5. Statistical Signal Processing

5.1 Sufficient Statistic

Let $\mathbf{X} \in \mathbb{C}^m$ be the random observation with the PDF $\mathrm{f}_{\mathbf{X}}(\boldsymbol{x};\boldsymbol{\theta})$ depending on the parameter $\boldsymbol{\theta} \in \mathbb{C}^n$. Construct the statistic

$$\mathbf{T} = \boldsymbol{g}(\mathbf{X})$$

with the function $\boldsymbol{g} : \mathbb{C}^m \to \mathbb{C}^k$. The statistic $\mathbf{T} \in \mathbb{C}^k$ is sufficient for $\boldsymbol{\theta}$ if $\mathbf{X}$ given $\mathbf{T}$ is independent of $\boldsymbol{\theta}$ ([Sch91, Kay93]), i.e.,

$$\mathrm{f}_{\mathbf{X}|\mathbf{T}}(\boldsymbol{x} \,|\, \boldsymbol{t}) \neq f(\boldsymbol{\theta}). \tag{5.1}$$

In other words, when the randomness of $\mathbf{X}$ is reduced by fixing $\mathbf{T}$ to be $\boldsymbol{t}$, no information about $\boldsymbol{\theta}$ can be found in $\mathbf{X}$ anymore. In contrast, all information about $\boldsymbol{\theta}$ is captured by $\mathbf{T}$. Therefore, when we are only interested in $\boldsymbol{\theta}$, it suffices to store the statistic $\boldsymbol{t} = \boldsymbol{g}(\boldsymbol{x})$ instead of the original measurement $\boldsymbol{x}$.

In particular, if the PDF of $\mathbf{X}$ can be factorized as

$$\mathrm{f}_{\mathbf{X}}(\boldsymbol{x};\boldsymbol{\theta}) = a(\boldsymbol{x}) b(\boldsymbol{g}(\boldsymbol{x}), \boldsymbol{\theta}) \tag{5.2}$$

where $a(\boldsymbol{x})$ is independent of the parameter $\boldsymbol{\theta}$, the statistic $\mathbf{T} = \boldsymbol{g}(\mathbf{X})$ is sufficient for $\boldsymbol{\theta}$.

The power of the definition of the sufficient statistic is illustrated by considering Bernoulli trials. A binary source generates the binary X_i, where X_i is 1 with probability θ and 0 with probability $1-\theta$. Consider that N independent trials were performed, leading to

$$\mathbf{X} = [\mathrm{X}_1, \ldots, \mathrm{X}_N]^{\mathrm{T}} \in \{0,1\}^N.$$

The probability for a particular result $\boldsymbol{x} = [x_1, \ldots, x_N]^{\mathrm{T}}$ of the N trials is given by the probability mass function (PMF) of $\mathbf{X}$, i.e.,

$$\mathrm{P}[\mathbf{X} = \boldsymbol{x}] = \mathrm{p}_{\mathbf{X}}(\boldsymbol{x};\theta) = \prod_{i=1}^{N} \theta^{x_i}(1-\theta)^{1-x_i}.$$

Remember that X_i is either 1 or 0. Therefore,

$$\mathrm{p}_{\mathbf{X}}(\boldsymbol{x};\theta) = \theta^s (1-\theta)^{N-s}$$

with the sum over the outcomes of all trials $s = \sum_{i=1}^N x_i$. Based on the trivial factorization $a(\boldsymbol{x}) = 1$ and $b(s,\theta) = \theta^s(1-\theta)^{1-s}$, it can be inferred that

$$\mathrm{S} = \sum_{i=1}^{N} \mathrm{X}_i$$

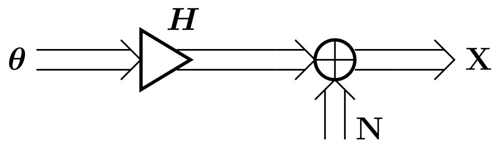

Fig. 5.1. Linear Gaussian Model with the Parameter $\boldsymbol{\theta}$

is a sufficient statistic for θ. The information contained in the outcomes of N trials is compressed into a scalar and no information about the quantity θ is lost.

For another example, let $\mathrm{X}_i \sim \mathcal{N}_{\mathbb{C}}(0, \sigma_{\mathrm{X}}^2)$ and N independent measurements are performed. The joint PDF of the N measurements comprised in $\mathbf{X} = [\mathrm{X}_1, \ldots, \mathrm{X}_N] \in \mathbb{C}^N$ can be written as

$$\mathrm{f}_{\mathbf{X}}(\boldsymbol{x}; \sigma_{\mathrm{X}}^2) = \prod_{i=1}^{N} \mathrm{f}_{\mathrm{X}_i}(x_i; \sigma_{\mathrm{X}}^2) = \frac{1}{\pi^N \sigma_{\mathrm{X}}^{2N}} \exp\left(-\sum_{i=1}^{N} \frac{|x_i|^2}{\sigma_{\mathrm{X}}^2} \right).$$

With $t = \sum_{i=1}^{N} |x_i|^2 = \|\boldsymbol{x}\|_2^2$, the exponential function becomes $b(t, \sigma_{\mathrm{X}}^2)$ and $\mathrm{T} = \|\mathbf{X}\|_2^2$ is a sufficient statistic for σ_{X}^2.

5.1.1 Linear Gaussian Model

In the linear Gaussian model, the observation $\mathbf{X} \in \mathbb{C}^m$ is the superposition of the parameter $\boldsymbol{\theta} \in \mathbb{C}^n$ transformed by the constant matrix $\boldsymbol{H} \in \mathbb{C}^{m \times n}$ and the additive Gaussian noise $\mathbf{N} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{C}_{\mathbf{N}})$, that is,

$$\mathbf{X} = \boldsymbol{H}\boldsymbol{\theta} + \mathbf{N}. \tag{5.3}$$

Correspondingly, also the observation $\mathbf{X}$ is Gaussian with the PDF

$$\mathrm{f}_{\mathbf{X}}(\boldsymbol{x}; \boldsymbol{\theta}) = \frac{1}{\pi^m \det(\boldsymbol{C}_{\mathbf{N}})} \exp\left(-(\boldsymbol{x} - \boldsymbol{H}\boldsymbol{\theta})^{\mathrm{H}} \boldsymbol{C}_{\mathbf{N}}^{-1} (\boldsymbol{x} - \boldsymbol{H}\boldsymbol{\theta}) \right).$$

With the factorization

$$\mathrm{f}_{\mathbf{X}}(\boldsymbol{x}; \boldsymbol{\theta}) = \frac{1}{\pi^m \det(\boldsymbol{C}_{\mathbf{N}})} \exp\left(-\boldsymbol{x}^{\mathrm{H}} \boldsymbol{C}_{\mathbf{N}}^{-1} \boldsymbol{x} \right) \exp\left(\boldsymbol{x}^{\mathrm{H}} \boldsymbol{C}_{\mathbf{N}}^{-1} \boldsymbol{H}\boldsymbol{\theta} + \boldsymbol{\theta}^{\mathrm{H}} \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathbf{N}}^{-1} \boldsymbol{x} - \boldsymbol{\theta}^{\mathrm{H}} \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathbf{N}}^{-1} \boldsymbol{H}\boldsymbol{\theta} \right)$$

it can be concluded [see (5.2)] that $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathbf{N}}^{-1} \boldsymbol{x}$ and

$$\mathbf{T} = \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathbf{N}}^{-1} \mathbf{X}$$

is a sufficient statistic for $\boldsymbol{\theta}$. Note that the transform

$$\boldsymbol{G}_{\mathrm{MF}} = \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathbf{N}}^{-1} \tag{5.4}$$

applied to the observation $\mathbf{X}$ to get the statistic $\mathbf{T}$ is called the matched filter (MF). Note that the application of any invertible transformation to $\mathbf{T}$ leads again to a sufficient statistic. In particular,

$$\mathbf{T}' = \boldsymbol{A}^{-1} \mathbf{T}$$

with invertible $\boldsymbol{A}$, is again a sufficient statistic since $\mathbf{T}$ can easily be obtained from $\mathbf{T}'$ via $\mathbf{T} = \boldsymbol{A}\mathbf{T}'$. Therefore, the application of any filter of the form

$$\boldsymbol{G}_{\text{sufficient}} = \boldsymbol{A}^{-1} \boldsymbol{G}_{\mathrm{MF}}$$

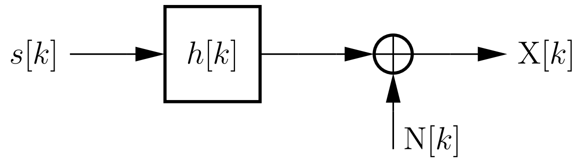

Fig. 5.2. Linear Gaussian FIR Model

to the observation $\mathbf{X}$ delivers a sufficient statistic in the linear Gaussian model (5.3).

The linear model based on the sufficient statistic $\mathbf{T}$ is

$$\mathbf{T} = \boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\boldsymbol{\theta} + \mathbf{N}'$$

with the transformed noise $\mathbf{N}' = \boldsymbol{G}_{\mathrm{MF}}\mathbf{N}$. Due to

$$\mathrm{E}[\mathbf{N}'] = \mathrm{E}[\boldsymbol{G}_{\mathrm{MF}}\mathbf{N}] = \boldsymbol{G}_{\mathrm{MF}}\mathrm{E}[\mathbf{N}] = \mathbf{0}$$

and

$$\begin{aligned}\mathrm{E}[\mathbf{N}'\mathbf{N}'^{,\mathrm{H}}] &= \mathrm{E}\left[\boldsymbol{G}_{\mathrm{MF}}\mathbf{N}\mathbf{N}^{\mathrm{H}}\boldsymbol{G}_{\mathrm{MF}}^{\mathrm{H}}\right] = \boldsymbol{G}_{\mathrm{MF}}\mathrm{E}[\mathbf{N}\mathbf{N}^{\mathrm{H}}]\boldsymbol{G}_{\mathrm{MF}}^{\mathrm{H}} \\ &= \boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{C}_{\mathbf{N}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H} = \boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\end{aligned}$$

we have that $\mathbf{N}' \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H})$.

5.1.1.1 Linear Gaussian FIR Channel

Suppose that an infinitely long sequence $s[k]$ is the input to the FIR channel

$$h[k] = \sum_{i=0}^{K} h_i \delta[k-i]$$

which is assumed to be constant. Accordingly, the output of the channel is given by (see Fig. 5.2)

$$\mathrm{X}[k] = (h * s)[k] + \mathrm{N}[k] = \sum_{i=0}^{K} h_i s[k-i] + \mathrm{N}[k]$$

with the additive Gaussian noise $\mathrm{N}[k]$ which is assumed to be zero-mean and stationary, i.e., $\mathrm{E}[\mathrm{N}[k]\mathrm{N}^*[k+\kappa]]$ only depends on κ but is independent of k. When comprising the coefficients of $h[k]$ in the vector

$$\boldsymbol{h} = [h_0, h_1, \ldots, h_K]^{\mathrm{T}} \in \mathbb{C}^{K+1}$$

the output of the channel can be rewritten as

$$\mathrm{X}[k] = \boldsymbol{h}^{\mathrm{T}} \begin{bmatrix} s[k] \\ s[k-1] \\ \vdots \\ s[k-K] \end{bmatrix} + \mathrm{N}[k].$$

After collecting $L+1$ consecutive outputs in the vector signal

$$\mathbf{X}[k] = \left[\mathrm{X}[k], \ldots, \mathrm{X}[k-L]\right]^{\mathrm{T}} \in \mathbb{C}^{L+1}$$

where we assume that $L \geq K$, we arrive at the linear Gaussian model [cf. (5.3)]

$$\mathbf{X}[k] = \boldsymbol{H}\boldsymbol{s}[k] + \mathbf{N}[k] \tag{5.5}$$

with the convolution matrix

$$\boldsymbol{H} = \begin{bmatrix} h_0 & h_1 & h_2 & \dots & h_K & 0 & \dots & 0 \\ 0 & h_0 & h_1 & \dots & h_{K-1} & h_K & \dots & 0 \\ & & \ddots & & & & \ddots & \\ 0 & \dots & 0 & h_0 & & \dots & & h_K \end{bmatrix} \in \mathbb{C}^{L+1 \times K+L+1} \tag{5.6}$$

which exhibits a Toeplitz structure. The samples of the sequence $s[k]$, which contribute to the entries of $\mathbf{X}[k]$, are collected in the vector signal

$$\boldsymbol{s}[k] = \big[s[k], \dots, s[k-K-L]\big]^{\mathrm{T}} \in \mathbb{C}^{K+L+1}.$$

Due to the stationarity of $\mathrm{N}[k]$, the noise signal

$$\mathbf{N}[k] = \big[\mathrm{N}[k], \dots, \mathrm{N}[k-L]\big]^{\mathrm{T}} \in \mathbb{C}^{L+1}$$

can be modelled as

$$\mathbf{N}[k] \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{C}_{\mathbf{N}})$$

with the noise covariance matrix $\boldsymbol{C}_{\mathbf{N}} = \mathrm{E}[\mathbf{N}[k]\mathbf{N}^{\mathrm{H}}[k]]$ which is independent of the time index k.

