rank of the matrix",rank)

# Find the null space of the matrix A
ns = null_space(A)
print("Null space of the matrix",ns)
# Find the dimension of the null space
nullity = ns.shape[1]
print("Null space of the matrix",nullity)
# Verify the rank-nullity theorem
if rank + nullity == A.shape[1]:
    print("Rank-nullity theorem holds.")
else:
    print("Rank-nullity theorem does not hold.")
```

```
Rank of the matrix 2
Null space of the matrix [[-0.40824829]
 [ 0.81649658]
 [-0.40824829]]
Null space of the matrix 1
Rank-nullity theorem holds.
```

4.3 Dimension of Vector Space

Find the dimension of subspace spanned by the vectors $(1, 2, 3)$, $(2, 3, 1)$ and $(3, 1, 2)$.

```
import numpy as np

# Define the vector space V
V = np.array([
    [1, 2, 3],
    [2, 3, 1],
    [3, 1, 2]])
# Find the dimension and basis of V
basis = np.linalg.matrix_rank(V)
dimension = V.shape[0]
print("Basis of the matrix",basis)
print("Dimension of the matrix",dimension)
```

```
Basis of the matrix 3
Dimension of the matrix 3
```

Extract the linearly independent rows in given matrix : Basis of Row space

```
from numpy import *
import sympy as sp
A=[[1,-1,1,1],[2,-5,2,2],[3,-3,5,3],[4,-4,4,4]]
AB=array(A)
S=shape(A)
n=len(A)
for i in range(n):
    if AB[i,i]==0:
        ab=copy(AB)
        for k in range(i+1,S[0]):
            if ab[k,i]!=0:
                ab[i,:]=AB[k,:]
                ab[k,:]=AB[i,:]
                AB=copy(ab)
    for j in range(i+1,n):
        Fact=AB[j,i]/AB[i,i]
        for k in range(i,n):
            AB[j,k]=AB[j,k]-Fact*AB[i,k]
display("REF of given matrix: ",sp.Matrix(AB))
temp = {(0, 0, 0,0)}
result = []
for idx, row in enumerate(map(tuple, AB)):
    if row not in temp:
        result.append(idx)
print("\n Basis are non-zero rows of A:")
display(sp.Matrix(AB[result]))
```

```
'REF of given matrix: '
```

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

```
Basis are non-zero rows of A:
```

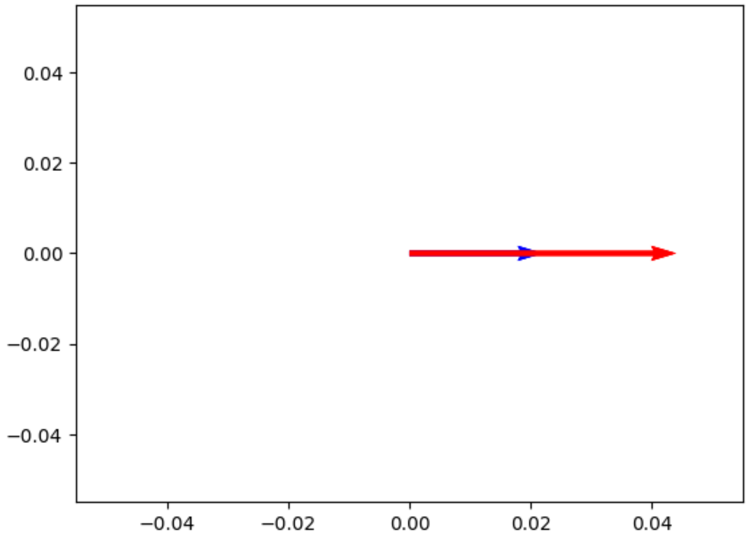
$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

4.4 Graphical representation of a transformation

4.4.1 Horizontal stretch:

Represent the horizontal stretch transformation $T : R^2ßR^2$ geometrically
Find the image of vector $(10, 0)$ when it is stretched horizontally by 2 units.

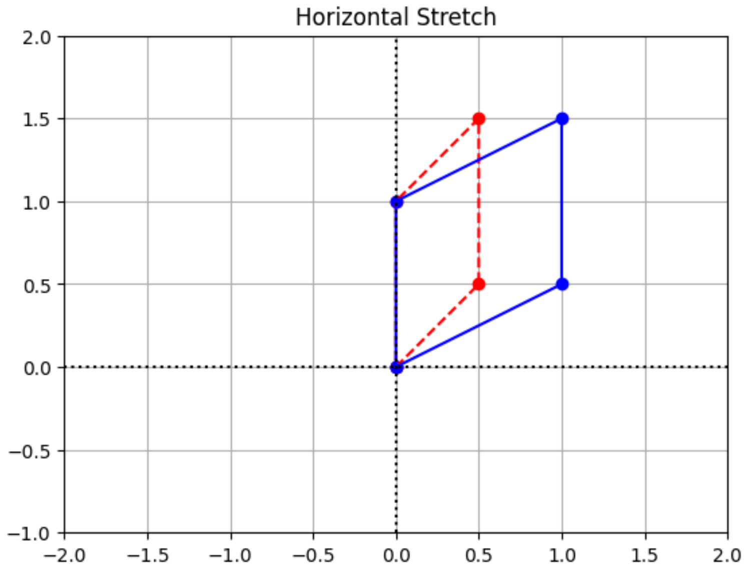
```
import numpy as np
import matplotlib.pyplot as plt
V = np.array([[10,0]])
origin = np.array([[0, 0, 0],[0, 0, 0]]) # origin point
A=np.matrix([[2,0],[0,1]])
V1=np.matrix(V)
V2=A*np.transpose(V1)
V2=np.array(V2)
plt.quiver(*origin, V[:,0], V[:,1], color=['b'], scale=50)
plt.quiver(*origin, V2[0,:], V2[1,:], color=['r'], scale=50)
plt.show()
```



Another example.

```
from math import pi, sin, cos
```

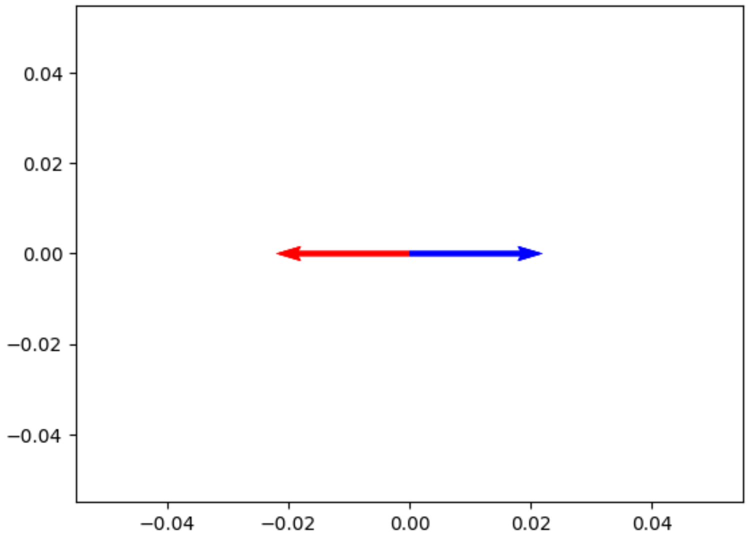
```
import matplotlib.pyplot as plt
import numpy as np

coords = np.array([[0,0],[0.5,0.5],[0.5,1.5],[0,1],[0,0]])
coords = coords.transpose()
coords
x = coords[0,:]
y = coords[1,:]

A = np.array([[2,0],[0,1]])
A_coords = A@coords
x_LT1 = A_coords[0,:]
y_LT1 = A_coords[1,:]

# Create the figure and axes objects
fig, ax = plt.subplots()

# Plot the points.  x and y are original vectors, x_LT1 and y_LT1 are
                                        images
ax.plot(x,y,'ro')
ax.plot(x_LT1,y_LT1,'bo')