Since the sequence $s[k]$ is infinitely long, it is impossible to estimate the whole sequence at once. Instead, we try to estimate only a single element of $s[k]$ in each time step. Therefore, the unknown parameter is

$$\theta = s[k-d]$$

with the delay d inherently necessary due to the FIR structure of $h[k]$. Note that d is a design parameter and will be discussed later. The PDF of the observation at time k reads as

$$\begin{aligned} \mathrm{f}_{\mathbf{X}[k]}(\boldsymbol{x};\theta) &= \mathrm{f}_{\mathbf{N}[k]}(\boldsymbol{x} - \boldsymbol{H}\boldsymbol{s}[k]) \\ &= \frac{1}{\pi^{L+1}\det(\boldsymbol{C}_{\mathbf{N}})} \exp\Big(-(\boldsymbol{x} - \boldsymbol{H}\boldsymbol{s}[k])^{\mathrm{H}} \boldsymbol{C}_{\mathbf{N}}^{-1} (\boldsymbol{x} - \boldsymbol{H}\boldsymbol{s}[k]) \Big). \end{aligned}$$

It is useful to rewrite the unknown sample as

$$\theta = \boldsymbol{e}_{d+1}^{\mathrm{T}} \boldsymbol{s}[k]$$

with the $d{+}1$-th canonical unit vector $\boldsymbol{e}_{d+1}$ whose elements are zero except for the $d{+}1$-th element, which is one. Therefore, above operation picks $s[k-d]$ out of $\boldsymbol{s}[k]$. Based on above expression for θ, the noise-free channel output can be decomposed as

$$\boldsymbol{H}\boldsymbol{s}[k] = \boldsymbol{H}\boldsymbol{e}_{d+1}\theta + \bar{\boldsymbol{H}}\bar{\boldsymbol{s}}[k]$$

where $\bar{\boldsymbol{H}} \in \mathbb{C}^{L+1 \times K+L}$ results from $\boldsymbol{H}$ by dropping the $d+1$-th column and $\bar{\boldsymbol{s}}[k] \in \mathbb{C}^{K+L}$ is $\boldsymbol{s}[k]$ after removing the $d+1$-th element $s[k-d]$. Accordingly, it is possible to write

$$\mathrm{f}_{\mathbf{X}[k]}(\boldsymbol{x};\theta) = a(\boldsymbol{x}) \exp\left(\theta^* \boldsymbol{e}_{d+1}^{\mathrm{T}} \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathbf{N}}^{-1} \boldsymbol{x} + \boldsymbol{x}^{\mathrm{H}} \boldsymbol{C}_{\mathbf{N}}^{-1} \boldsymbol{H}\boldsymbol{e}_{d+1}\theta - \boldsymbol{s}^{\mathrm{H}}[k] \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathbf{N}}^{-1} \boldsymbol{H}\boldsymbol{s}[k]\right)$$

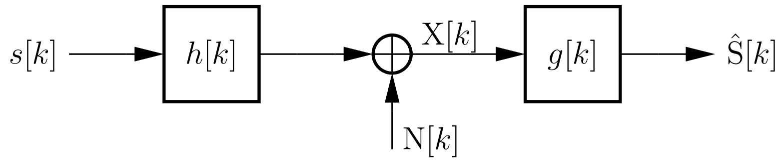

Fig. 5.3. Linear Gaussian FIR Model with Equalizer

with the function

$$a(\boldsymbol{x}) = \frac{1}{\pi^{L+1}\det(\boldsymbol{C}_{\mathbf{N}})} \exp\left(-\boldsymbol{x}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{x} + \boldsymbol{x}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\bar{\boldsymbol{H}}\bar{\boldsymbol{s}}[k] + \bar{\boldsymbol{s}}^{\mathrm{H}}[k]\bar{\boldsymbol{H}}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{x}\right).$$

which is independent of θ. Based on (5.2), it can be inferred that

$$\mathrm{T}[k] = \boldsymbol{e}_{d+1}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\mathbf{X}[k]$$

is a sufficient statistic for θ. Note that the transform

$$\boldsymbol{g}^{\mathrm{H}} = \left[g_0, \ldots, g_L\right] = \boldsymbol{e}_{d+1}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}$$

is independent of k. Therefore, it holds that

$$\mathrm{T}[k] = \boldsymbol{g}^{\mathrm{H}}\mathbf{X}[k] = \sum_{\ell=0}^{L} g_\ell \mathrm{X}[k-\ell]$$

and this operation can be performed by the FIR filter

$$g[k] = \sum_{\ell=0}^{L} g_\ell \delta[k-\ell]$$

that is, $\mathrm{T}[k] = (g * \mathrm{X})[k]$ (see Fig. 5.3).

If d is chosen such that the corresponding column of the convolution matrix comprises all $K+1$ coefficients of $h[k]$, the resulting equalizer is the matched filter, i.e.,

$$\boldsymbol{g}_{\mathrm{MF}}^{\mathrm{H}} = \boldsymbol{e}_{d+1}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}$$

for $K \leq d \leq L$.

In the following, a white noise sequence is assumed, that is, $\mathrm{E}[\mathrm{N}[k]\mathrm{N}^*[k+\kappa]] = \sigma_{\mathrm{N}}^2\delta[\kappa]$ and consequently, $\boldsymbol{C}_{\mathbf{N}} = \sigma_{\mathrm{N}}^2\mathbf{I}$. In this case, the matched filter coefficients can be found to be

$$\boldsymbol{g}_{\mathrm{MF}}^{\mathrm{H}} = \boldsymbol{e}_{d+1}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{H}}\frac{1}{\sigma_{\mathrm{N}}^2}.$$

Note that the vector $\boldsymbol{g}_{\mathrm{MF}}$ contains leading zero elements at both ends and only $K+1$ elements are non-zero. Reducing L to K yields

$$\boldsymbol{g}_{\mathrm{MF}}^{\mathrm{H}} = \frac{1}{\sigma_{\mathrm{N}}^2}\left[h_K^*, \ldots, h_0^*\right]$$

and thus,

$$g_{\mathrm{MF}}[k] = \frac{1}{\sigma_{\mathrm{N}}^2}\sum_{\ell=0}^{K} h_{K-\ell}^*\delta[k-\ell] = \frac{1}{\sigma_{\mathrm{N}}^2}h^*[K-k]. \tag{5.7}$$

We observe that for $\boldsymbol{C}_{\mathbf{N}} = \sigma_{\mathrm{N}}^2\mathbf{I}$, the impulse response of the matched filter is given by the time-reverted, complex conjugate channel impulse response weighted by the noise variance. The corresponding delay is given by $d_{\mathrm{MF}} = K$.

5.2 Maximum Likelihood Estimation

Consider the random variable X with the cumulative distribution function (CDF)

$$\mathrm{F_X}(\xi) = \mathrm{P}[\mathrm{X} \leq \xi] = \int_{-\infty}^{\xi} \mathrm{f_X}(x)\,\mathrm{d}x$$

and the corresponding PDF $\mathrm{f_X}(x)$. Thus, we have

$$\mathrm{P}[\xi < \mathrm{X} \leq \xi + \Delta\xi] = \int_{\xi}^{\xi+\Delta\xi} \mathrm{f_X}(x)\,\mathrm{d}x \approx \mathrm{f_X}(\xi)\Delta\xi$$

for appropriately small $\Delta\xi$. In other words, the PDF at the point ξ is proportional to the probability that a realization of the random variable X lies in the $\Delta\xi$-vicinity of ξ.

Above observation motivates the maximum likelihood (ML) principle (see [Sch91, Kay93]). For a given realization $\hat{\boldsymbol{x}}$ of the observation $\mathbf{X}$, the PDF of $\mathbf{X}$ is maximized w.r.t. the parameter $\boldsymbol{\theta}$, i.e.,

$$\hat{\boldsymbol{\theta}}_{\mathrm{ML}} = \operatorname*{argmax}_{\boldsymbol{\theta}} \mathrm{f}_{\mathbf{X}}(\hat{\boldsymbol{x}};\boldsymbol{\theta}). \tag{5.8}$$

In other words, the parameter $\boldsymbol{\theta}$ is chosen such that it is most likely that the experiment led to the outcome $\hat{\boldsymbol{x}}$. To highlight the different interpretation of the PDF $\mathrm{f}_{\mathbf{X}}(\hat{\boldsymbol{x}};\boldsymbol{\theta})$ for ML estimation, we define the likelihood function (see [Sch91, Kay93, Pap91])

$$l(\boldsymbol{\theta};\hat{\boldsymbol{x}}) = \mathrm{f}_{\mathbf{X}}(\hat{\boldsymbol{x}};\boldsymbol{\theta}).$$

Whereas the PDF $\mathrm{f}_{\mathbf{X}}(\boldsymbol{x};\boldsymbol{\theta})$ describes the statistical properties of the random variable $\mathbf{X}$ for given parameter $\boldsymbol{\theta}$, the likelihood function $l(\boldsymbol{\theta};\hat{\boldsymbol{x}})$ represents the dependence of the unknown parameter $\boldsymbol{\theta}$ on the realization $\hat{\boldsymbol{x}}$. Due to the monotonicity of the logarithm, also the log-likelihood function

$$L(\boldsymbol{\theta};\hat{\boldsymbol{x}}) = \ln l(\boldsymbol{\theta};\hat{\boldsymbol{x}}) = \ln \mathrm{f}_{\mathbf{X}}(\hat{\boldsymbol{x}};\boldsymbol{\theta}) \tag{5.9}$$

can be maximized to find the ML estimate

$$\hat{\boldsymbol{\theta}}_{\mathrm{ML}} = \operatorname*{argmax}_{\boldsymbol{\theta}} L(\boldsymbol{\theta};\hat{\boldsymbol{x}}). \tag{5.10}$$

From the first-order optimality condition, it is necessary that

$$\frac{\partial}{\partial\boldsymbol{\theta}} L(\hat{\boldsymbol{\theta}}_{\mathrm{ML}};\hat{\boldsymbol{x}}) = \boldsymbol{s}(\hat{\boldsymbol{\theta}}_{\mathrm{ML}};\hat{\boldsymbol{x}}) = \mathbf{0}$$

with the score function

$$\boldsymbol{s}(\boldsymbol{\theta};\hat{\boldsymbol{x}}) = \frac{\partial}{\partial\boldsymbol{\theta}} L(\boldsymbol{\theta};\hat{\boldsymbol{x}}).$$

5.2.1 Maximum A-Posteriori Estimation

Suppose that also the unknown quantity is random. Thus, $\boldsymbol{\Theta}$ exhibits the PDF $\mathrm{f}_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ and statistical bindings between the observation $\mathbf{X}$ and the unknown $\boldsymbol{\Theta}$ are represented by the joint PDF $\mathrm{f}_{\mathbf{X},\boldsymbol{\Theta}}(\boldsymbol{x},\boldsymbol{\theta})$. For the maximum a-posteriori (MAP) estimation ([Sch91, Kay93, MW95]), the joint PDF is maximized w.r.t. $\boldsymbol{\theta}$, i.e.,

$$\hat{\boldsymbol{\theta}}_{\mathrm{MAP}} = \operatorname*{argmax}_{\boldsymbol{\theta}} \mathrm{f}_{\mathbf{X},\boldsymbol{\Theta}}(\hat{\boldsymbol{x}},\boldsymbol{\theta}) \tag{5.11}$$

based on the outcome $\hat{\boldsymbol{x}}$ of the experiment, that is, the realization of the observation. MAP estimation finds the estimate that in combination with the outcome of the experiment has the largest likelihood.

Based on Bayes' rule, two interpretations of the MAP optimization (5.11) are possible. First,

$$\hat{\boldsymbol{\theta}}_{\text{MAP}} = \underset{\boldsymbol{\theta}}{\operatorname{argmax}}\, \mathrm{f}_{\mathbf{X}|\boldsymbol{\Theta}}(\hat{\boldsymbol{x}} \mid \boldsymbol{\theta})\, \mathrm{f}_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$$

holds. As $\mathrm{f}_{\mathbf{X}|\boldsymbol{\Theta}}(\boldsymbol{x}|\boldsymbol{\theta})$ is equivalent to $\mathrm{f}_{\mathbf{X}}(\boldsymbol{x};\boldsymbol{\theta})$ with deterministic parameter $\boldsymbol{\theta}$ as used for maximum likelihood estimation, MAP is like ML except that the objective function is extended by incorporating the a-priori information from the marginal PDF $\mathrm{f}_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$.

Second, we have that

$$\hat{\boldsymbol{\theta}}_{\text{MAP}} = \underset{\boldsymbol{\theta}}{\operatorname{argmax}}\, \mathrm{f}_{\boldsymbol{\Theta}|\mathbf{X}}(\boldsymbol{\theta} \mid \hat{\boldsymbol{x}})\, \mathrm{f}_{\mathbf{X}}(\hat{\boldsymbol{x}}) = \underset{\boldsymbol{\theta}}{\operatorname{argmax}}\, \mathrm{f}_{\boldsymbol{\Theta}|\mathbf{X}}(\boldsymbol{\theta} \mid \hat{\boldsymbol{x}})$$

since $\mathrm{f}_{\mathbf{X}}(\hat{\boldsymbol{x}})$ is independent of $\boldsymbol{\theta}$. This reformulation of the MAP optimization (5.11) illustrates that the estimate is chosen to maximize the likelihood for given outcome $\hat{\boldsymbol{x}}$ of the experiment by maximizing the a-posteriori PDF $\mathrm{f}_{\boldsymbol{\Theta}|\mathbf{X}}(\boldsymbol{\theta} \mid \hat{\boldsymbol{x}})$ w.r.t. $\boldsymbol{\theta}$.

5.2.2 Linear Gaussian Model

Remember that [see (5.3)]

$$\mathbf{X} = \boldsymbol{H}\boldsymbol{\theta} + \mathbf{N}$$

in the linear Gaussian model, where the noise is zero-mean Gaussian with covariance matrix $\boldsymbol{C}_{\mathbf{N}}$. Based on this assumption, the log-likelihood function can be written as

$$L(\boldsymbol{\theta};\boldsymbol{x}) = -(\boldsymbol{x} - \boldsymbol{H}\boldsymbol{\theta})^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}(\boldsymbol{x} - \boldsymbol{H}\boldsymbol{\theta}) + c$$

with the realization $\boldsymbol{x}$ of $\mathbf{X}$ and the terms independent of $\boldsymbol{\theta}$ have been collected in $c \in \mathbb{R}$. The ML estimate results from maximizing the log-likelihood function. Alternatively,

$$\hat{\boldsymbol{\theta}}_{\text{ML}} = \underset{\boldsymbol{\theta}}{\operatorname{argmin}} (\boldsymbol{x} - \boldsymbol{H}\boldsymbol{\theta})^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}(\boldsymbol{x} - \boldsymbol{H}\boldsymbol{\theta}) \tag{5.12}$$

where the maximization has been replaced by a minimization via taking the negative and the constant term c has been dropped. In the optimum, the first derivative of the cost function must be zero. Hence, we obtain the condition [see Appendix A9]

$$\frac{\partial}{\partial \boldsymbol{\theta}^*} (\boldsymbol{x} - \boldsymbol{H}\boldsymbol{\theta})^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}(\boldsymbol{x} - \boldsymbol{H}\boldsymbol{\theta}) = -\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}(\boldsymbol{x} - \boldsymbol{H}\boldsymbol{\theta}) = \mathbf{0}$$

from which we can conclude that

$$\hat{\boldsymbol{\theta}}_{\text{ML}} = \left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\right)^{-1} \boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{x} \tag{5.13}$$

where inherently the assumption has been made that $m \geq n$ such that $\boldsymbol{H} \in \mathbb{C}^{m \times n}$ is either square or tall such that $\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}$ is invertible. Note that the ML estimate is unbiased since

$$\begin{aligned}
\mathrm{E}\left[\hat{\boldsymbol{\Theta}}_{\text{ML}}\right] &= \mathrm{E}\left[\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\right)^{-1} \boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\mathbf{X}\right] \\
&= \mathrm{E}\left[\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\right)^{-1} \boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\boldsymbol{\theta}\right] + \mathrm{E}\left[\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\right)^{-1} \boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\mathbf{N}\right] \\
&= \mathrm{E}[\boldsymbol{\theta}] + \mathbf{0} = \boldsymbol{\theta}
\end{aligned}$$

and can be expressed as

$$\hat{\Theta}_{\mathrm{ML}} = \boldsymbol{\theta} + \mathrm{N}_{\mathrm{ML}} \tag{5.14}$$

with the zero-mean Gaussian noise $\mathrm{N}_{\mathrm{ML}} = (\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H})^{-1}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\mathrm{N}$ whose covariance matrix is

$$\begin{aligned}\boldsymbol{C}_{\mathrm{N}_{\mathrm{ML}}} &= \mathrm{E}\left[\mathrm{N}_{\mathrm{ML}}\mathrm{N}_{\mathrm{ML}}^{\mathrm{H}}\right] = \mathrm{E}\left[\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\right)^{-1}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\mathrm{N}\mathrm{N}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\right)^{-1}\right]\\ &= \left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\right)^{-1}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\mathrm{E}[\mathrm{N}\mathrm{N}^{\mathrm{H}}]\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\right)^{-1}\\ &= \left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\right)^{-1}.\end{aligned} \tag{5.15}$$

Additionally, note that the frontend of the ML estimate is the matched filter $\boldsymbol{G}_{\mathrm{MF}} = \boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}$ [see (5.4)] followed by the inverse $(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H})^{-1}$ to remove the interference to ensure unbiasedness [cf. (5.13)].