# Connect the points by lines
ax.plot(x,y,'r',ls="--")
ax.plot(x_LT1,y_LT1,'b')

# Edit some settings
ax.axvline(x=0,color="k",ls=":")
ax.axhline(y=0,color="k",ls=":")
ax.grid(True)
ax.axis([-2,2,-1,2])
ax.set_aspect('equal')
ax.set_title("Horizontal Stretch");
```



4.4.2 Reflection:

Represent the reflection transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ geometrically.
Find the image of vector $(10, 0)$ when it is reflected about y axis.

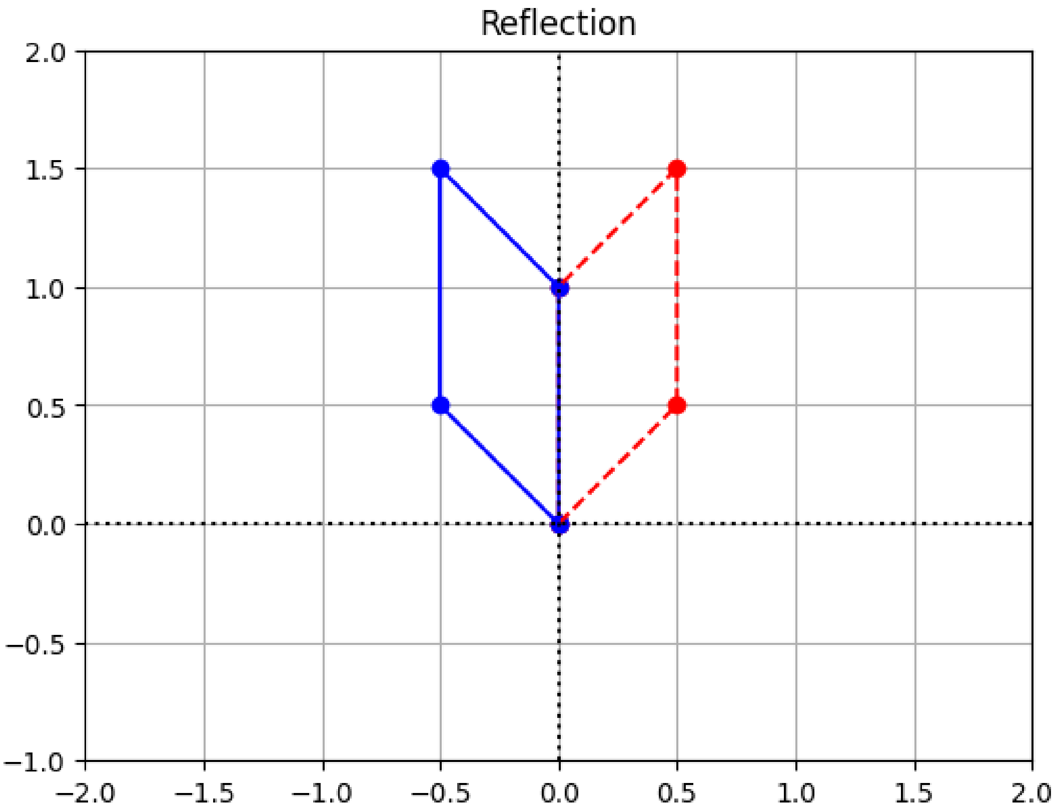
```
import numpy as np
import matplotlib.pyplot as plt
V = np.array([[10,0]])
origin = np.array([[0, 0, 0],[0, 0, 0]]) # origin point
A=np.matrix([[-1,0],[0,1]])
V1=np.matrix(V)
V2=A*np.transpose(V1)
V2=np.array(V2)
plt.quiver(*origin, V[:,0], V[:,1], color=['b'], scale=50)
plt.quiver(*origin, V2[0,:], V2[1,:], color=['r'], scale=50)
plt.show()
```



Another example.

```
B = np.array([[-1,0],[0,1]])
B_coords = B@coords

x_LT2 = B_coords[0,:]
y_LT2 = B_coords[1,:]

# Create the figure and axes objects
fig, ax = plt.subplots()

# Plot the points.  x and y are original vectors, x_LT1 and y_LT1 are
                        images
ax.plot(x,y,'ro')
ax.plot(x_LT2,y_LT2,'bo')

# Connect the points by lines
ax.plot(x,y,'r',ls="--")
ax.plot(x_LT2,y_LT2,'b')

# Edit some settings
ax.axvline(x=0,color="k",ls=":")
```

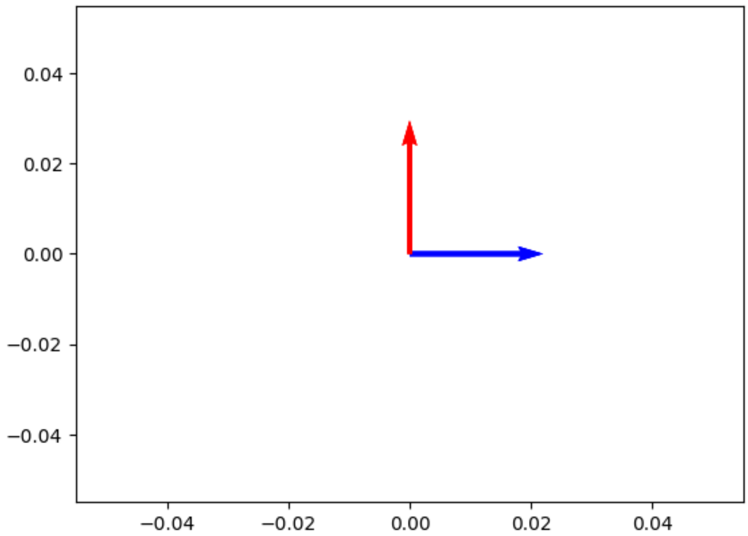
```
ax.axhline(y=0,color="k",ls=":")
ax.grid(True)
ax.axis([-2,2,-1,2])
ax.set_aspect('equal')
ax.set_title("Reflection");
```



4.4.3 Rotation:

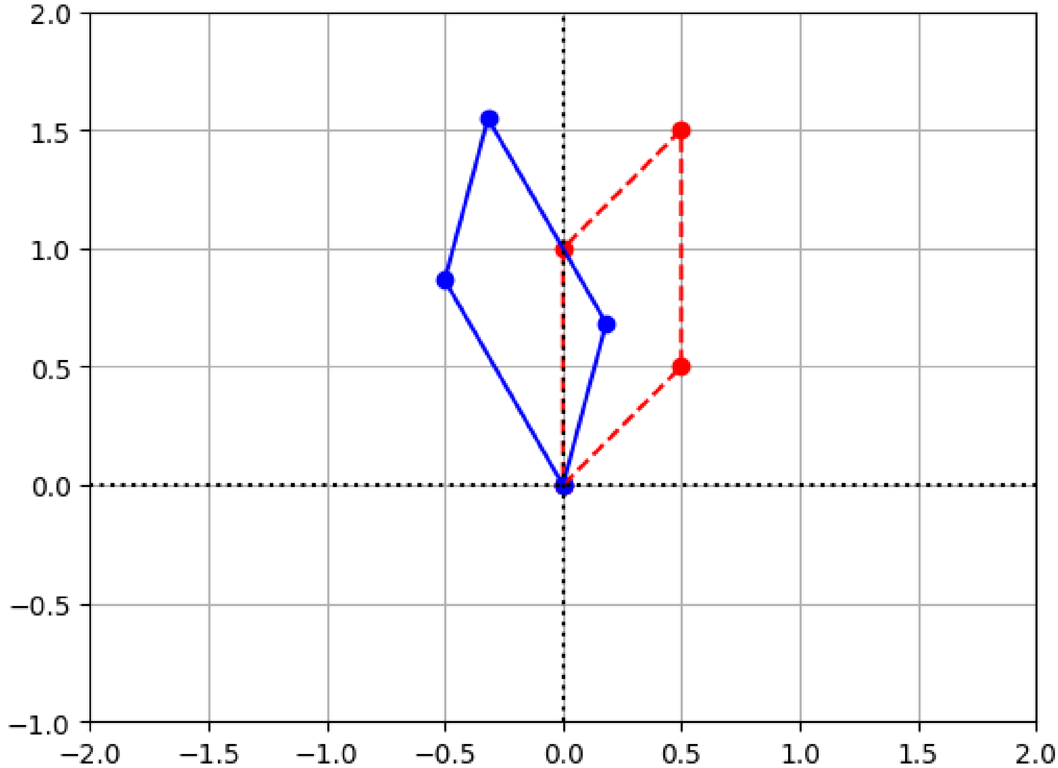
Represent the rotation transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ geometrically.
Find the image of vector $(10, 0)$ when it is rotated by $\pi/2$ radians.