Also in the linear Gaussian model, we can model $\boldsymbol{\Theta}$ to be random, i.e.,

$$\mathbf{X} = \boldsymbol{H}\boldsymbol{\Theta} + \mathbf{N}.$$

As we consider the linear Gaussian model, $\boldsymbol{\Theta}$ must be complex Gaussian with the PDF

$$\mathrm{f}_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) = \frac{1}{\pi^n \det(\boldsymbol{C}_{\boldsymbol{\Theta}})} \exp\left(-\boldsymbol{\theta}^{\mathrm{H}}\boldsymbol{C}_{\boldsymbol{\Theta}}^{-1}\boldsymbol{\theta}\right)$$

where it is assumed that $\boldsymbol{\Theta}$ is zero-mean. With Bayes' rule, the joint PDF $\mathrm{f}_{\mathbf{X},\boldsymbol{\Theta}}(\boldsymbol{x},\boldsymbol{\theta})$, that is maximized w.r.t. $\boldsymbol{\theta}$ according to the MAP criterion, reads as $\mathrm{f}_{\mathbf{X},\boldsymbol{\Theta}}(\boldsymbol{x},\boldsymbol{\theta}) = \mathrm{f}_{\mathbf{X}|\boldsymbol{\Theta}}(\boldsymbol{x}|\boldsymbol{\theta})\,\mathrm{f}_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$. Consequently, we get for the corresponding log-likelihood function

$$L(\boldsymbol{\theta},\boldsymbol{x}) = -(\boldsymbol{x}-\boldsymbol{H}\boldsymbol{\theta})^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}(\boldsymbol{x}-\boldsymbol{H}\boldsymbol{\theta}) - \boldsymbol{\theta}^{\mathrm{H}}\boldsymbol{C}_{\boldsymbol{\Theta}}^{-1}\boldsymbol{\theta} + \gamma.$$

Here, γ comprises all terms independent of $\boldsymbol{\theta}$. The resulting MAP formulation can be written as

$$\hat{\boldsymbol{\theta}}_{\mathrm{MAP}} = \underset{\boldsymbol{\theta}}{\operatorname{argmin}}(\boldsymbol{x}-\boldsymbol{H}\boldsymbol{\theta})^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}(\boldsymbol{x}-\boldsymbol{H}\boldsymbol{\theta}) + \boldsymbol{\theta}^{\mathrm{H}}\boldsymbol{C}_{\boldsymbol{\Theta}}^{-1}\boldsymbol{\theta}. \tag{5.16}$$

From [see Appendix A9]

$$\frac{\partial}{\partial\boldsymbol{\theta}^*}(\boldsymbol{x}-\boldsymbol{H}\boldsymbol{\theta})^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}(\boldsymbol{x}-\boldsymbol{H}\boldsymbol{\theta}) + \boldsymbol{\theta}^{\mathrm{H}}\boldsymbol{C}_{\boldsymbol{\Theta}}^{-1}\boldsymbol{\theta} = -\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}(\boldsymbol{x}-\boldsymbol{H}\boldsymbol{\theta}) + \boldsymbol{C}_{\boldsymbol{\Theta}}^{-1}\boldsymbol{\theta} = \mathbf{0}$$

we get the MAP estimate in the linear Gaussian model

$$\hat{\boldsymbol{\theta}}_{\mathrm{MAP}} = \left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H} + \boldsymbol{C}_{\boldsymbol{\Theta}}^{-1}\right)^{-1}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{x}. \tag{5.17}$$

Note that

$$\hat{\boldsymbol{\Theta}}_{\mathrm{MAP}} \neq \boldsymbol{\Theta} + \mathrm{N}_{\mathrm{MAP}}$$

i.e., it cannot be decomposed into the sum of the desired quantity $\boldsymbol{\Theta}$ and an independent noise component $\mathrm{N}_{\mathrm{MAP}}$ because of the regularization term $\boldsymbol{C}_{\boldsymbol{\Theta}}^{-1}$ in the inverse. Additionally, note that the front end of the MAP estimator is the matched filter $\boldsymbol{G}_{\mathrm{MF}} = \boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}$ [see (5.4)].

5.3 Least Squares Estimation

In the linear Gaussian model, the observation reads as

$$\mathrm{X} = \boldsymbol{H}\boldsymbol{\theta} + \mathrm{N}$$

with the additive Gaussian noise N. Consider that the model is simplified as follows. The randomness of the observation X is neglected and due to measurement errors (generated by the noise N), the resulting equation system can only be solved approximately. The corresponding least squares (LS) problem reads as (see [TB97])

$$\boldsymbol{\theta}_{\mathrm{LS}} = \underset{\boldsymbol{\theta}}{\operatorname{argmin}} \|\boldsymbol{x} - \boldsymbol{H}\boldsymbol{\theta}\|_2^2 . \tag{5.18}$$

Setting the derivative of the LS cost function to zero leads to [see Appendix A9]

$$\begin{aligned}\frac{\partial}{\partial \boldsymbol{\theta}^*} \|\boldsymbol{x} - \boldsymbol{H}\boldsymbol{\theta}\|_2^2 &= \frac{\partial}{\partial \boldsymbol{\theta}^*} \left(\|\boldsymbol{x}\|_2^2 - \boldsymbol{x}^{\mathrm{H}}\boldsymbol{H}\boldsymbol{\theta} - \boldsymbol{\theta}^{\mathrm{H}}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{x} + \boldsymbol{\theta}^{\mathrm{H}}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\boldsymbol{\theta}\right)\\ &= -\boldsymbol{H}^{\mathrm{H}}\boldsymbol{x} + \boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\boldsymbol{\theta} = \mathbf{0}.\end{aligned}$$

Solving for $\boldsymbol{\theta}$ gives the least squares estimate

$$\hat{\boldsymbol{\theta}}_{\mathrm{LS}} = \left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\right)^{-1} \boldsymbol{H}^{\mathrm{H}}\boldsymbol{x} = \boldsymbol{H}^{+}\boldsymbol{x}. \tag{5.19}$$

Taking into account the randomness of the observation X again, we see that the least squares estimate is unbiased since

$$\mathrm{E}\big[\hat{\Theta}_{\mathrm{LS}}\big] = \mathrm{E}\left[\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\right)^{-1} \boldsymbol{H}^{\mathrm{H}}\mathrm{X}\right] = \mathrm{E}[\boldsymbol{\theta}] + \mathbf{0} = \boldsymbol{\theta}$$

due the pseudoinverse in the expression for $\hat{\Theta}_{\mathrm{LS}}$ and based on the assumption that $m \geq n$ such that $\boldsymbol{H} \in \mathbb{C}^{m\times n}$ has a left-inverse.

Note that the LS estimate is closely related to the ML estimate. When restricting the noise covariance in (5.13) to be a weighted identity matrix, i.e., $\boldsymbol{C}_{\mathrm{N}} = \sigma_{\mathrm{N}}^2\mathbf{I}$, we find for the ML estimate in the linear Gaussian model

$$\begin{aligned}\hat{\boldsymbol{\theta}}_{\mathrm{ML}} &= \left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathrm{N}}^{-1}\boldsymbol{H}\right)^{-1} \boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathrm{N}}^{-1}\boldsymbol{x}\\ &= \left(\boldsymbol{H}^{\mathrm{H}}\sigma_{\mathrm{N}}^{-2}\boldsymbol{H}\right)^{-1} \boldsymbol{H}^{\mathrm{H}}\sigma_{\mathrm{N}}^{-2}\boldsymbol{x}\\ &= \left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\right)^{-1} \boldsymbol{H}^{\mathrm{H}}\boldsymbol{x} = \hat{\boldsymbol{\theta}}_{\mathrm{LS}}.\end{aligned}$$

In other words, the ML estimate reduces to the LS estimate in the linear Gaussian model for a noise covariance matrix equal to a weighted identity matrix. Both estimates are unbiased. Therefore, as the ML estimate in (5.14), the LS estimate is given by

$$\hat{\Theta}_{\mathrm{LS}} = \boldsymbol{\theta} + \mathrm{N}_{\mathrm{LS}}$$

with the zero-mean noise N_{LS} whose covariance matrix can be found to be

$$\boldsymbol{C}_{\mathrm{N}_{\mathrm{LS}}} = \mathrm{E}\left[\mathrm{N}_{\mathrm{LS}}\mathrm{N}_{\mathrm{LS}}^{\mathrm{H}}\right] = \left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\right)^{-1} \boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathrm{N}}\boldsymbol{H}\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\right)^{-1}.$$

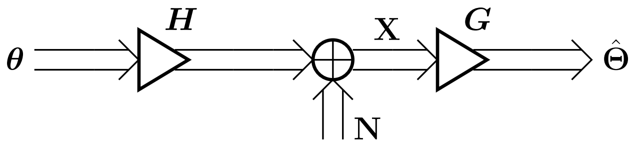

Fig. 5.4. Linear Model with the Linear Estimator $\boldsymbol{G}$

In the following, let us investigate the difference of the noise covariance matrix of the LS estimate and that of the ML estimate [see (5.15)], i.e.,

$$\begin{aligned}\boldsymbol{C}_{\mathrm{N}_{\mathrm{LS}}} - \boldsymbol{C}_{\mathrm{N}_{\mathrm{ML}}} &= \left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\right)^{-1}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathrm{N}}\boldsymbol{H}\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\right)^{-1} - \left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathrm{N}}^{-1}\boldsymbol{H}\right)^{-1}\\ &= \left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\right)^{-1}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathrm{N}}\boldsymbol{H}\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\right)^{-1} - \left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathrm{N}}^{-1}\boldsymbol{H}\right)^{-1} - \left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathrm{N}}^{-1}\boldsymbol{H}\right)^{-1} + \left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathrm{N}}^{-1}\boldsymbol{H}\right)^{-1}\end{aligned}$$

where the trivial extension by a zero matrix $-(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathrm{N}}^{-1}\boldsymbol{H})^{-1} + (\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathrm{N}}^{-1}\boldsymbol{H})^{-1} = \mathbf{0}$ was made in the second line. Equivalently, we have

$$\begin{aligned}\boldsymbol{C}_{\mathrm{N}_{\mathrm{LS}}} - \boldsymbol{C}_{\mathrm{N}_{\mathrm{ML}}} = &\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\right)^{-1}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathrm{N}}\boldsymbol{H}\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\right)^{-1} - \left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\right)^{-1}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathrm{N}}^{-1}\boldsymbol{H}\right)^{-1}\\ &- \left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathrm{N}}^{-1}\boldsymbol{H}\right)^{-1}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\right)^{-1} + \left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathrm{N}}^{-1}\boldsymbol{H}\right)^{-1}\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathrm{N}}^{-1}\boldsymbol{H}\right)\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathrm{N}}^{-1}\boldsymbol{H}\right)^{-1}\end{aligned}$$

With

$$\boldsymbol{\Delta} = \boldsymbol{H}\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\right)^{-1} - \boldsymbol{C}_{\mathrm{N}}^{-1}\boldsymbol{H}\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathrm{N}}^{-1}\boldsymbol{H}\right)^{-1}$$

it can be shown that

$$\boldsymbol{C}_{\mathrm{N}_{\mathrm{LS}}} - \boldsymbol{C}_{\mathrm{N}_{\mathrm{ML}}} = \boldsymbol{\Delta}^{\mathrm{H}}\boldsymbol{C}_{\mathrm{N}}\boldsymbol{\Delta} \succeq \mathbf{0}$$

since $\boldsymbol{C}_{\mathrm{N}} \succ \mathbf{0}$. This result implies that

$$\boldsymbol{C}_{\mathrm{N}_{\mathrm{LS}}} \succeq \boldsymbol{C}_{\mathrm{N}_{\mathrm{ML}}} \tag{5.20}$$

i.e., the noise of the LS estimate is larger than that of the ML estimate in any dimension.

5.4 Linear Estimation

In this section, the estimator is restricted to be linear in a linear model (see Fig. 5.4). For the zero-mean noise N, however, we do not employ the assumption that it is Gaussian. Only the covariance matrix $\boldsymbol{C}_{\mathrm{N}} = \mathrm{E}[\mathrm{N}\mathrm{N}^{\mathrm{H}}]$ is assumed to be known.

The estimate is the output of the linear estimator $\boldsymbol{G} \in \mathbb{C}^{n\times m}$ applied to the observation X, i.e.,

$$\hat{\Theta} = \boldsymbol{G}\mathrm{X} = \boldsymbol{G}\boldsymbol{H}\boldsymbol{\theta} + \boldsymbol{G}\mathrm{N}. \tag{5.21}$$

The goal of the designs discussed in the following is to find the filter operation $\boldsymbol{G}$ of the linear estimator.

5.4.1 Best Linear Unbiased Estimator

The main goal of the best linear unbiased estimator (BLUE) is to ensure unbiasedness of the estimate. Therefore, we require that

$$\mathrm{E}\big[\hat{\Theta}\big] = \boldsymbol{G}\mathrm{E}[\mathrm{X}] = \boldsymbol{\theta}$$

and with (5.21),

$$\begin{aligned}\boldsymbol{G}\mathrm{E}[\boldsymbol{H\theta}+\mathrm{N}] &= \boldsymbol{\theta}\\ \boldsymbol{GH\theta}+\boldsymbol{G}0 &= \boldsymbol{\theta}\\ \boldsymbol{GH\theta} &= \boldsymbol{\theta}.\end{aligned}$$

The value of $\boldsymbol{\theta}$ is unknown. In other words, the equation $\boldsymbol{GH\theta} = \boldsymbol{\theta}$ must be fulfilled for any $\boldsymbol{\theta} \in \mathbb{C}^n$ and the unbiasedness requirement finally reads as

$$\boldsymbol{GH} = \mathbf{I}. \tag{5.22}$$

As the interference between the entries in $\boldsymbol{\theta}$ is forced to be zero by this equation, (5.22) is often called the zero-forcing (ZF) condition.

The BLUE is the linear estimator that minimizes the variance of the estimator under the constraint of unbiasedness, i.e.,

$$\boldsymbol{G}_{\mathrm{BLUE}} = \underset{\boldsymbol{G}}{\operatorname{argmin}}\, \mathrm{var}\left[\hat{\boldsymbol{\Theta}}\right] \qquad \text{s.t.:} \quad \mathrm{E}\big[\hat{\boldsymbol{\Theta}}\big] = \boldsymbol{\theta}. \tag{5.23}$$

The variance of the estimator can be written as

$$\begin{aligned}\mathrm{var}\big[\hat{\boldsymbol{\Theta}}\big] &= \mathrm{E}\left[\big\|\hat{\boldsymbol{\Theta}} - \mathrm{E}\big[\hat{\boldsymbol{\Theta}}\big]\big\|_2^2\right]\\ &= \mathrm{E}\left[\|\boldsymbol{GH\theta}+\boldsymbol{G}\mathrm{N}-\boldsymbol{GH\theta}\|_2^2\right]\\ &= \mathrm{E}\left[\mathrm{N}^{\mathrm{H}}\boldsymbol{G}^{\mathrm{H}}\boldsymbol{G}\mathrm{N}\right]\\ &= \mathrm{E}\left[\mathrm{tr}\left(\boldsymbol{G}\mathrm{N}\mathrm{N}^{\mathrm{H}}\boldsymbol{G}^{\mathrm{H}}\right)\right]\\ &= \mathrm{tr}(\boldsymbol{G}\boldsymbol{C}_{\mathrm{N}}\boldsymbol{G}^{\mathrm{H}})\end{aligned}$$

where we exploited that $\mathrm{N}^{\mathrm{H}}\boldsymbol{G}^{\mathrm{H}}\boldsymbol{G}\mathrm{N} = \mathrm{tr}(\mathrm{N}^{\mathrm{H}}\boldsymbol{G}^{\mathrm{H}}\boldsymbol{G}\mathrm{N})$ since it is a scalar, used (A35), and exchanged the linear operators expectation and trace. With this result and (5.22), (5.23) can be reformulated as

$$\boldsymbol{G}_{\mathrm{BLUE}} = \underset{\boldsymbol{G}}{\operatorname{argmin}}\, \mathrm{tr}(\boldsymbol{G}\boldsymbol{C}_{\mathrm{N}}\boldsymbol{G}^{\mathrm{H}}) \qquad \text{s.t.:} \quad \boldsymbol{GH} = \mathbf{I}. \tag{5.24}$$

This optimization w.r.t. the matrix $\boldsymbol{G}$ can be split into separate optimizations w.r.t. the rows of $\boldsymbol{G}$. To this end, let $\boldsymbol{g}_k^{\mathrm{H}} \in \mathbb{C}^{1\times m}$ denote the k-th row of the estimator $\boldsymbol{G}$, i.e.,

$$\boldsymbol{G} = \begin{bmatrix}\boldsymbol{g}_1^{\mathrm{H}}\\ \vdots\\ \boldsymbol{g}_n^{\mathrm{H}}\end{bmatrix} = [\boldsymbol{g}_1,\ldots,\boldsymbol{g}_n]^{\mathrm{H}} \in \mathbb{C}^{n\times m}. \tag{5.25}$$

Note that $\boldsymbol{g}_k^{\mathrm{H}} = \boldsymbol{e}_k^{\mathrm{T}}\boldsymbol{G}$ and $\boldsymbol{g}_k = \boldsymbol{G}^{\mathrm{H}}\boldsymbol{e}_k$ with the canonical unit vector $\boldsymbol{e}_k \in \{0,1\}^n$ whose k-th element is one and the other elements are zero. As the trace of a matrix is the sum of its diagonal elements, it holds that

$$\mathrm{tr}(\boldsymbol{G}\boldsymbol{C}_{\mathrm{N}}\boldsymbol{G}^{\mathrm{H}}) = \sum_{k=1}^{n} \boldsymbol{e}_k^{\mathrm{T}}\boldsymbol{G}\boldsymbol{C}_{\mathrm{N}}\boldsymbol{G}^{\mathrm{H}}\boldsymbol{e}_k = \sum_{k=1}^{n} \boldsymbol{g}_k^{\mathrm{H}}\boldsymbol{C}_{\mathrm{N}}\boldsymbol{g}_k.$$

Moreover, the k-th row of the zero-forcing constraint in (5.24) simply reads as

$$\begin{aligned} \boldsymbol{e}_k^{\mathrm{T}} \boldsymbol{G}\boldsymbol{H} &= \boldsymbol{e}_k^{\mathrm{T}} \mathbf{I} \\ \boldsymbol{g}_k^{\mathrm{H}} \boldsymbol{H} &= \boldsymbol{e}_k^{\mathrm{T}}. \end{aligned}$$

It can be observed that the k-th row of the unbiasedness constraint only depends on $\boldsymbol{g}_k^{\mathrm{H}}$ and is independent of the other rows of $\boldsymbol{G}$. Consequently, (5.24) can be divided into n optimizations, each for the corresponding row of $\boldsymbol{G}$, that is, for $k = 1, \dots, n$,

$$\boldsymbol{g}_{\mathrm{BLUE},k}^{\mathrm{H}} = \underset{\boldsymbol{g}_k^{\mathrm{H}}}{\operatorname{argmin}}\, \boldsymbol{g}_k^{\mathrm{H}} \boldsymbol{C}_{\mathbf{N}} \boldsymbol{g}_k \qquad \text{s.t.:} \quad \boldsymbol{g}_k^{\mathrm{H}} \boldsymbol{H} = \boldsymbol{e}_k^{\mathrm{T}}. \tag{5.26}$$

This is a convex optimization problem since the cost function is a Hermitian form with the positive definite $\boldsymbol{C}_{\mathbf{N}}$ and the constraint is linear. Thus, the first order optimality conditions are sufficient to find the global optimum $\boldsymbol{g}_{\mathrm{BLUE},k}^{\mathrm{H}}$. First, the Lagrangian function must be formed that is the sum of the objective function and a weighted version of the constraint (reformulated such that one side of the equation is zero), i.e.,

$$L\left(\boldsymbol{g}_k^{\mathrm{H}}, \boldsymbol{\lambda}\right) = \boldsymbol{g}_k^{\mathrm{H}} \boldsymbol{C}_{\mathbf{N}} \boldsymbol{g}_k + 2\mathrm{Re}\left\{\left(\boldsymbol{g}_k^{\mathrm{H}} \boldsymbol{H} - \boldsymbol{e}_k^{\mathrm{T}}\right) \boldsymbol{\lambda}\right\}$$

where the real-part operator must be put around the weighted constraint because it is complex valued [see Appendix A10]. The optimal $\boldsymbol{g}_{\mathrm{BLUE},k}^{\mathrm{H}}$ must obey the constraint of (5.26), but also the derivative of the Lagrangian function $L(\boldsymbol{g}_k^{\mathrm{H}}, \boldsymbol{\lambda})$ w.r.t. $\boldsymbol{g}_k^{\mathrm{H}}$ must be zero. The condition for the derivative can be expressed as [see Appendix A9]

$$\frac{\partial L\left(\boldsymbol{g}_k^{\mathrm{H}}, \boldsymbol{\lambda}\right)}{\partial \boldsymbol{g}_k^{*}} = \boldsymbol{C}_{\mathbf{N}} \boldsymbol{g}_k + \boldsymbol{H}\boldsymbol{\lambda} = \mathbf{0}.$$

Therefore, we find that

$$\boldsymbol{g}_k = \boldsymbol{C}_{\mathbf{N}}^{-1} \boldsymbol{H} \boldsymbol{\lambda}.$$

Substituting this result into the constraint yields

$$\boldsymbol{\lambda}^{\mathrm{H}} \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathbf{N}}^{-1} \boldsymbol{H} = \boldsymbol{e}_k^{\mathrm{T}}$$

and thus,

$$\boldsymbol{g}_{\mathrm{BLUE},k}^{\mathrm{H}} = \boldsymbol{e}_k^{\mathrm{T}} \left(\boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathbf{N}}^{-1} \boldsymbol{H}\right)^{-1} \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathbf{N}}^{-1}. \tag{5.27}$$

Collecting the results for $k = 1, \dots, K$ into a matrix gives

$$\boldsymbol{G}_{\mathrm{BLUE}} = \left(\boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathbf{N}}^{-1} \boldsymbol{H}\right)^{-1} \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathbf{N}}^{-1}. \tag{5.28}$$

Note that the front-end (first stage) of the BLUE is the matched filter $\boldsymbol{G}_{\mathrm{MF}} = \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathbf{N}}^{-1}$. Therefore, the output of $\boldsymbol{G}_{\mathrm{BLUE}}$ is a statistic resulting from an invertible transform of the matched filter output. In other words, the output of the BLUE is a sufficient statistic for $\boldsymbol{\theta}$ in the linear Gaussian model. Additionally, note that the resulting BLUE estimate

$$\hat{\Theta}_{\mathrm{BLUE}} = \boldsymbol{G}_{\mathrm{BLUE}} \mathbf{X} = \left(\boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathbf{N}}^{-1} \boldsymbol{H}\right)^{-1} \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathbf{N}}^{-1} \mathbf{X} \tag{5.29}$$

is the same as the ML estimate for the linear Gaussian model [cf. (5.13)]. This observation leads to the conclusion that the ML estimate in linear Gaussian model is not only unbiased but also exhibits minimum variance at the same time. Note, however, that such an observation cannot be made anymore for non-Gaussian noise in the linear model (5.21). In that case, the BLUE still has got the form in (5.28) leading to the estimate (5.29), but the ML estimate is different from (5.13), e.g., it might even be a non-linear function of the observation $\mathbf{X}$.

5.4.2 Linear Minimum Mean Square Error Estimator

The linear minimum mean square error (MMSE) estimator results from the minimization of the mean square error (MSE) where the desired vector $\mathbf{\Theta}$ is modelled to be random. We assume that $\mathbf{\Theta}$ is zero-mean, its covariance matrix is known to be $\boldsymbol{C}_{\mathbf{\Theta}} = \mathrm{E}[\mathbf{\Theta}\mathbf{\Theta}^{\mathrm{H}}]$, and it is independent of the noise $\mathbf{N}$. Thus, $\mathrm{E}[\mathbf{\Theta}\mathbf{N}^{\mathrm{H}}] = \mathbf{0}$ and $\mathrm{E}[\mathbf{N}\mathbf{\Theta}^{\mathrm{H}}] = \mathbf{0}$ hold.

Define the error

$$\mathbf{E} = \mathbf{\Theta} - \hat{\mathbf{\Theta}}$$

which is obviously random like $\mathbf{\Theta}$ and $\hat{\mathbf{\Theta}}$. Therefore, the second moment of the error is employed for the design of the estimator $\boldsymbol{G}$, i.e., the MSE

$$\mathrm{MSE} = \mathrm{var}[\mathbf{E}] = \mathrm{E}[\|\mathbf{E}\|_2^2] = \mathrm{E}\left[\left\|\mathbf{\Theta} - \hat{\mathbf{\Theta}}\right\|_2^2\right]. \tag{5.30}$$

Incorporating the linear model (5.21) yields

$$\begin{aligned}
\mathrm{MSE} &= \mathrm{E}\left[\|\mathbf{\Theta} - \boldsymbol{G}\boldsymbol{H}\mathbf{\Theta} - \boldsymbol{G}\mathbf{N}\|_2^2\right] \\
&= \mathrm{E}\left[(\mathbf{\Theta} - \boldsymbol{G}\boldsymbol{H}\mathbf{\Theta} - \boldsymbol{G}\mathbf{N})^{\mathrm{H}}(\mathbf{\Theta} - \boldsymbol{G}\boldsymbol{H}\mathbf{\Theta} - \boldsymbol{G}\mathbf{N})\right] \\
&= \mathrm{tr}\left(\mathrm{E}\left[(\mathbf{\Theta} - \boldsymbol{G}\boldsymbol{H}\mathbf{\Theta} - \boldsymbol{G}\mathbf{N})(\mathbf{\Theta} - \boldsymbol{G}\boldsymbol{H}\mathbf{\Theta} - \boldsymbol{G}\mathbf{N})^{\mathrm{H}}\right]\right)
\end{aligned} \tag{5.31}$$

where $\|\boldsymbol{y}\|_2^2 = \boldsymbol{y}^{\mathrm{H}}\boldsymbol{y}$ was employed for the second line. Additionally, the last line was obtained by exploiting that $\boldsymbol{y}^{\mathrm{H}}\boldsymbol{y}$ is scalar, the trace of a scalar gives the scalar, and that the trace and the expectation are linear operators. By incorporating the independence of $\mathbf{\Theta}$ and $\mathbf{N}$, we obtain

$$\begin{aligned}
\mathrm{MSE} &= \mathrm{tr}\left(\mathrm{E}\left[(\mathbf{\Theta} - \boldsymbol{G}\boldsymbol{H}\mathbf{\Theta})(\mathbf{\Theta} - \boldsymbol{G}\boldsymbol{H}\mathbf{\Theta})^{\mathrm{H}}\right]\right) + \mathrm{tr}\left(\mathrm{E}\left[\boldsymbol{G}\mathbf{N}\mathbf{N}^{\mathrm{H}}\boldsymbol{G}^{\mathrm{H}}\right]\right) \\
&= \mathrm{tr}\left((\mathbf{I} - \boldsymbol{G}\boldsymbol{H})\boldsymbol{C}_{\mathbf{\Theta}}(\mathbf{I} - \boldsymbol{G}\boldsymbol{H})^{\mathrm{H}}\right) + \mathrm{tr}\left(\boldsymbol{G}\boldsymbol{C}_{\mathbf{N}}\boldsymbol{G}^{\mathrm{H}}\right).
\end{aligned}$$

The MMSE estimator minimizes the MSE, i.e.,

$$\boldsymbol{G}_{\mathrm{MMSE}} = \underset{\boldsymbol{G}}{\mathrm{argmin}}\,\mathrm{MSE}. \tag{5.32}$$