```
import numpy as np
import matplotlib.pyplot as plt
V = np.array([[10,0]])
origin = np.array([[0, 0, 0],[0, 0, 0]]) # origin point
A=np.matrix([[0,-1],[1,1]])
V1=np.matrix(V)
V2=A*np.transpose(V1)
V2=np.array(V2)
plt.quiver(*origin, V[:,0], V[:,1], color=['b'], scale=50)
plt.quiver(*origin, V2[0,:], V2[1,:], color=['r'], scale=50)
plt.show()
```



Another example.

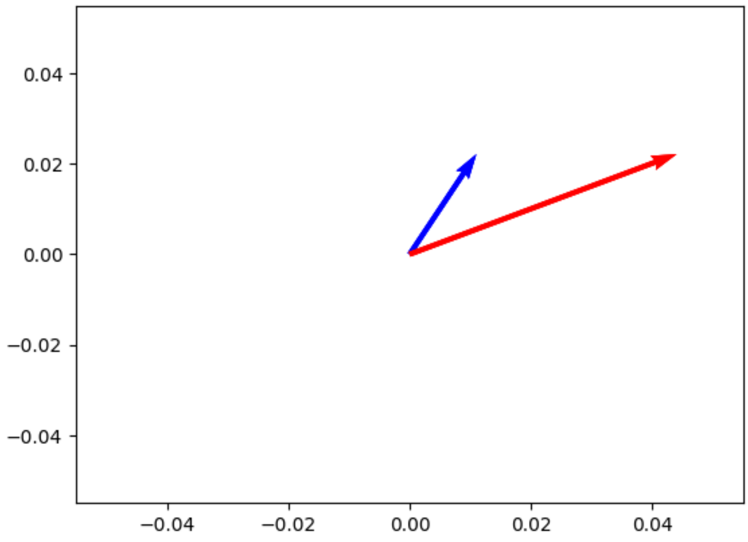
```
theta = pi/6
R = np.array([[cos(theta),-sin(theta)],[sin(theta),cos(theta)]])
R_coords = R@coords

x_LT3 = R_coords[0,:]
y_LT3 = R_coords[1,:]

# Create the figure and axes objects
fig, ax = plt.subplots()

# Plot the points.  x and y are original vectors, x_LT1 and y_LT1 are
                                     images
ax.plot(x,y,'ro')
ax.plot(x_LT3,y_LT3,'bo')

# Connect the points by lines
ax.plot(x,y,'r',ls="--")
ax.plot(x_LT3,y_LT3,'b')

# Edit some settings
ax.axvline(x=0,color="k",ls=":")
ax.axhline(y=0,color="k",ls=":")
ax.grid(True)
ax.axis([-2,2,-1,2])
ax.set_aspect('equal')
```



4.4.4 Shear Transformation

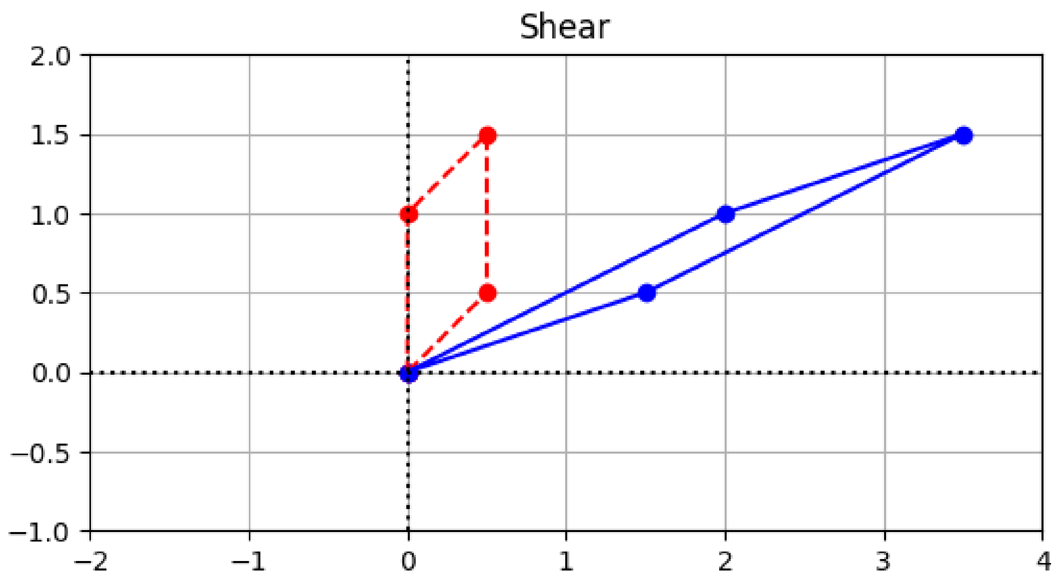
Represent the Shear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ geometrically.
Find the image of $(2, 3)$ under shear transformation.

```
import numpy as np
import matplotlib.pyplot as plt
V = np.array([[2,3]])
origin = np.array([[0, 0, 0],[0, 0, 0]]) # origin point
A=np.matrix([[1,2],[0,1]])
V1=np.matrix(V)
V2=A*np.transpose(V1)
V2=np.array(V2)
print("Image of given vectors is:", V2)
plt.quiver(*origin, V[:,0], V[:,1], color=['b'], scale=20)
plt.quiver(*origin, V2[0,:], V2[1,:], color=['r'], scale=20)
plt.show()
```



Another example.

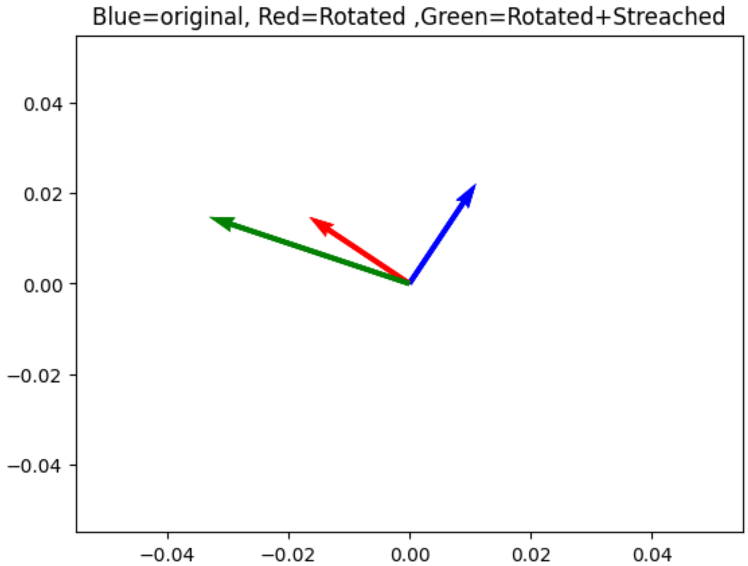
```
S = np.array([[1,2],[0,1]])
S_coords = S@coords

x_LT4 = S_coords[0,:]
y_LT4 = S_coords[1,:]


# Create the figure and axes objects
fig, ax = plt.subplots()

# Plot the points.  x and y are original vectors, x_LT1 and y_LT1 are
                                    images
ax.plot(x,y,'ro')
ax.plot(x_LT4,y_LT4,'bo')

# Connect the points by lines
ax.plot(x,y,'r',ls="--")
ax.plot(x_LT4,y_LT4,'b')

# Edit some settings
ax.axvline(x=0,color="k",ls=":")
ax.axhline(y=0,color="k",ls=":")
ax.grid(True)
ax.axis([-2,4,-1,2])
ax.set_aspect('equal')
ax.set_title("Shear");
```



4.4.5 Composition

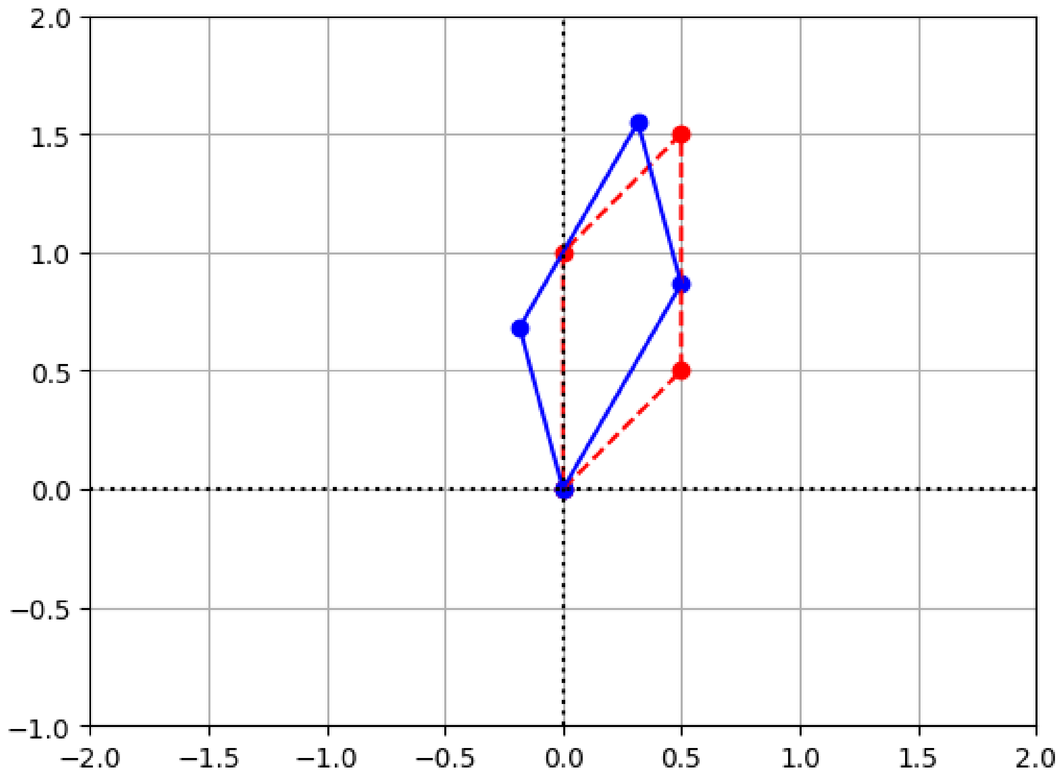
Represent the composition of two 2D transformations.
Find the image of vector $(10, 0)$ when it is rotated by $\pi/2$ radians then stretched horizontally 2 units.

```
import numpy as np
import matplotlib.pyplot as plt
V = np.array([[2,3]])
origin = np.array([[0, 0, 0],[0, 0, 0]]) # origin point
A=np.matrix([[0,-1],[1,0]])
B=np.matrix([[2,0],[0,1]])
V1=np.matrix(V)
V2=A*np.transpose(V1)
V3=B*V2
V2=np.array(V2)
V3=np.array(V3)
print("Image of given vectors is:", V3)
plt.quiver(*origin, V[:,0], V[:,1], color=['b'], scale=20)
plt.quiver(*origin, V2[0,:], V2[1,:], color=['r'], scale=20)
plt.quiver(*origin, V3[0,:], V3[1,:], color=['g'], scale=20)
plt.title('Blue=original, Red=Rotated ,Green=Rotated+Streached')
plt.show()
```



Another example.

```
C = np.array([[-cos(theta),sin(theta)],[sin(theta),cos(theta)]])
C_coords = C@coords

x_LT5 = C_coords[0,:]
y_LT5 = C_coords[1,:]


# Create the figure and axes objects
fig, ax = plt.subplots()

# Plot the points.  x and y are original vectors, x_LT1 and y_LT1 are
                                    images
ax.plot(x,y,'ro')
ax.plot(x_LT5,y_LT5,'bo')

# Connect the points by lines
ax.plot(x,y,'r',ls="--")
ax.plot(x_LT5,y_LT5,'b')

# Edit some settings
ax.axvline(x=0,color="k",ls=":")
ax.axhline(y=0,color="k",ls=":")
ax.grid(True)
ax.axis([-2,2,-1,2])
ax.set_aspect('equal')
```



4.5 Exercise:

1. Verify the rank nullity theorem for the following linear transformation

   a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x + 4y, 2x + 5y, 3x + 6y)$.
   Ans: Rank=2, Nullity=1, RNT verified

   b) $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by $T(x, y, z) = (x+4y-z, 2x+5y+8z, 3x+y+2z, x+y+z)$.
   Ans: Rank=3, Nullity=1, RNT verified

2. Find the dimension of the subspace spanned following set of vectors

   a) $S = (1, 2, 3, 4), (2, 4, 6, 8), (1, 1, 1, 1)$
   Ans: Dimension of subspace is 2

   b) $S = (1, -1, 3, 4), (2, 1, 6, 8), (1, 1, 1, 1), (3, 3, 3, 3)$
   Ans: Dimension of subspace is 3

3. Find the image of $(1, 3)$ under following $2D$ transformations

   a) Horizontal stretch

   b) Reflection

   c) Shear

   d) Rotation

LAB 5: Computing the inner product and orthogonality

5.1 Objectives:

Use python

1. to compute the inner product of two vectors.
2. to check whether the given vectors are orthogonal.

5.2 Inner Product of two vectors

Find the inner product of the vectors $(2, 1, 5, 4)$ and $(3, 4, 7, 8)$.

```
import numpy as np

#initialize arrays
A = np.array([2, 1, 5, 4])
B = np.array([3, 4, 7, 8])

#dot product
output = np.dot(A, B)

print(output)
```

```
77
```

5.3 Checking orthogonality

Verify whether the following vectors $(2, 1, 5, 4)$ and $(3, 4, 7, 8)$ are orthogonal.

```
import numpy as np

#initialize arrays
A = np.array([2, 1, 5, 4])
B = np.array([3, 4, 7, 8])

#dot product
output = np.dot(A, B)
print('Inner product is :',output)
if output==0:
  print('given vectors are orthognal ')
else:
  print('given vectors are not orthognal ')
```

```
Inner product is : 77
given vectors are not orthognal
```

5.4 Exercise:

1. Find the inner product of $(1, 2, 3)$ and $(3, 4, 5)$.

   Ans: 26

2. Find the inner product of $(1, -1, 2, 1)$ and $(4, 2, 1, 0)$.

   Ans: 4

3. Check whether the following vectors are orthogonal or not

   a) $(1, 1, -1)$ and $(2, 3, 5)$. Ans: True

   b) $(1, 0, 2, 0)$ and $(4, 2, -2, 5)$. Ans: True

   c) $(1, 2, 3, 4)$ and $(2, 3, 4, 5)$. Ans: False

LAB 6: Solution of algebraic and transcendental equation by Regula-Falsi and Newton-Raphson method

6.1 Objectives:

Use python

1. to solve algebraic and transcendental equation by Regula-Falsi method.

2. to solve algebraic and transcendental equation by Newton-Raphson method.

6.2 Regula-Falsi method to solve a transcendental equation

Obtain a root of the equation $x^3 - 2x - 5 = 0$ between 2 and 3 by regula-falsi method. Perform 5 iterations.

```
# Regula Falsi method
from sympy import *
x=Symbol('x')
g =input('Enter the function  ')  #%x^3-2*x-5;    %function
f=lambdify(x,g)
a=float(input('Enter a valus :')) #2
b=float(input('Enter b valus :')) # 3
N=int(input('Enter number of iterations :')) #5

for i in range(1,N+1):
    c=(a*f(b)-b*f(a))/(f(b)-f(a))
    if((f(a)*f(c)<0)):
        b=c
    else:
        a=c
    print('itration %d  \t  the root %0.3f \t function value %0.3f \n'%
                                        (i,c,f(c)));
```

```
Enter the function  x**3-2*x-5
Enter a valus :2
Enter b valus :3
Enter number of iterations :5
itration 1        the root 2.059        function value -0.391

itration 2        the root 2.081        function value -0.147

itration 3        the root 2.090        function value -0.055

itration 4        the root 2.093        function value -0.020

itration 5        the root 2.094        function value -0.007
```

Using tolerance value we can write the same program as follows:
Obtain a root of the equation $x^3 - 2x - 5 = 0$ between 2 and 3 by regula-falsi method. Correct to 3 decimal places.