As shown in (5.25), we denote the k-th row of $\boldsymbol{G} \in \mathbb{C}^{n\times m}$ as $\boldsymbol{g}_k^{\mathrm{H}} \in \mathbb{C}^{1\times m}$ where it holds that $\boldsymbol{g}_k^{\mathrm{H}} = \boldsymbol{e}_k^{\mathrm{T}}\boldsymbol{G}$ with the canonical unit vector $\boldsymbol{e}_k \in \{0,1\}^n$ whose k-th element is one and the other elements are zero. Due to $\mathrm{tr}(\boldsymbol{A}) = \sum_{k=1}^n \boldsymbol{e}_k^{\mathrm{T}}\boldsymbol{A}\boldsymbol{e}_k$ for $\boldsymbol{A} \in \mathbb{C}^{n\times n}$, the MSE can be split into partial MSEs, i.e.,

$$\mathrm{MSE} = \sum_{k=1}^{n} \mathrm{MSE}_k$$

with

$$\begin{aligned}
\mathrm{MSE}_k &= \boldsymbol{e}_k^{\mathrm{T}}(\mathbf{I} - \boldsymbol{G}\boldsymbol{H})\boldsymbol{C}_{\mathbf{\Theta}}(\mathbf{I} - \boldsymbol{G}\boldsymbol{H})^{\mathrm{H}}\boldsymbol{e}_k + \boldsymbol{e}_k^{\mathrm{T}}\boldsymbol{G}\boldsymbol{C}_{\mathbf{N}}\boldsymbol{G}^{\mathrm{H}}\boldsymbol{e}_k \\
&= \boldsymbol{e}_k^{\mathrm{T}}\boldsymbol{C}_{\mathbf{\Theta}}\boldsymbol{e}_k - 2\mathrm{Re}\{\boldsymbol{e}_k^{\mathrm{T}}\boldsymbol{G}\boldsymbol{H}\boldsymbol{C}_{\mathbf{\Theta}}\boldsymbol{e}_k\} + \boldsymbol{e}_k^{\mathrm{T}}\left(\boldsymbol{G}\boldsymbol{H}\boldsymbol{C}_{\mathbf{\Theta}}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{G}^{\mathrm{H}} + \boldsymbol{G}\boldsymbol{C}_{\mathbf{N}}\boldsymbol{G}^{\mathrm{H}}\right)\boldsymbol{e}_k \\
&= \boldsymbol{e}_k^{\mathrm{T}}\boldsymbol{C}_{\mathbf{\Theta}}\boldsymbol{e}_k - 2\mathrm{Re}\{\boldsymbol{g}_k^{\mathrm{H}}\boldsymbol{H}\boldsymbol{C}_{\mathbf{\Theta}}\boldsymbol{e}_k\} + \boldsymbol{g}_k^{\mathrm{H}}\left(\boldsymbol{H}\boldsymbol{C}_{\mathbf{\Theta}}\boldsymbol{H}^{\mathrm{H}} + \boldsymbol{C}_{\mathbf{N}}\right)\boldsymbol{g}_k.
\end{aligned} \tag{5.33}$$

Note that the k-th partial MSE only depends on $\boldsymbol{g}_k^{\mathrm{H}}$ but not on the other rows of $\boldsymbol{G}$. Consequently, the k-th row of $\boldsymbol{G}_{\mathrm{MMSE}}$ can be found via

$$\begin{aligned}\boldsymbol{g}_{\mathrm{MMSE},k}^{\mathrm{H}} &= \underset{\boldsymbol{g}_k^{\mathrm{H}}}{\operatorname{argmin}}\, \mathrm{MSE}_k \\ &= \underset{\boldsymbol{g}_k^{\mathrm{H}}}{\operatorname{argmin}}\, \boldsymbol{e}_k^{\mathrm{T}} \boldsymbol{C}_{\boldsymbol{\Theta}} \boldsymbol{e}_k - 2\mathrm{Re}\{\boldsymbol{g}_k^{\mathrm{H}} \boldsymbol{H} \boldsymbol{C}_{\boldsymbol{\Theta}} \boldsymbol{e}_k\} + \boldsymbol{g}_k^{\mathrm{H}} \left(\boldsymbol{H} \boldsymbol{C}_{\boldsymbol{\Theta}} \boldsymbol{H}^{\mathrm{H}} + \boldsymbol{C}_{\mathrm{N}}\right) \boldsymbol{g}_k.\end{aligned} \tag{5.34}$$

In the optimum, the derivative of MSE_k w.r.t. $\boldsymbol{g}_k$ must be zero. Thus, we get the condition [see Appendix A9]

$$\frac{\partial \mathrm{MSE}_k}{\partial \boldsymbol{g}_k^*} = -\boldsymbol{H} \boldsymbol{C}_{\boldsymbol{\Theta}} \boldsymbol{e}_k + \left(\boldsymbol{H} \boldsymbol{C}_{\boldsymbol{\Theta}} \boldsymbol{H}^{\mathrm{H}} + \boldsymbol{C}_{\mathrm{N}}\right) \boldsymbol{g}_k = \mathbf{0}$$

which leads to

$$\boldsymbol{g}_{\mathrm{MMSE},k}^{\mathrm{H}} = \boldsymbol{e}_k^{\mathrm{T}} \boldsymbol{C}_{\boldsymbol{\Theta}} \boldsymbol{H}^{\mathrm{H}} \left(\boldsymbol{H} \boldsymbol{C}_{\boldsymbol{\Theta}} \boldsymbol{H}^{\mathrm{H}} + \boldsymbol{C}_{\mathrm{N}}\right)^{-1}. \tag{5.35}$$

Combining the results for $k = 1, \ldots, K$ yields

$$\boldsymbol{G}_{\mathrm{MMSE}} = [\boldsymbol{g}_{\mathrm{MMSE},1}, \ldots, \boldsymbol{g}_{\mathrm{MMSE},n}]^{\mathrm{H}} = \boldsymbol{C}_{\boldsymbol{\Theta}} \boldsymbol{H}^{\mathrm{H}} \left(\boldsymbol{H} \boldsymbol{C}_{\boldsymbol{\Theta}} \boldsymbol{H}^{\mathrm{H}} + \boldsymbol{C}_{\mathrm{N}}\right)^{-1} \tag{5.36}$$

which is the linear estimator that gives the MMSE estimate

$$\hat{\boldsymbol{\Theta}}_{\mathrm{MMSE}} = \boldsymbol{G}_{\mathrm{MMSE}} \mathbf{X} = \boldsymbol{C}_{\boldsymbol{\Theta}} \boldsymbol{H}^{\mathrm{H}} \left(\boldsymbol{H} \boldsymbol{C}_{\boldsymbol{\Theta}} \boldsymbol{H}^{\mathrm{H}} + \boldsymbol{C}_{\mathrm{N}}\right)^{-1} \mathbf{X}. \tag{5.37}$$

With the matrix inversion lemma (A25), the solution for the linear MMSE estimator in (5.36) can equivalently be rewritten as

$$\begin{aligned}\boldsymbol{G}_{\mathrm{MMSE}} &= \boldsymbol{C}_{\boldsymbol{\Theta}} \boldsymbol{H}^{\mathrm{H}} \left(\boldsymbol{C}_{\mathrm{N}}^{-1} - \boldsymbol{C}_{\mathrm{N}}^{-1} \boldsymbol{H} \left(\boldsymbol{C}_{\boldsymbol{\Theta}}^{-1} + \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathrm{N}}^{-1} \boldsymbol{H}\right)^{-1} \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathrm{N}}^{-1}\right) \\ &= \boldsymbol{C}_{\boldsymbol{\Theta}} \left(\boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathrm{N}}^{-1} - \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathrm{N}}^{-1} \boldsymbol{H} \left(\boldsymbol{C}_{\boldsymbol{\Theta}}^{-1} + \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathrm{N}}^{-1} \boldsymbol{H}\right)^{-1} \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathrm{N}}^{-1}\right) \\ &= \boldsymbol{C}_{\boldsymbol{\Theta}} \left(\boldsymbol{C}_{\boldsymbol{\Theta}}^{-1} + \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathrm{N}}^{-1} \boldsymbol{H} - \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathrm{N}}^{-1} \boldsymbol{H}\right) \left(\boldsymbol{C}_{\boldsymbol{\Theta}}^{-1} + \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathrm{N}}^{-1} \boldsymbol{H}\right)^{-1} \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathrm{N}}^{-1} \\ &= \left(\boldsymbol{C}_{\boldsymbol{\Theta}}^{-1} + \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathrm{N}}^{-1} \boldsymbol{H}\right)^{-1} \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathrm{N}}^{-1}.\end{aligned} \tag{5.38}$$

Hence, the linear MMSE estimate can alternatively be expressed as [cf. (5.37)]

$$\hat{\boldsymbol{\Theta}}_{\mathrm{MMSE}} = \left(\boldsymbol{C}_{\boldsymbol{\Theta}}^{-1} + \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathrm{N}}^{-1} \boldsymbol{H}\right)^{-1} \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathrm{N}}^{-1} \mathbf{X}. \tag{5.39}$$

Note that this expression for the linear MMSE estimate $\hat{\boldsymbol{\Theta}}_{\mathrm{MMSE}}$ is the same as that of the MAP esimate in (5.17) for the linear Gaussian model. However, this equivalence only holds in the linear Gaussian model. For a non-Gaussian noise $\mathbf{N}$ and/or non-Gaussian desired quantity $\boldsymbol{\Theta}$ in the linear model, the result for the MAP estimate is different from (5.17), whereas the linear MMSE estimate is still given by (5.39).

For the following discussion, we assume that the entries of the desired vector $\boldsymbol{\Theta}$ are uncorrelated, i.e.,

$$\boldsymbol{C}_{\boldsymbol{\Theta}} = \mathrm{E}[\boldsymbol{\Theta}\boldsymbol{\Theta}^{\mathrm{H}}] = \sigma_{\boldsymbol{\Theta}}^2 \mathbf{I}.$$

Therefore, the solution for the MMSE filter can be rewritten as [cf. (5.38)]

$$\begin{aligned}\boldsymbol{G}_{\mathrm{MMSE}} &= \left(\sigma_{\boldsymbol{\Theta}}^{-2} \mathbf{I} + \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathrm{N}}^{-1} \boldsymbol{H}\right)^{-1} \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathrm{N}}^{-1} \\ &= \sigma_{\boldsymbol{\Theta}}^2 \left(\mathbf{I} + \sigma_{\boldsymbol{\Theta}}^2 \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathrm{N}}^{-1} \boldsymbol{H}\right)^{-1} \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\mathrm{N}}^{-1}.\end{aligned} \tag{5.40}$$

The two expressions allow the investigation of the MMSE filter depending on the variance σ_{Θ}^2.

For very small values for the variance σ_{Θ}^2, we get [see second line of (5.40)]

$$\lim_{\sigma_{\Theta}^2 \to 0} \boldsymbol{G}_{\text{MMSE}} = \lim_{\sigma_{\Theta}^2 \to 0} \sigma_{\Theta}^2 \boldsymbol{H}^{\text{H}} \boldsymbol{C}_{\mathbf{N}}^{-1} = \lim_{\sigma_{\Theta}^2 \to 0} \sigma_{\Theta}^2 \boldsymbol{G}_{\text{MF}} = \mathbf{0}$$

with the matched filter $\boldsymbol{G}_{\text{MF}}$ [see (5.4)]. In other words, the MMSE filter converges to the all-zeros matrix if the variance of the entries of the desired $\mathbf{\Theta}$ is small compared to the variances of the entries of the noise $\mathbf{N}$. However, we can observe that the structure is given by that of the matched filter. Thus, this result highlights that the application of the matched filter $\boldsymbol{G}_{\text{MF}}$ is advisable in the MMSE sense, if the desired signal is weak.

When the variance σ_{Θ}^2 is large, the resulting MMSE filter reads as [see first line of (5.40)]

$$\lim_{\sigma_{\Theta}^2 \to \infty} \boldsymbol{G}_{\text{MMSE}} = \left(\boldsymbol{H}^{\text{H}} \boldsymbol{C}_{\mathbf{N}}^{-1} \boldsymbol{H}\right)^{-1} \boldsymbol{H}^{\text{H}} \boldsymbol{C}_{\mathbf{N}}^{-1} = \boldsymbol{G}_{\text{BLUE}}.$$

We can observe that the MMSE filter converges to the BLUE for negligible noise, that is, it fulfills the unbiasedness condition $\boldsymbol{G}\boldsymbol{H} = \mathbf{I}$.

As can be inferred from (5.32), the linear MMSE estimator leads to the minimum MSE when restricting to linear estimators. The MSE is the trace of the MSE matrix [cf. (5.31)]

$$\boldsymbol{M}(\boldsymbol{G}) = \mathrm{E}\left[(\mathbf{\Theta} - \boldsymbol{G}\boldsymbol{H}\mathbf{\Theta} - \boldsymbol{G}\mathbf{N})(\mathbf{\Theta} - \boldsymbol{G}\boldsymbol{H}\mathbf{\Theta} - \boldsymbol{G}\mathbf{N})^{\text{H}}\right]$$

where it is highlighted that the MSE matrix is a function of the linear estimator $\boldsymbol{G}$. Due to the independence of $\mathbf{\Theta}$ and $\mathbf{N}$,

$$\begin{aligned}\boldsymbol{M}(\boldsymbol{G}) &= (\mathbf{I} - \boldsymbol{G}\boldsymbol{H})\boldsymbol{C}_{\Theta}(\mathbf{I} - \boldsymbol{G}\boldsymbol{H})^{\text{H}} + \boldsymbol{G}\boldsymbol{C}_{\mathbf{N}}\boldsymbol{G}^{\text{H}} \\ &= \boldsymbol{C}_{\Theta} - \boldsymbol{G}\boldsymbol{H}\boldsymbol{C}_{\Theta} - \boldsymbol{C}_{\Theta}\boldsymbol{H}^{\text{H}}\boldsymbol{G}^{\text{H}} + \boldsymbol{G}\boldsymbol{H}\boldsymbol{C}_{\Theta}\boldsymbol{H}^{\text{H}}\boldsymbol{G}^{\text{H}} + \boldsymbol{G}\boldsymbol{C}_{\mathbf{N}}\boldsymbol{G}^{\text{H}}.\end{aligned} \tag{5.41}$$

With the result for the MMSE estimator $\boldsymbol{G}_{\text{MMSE}}$ in (5.36), it can be found that

$$\begin{aligned}\boldsymbol{M}(\boldsymbol{G}_{\text{MMSE}}) &= \boldsymbol{C}_{\Theta} - \boldsymbol{G}_{\text{MMSE}}\boldsymbol{H}\boldsymbol{C}_{\Theta} \\ &= \boldsymbol{C}_{\Theta} - \boldsymbol{C}_{\Theta}\boldsymbol{H}^{\text{H}}\left(\boldsymbol{H}\boldsymbol{C}_{\Theta}\boldsymbol{H}^{\text{H}} + \boldsymbol{C}_{\mathbf{N}}\right)^{-1}\boldsymbol{H}\boldsymbol{C}_{\Theta}.\end{aligned} \tag{5.42}$$