```
# Regula Falsi method while loop2
from sympy import *
x=Symbol('x')
g =input('Enter the function  ') #%x^3-2*x-5;    %function
f=lambdify(x,g)
a=float(input('Enter a valus :')) # 2
b=float(input('Enter b valus :')) # 3
N=float(input('Enter tolarence  :')) # 0.001
x=a;
c=b;
i=0
while (abs(x-c)>=N):
    x=c
    c=((a*f(b)-b*f(a))/(f(b)-f(a)));
    if((f(a)*f(c)<0)):
        b=c
    else:
        a=c
        i=i+1
    print('itration %d  \t  the root %0.3f \t function value %0.3f \n'%
                                        (i,c,f(c)));
print('final value of the root is %0.5f'%c)
```

```
Enter the function  x**3-2*x-5
Enter a valus :2
Enter b valus :3
Enter tolarence  :0.001
itration 1         the root 2.059        function value -0.391

itration 2         the root 2.081        function value -0.147

itration 3         the root 2.090        function value -0.055

itration 4         the root 2.093        function value -0.020

itration 5         the root 2.094        function value -0.007

itration 6         the root 2.094        function value -0.003

final value of the root is 2.09431
```

6.3 Newton-Raphson method to solve a transcendental equation

Find a root of the equation $3x = \cos x + 1$, near 1, by Newton Raphson method. Perform 5 iterations

```
from sympy import *
x=Symbol('x')
g =input('Enter the function  ') #%3x-cos(x)-1;    %function
f=lambdify(x,g)
dg = diff(g);
```

```
df=lambdify(x,dg)
x0= float(input('Enter the intial approximation  ')); # x0=1
n= int(input('Enter the number of iterations  '));   #n=5;
for i in range(1,n+1):
    x1 =(x0 - (f(x0)/df(x0)))
    print('itration %d  \t  the root %0.3f \t function value %0.3f \n'%
                                    (i, x1,f(x1))); #print all
                                    iteration value
    x0 = x1
```

```
Enter the function  3*x-cos(x)-1
Enter the intial approximation  1
Enter the number of iterations  5
itration 1        the root 0.620        function value 0.046

itration 2        the root 0.607        function value 0.000

itration 3        the root 0.607        function value 0.000

itration 4        the root 0.607        function value 0.000

itration 5        the root 0.607        function value 0.000
```

6.4 Exercise:

1. Find a root of the equation $3x = \cos x + 1$, between 0 and 1, by Regula-falsi method. Perform 5 iterations.

   Ans: 0.607

2. Find a root of the equation $xe^x = 2$, between 0 and 1, by Regula-falsi method. Correct to 3 decimal places.

   Ans: 0.853

3. Obtain a real positive root of $x^4 - x = 0$, near 1, by Newton-Raphson method. Perform 4 iterations.

   Ans: 1.856

4. Obtain a real positive root of $x^4 + x^3 - 7x^2 - x + 5 = 0$, near 3, by Newton-Raphson method. Perform 7 iterations.

   Ans: 2.061

LAB 7: Interpolation /Extrapolation using Newton's forward and backward difference formula

7.1 Objectives:

Use python

1. to interpolate using Newton's Forward interpolation method.
2. to interpolate using Newton's backward interpolation method.
3. to extrapolate using Newton's backward interpolation method.

1. Use Newtons forward interpolation to obtain the interpolating polynomial and hence calculate y(2) for the following:

x:	1	3	5	7	9
y:	6	10	62	210	502

```
from sympy import *
import numpy as np
n = int(input('Enter number of data points: '))
210
x = np.zeros((n))
y = np.zeros((n,n))


# Reading data points
print('Enter data for x and y: ')
for i in range(n):
    x[i] = float(input( 'x['+str(i)+']='))
    y[i][0] = float(input( 'y['+str(i)+']='))

# Generating forward difference table
for i in range(1,n):
    for j in range(0,n-i):
        y[j][i] = y[j+1][i-1] - y[j][i-1]

print('\nFORWARD DIFFERENCE TABLE\n');

for i in range(0,n):
    print('%0.2f' %(x[i]), end='')
    for j in range(0, n-i):
        print('\t\t%0.2f' %(y[i][j]), end='')
    print()
 # obtaining the polynomial
t=symbols('t')
f=[] # f is a list type data


p=(t-x[0])/(x[1]-x[0])
f.append(p)
for i in range(1,n-1):
    f.append(f[i-1]*(p-i)/(i+1))
    poly=y[0][0]
for i in range(n-1):
    poly=poly+y[0][i+1]*f[i]
```

```
simp_poly=simplify(poly)
print('\nTHE INTERPOLATING POLYNOMIAL IS\n');
pprint(simp_poly)
# if you want to interpolate at some point the next session will help
inter=input('Do you want to interpolate at a point(y/n)? ') # y
if inter=='y':
    a=float(input('enter the point ')) #2
    interpol=lambdify(t,simp_poly)
    result=interpol(a)
    print('\nThe  value of the function at' ,a,'is\n',result);
```

```
Enter number of data points: 5
Enter data for x and y:
x[0]=1
y[0]=6
x[1]=3
y[1]=10
x[2]=5
y[2]=62
x[3]=7
y[3]=210
x[4]=9
y[4]=502

FORWARD DIFFERENCE TABLE

1.00        6.00        4.00        48.00       48.00       0.00
3.00        10.00       52.00       96.00       48.00
5.00        62.00       148.00      144.00
7.00        210.00      292.00
9.00        502.00

THE INTERPOLATING POLYNOMIAL IS

     3        2
1.0⋅t  - 3.0⋅t  + 1.0⋅t + 7.0
Do you want to interpolate at a point(y/n)? y
enter the point 2

The  value of the function at 2.0 is
 5.0
```

2. Use Newtons backward interpolation to obtain the interpolating polynomial and hence calculate y(8) for the following data:

x:	1	3	5	7	9
y:	6	10	62	210	502

```
from sympy import *
import numpy as np
import sys
print("This will use Newton's backword intepolation formula ")
# Reading number of unknowns
n = int(input('Enter number of data points: '))

# Making numpy array of n & n x n size and initializing
# to zero for storing x and y value along with differences of y
x = np.zeros((n))
y = np.zeros((n,n))


# Reading data points
```

```
print('Enter data for x and y: ')
for i in range(n):
    x[i] = float(input( 'x['+str(i)+']='))
    y[i][0] = float(input( 'y['+str(i)+']='))

# Generating backward difference table
for i in range(1,n):
    for j in range(n-1,i-2,-1):
        y[j][i] = y[j][i-1] - y[j-1][i-1]


print('\nBACKWARD DIFFERENCE TABLE\n');

for i in range(0,n):
    print('%0.2f' %(x[i]), end='')
    for j in range(0, i+1):
        print('\t%0.2f' %(y[i][j]), end='')
    print()


# obtaining the polynomial
t=symbols('t')
f=[]


p=(t-x[n-1])/(x[1]-x[0])
f.append(p)
for i in range(1,n-1):
       f.append(f[i-1]*(p+i)/(i+1))

poly=y[n-1][0]
print(poly)
for i in range(n-1):
       poly=poly+y[n-1][i+1]*f[i]
       simp_poly=simplify(poly)
print('\nTHE INTERPOLATING POLYNOMIAL IS\n');
pprint(simp_poly)
# if you want to interpolate at some point the next session will help
inter=input('Do you want to interpolate at a point(y/n)? ')
if inter=='y':
       a=float(input('enter the point '))
       interpol=lambdify(t,simp_poly)
       result=interpol(a)
       print('\nThe  value of the function at' ,a,'is\n',result);
```

```
This will use Newton's backword intepolation formula
Enter number of data points: 5
Enter data for x and y:
x[0]=1
y[0]=6
x[1]=3
y[1]=10
x[2]=5
y[2]=62
x[3]=7
y[3]=210
x[4]=9
y[4]=502

BACKWARD DIFFERENCE TABLE

1.00    6.00
3.00    10.00   4.00
5.00    62.00   52.00   48.00
7.00    210.00  148.00  96.00   48.00
9.00    502.00  292.00  144.00  48.00   0.00
502.0

THE INTERPOLATING POLYNOMIAL IS

     3        2
1.0·t  - 3.0·t  + 1.0·t + 7.0
Do you want to interpolate at a point(y/n)? y
enter the point 8

The  value of the function at 8.0 is
 335.0
```

7.2 Exercise:

1. Obtain the interpolating polynomial for the following data

x:	0	1	2	3
y:	1	2	1	10

Ans: $2x^3 - 7x^2 + 6x + 1$

2. Find the number of men getting wage Rs. 100 from the following table:

wage:	50	150	250	350
No. of men:	9	30	35	42

Ans: 23 men

3. Using Newton's backward interpolation method obtain y(160) for the following data

x :	100	150	200	250	300
y :	10	13	15	17	18

Ans: 13.42

4. Using Newtons forward interpolation polynomial and calculate y(1) and y(10).

x :	3	4	5	6	7	8	9
y :	4.8	8.4	14.5	23.6	36.2	52.8	73.9

Ans: 3.1 and 100

LAB 8: Computation of area under the curve using Trapezoidal, Simpson's $\left(\frac{1}{3}\right)^{\text{rd}}$ and Simpsons $\left(\frac{3}{8}\right)^{\text{th}}$ rule

8.1 Objectives:

Use python

1. to find area under the curve represented by a given function using Trapezoidal rule.
2. to find area under the curve represented by a given function using Simpson's $\left(\frac{1}{3}\right)^{\text{rd}}$ rule.
3. to find area under the curve represented by a given function using Simpson's $\left(\frac{3}{8}\right)^{\text{th}}$ rule.
4. to find the area below the curve when discrete points on the curve are given.

8.2 Trapezoidal Rule

Evaluate $\int_0^5 \frac{1}{1+x^2}$.

```
# Definition of the function to integrate
def my_func(x):
    return 1 / (1 + x ** 2)
```

```
# Function to implement trapezoidal method
def trapezoidal(x0, xn, n):
  h = (xn - x0) / n                              # Calculating step
                                                 size
  # Finding sum
  integration = my_func(x0) + my_func(xn)        # Adding first and
                                                 last terms
  for i in range(1, n):
    k = x0 + i * h                               # i-th step value
    integration = integration + 2 * my_func(k)   # Adding areas of the
                                                    trapezoids
  # Proportioning sum of trapezoid areas
  integration = integration * h / 2
  return integration
```

```
# Input section
lower_limit = float(input("Enter lower limit of integration: "))
upper_limit = float(input("Enter upper limit of integration: "))
sub_interval = int(input("Enter number of sub intervals: "))

# Call trapezoidal() method and get result
result = trapezoidal(lower_limit, upper_limit, sub_interval)

# Print result
print("Integration result by Trapezoidal method is: " , result)
```

```
Enter lower limit of integration: 0
Enter upper limit of integration: 5
Enter number of sub intervals: 10
Integration result by Trapezoidal method is:  1.3731040812301099
```

8.3 Simpson's $\left(\frac{1}{3}\right)^{\text{rd}}$ Rule

Evaluate $\int\limits_0^5 \frac{1}{1+x^2}$.

```
# Definition of the function to integrate
def my_func(x):
    return 1 / (1 + x ** 2)
```

```
# Function to implement the Simpson's one-third rule

def simpson13(x0,xn,n):
  h = (xn - x0) / n                  # calculating step size
  # Finding sum
  integration = (my_func(x0) + my_func(xn))
  k = x0
  for i in range(1,n):
    if i%2 == 0:
      integration = integration + 4 * my_func(k)
    else:
      integration = integration + 2 * my_func(k)
    k += h
  # Finding final integration value
  integration = integration * h * (1/3)
  return integration

# Input section
lower_limit = float(input("Enter lower limit of integration: "))
upper_limit = float(input("Enter upper limit of integration: "))
sub_interval = int(input("Enter number of sub intervals: "))

# Call trapezoidal() method and get result
result = simpson13(lower_limit, upper_limit, sub_interval)
print("Integration result by Simpson's 1/3 method is: %0.6f" % (result)
                                       )
```

```
Enter lower limit of integration: 0
Enter upper limit of integration: 5
Enter number of sub intervals: 100
Integration result by Simpson's 1/3 method is: 1.404120
```

8.4 Simpson's 3/8th rule

Evaluate $\int_0^6 \frac{1}{1+x^2}dx$ using Simpson's 3/8 th rule, taking 6 sub intervals

```
def simpsons_3_8_rule(f, a, b, n):
```

```
    h = (b - a) / n
    s = f(a) + f(b)
    for i in range(1, n, 3):
        s += 3 * f(a + i * h)
    for i in range(3, n-1, 3):
        s += 3 * f(a + i * h)
    for i in range(2, n-2, 3):
        s += 2 * f(a + i * h)
    return s * 3 * h / 8

def f(x):
    return 1/(1+x**2) # function here

a = 0   # lower limit
b = 6 #  upper limit
n = 6 # number of sub intervals

result = simpsons_3_8_rule(f, a, b, n)
print('%3.5f'%result)
```

```
1.27631
```

8.5 Exercise:

1. Evaluate the integral $\int_0^1 \frac{x^2}{1+x^3}dx$ using Simpson's $\frac{1}{3}$ rule.

   Ans: 0.23108

2. Use Simpson's $\frac{3}{8}$ rule to find $\int_0^{0.6} e^{-x^2}dx$ by taking seven ordinates.

   Ans: 0.5351

3. Evaluate using trapezoidal rule $\int_0^{\pi} sin^2 x dx$. Take $n = 6$.

   Ans: $\pi/2$

4. A solid of revolution is formed by rotating about the x-axis, the area between the x-axis, the lines $x = 0$ and $x = 1$, and a curve through the points with the following co-ordinates:

x	y
0.00	1.0000
0.25	0.9896
0.50	0.9589
0.75	0.9089
1.00	0.8415

Estimate the volume of the solid formed using Simpson's $\frac{1}{3}$rd rule. Hint: Required volume is $\int_0^1 y^2 * \pi dx$. **[Ans: 2.8192]**

5. The velocity v(km/min) of a moped which starts from rest, is given at fixed intervals of time t(min) as follows:

t:	2	4	6	8	10	12	14	16	18	20
v:	10	18	25	29	32	20	11	5	2	0

Estimate approximately the distance covered in twenty minutes.

Answer for 5.

We know that ds/dt=v. So to get distance (s) we have to integrate.
Here $h = 2.2$, $v_0 = 0$, $v_1 = 10$, $v_2 = 18$, $v_3 = 25$ etc.

```
# we shall use simpson's 1/3 rule directly to estimate

h=2
y= [0, 10 ,18, 25, 29,32 ,20, 11 ,5 ,2 , 0]
result=(h/3)*((y[0]+y[10])+4*(y[1]+y[3]+y[5]+y[7]+y[9])+2*(y[2]+y[4]+y[
                                        6]+y[8]))

print('%3.5f'%result,'km.')
```

```
309.33333 km.
```

LAB 9: Solution of ODE of first order and first degree by Taylor's series and Modified Euler's method

9.1 Objectives:

Use python

1. to solve ODE by Taylor series method.
2. to solve ODE by Modified Euler method.
3. to trace the solution curves.

9.2 Taylor series method to solve ODE

Solve: $\frac{dy}{dx} - 2y = 3e^x$ with $y(0) = 0$ using Taylor series method at $x = 0.1(0.1)0.3$.

```
## module taylor
'''X,Y = taylor(deriv,x,y,xStop,h).
4th-order Taylor series method for solving the initial value problem {y
                                    }' = {F(x,{y})}, where
{y} = {y[0],y[1],...y[n-1]}.
x,y = initial conditions
xStop = terminal value of x
h = increment of x
'''
from numpy import array
def taylor(deriv,x,y,xStop,h):
    X = []
    Y = []
    X.append(x)
    Y.append(y)
    while x < xStop:                    # Loop over integration steps
        D = deriv(x,y)                 # Derivatives of y
        H = 1.0
        for j in range(3):             # Build Taylor series
            H = H*h/(j + 1)
            y = y + D[j]*H             # H = h^j/j!
        x = x + h
        X.append(x) # Append results to
        Y.append(y) # lists X and Y

    return array(X),array(Y) # Convert lists into arrays

# deriv = user-supplied function that returns derivatives in the 4 x n
                                    array
'''
[y'[0] y'[1] y'[2] ... y'[n-1]
y''[0] y''[1] y''[2] ... y''[n-1]
y'''[0] y'''[1] y'''[2] ... y'''[n-1]
y''''[0] y''''[1] y''''[2] ... y''''[n-1]]
'''
def deriv(x,y):
    D = zeros((4,1))
```

```
    D[0] = [2*y[0] + 3*exp(x)]
    D[1] = [4*y[0]+ 9*exp(x)]
    D[2] = [8*y[0]+ 21*exp(x)]
    D[3] = [16*y[0]+ 45*exp(x)]
    return D

x = 0.0          # Initial value of x
xStop = 0.3       # last value
y = array([0.0])           # Initial values of y
h = 0.1               # Step size
X,Y = taylor(deriv,x,y,xStop,h)

print("The required values are :at x= %0.2f, y=%0.5f, x=%0.2f, y=%0.5f,
                                     x = %0.2f, y=%0.5f,x = %0.2f, y=%0
                                     .5f"%(X[0],Y[0],X[1],Y[1],X[2],Y[2]
                                     ,X[3],Y[3] ))
```

```
The required values are :at x= 0.00, y=0.00000, x=0.10, y=0.34850,
x = 0.20, y=0.81079,x = 0.30, y=1.41590
```

Solve $y' + 4y = x^2$ with initial conditions $y(0) = 1$ using Taylor series method at $x = 0.1, 0.2$.