The other expression for $\boldsymbol{G}_{\text{MMSE}}$ in (5.38) finally leads to the MMSE matrix

$$\begin{aligned}\boldsymbol{M}_{\text{MMSE}} = \boldsymbol{M}(\boldsymbol{G}_{\text{MMSE}}) &= \boldsymbol{C}_{\Theta} - \left(\boldsymbol{C}_{\Theta}^{-1} + \boldsymbol{H}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\right)^{-1}\boldsymbol{H}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\boldsymbol{C}_{\Theta} \\ &= \left(\boldsymbol{C}_{\Theta}^{-1} + \boldsymbol{H}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\right)^{-1}.\end{aligned} \tag{5.43}$$

It is interesting to investigate the difference of the MSE matrix for an arbitrary linear estimator $\boldsymbol{G}$ [see (5.41)] and the MMSE matrix [see (5.42)], that is,

$$\begin{aligned}\boldsymbol{M}(\boldsymbol{G}) - \boldsymbol{M}_{\text{MMSE}} &= \boldsymbol{C}_{\Theta} - \boldsymbol{G}\boldsymbol{H}\boldsymbol{C}_{\Theta} - \boldsymbol{C}_{\Theta}\boldsymbol{H}^{\text{H}}\boldsymbol{G}^{\text{H}} + \boldsymbol{G}\boldsymbol{H}\boldsymbol{C}_{\Theta}\boldsymbol{H}^{\text{H}}\boldsymbol{G}^{\text{H}} + \boldsymbol{G}\boldsymbol{C}_{\mathbf{N}}\boldsymbol{G}^{\text{H}} \\ &\quad - \boldsymbol{C}_{\Theta} + \boldsymbol{C}_{\Theta}\boldsymbol{H}^{\text{H}}\left(\boldsymbol{H}\boldsymbol{C}_{\Theta}\boldsymbol{H}^{\text{H}} + \boldsymbol{C}_{\mathbf{N}}\right)^{-1}\boldsymbol{H}\boldsymbol{C}_{\Theta} \\ &= \boldsymbol{G}\left(\boldsymbol{H}\boldsymbol{C}_{\Theta}\boldsymbol{H}^{\text{H}} + \boldsymbol{C}_{\mathbf{N}}\right)\boldsymbol{G}^{\text{H}} - \boldsymbol{G}\boldsymbol{H}\boldsymbol{C}_{\Theta} - \boldsymbol{C}_{\Theta}\boldsymbol{H}^{\text{H}}\boldsymbol{G}^{\text{H}} \\ &\quad + \boldsymbol{C}_{\Theta}\boldsymbol{H}^{\text{H}}\left(\boldsymbol{H}\boldsymbol{C}_{\Theta}\boldsymbol{H}^{\text{H}} + \boldsymbol{C}_{\mathbf{N}}\right)^{-1}\boldsymbol{H}\boldsymbol{C}_{\Theta} \\ &= (\boldsymbol{G} - \boldsymbol{G}_{\text{MMSE}})\left(\boldsymbol{H}\boldsymbol{C}_{\Theta}\boldsymbol{H}^{\text{H}} + \boldsymbol{C}_{\mathbf{N}}\right)\left(\boldsymbol{G}^{\text{H}} - \boldsymbol{G}_{\text{MMSE}}^{\text{H}}\right)\end{aligned}$$

with the linear MMSE estimator $\boldsymbol{G}_{\text{MMSE}}$ given in (5.36). Note that $\boldsymbol{H}\boldsymbol{C}_{\boldsymbol{\Theta}}\boldsymbol{H}^{\text{H}} + \boldsymbol{C}_{\text{N}} \succ \mathbf{0}$ since the noise covariance matrix $\boldsymbol{C}_{\text{N}}$ is positive definite. Consequently, $\boldsymbol{M}(\boldsymbol{G}) - \boldsymbol{M}_{\text{MMSE}} \succeq \mathbf{0}$ or equivalently,

$$\boldsymbol{M}(\boldsymbol{G}) \succeq \boldsymbol{M}_{\text{MMSE}} \tag{5.44}$$

irrespective of the choice for $\boldsymbol{G}$. In other words, the MMSE estimator $\boldsymbol{G}_{\text{MMSE}}$ not only minimizes the MSE, i.e., the trace of $\boldsymbol{M}(\boldsymbol{G})$, but also the resulting MMSE matrix $\boldsymbol{M}_{\text{MMSE}}$ is the minimum in the sense of the matrix definiteness order.

5.4.2.1 Bias-Variance Trade-Off

In this discussion, we first consider the conditioned MSE (CMSE)

$$\text{CMSE} = \text{E}\left[\left\|\boldsymbol{\Theta} - \hat{\boldsymbol{\Theta}}\right\|_2^2 \,\middle|\, \boldsymbol{\Theta} = \boldsymbol{\theta}\right] \tag{5.45}$$

instead of the unconditioned MSE in (5.30), where it holds that

$$\text{MSE} = \text{E}[\text{CMSE}] = \text{E}\left[\text{E}\left[\left\|\boldsymbol{\Theta} - \hat{\boldsymbol{\Theta}}\right\|_2^2 \,\middle|\, \boldsymbol{\Theta}\right]\right] = \text{E}\left[\left\|\boldsymbol{\Theta} - \hat{\boldsymbol{\Theta}}\right\|_2^2\right].$$

The CMSE can be interpreted as the squared error averaged over many experiments for a fixed value of the desired but unknown quantity $\boldsymbol{\Theta} = \boldsymbol{\theta}$. Extend the error in the CMSE by the trivial operation of subtracting and adding the conditional mean $\text{E}[\hat{\boldsymbol{\Theta}}|\boldsymbol{\Theta} = \boldsymbol{\theta}]$, i.e.,

$$\text{CMSE} = \text{E}\left[\left\|\boldsymbol{\Theta} - \text{E}\big[\hat{\boldsymbol{\Theta}} \,\big|\, \boldsymbol{\Theta} = \boldsymbol{\theta}\big] - \big(\hat{\boldsymbol{\Theta}} - \text{E}\big[\hat{\boldsymbol{\Theta}} \,\big|\, \boldsymbol{\Theta} = \boldsymbol{\theta}\big]\big)\right\|_2^2 \,\middle|\, \boldsymbol{\Theta} = \boldsymbol{\theta}\right].$$

Multiplying out yields

$$\begin{aligned}
\text{CMSE} &= \text{E}\left[\left\|\boldsymbol{\Theta} - \text{E}\big[\hat{\boldsymbol{\Theta}} \,\big|\, \boldsymbol{\Theta} = \boldsymbol{\theta}\big]\right\|_2^2 \,\middle|\, \boldsymbol{\Theta} = \boldsymbol{\theta}\right] + \text{E}\left[\left\|\hat{\boldsymbol{\Theta}} - \text{E}\big[\hat{\boldsymbol{\Theta}} \,\big|\, \boldsymbol{\Theta} = \boldsymbol{\theta}\big]\right\|_2^2 \,\middle|\, \boldsymbol{\Theta} = \boldsymbol{\theta}\right] \\
&\quad - 2\text{Re}\left\{\text{E}\left[\big(\boldsymbol{\Theta} - \text{E}\big[\hat{\boldsymbol{\Theta}} \,\big|\, \boldsymbol{\Theta} = \boldsymbol{\theta}\big]\big)^{\text{H}}\big(\hat{\boldsymbol{\Theta}} - \text{E}\big[\hat{\boldsymbol{\Theta}} \,\big|\, \boldsymbol{\Theta} = \boldsymbol{\theta}\big]\big) \,\big|\, \boldsymbol{\Theta} = \boldsymbol{\theta}\right]\right\} \\
&= \text{E}\left[\left\|\boldsymbol{\Theta} - \text{E}\big[\hat{\boldsymbol{\Theta}} \,\big|\, \boldsymbol{\Theta} = \boldsymbol{\theta}\big]\right\|_2^2 \,\middle|\, \boldsymbol{\Theta} = \boldsymbol{\theta}\right] + \text{E}\left[\left\|\hat{\boldsymbol{\Theta}} - \text{E}\big[\hat{\boldsymbol{\Theta}} \,\big|\, \boldsymbol{\Theta} = \boldsymbol{\theta}\big]\right\|_2^2 \,\middle|\, \boldsymbol{\Theta} = \boldsymbol{\theta}\right] \\
&\quad - 2\text{Re}\left\{\big(\boldsymbol{\theta} - \text{E}\big[\hat{\boldsymbol{\Theta}} \,\big|\, \boldsymbol{\Theta} = \boldsymbol{\theta}\big]\big)^{\text{H}}\text{E}\big[\hat{\boldsymbol{\Theta}} - \text{E}\big[\hat{\boldsymbol{\Theta}} \,\big|\, \boldsymbol{\Theta} = \boldsymbol{\theta}\big] \,\big|\, \boldsymbol{\Theta} = \boldsymbol{\theta}\big]\right\}.
\end{aligned}$$

Note that the last term is zero because

$$\text{E}\big[\hat{\boldsymbol{\Theta}} - \text{E}\big[\hat{\boldsymbol{\Theta}} \,\big|\, \boldsymbol{\Theta} = \boldsymbol{\theta}\big] \,\big|\, \boldsymbol{\Theta} = \boldsymbol{\theta}\big] = \text{E}\big[\hat{\boldsymbol{\Theta}} \,\big|\, \boldsymbol{\Theta} = \boldsymbol{\theta}\big] - \text{E}\big[\hat{\boldsymbol{\Theta}} \,\big|\, \boldsymbol{\Theta} = \boldsymbol{\theta}\big] = \mathbf{0}.$$

Therefore, the MSE reads as

$$\text{MSE} = \text{E}[\text{CMSE}] = \text{bias}^2\left[\hat{\boldsymbol{\Theta}}\right] + \text{var}\left[\hat{\boldsymbol{\Theta}}\right] \tag{5.46}$$

with

$$\begin{aligned}
\text{bias}^2\left[\hat{\boldsymbol{\Theta}}\right] &= \text{E}\left[\left\|\boldsymbol{\Theta} - \text{E}\big[\hat{\boldsymbol{\Theta}} \,\big|\, \boldsymbol{\Theta}\big]\right\|_2^2\right] \\
\text{var}\left[\hat{\boldsymbol{\Theta}}\right] &= \text{E}\left[\left\|\hat{\boldsymbol{\Theta}} - \text{E}\big[\hat{\boldsymbol{\Theta}} \,\big|\, \boldsymbol{\Theta}\big]\right\|_2^2\right].
\end{aligned}$$

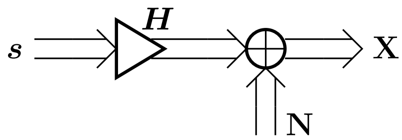

Fig. 5.5. MIMO System

The result in (5.46) shows that there is a trade-off between the variance of the estimator and its bias, i.e., the systematic error, when designing the estimator to minimize the mean square error. When trying to minimize the variance of the estimator by choosing a fixed value for the estimate, e.g., $\hat{\mathbf{\Theta}} = \boldsymbol{q}$, that leads to $\mathrm{var}[\hat{\mathbf{\Theta}}] = \mathbf{0}$, the systematic error expressed by $\mathrm{bias}^2[\hat{\mathbf{\Theta}}]$ is poor. On the other hand, when trying to minimize the bias of the estimator by restricting it to be unbiased, e.g., by using a least squares estimator or the BLUE, we get $\mathrm{bias}^2[\hat{\mathbf{\Theta}}] = \mathbf{0}$ but the resulting variance $\mathrm{var}[\hat{\mathbf{\Theta}}]$ of the estimator is large. In contrast, the best choice in the minimum mean square error sense finds the optimum trade-off between the bias and the variance of the estimator.

5.5 MIMO Detection

In the setups of a single transmitter or multiple transmitters, the receiver must separate the discrete-valued data symbols and retrieve the data by means of detection. Based on the knowledge of the channel that can be easily obtained during a training phase, the receiver exploits the dependence of the statistical properties of the received signal on the transmitted data for detection. This dependence is optimally utilized by maximum likelihood (ML) detection.

We consider the multiple-input multiple-output (MIMO) setup depicted in Fig. 5.5, where the data signal $\boldsymbol{s} \in \mathbb{A}^n$ is discrete-valued, e.g., $\mathbb{A}_{\text{BPSK}} = \{-1, +1\}$ or $\mathbb{A}_{\text{QPSK}} = \{e^{j\frac{\pi}{4}}, e^{j\frac{3\pi}{4}}, e^{j\frac{5\pi}{4}}, e^{j\frac{7\pi}{4}}\}$. The signal $\boldsymbol{s}$ propagates over the MIMO channel $\boldsymbol{H} \in \mathbb{C}^{m\times n}$ and is perturbed by the circularly symmetric complex Gaussian noise signal $\mathbf{N} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{C}_{\mathbf{N}})$ to form the observation

$$\mathbf{X} = \boldsymbol{H}\boldsymbol{s} + \mathbf{N} \in \mathbb{C}^m \tag{5.47}$$

with the PDF

$$\mathrm{f}_{\mathbf{X}}(\boldsymbol{x}; \boldsymbol{s}) = \frac{1}{\pi^m \det(\boldsymbol{C}_{\mathbf{N}})} \exp\left(-(\boldsymbol{x} - \boldsymbol{H}\boldsymbol{s})^{\mathrm{H}} \boldsymbol{C}_{\mathbf{N}}^{-1} (\boldsymbol{x} - \boldsymbol{H}\boldsymbol{s})\right).$$

5.5.1 Maximum Likelihood Detection

The ML detector maximizes the likelihood $\mathrm{f}_{\mathbf{X}}(\boldsymbol{x}; \boldsymbol{s})$, i.e., the probability that an assumed $\boldsymbol{s} \in \mathbb{A}^n$ led to the received signal $\boldsymbol{x} \in \mathbb{C}^m$, that is,

$$\tilde{\boldsymbol{s}}_{\text{ML}} = \underset{\boldsymbol{s}\in\mathbb{A}^n}{\operatorname{argmax}}\, \mathrm{f}_{\mathbf{X}}(\boldsymbol{x}; \boldsymbol{s}) = \underset{\boldsymbol{s}\in\mathbb{A}^n}{\operatorname{argmax}}\, L(\boldsymbol{s}; \boldsymbol{x}) \tag{5.48}$$

with the log-likelihood function

$$L(\boldsymbol{s}; \boldsymbol{x}) = -\left(\boldsymbol{x} - \boldsymbol{H}\boldsymbol{s}\right)^{\mathrm{H}} \boldsymbol{C}_{\mathbf{N}}^{-1} \left(\boldsymbol{x} - \boldsymbol{H}\boldsymbol{s}\right) + c$$

where $c = -\ln(\pi^m \det(\boldsymbol{X}_{\mathbf{N}}))$ is constant and independent of $\boldsymbol{s}$. Clearly, c can be dropped, since it has no influence on the result of (5.48). Since maximizing a quantity is equivalent to minimizing the negative of the same quantity, the rule for ML detection can equivalently be written as

$$\tilde{\boldsymbol{s}}_{\mathrm{ML}} = \underset{\boldsymbol{s}\in\mathbb{A}^n}{\operatorname{argmin}}\,(\boldsymbol{x}-\boldsymbol{H}\boldsymbol{s})^{\mathrm{H}}\,\boldsymbol{C}_{\mathbf{N}}^{-1}\,(\boldsymbol{x}-\boldsymbol{H}\boldsymbol{s})\,. \tag{5.49}$$

This is a least squares problem in $\boldsymbol{s}$, where a Hermitian form with the inverse of the noise covariance matrix $\boldsymbol{C}_{\mathbf{N}}^{-1}$ is minimized. Note, however, that $\boldsymbol{s}\in\mathbb{A}^n$, i.e., $\boldsymbol{s}$ can only have discrete values out of the discrete alphabet $\mathbb{A}^n$.