```
from numpy import array
def taylor(deriv,x,y,xStop,h):
    X = []
    Y = []
    X.append(x)
    Y.append(y)
    while x < xStop:                    # Loop over integration steps
        D = deriv(x,y)                 # Derivatives of y
        H = 1.0
        for j in range(3):            # Build Taylor series
            H = H*h/(j + 1)
            y = y + D[j]*H            # H = h^j/j!
        x = x + h
        X.append(x) # Append results to
        Y.append(y) # lists X and Y

    return array(X),array(Y) # Convert lists into arrays

# deriv = user-supplied function that returns derivatives in the 4 x n
                                     array
'''
[y'[0] y'[1] y'[2] ... y'[n-1]
y"[0] y"[1] y"[2] ... y"[n-1]
y'''[0] y'''[1] y'''[2] ... y'''[n-1]
y""[0] y""[1] y""[2] ... y""[n-1]]
'''
def deriv(x,y):
    D = zeros((4,1))
    D[0] = [x**2-4*y[0]]
    D[1] = [2*x-4*x**2+16*y[0]]
    D[2] = [2-8*x+16*x**2-64*y[0]]
    D[3] = [-8+32*x-64*x**2+256*y[0]]
```

```
    return D

x = 0.0          # Initial value of x
xStop = 0.2       # last value
y = array([1.0])           # Initial values of y
h = 0.1               # Step size
X,Y = taylor(deriv,x,y,xStop,h)

print("The required values are :at x= %0.2f, y=%0.5f, x=%0.2f, y=%0.5f,
                                     x = %0.2f, y=%0.5f"%(X[0],Y[0],X[1
                                    ],Y[1],X[2],Y[2]))
```

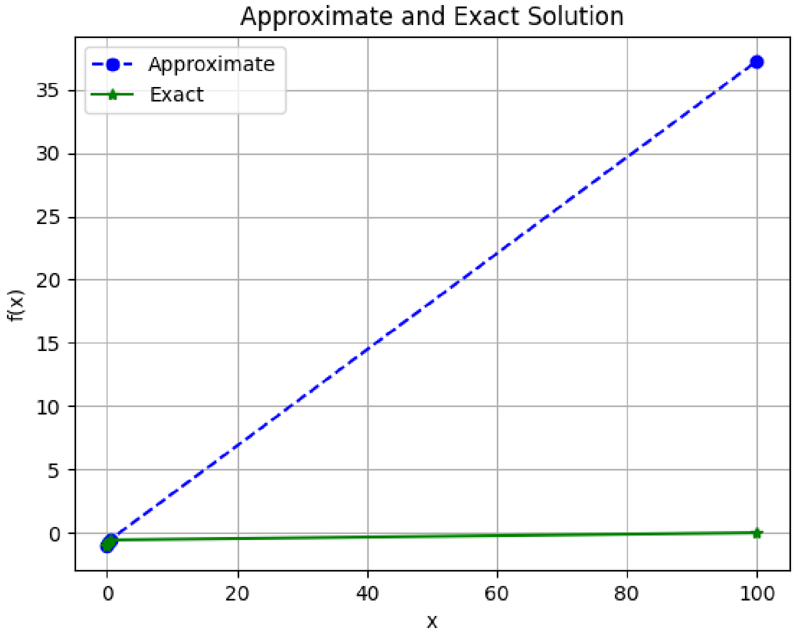
```
The required values are :at x= 0.00, y=1.00000, x=0.10, y=0.66967,
x = 0.20, y=0.45026
```

9.3 Euler's method to solve ODE:

To solve the ODE of the form $\frac{dy}{dx} = f(x, y)$ with initial conditions $y(x_0) = y_0$. The iterative formula is given by : $y(x_{(i+1)} = y(x_i) + hf(x_i, y(x_i))$.
Solve: $y' = e^{-x}$ with $y(0) = -1$ using Euler's method at $x = 0.2(0.2)0.6$.

```
import numpy as np
import matplotlib.pyplot as plt

# Define parameters
f = lambda x, y: np.exp(-x) # ODE
h = 0.2 # Step size
y0 = -1 # Initial Condition
n=3
# Explicit Euler Method

y[0] = y0
x[0]=0

for i in range(0, n):
    x[i+1]=x[i]+h
    y[i + 1] = y[i] + h*f(x[i], y[i])

print("The required values are at x= %0.2f, y=%0.5f, x=%0.2f, y=%0.5f,
                                    x = %0.2f, y=%0.5f,x = %0.2f, y=%0.
                                    5f"%(x[0],y[0],x[1],y[1],x[2],y[2],
                                     x[3],y[3]))
print("\n\n")

plt.plot(x, y, 'bo--', label='Approximate')
plt.plot(x, -np.exp(-x), 'g*-', label='Exact')
plt.title("Approximate and Exact Solution" )
plt.xlabel('x')
plt.ylabel('f(x)')
plt.grid()
plt.legend(loc='best')
plt.show()
```

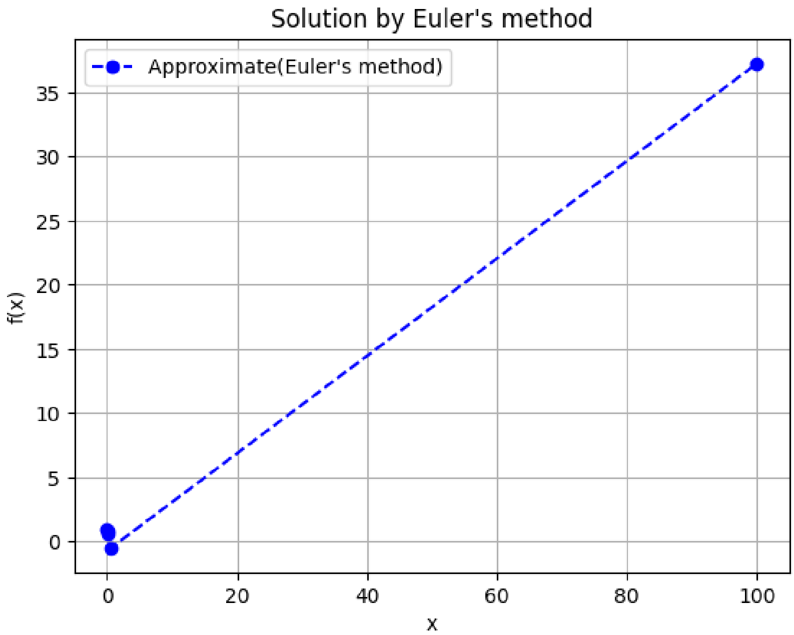
```
The required values are at x= 0.00, y=-1.00000, x=0.20, y=-0.80000,
x = 0.40, y=-0.63625,x = 0.60, y=-0.50219
```



Solve: $y' = -2y + x^3e^{-2x}$ with $y(0) = 1$ using Euler's method at $x = 0.1, 0.2$.

```
import numpy as np
import matplotlib.pyplot as plt

# Define parameters
f = lambda x, y: -2*y+(x**3)*np.exp(-2*x) # ODE
h = 0.1 # Step size
y0 = 1 # Initial Condition
n=2
# Explicit Euler Method

y[0] = y0
x[0]=0

for i in range(0, n):
    x[i+1]=x[i]+h
    y[i + 1] = y[i] + h*f(x[i], y[i])


print("The required values are at x= %0.2f, y=%0.5f, x=%0.2f, y=%0.5f,x
                                    =%0.2f, y=%0.5f\n\n"%(x[0],y[0],x[1
                                    ],y[1],x[2],y[2]))

plt.plot(x, y, 'bo--', label="Approximate(Euler's method)")

plt.title("Solution by Euler's method")
plt.xlabel('x')
plt.ylabel('f(x)')
plt.grid()
plt.legend(loc='best')
plt.show()
```