A naive solution of (5.49) is as follows. For every possible value of $\boldsymbol{s}\in\mathbb{A}^n$, compute the part of the ML metric that depends on $\boldsymbol{s}$, i.e., $-2\mathrm{Re}\left(\boldsymbol{s}^{\mathrm{H}}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{x}\right)+\boldsymbol{s}^{\mathrm{H}}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\boldsymbol{s}$. Compare all metrics and $\tilde{\boldsymbol{s}}_{\mathrm{ML}}$ is the $\boldsymbol{s}$ with the smallest metric. For this procedure, $|\mathbb{A}|^n$ values of $\boldsymbol{s}$ must be tested and the complexity of every test is determined by the product $\boldsymbol{s}^{\mathrm{H}}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\boldsymbol{s}$.[1] Therefore, the order of complexity of ML detection is $\mathrm{O}(n^2|\mathbb{A}|^n)$ per data vector. We see that ML detection has a complexity that is exponential in the number of data streams n, i.e., it is an NP-hard problem. Note that also the sphere decoder which is discussed in Subsection 5.5.4 has exponential complexity.

5.5.2 Suboptimality of Symbol-by-Symbol Detection

The data signal $\boldsymbol{s}$ can only have discrete values. Therefore, a solution of (5.49) via differentiation is impossible. Let us, nevertheless, solve (5.49) assuming $\mathbb{A}=\mathbb{C}$ although $\mathbb{A}$ is a discrete set. Since the resulting values for $\boldsymbol{s}$ are out of $\mathbb{C}^n$ instead of $\mathbb{A}^n$, we simply round to the nearest member of $\mathbb{A}^n$. So, we differentiate the cost of (5.49) with respect to $\boldsymbol{s}^*$ and set the result to zero, i.e.,

$$\frac{\partial}{\partial \boldsymbol{s}^*}\,(\boldsymbol{x}-\boldsymbol{H}\boldsymbol{s})^{\mathrm{H}}\,\boldsymbol{C}_{\mathbf{N}}^{-1}\,(\boldsymbol{x}-\boldsymbol{H}\boldsymbol{s}) = -\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\,(\boldsymbol{x}-\boldsymbol{H}\boldsymbol{s}) = \mathbf{0}.$$

The resulting estimate of the data signal is

$$\hat{\boldsymbol{s}}_{\mathrm{BLUE}} = \left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\right)^{-1}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{x}\in\mathbb{C}^n. \tag{5.50}$$

Note that $(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H})^{-1}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}$ is the best linear unbiased estimator (BLUE) or zero-forcing equalizer (see Subsection 5.4.1) which is the matched filter followed by an invertible transformation, that is, it delivers a sufficient statistic (see Subsection 5.1.1). Therefore, the ML detection can be based on $\hat{\mathrm{S}}_{\mathrm{BLUE}}$ instead of $\mathrm{X}=\boldsymbol{H}\boldsymbol{s}+\mathrm{N}$. The BLUE estimate can be written as

$$\hat{\mathrm{S}}_{\mathrm{BLUE}} = \boldsymbol{s}+\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\right)^{-1}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\mathrm{N}$$

i.e., $\hat{\mathrm{S}}_{\mathrm{BLUE}}\sim\mathcal{N}_{\mathbb{C}}(\boldsymbol{s},(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H})^{-1})$ with the unknown parameter $\boldsymbol{s}$. First, remember that it holds for the observation that $\mathrm{X}\sim\mathcal{N}_{\mathbb{C}}(\boldsymbol{H}\boldsymbol{s},\boldsymbol{C}_{\mathbf{N}})$ with the parameter $\boldsymbol{s}$. Second, recall the ML rule (5.49), where the ML metric has the form $(\boldsymbol{x}-\boldsymbol{H}\boldsymbol{s})^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}(\boldsymbol{x}-\boldsymbol{H}\boldsymbol{s})$, i.e., it is a Hermitian form with the inverse of the noise covariance matrix $\boldsymbol{C}_{\mathbf{N}}$ and the mean $\boldsymbol{H}\boldsymbol{s}$ is subtracted from $\boldsymbol{x}$. Thus, we can conclude that ML detection employing the sufficient statistic $\hat{\mathrm{S}}_{\mathrm{BLUE}}$ is based on a metric that is a quadratic form with the inverse of the noise covariance matrix $(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H})^{-1}$ and the mean $\boldsymbol{s}$ is subtracted from $\hat{\boldsymbol{s}}_{\mathrm{BLUE}}$, i.e.,

$$\tilde{\boldsymbol{s}}_{\mathrm{ML}} = \underset{\boldsymbol{s}\in\mathbb{A}^n}{\operatorname{argmin}}\,(\hat{\boldsymbol{s}}_{\mathrm{BLUE}}-\boldsymbol{s})^{\mathrm{H}}\,\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\,(\hat{\boldsymbol{s}}_{\mathrm{BLUE}}-\boldsymbol{s})\,. \tag{5.51}$$

[1]The matched filter output $\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{x}$ only must be computed once. Additionally, the matrix product $\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}$ can be decomposed into the product of two triangular matrices with the Cholesky factorization.

The above mentioned strategy of rounding the estimate $\hat{\boldsymbol{s}}_{\text{BLUE}}$ to the nearest member of $\mathbb{A}^n$ can be understood better, when considering the special case that the columns of $\boldsymbol{H}$ are $\boldsymbol{C}_{\mathbf{N}}^{-1}$-conjugate, i.e., $\boldsymbol{H}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}$ is diagonal. In that case,

$$\tilde{\boldsymbol{s}}_{\text{ML}} = \underset{\boldsymbol{s}\in\mathbb{A}^n}{\operatorname{argmin}} \sum_{i=1}^{n} \alpha_i \left|\hat{s}_{\text{BLUE},i} - s_i\right|^2$$

where b_i denotes the i-th element of the vector $\boldsymbol{b}$ and α_i is the i-th diagonal element of $\boldsymbol{H}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}$. Clearly, the sum is minimized, when every summand is minimized separately. Therefore,

$$\tilde{\boldsymbol{s}}_{\text{ML}} = \left[\text{Q}(\hat{s}_{\text{BLUE},1}), \ldots, \text{Q}(\hat{s}_{\text{BLUE},n})\right]^{\text{T}} \in \mathbb{A}^n$$

for diagonal $\boldsymbol{H}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}$. Here, we introduced the nearest neighbor quantizer

$$\text{Q}(x) = \underset{a\in\mathbb{A}}{\operatorname{argmin}} \left|x - a\right|^2 \tag{5.52}$$

that rounds to the nearest element in $\mathbb{A}$. Note that rounding a vector $\boldsymbol{y} \in \mathbb{C}^n$ to the nearest element of $\mathbb{A}^n$ is equivalent to applying $\text{Q}(\bullet)$ to every element of $\boldsymbol{y}$ separately, i.e.,

$$\text{Q}(\boldsymbol{y}) = \underset{\boldsymbol{a}\in\mathbb{A}^n}{\operatorname{argmin}} \|\boldsymbol{y} - \boldsymbol{a}\|_2^2 = \left[\text{Q}(y_1), \ldots, \text{Q}(y_n)\right]^{\text{T}} \in \mathbb{A}^n. \tag{5.53}$$

Therefore, the above mentioned strategy of rounding the estimate $\hat{\boldsymbol{s}}_{\text{BLUE}}$ to the nearest member of $\mathbb{A}^n$ can only be ML-optimal, if $\boldsymbol{H}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}$ is diagonal. Since $\boldsymbol{H}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}$ is not diagonal for almost all channels, the scheme with a linear transformation applied to the observation $\mathbf{X}$ followed by the symbol-by-symbol quantizer $\text{Q}(\bullet)$ [see (5.53)] is suboptimal in general, that is, linear detection is suboptimal.

5.5.3 MMSE Metric

The ML detector minimizes the ML metric

$$\mu_{\text{ML}}(\boldsymbol{s}) = \left(\hat{\boldsymbol{s}}_{\text{BLUE}} - \boldsymbol{s}\right)^{\text{H}} \boldsymbol{H}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H} \left(\hat{\boldsymbol{s}}_{\text{BLUE}} - \boldsymbol{s}\right) \tag{5.54}$$

by the choice of $\boldsymbol{s} \in \mathbb{A}^n$ [see (5.51)], where $\hat{\text{S}}_{\text{BLUE}} = (\boldsymbol{H}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H})^{-1}\boldsymbol{H}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\mathbf{X}$ is the output of the BLUE or linear zero-forcing filter (see Subsection 5.4.1).

The *minimum mean square error* (MMSE) metric can be obtained from the ML metric as follows

$$\begin{aligned}
\mu_{\text{MMSE}}(\boldsymbol{s}) &= \mu_{\text{ML}}(\boldsymbol{s}) + \boldsymbol{s}^{\text{H}}\boldsymbol{s} - \boldsymbol{x}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\left(\boldsymbol{H}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H} + \left(\boldsymbol{H}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\right)^2\right)^{-1}\boldsymbol{H}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{x} \\
&= \boldsymbol{x}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\left(\left(\boldsymbol{H}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\right)^{-1} - \left(\boldsymbol{H}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H} + \left(\boldsymbol{H}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\right)^2\right)^{-1}\right)\boldsymbol{H}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{x} \\
&\quad - \boldsymbol{x}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\boldsymbol{s} - \boldsymbol{s}^{\text{H}}\boldsymbol{H}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{x} + \boldsymbol{s}^{\text{H}}\left(\mathbf{I} + \boldsymbol{H}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\right)\boldsymbol{s}.
\end{aligned}$$

Note that $\boldsymbol{x}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}(\boldsymbol{H}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H} + (\boldsymbol{H}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H})^2)^{-1}\boldsymbol{H}^{\text{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{x}$ is independent of $\boldsymbol{s}$ but $\boldsymbol{s}^{\text{H}}\boldsymbol{s}$ depends on the choice of $\boldsymbol{s}$. Therefore, the MMSE metric and the ML metric are different in general. However, for the special case of constant-modulus alphabets, e.g., QPSK and 8PSK, $\boldsymbol{s}^{\text{H}}\boldsymbol{s}$ is also independent of $\boldsymbol{s}$ and the MMSE metric is equivalent to the ML metric (e.g., [ZF06]). With the matrix inversion lemma (A25), it can be shown that

$$\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\right)^{-1}-\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}+\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\right)^{2}\right)^{-1}=\left(\mathbf{I}+\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\right)^{-1}.$$

Employing this result in above expression for the MMSE metric leads to

$$\mu_{\mathrm{MMSE}}(\boldsymbol{s})=\left(\hat{\boldsymbol{s}}_{\mathrm{MMSE}}-\boldsymbol{s}\right)^{\mathrm{H}}\left(\mathbf{I}+\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\right)\left(\hat{\boldsymbol{s}}_{\mathrm{MMSE}}-\boldsymbol{s}\right) \tag{5.55}$$

with the output $\hat{\mathrm{S}}_{\mathrm{MMSE}}=(\mathbf{I}+\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H})^{-1}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\mathbf{X}$ of the linear MMSE equalizer (see Subsection 5.4.2) under the assumption that the covariance matrix of $\mathbf{S}$ is $\boldsymbol{C}_{\mathbf{S}}=\mathbf{I}$.

The ML metric (5.54) is a Hermitian form with the inverse of the covariance matrix $(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H})^{-1}$ of the Gaussian noise and the mean $\boldsymbol{s}$ is subtracted from $\hat{\boldsymbol{s}}_{\mathrm{BLUE}}$. The MMSE metric (5.55) can be interpreted similarly. It is a Hermitian form with the inverse of the MMSE matrix $(\mathbf{I}+\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H})^{-1}$ [cf. (5.43)] and $\boldsymbol{s}$ is subtracted from $\hat{\boldsymbol{s}}_{\mathrm{MMSE}}$. Note, however, that the interference is not Gaussian as it results from the symbols $\boldsymbol{s}\in\mathbb{A}^n$ and the mean of $\hat{\boldsymbol{s}}_{\mathrm{MMSE}}$ is $(\mathbf{I}+\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H})^{-1}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\boldsymbol{s}\neq\boldsymbol{s}$. Therefore, it is surprising that the detection based on the MMSE metric

$$\tilde{\boldsymbol{s}}_{\mathrm{MMSE}}=\underset{\boldsymbol{s}\in\mathbb{A}^n}{\operatorname{argmin}}\left(\hat{\boldsymbol{s}}_{\mathrm{MMSE}}-\boldsymbol{s}\right)^{\mathrm{H}}\left(\mathbf{I}+\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{H}\right)\left(\hat{\boldsymbol{s}}_{\mathrm{MMSE}}-\boldsymbol{s}\right) \tag{5.56}$$

leads to the same result as the ML detection (5.51), if $\mathbb{A}$ is a constant-modulus alphabet. Moreover, also the results of the MMSE metric based detection for non-constant modulus alphabets are very close to that of ML detection.

When comparing (5.54) and (5.55), we observe that the outputs of different filters are used for the ML metric and the MMSE metric. Additionally, the matrices of the quadratic forms are different, i.e., the matrix of the MMSE metric is the matrix of the ML metric plus an identity matrix. This regularization of the MMSE metric compared to the ML metric is advantageous for an algorithmic solution of the ML rule (5.51), where $\mu_{\mathrm{MMSE}}(\boldsymbol{s})$ is used instead of $\mu_{\mathrm{ML}}(\boldsymbol{s})$. The sphere decoder (see Subsection 5.5.4) needs less complexity to find the optimal vector, when it is based on the MMSE metric than for the ML metric (e.g., [MEDC06]). Moreover, the regularization improves suboptimal detectors such as linear detection for example (linear detection based on the MMSE equalizer outperforms that based on the zero-forcing equalizer).

5.5.4 Sphere Decoder

The ML metric (5.54) and the MMSE metric (5.55) can both be expressed as

$$\mu(\boldsymbol{s})=\left(\hat{\boldsymbol{s}}-\boldsymbol{s}\right)^{\mathrm{H}}\boldsymbol{A}\left(\hat{\boldsymbol{s}}-\boldsymbol{s}\right)$$

where $\boldsymbol{A}\in\mathbb{C}^{n\times n}$ is non-negative definite. Therefore, the Cholesky factorization $\boldsymbol{A}=\boldsymbol{L}^{\mathrm{H}}\boldsymbol{D}\boldsymbol{L}$ exists with unit lower-triangular

$$\boldsymbol{L}=\begin{bmatrix}1 & 0 & & \cdots & 0\\ \times & 1 & 0 & \cdots & 0\\ & & \ddots & & \vdots\\ \times & \cdots & \times & 1 & 0\\ \times & \cdots & & \times & 1\end{bmatrix}\in\mathbb{C}^{n\times n}$$

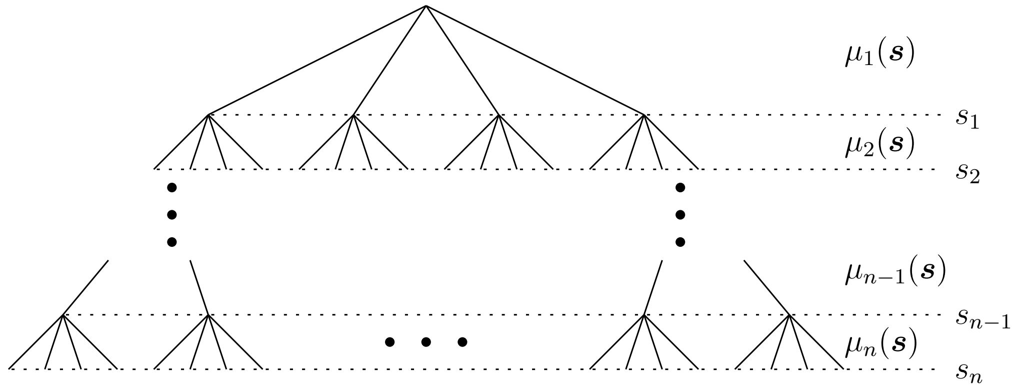

Fig. 5.6. ML Detection as a Search in a Tree with the Cost μ_i to Come from s_{i-1} to s_i and $|\mathbb{A}| = 4$

and non-negative diagonal $\boldsymbol{D} \in \mathbb{R}_{+,0}^{n\times n}$. Substituting the Cholesky factorization of $\boldsymbol{A}$ into above general expression for the metrics leads to

$$\mu(\boldsymbol{s}) = \left\| \boldsymbol{D}^{1/2} \left(\boldsymbol{L}\hat{\boldsymbol{s}} - \boldsymbol{L}\boldsymbol{s} \right) \right\|_2^2 = \left\| \boldsymbol{D}^{1/2} \left(\hat{\boldsymbol{x}} - \boldsymbol{L}\boldsymbol{s} \right) \right\|_2^2$$

with $\hat{\boldsymbol{x}} = \boldsymbol{L}\hat{\boldsymbol{s}}$. Due to the unit triangular structure of $\boldsymbol{L}$, we get

$$\mu(\boldsymbol{s}) = \sum_{i=1}^{n} d_{i,i} \left| \hat{x}_i - s_i - \sum_{j=1}^{i-1} \ell_{i,j} s_j \right|^2 .$$

Note that the i-th summand $\mu_i(\boldsymbol{s}) = d_{i,i} |\hat{x}_i - s_i - \sum_{j=1}^{i-1} \ell_{i,j} s_j|^2$ only depends on $s_1, \ldots, s_i$. Thus, the minimization of $\mu(\boldsymbol{s})$ can be interpreted as a search for the path with the minimum total metric $\sum_{i=1}^{n} \mu_i(\boldsymbol{s})$ in a tree (see Fig. 5.6).

The idea of the sphere decoder [VB93] is to search not over all $\boldsymbol{s} \in \mathbb{A}^n$ (i.e., search over all possible paths in the tree) but over all points out of $\mathbb{A}^n$ that lie inside a sphere with radius r around $\hat{\boldsymbol{x}}$ (i.e., only the parts of the tree with total metric smaller than r^2 are searched over). The restriction to points inside the sphere does not have any impact on the optimality of the result as long as r is large enough. Suppose that there is at least one point inside the sphere. In this case, also the ML optimal point $\tilde{\boldsymbol{s}}_{\mathrm{ML}}$ (or the point $\tilde{\boldsymbol{s}}_{\mathrm{MMSE}}$ for the MMSE metric) must lie inside the sphere, because $\tilde{\boldsymbol{s}}_{\mathrm{ML}}$ (or $\tilde{\boldsymbol{s}}_{\mathrm{MMSE}}$) is the point out of $\mathbb{A}^n$ that lies closest to $\hat{\boldsymbol{x}}$. However, if r is too small such that no point of $\mathbb{A}^n$ lies inside the sphere, the search fails and r must be increased.

Suppose we have a value for r that is large enough such that at least one point (the optimal point) of $\mathbb{A}^n$ is inside the sphere. Then, we try to find the point inside the sphere with radius r that is closest to $\hat{\boldsymbol{x}}$. All points inside the sphere fulfill

$$\mu(\boldsymbol{s}) = \sum_{i=1}^{n} d_{i,i} \left| \hat{x}_i - s_i - \sum_{j=1}^{i-1} \ell_{i,j} s_j \right|^2 \leq r^2.$$

If the whole sum is smaller than or equal to r^2, also the first summand must obey this inequality:

$$\mu_1(\boldsymbol{s}) = d_{1,1} \left| \hat{x}_1 - s_1 \right|^2 \leq r^2.$$

Alternatively,

$$\left| \hat{x}_1 - s_1 \right|^2 \leq \frac{r^2}{d_{1,1}}$$

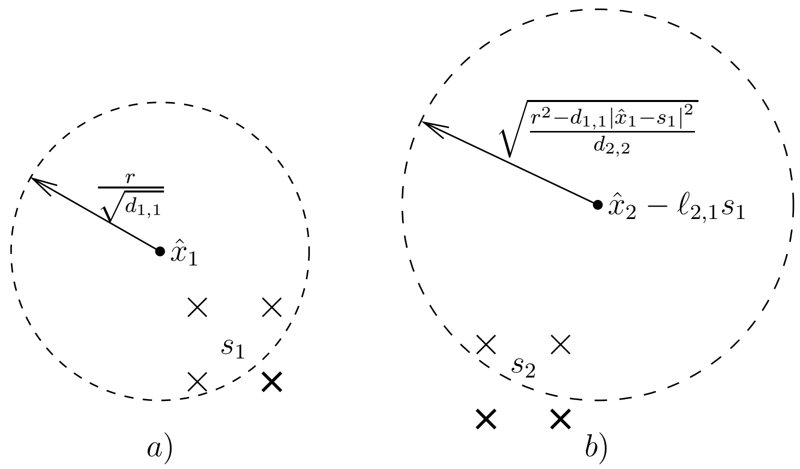

Fig. 5.7. Examples for Finding Possible Symbol Values with the Sphere Decoder, $|\mathbb{A}| = 4$

that is, only values for s_1 are allowed that lie inside the circle around $\hat{x}_1$ with radius $r/\sqrt{d_{1,1}}$. For the example depicted in Fig. 5.7a), the point of $\mathbb{A}$ outside the circle (bold cross) is not allowed, if we'd like to end up with an element of $\mathbb{A}^n$ inside the sphere with radius r. Therefore, one branch starting from the root of the tree in Fig. 5.6 is removed (about one quarter of the complexity is saved). Now, some value for s_1 is chosen and due to $\mu(\boldsymbol{s}) \le r^2$, we must have that

$$\mu_1(\boldsymbol{s}) + \mu_2(\boldsymbol{s}) = d_{1,1}\,|\hat{x}_1 - s_1|^2 + d_{2,2}\,|\hat{x}_2 - \ell_{2,1}s_1 - s_2|^2 \le r^2$$

or equivalently,

$$|\hat{x}_2 - \ell_{2,1}s_1 - s_2|^2 \le \frac{r^2 - d_{1,1}\,|\hat{x}_1 - s_1|^2}{d_{2,2}}.$$

So, $s_2 \in \mathbb{A}$ must lie inside the circle around $\hat{x}_2 - \ell_{2,1}s_1$ with radius $\sqrt{(r^2 - d_{1,1}\,|\hat{x}_1 - s_1|^2)/d_{2,2}}$. For the example in Fig. 5.7b), only two elements of $\mathbb{A}$ fulfill above inequality. The other two elements (bold crosses) can be ignored. This means that two branches of the tree in Fig. 5.6 originating from the chosen value of s_1 can be dropped (about one eighth of complexity is saved). The next steps of the sphere decoder are similar. At the i-th step, the inequality $\sum_{k=1}^{i} \mu_k(\boldsymbol{s}) \le r^2$ is used to find a condition for s_i, when the values of $s_1, \ldots, s_{i-1}$ are fixed. Again, some branches of the tree can be dropped, since the corresponding elements of $\mathbb{A}$ do not fulfill the resulting condition for s_i. When a leaf of the tree is reached with $\mu(\boldsymbol{s}) = \sum_{i=1}^{n} \mu_i(\boldsymbol{s}) \le r^2$, i.e., the corresponding $\boldsymbol{s} \in \mathbb{A}^n$ lies inside the sphere with radius r around $\hat{\boldsymbol{x}}$, the radius can be reduced to $\sqrt{\mu(\boldsymbol{s})}$, since the optimal point must be closer to $\hat{\boldsymbol{x}}$ than the obtained $\boldsymbol{s}$.

The procedure of the sphere decoder can be described as follows. It is assumed that the Cholesky factorization $\boldsymbol{A} = \boldsymbol{L}^{\mathrm{H}}\boldsymbol{D}\boldsymbol{L}$ has already been computed. The initial radius r and the received signal $\boldsymbol{x}$ are the inputs. At every step, $\boldsymbol{s} = [s_1, \ldots, s_n]^{\mathrm{T}} \in \mathbb{A}^n$.

1) Compute $\hat{\boldsymbol{x}} \leftarrow \boldsymbol{D}^{-1}\boldsymbol{L}^{\mathrm{H},-1}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\mathbf{N}}^{-1}\boldsymbol{x}$.
2) Initialize the level of the tree: $i \leftarrow 1$.
3) Find the set $\mathbb{A}_i \subseteq \mathbb{A}$ for whose members $s_i \in \mathbb{A}_i$ it holds that
   $|\hat{x}_i - s_i - \sum_{j=1}^{i-1} \ell_{i,j}s_j|^2 \le (r^2 - \sum_{j=1}^{i-1} \mu_j(\boldsymbol{s}))/d_{i,i}$.
4) If $\mathbb{A}_i = \emptyset$, then this branch of the tree lies outside the sphere
   Move back one level of the tree: $i \leftarrow i - 1$.
   If $i = 0$, then all points inside the sphere have been tested.

Exit and the saved $\tilde{\boldsymbol{s}}$ is the solution.
Else go to 4).
5) Pick the s_i from $\mathbb{A}_i$ with the smallest $\mu_i(\boldsymbol{s})$. Remove s_i from $\mathbb{A}_i$.
6) Go to next level of the tree: $i \leftarrow i+1$.
7) If $i \leq n$, then search this level of the tree:
Go to 3).
8) A leaf of the tree has been reached.
Update solution: $\tilde{\boldsymbol{s}} \leftarrow \boldsymbol{s}$.
Reduce sphere radius: $r \leftarrow \sqrt{\mu(\boldsymbol{s})}$.
9) Move back to second last level: $i \leftarrow n-1$.
Go to 4).

A very important class of modulation alphabets are square QAM alphabets, whose real and imaginary parts are weighted and shifted integer numbers:

$$\mathbb{A} = \left\{ s = \frac{1}{\alpha}(x + \mathrm{j}y) + \frac{1}{2\alpha} + \mathrm{j}\frac{1}{2\alpha} \,\middle|\, x \in \mathbb{Z}_{\mathbb{A}}, y \in \mathbb{Z}_{\mathbb{A}} \right\}$$

where $\mathbb{Z}_{\mathbb{A}} \subseteq \mathbb{Z}$ and $\alpha \neq 0$ is some constant for normalization. Two examples for this type of QAM alphabets are 4QAM ($\mathbb{Z}_{\mathbb{A}} = \{-1, 0\}$) and 16QAM ($\mathbb{Z}_{\mathbb{A}} = \{-2, -1, 0, +1\}$). Due to the structure of $\mathbb{A}$, we can switch to a real-valued representation of the signals. The real-valued representation of the symbols reads as

$$\boldsymbol{s}_{\text{real-valued}} = \frac{1}{\alpha}\left(\bar{\boldsymbol{s}} + \frac{1}{2}\mathbf{1}\right) \in \mathbb{R}^{2n}$$

with $\bar{\boldsymbol{s}} \in \mathbb{Z}_{\mathbb{A}}^{2n}$ and $\mathbf{1}$ is the $2n$-dimensional all ones vector.

The complex-valued detection problem is the minimization of the metric $\mu(\boldsymbol{s})$ by the choice of $\boldsymbol{s} \in \mathbb{A}^n$. Clearly, this minimization can be transformed to a real-valued problem, where the minimization is with respect to $\boldsymbol{s}_{\text{real-valued}}$. After substituting above expression for $\boldsymbol{s}_{\text{real-valued}}$ into the real-valued minimization problem, the detection rule can be expressed as

$$\tilde{\bar{\boldsymbol{s}}} = \underset{\bar{\boldsymbol{s}} \in \mathbb{Z}_{\mathbb{A}}^{2n}}{\operatorname{argmin}} \left\| \bar{\boldsymbol{x}} - \bar{\boldsymbol{D}}^{1/2} \bar{\boldsymbol{L}} \bar{\boldsymbol{s}} \right\|_2^2$$

and the real-valued representation of the result of the detection is $\tilde{\boldsymbol{s}}_{\text{real-valued}} = (\tilde{\bar{\boldsymbol{s}}} + \frac{1}{2}\mathbf{1})/\alpha$. We see that the structure of the problem is the same as for general alphabets and the complex-valued representation. However, above minimization is an integer least squares problem or a closest point search in a lattice (e.g., [GLS93, AEVZ02]), for which very efficient algorithms exist such as the Fincke-Pohst [FP85, VB93] and the Schnorr-Euchner strategy [SE94, DEC03].

The sphere decoder is a very efficient algorithm to solve the ML problem (5.49) which is NP-hard. In [JO06], it was shown that also the sphere decoder based on lattice search techniques is NP-hard. Note, however, that this high complexity is the worst case complexity. Depending on the properties of the received vector, the channel, the noise, the complexity can be very small—in the extreme case, the sphere decoder has a complexity nearly as low as decision feedback equalization. So, the sphere decoder has a variable complexity and the worst case complexity is very high. These properties, besides the problem that the algorithm moves up and down in the tree, make it difficult to apply the sphere decoder in practical systems. Nevertheless, an efficient implementation of the sphere decoder was reported in [BBW$^+$05].