```
The required values are at x= 0.00, y=1.00000, x=0.10, y=0.80000,
x=0.20, y=0.64008
```



9.4 Modified Euler's method

The iterative formula is:

$$y_1^{(n+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(n)})], \qquad n = 0, 1, 2, 3, \dots,$$

where, $y_1^{(n)}$ is the n^{th} approximation to y_1.

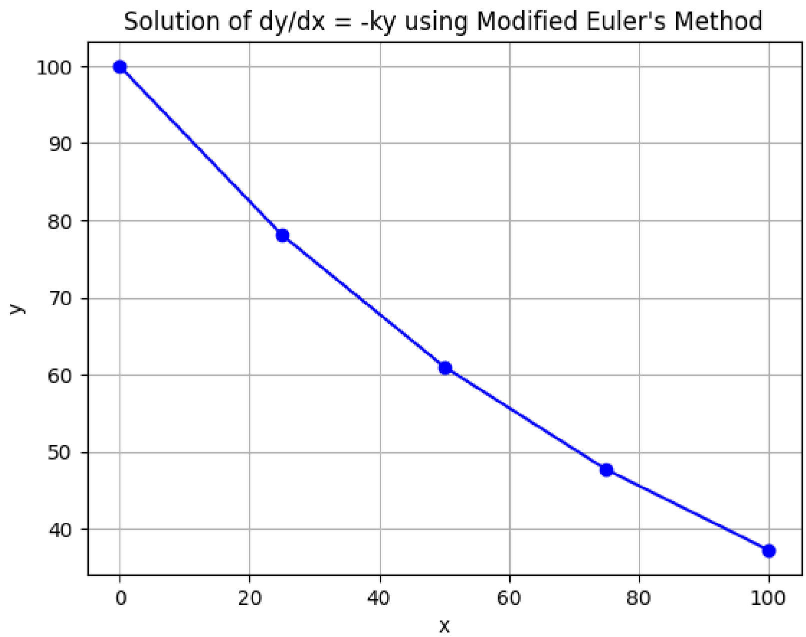
The first iteration will use Euler's method: $y_1^{(0)} = y_0 + hf(x_0, y_0)$.

Solve $y' = -ky$ with $y(0) = 100$ using modified Euler's method at $x = 100$, by taking $h = 25$.

```
import numpy as np
import matplotlib.pyplot as plt

def modified_euler(f, x0, y0, h, n):
    x = np.zeros(n+1)
    y = np.zeros(n+1)

    x[0] = x0
    y[0] = y0

    for i in range(n):
        x[i+1] = x[i] + h
        k1 = h * f(x[i], y[i])
        k2 = h * f(x[i+1], y[i] + k1)
        y[i+1] = y[i] + 0.5 * (k1 + k2)

    return x, y
```

```
def f(x, y):
    return -0.01 * y          # ODE dy/dx = -ky

x0 = 0.0
y0 = 100.0
h = 25
n = 4

x, y = modified_euler(f, x0, y0, h, n)

print("The required value at x= %0.2f, y=%0.5f"%(x[4],y[4]))
print("\n\n")

# Plotting the results
plt.plot(x, y, 'bo-')
plt.xlabel('x')
plt.ylabel('y')
plt.title('Solution of dy/dx = -ky using Modified Euler\'s Method')
plt.grid(True)
plt.show()
```

```
The required value at x= 100.00, y=37.25290
```



9.5 Exercise:

1. Find $y(0.1)$ by Taylor Series exapnsion when $y' = x - y^2, y(0) = 1$.

   Ans: $y(0.1) = 0.9138$

2. Find $y(0.2)$ by Taylor Series exapnsion when $y' = x^2y - 1, y(0) = 1, h = 0.1$.

   Ans: $y(0.2) = 0.80227$

3. Evaluate by modified Euler's method: $y' = ln(x + y), y(0) = 2$ at $x = 0(0.2)0.8$.
   Ans: $2.0656, 2.1416, 2.2272, 2.3217$
4. Solve by modified Euler's method: $y' = x + y, y(0) = 1, h = 0.1, x = 0(0.1)0.3$.
   Ans: $1.1105, 1.2432, 1.4004$

LAB 10: Solution of ODE of first order and first degree by Runge-Kutta 4th order method and Milne's predictor and corrector method

10.1 Objectives:

1. To write a python program to solve first order differential equation using 4th order Runge Kutta method.

2. To write a python program to solve first order differential equation using Milne's predictor and corrector method.

10.2 Runge-Kutta method

Apply the Runge Kutta method to find the solution of $dy/dx = 1 + (y/x)$ at $y(2)$ taking $h = 0.2$. Given that $y(1) = 2$.

```
from sympy import *
import numpy  as np
def RungeKutta(g,x0,h,y0,xn):

  x,y=symbols('x,y')
  f=lambdify([x,y],g)
  xt=x0+h
  Y=[y0]
  while xt<=xn:
      k1=h*f(x0,y0)
      k2=h*f(x0+h/2, y0+k1/2)
      k3=h*f(x0+h/2, y0+k2/2)
      k4=h*f(x0+h, y0+k3)
      y1=y0+(1/6)*(k1+2*k2+2*k3+k4)
      Y.append(y1)
      #print('y(%3.3f'%xt,') is %3.3f'%y1)
      x0=xt
      y0=y1
      xt=xt+h
  return np.round(Y,2)
RungeKutta('1+(y/x)',1,0.2,2,2)
```

```
array([2.  , 2.62, 3.27, 3.95, 4.66, 5.39])
```

10.3 Milne's predictor and corrector method

Apply Milne's predictor and corrector method to solve $dy/dx = x^2 + (y/2)$ at y(1.4). Given that y(1)=2, y(1.1)=2.2156, y(1.2)=2.4649, y(1.3)=2.7514. Use corrector formula thrice.

```
# Milne's method to solve first order DE
# Use corrector formula thrice
x0=1
y0=2
```

```
y1=2.2156
y2=2.4649
y3=2.7514
h=0.1
x1=x0+h
x2=x1+h
x3=x2+h
x4=x3+h
def f(x,y):
  return x**2+(y/2)

y10=f(x0, y0)
y11=f(x1,y1)
y12=f(x2,y2)
y13=f(x3,y3)
y4p=y0+(4*h/3)*(2*y11-y12+2*y13)
print('predicted value of y4 is %3.3f'%y4p)
y14=f(x4,y4p);
for i in range(1,4):
  y4=y2+(h/3)*(y14+4*y13+y12);
  print('corrected value of y4 after \t iteration  %d  is \t %3.5f\t '%
                                    (i,y4))
  y14=f(x4,y4);
```

```
predicted value of y4 is 3.079
corrected value of y4 after         iteration  1  is          3.07940
corrected value of y4 after         iteration  2  is          3.07940
corrected value of y4 after         iteration  3  is          3.07940
```

In the next program, function will take all the inputs from the user and display the answer.
Apply Milne's predictor and corrector method to solve $dy/dx = x^2 + (y/2)$ at y(1.4). Given that y(1)=2, y(1.1)=2.2156, y(1.2)=2.4649, y(1.3)=2.7514. Use corrector formula thrice.

```
from sympy import *
def Milne(g,x0,h,y0,y1,y2,y3):
    x,y=symbols('x,y')
    f=lambdify([x,y],g)
    x1=x0+h
    x2=x1+h
    x3=x2+h
    x4=x3+h

    y10=f(x0, y0)
    y11=f(x1,y1)
    y12=f(x2,y2)
    y13=f(x3,y3)
    y4p=y0+(4*h/3)*(2*y11-y12+2*y13)
    print('predicted value of y4',y4p)
    y14=f(x4,y4p)
    for i in range(1,4):
        y4=y2+(h/3)*(y14+4*y13+y12)
        print('corrected value of y4 , iteration  %d '%i,y4)
```

```
        y14=f(x4,y4)
Milne('x**2+y/2',1,0.1,2,2.2156,2.4649,2.7514)
```

```
predicted value of y4 3.0792733333333335
corrected value of y4 , iteration  1  3.0793962222222224
corrected value of y4 , iteration  2  3.079398270370371
corrected value of y4 , iteration  3  3.079398304506173
```

Apply Milne's predictor and corrector method to solve $dy/dx = x - y^2$, y(0)=2 obtain y(0.8). Take h=0.2. Use Runge-Kutta method to calculate required initial values.

```
Y=RungeKutta('x-y**2',0,0.2,0,0.8)
print('y values from Runge -Kutta method:',Y)
Milne('x-y**2',0,0.2,Y[0],Y[1],Y[2],Y[3])
```

```
y values from Runge -Kutta method: [0.   0.02 0.08 0.18 0.3 ]
predicted value of y4 0.3042133333333334
corrected value of y4 , iteration  1  0.3047636165214815
corrected value of y4 , iteration  2  0.3047412758696499
corrected value of y4 , iteration  3  0.3047421836520892
```

10.4 Exercise:

1. Find $y(0.1)$ by Runge Kutta method when $y' = x - y^2, y(0) = 1$.

   Ans: $y(0.1) = 0.91379$

2. Evaluate by Runge Kutta method : $y' = log(x + y), y(0) = 2$ at $x = 0(0.2)0.8$.

   Ans: $2.155, 2.3418, 2.557, 2.801$

3. Solve by Milnes method: $y' = x + y$, y(0)=1, h=0.1, Calculate $y(0.4)$. Calculate required initial values from Runge Kutta method.

   Ans: 1.583649219