probability and queueing ~~Method~~ Models

19MA222

Sample space = { HHH, HTH, HHT, THH, TTT, THT, TTH, HTT }

$n(S) = 8$

"Getting 2 heads one tail"

A = { HTH, HHT, THH }

$n(A) = 3$

$$\text{probability} = \frac{n(A)}{n(S)}$$

$$= 3/8$$

Unit - 1 Random variables and probabilities distributions

Unit - 2 Two dimensional rv's

Unit - 3 Random process

Unit 4 Queueing models

Unit 5 Adv models

Unit - I

Random variables

X

discrete random variable

$X = \{0, 1, 2, 3\}$

continuous random variable

$0 \le X \le 1$

probability distribution

$x =$	0	1	2	3
$p(x) =$	1/8	3/8	3/8	1/8

$$f(x) = \begin{cases} 3x^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

probability mass function

$0 \le P_i \le 1$

$$\boxed{\sum_i P_i = 1}$$

probability density function

$0 \le f(x) \le 1$

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$$

Mathematical Expectations

E(X)

Discrete

$E(X) = \sum_i x_i p_i$

Continuous

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$Var(X) = E(X^2) - E(X)^2$$

$\Rightarrow$ Standard deviation $= \sqrt{\text{Variance}}$

1 A) If x is continuous random variable with p.d.f $f(x) = \begin{cases} c(x-x^2), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$ find the value of 'c'.

soln:-

$$f(x) = \begin{cases} c(x-x^2), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Getting x is a constant random variable with p.d.f $f(x)$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{0}^{1} f(x)\,dx = 1$$

$$\Rightarrow \int_{0}^{1} c(x-x^2)\,dx = 1$$

$$= c\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$$

$$\Rightarrow c\left[\frac{1}{2}-\frac{1}{3}\right]=1$$

$$\Rightarrow c\,(1/6)=1$$

$$\Rightarrow \boxed{c=6}$$

2 A) If x is continuous random variable with p.d.f $f(x)=\begin{cases}3x^2, & 0<x<1\\ 0, & \text{otherwise}\end{cases}$ find the mean of 'x'.

soln:-

Getting x is continuous random variable with p.d.f

$$f(x)=\begin{cases}3x^2 & 0<x<1\\ 0, & \text{otherwise}\end{cases}$$

$$\text{Mean } E(X)=\int_{-\infty}^{\infty} x\,f(x)\,dx$$

$$=\int_0^1 x\cdot 3x^2\,dx$$

$$=3\int_0^1 x^3\,dx$$

$$=3\left[\frac{x^4}{4}\right]_0^1$$

$= 3\frac{1}{4}$

mean $= 3/4$

course outcome :-

Understand fundamental concepts of probability and standard distributions which can describe real life phenomena.

5 b) If the pdf of random variable x is

$f(x) = \begin{cases} kxe^{-x}, & x > 0 \\ 0, & x \le 0 \end{cases}$ then find the value of 'k' mean and variance of x.

soln:-

Getting x is constant r.v with p.d.f

$$f(x) = \begin{cases} kxe^{-x} & , x > 0 \\ 0 & , x \le 0 \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{0}^{\infty} kxe^{-x}\,dx = 1$$

$$K \int_0^{\infty} x^1 e^{-x}\, dx = 1$$

$$\boxed{\int_0^{\infty} x^n e^{-x}\, dx = n!}$$

$$\boxed{K = 1}$$

$$f(x) = \begin{cases} xe^{-x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

Mean $E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$= \int_0^{\infty} x \cdot x e^{-x}\, dx$$

$$= \int_0^{\infty} x^2 e^{-x}\, dx$$

$$= 2!$$

$$\boxed{\text{Mean } E(x) = 2}$$

Variance $= E(x^2) - (E(x))^2$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 \, f(x)\, dx$$

$$= \int_{0}^{\infty} x^2 . x e^{-x} dx$$

$$= \int_{0}^{\infty} x^3 e^{-x} dx = 3! = 6$$

$$\text{variance} = E(x^2) - E(x)^2$$

$$= 6 - 2^2$$

$$= 6 - 4$$

$$= 2$$

4(B) A r.v X has the following distribution

$X=x$	0	1	2	3	4	5	6	7	8
$P(X=x)$	a	$2a$	$3a$	$4a$	$7a$	$8a$	$9a$	$10a$	$11a$

i) find the value of a

ii) Evaulate $P(x<3)$, $P(x\geq 3)$

iii) Find cumulative distribution of x

iv) Find mean and variance of X.

Soln:-

i) Getting x is discrete distribution

$$\sum P_i = 1$$

$$a + 2a + 3a + 4a + 7a + 8a + 9a + 10a + 11a = 1$$

$$55a = 1$$

$$a = \frac{1}{55}$$

$X=x$:	0	1	2	3	4	5	6	7	8
$P(X=x)$:	$\frac{1}{55}$	$\frac{2}{55}$	$\frac{3}{55}$	$\frac{4}{55}$	$\frac{7}{55}$	$\frac{8}{55}$	$\frac{9}{55}$	$\frac{10}{55}$	$\frac{11}{55}$

ii) $P(X<3) = \frac{1}{55} + \frac{2}{55} + \frac{3}{55} = \frac{6}{55}$

$P(X \geq 3) = \frac{4}{55} + \frac{7}{55} + \frac{8}{55} + \frac{9}{55} + \frac{10}{55} + \frac{11}{55} = \frac{49}{55}$

iii) cumulative distribution of x

$x=x$:	0	1	2	3	4	5	6	7	8
$F(x=x)$:	$\frac{1}{55}$	$\frac{3}{55}$	$\frac{6}{55}$	$\frac{10}{55}$	$\frac{17}{55}$	$\frac{25}{55}$	$\frac{34}{55}$	$\frac{44}{55}$	$\frac{55}{55}$

iv) Mean and variance of x

Mean $E(x) = \sum x_i p_i$

$= 0 \times \frac{1}{55} + 1 \times \frac{2}{55} + 2 \times \frac{3}{55} + 3 \times \frac{4}{55}$

$+ 4 \times \frac{7}{55} + 5 \times \frac{8}{55} + 6 \times \frac{9}{55} + 7 \times \frac{10}{55} + 8 \times \frac{11}{55}$

$= \frac{2+6+12+28+40+54+70+88}{55}$

$= \frac{300}{55}$

Mean $E(x) = 5.455$

$E(x^2) = \sum x_i^2 p_i$

$= 0^2 \times \frac{1}{55} + 1^2 \times \frac{2}{55} + 2^2 \times \frac{3}{55} + 3^2 \times \frac{4}{55} + 4^2 \times \frac{7}{55} + 5^2 \times \frac{8}{55}$

$6^2 \times \frac{9}{55} + 7^2 \times \frac{10}{55} + 8^2 \times \frac{11}{55}$

$= \frac{2 + 12 + 36 + 112 + 200 + 324 + 490 + 704}{55}$

$E(x^2) = \frac{1880}{55}$

$= 34.1818$

$= 34.182$

variance $= E(x^2) - E(x)^2$

$= 34.182 - (5.455)^2$

$= 4.4249$

$= 4.425$ //.

5 (c) A r.v X has the following probability distribution

$X = x$:	0	1	2	3	4	5	6	7
$P(X=x)$:	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

find i) the value of k

ii) $P(X<5)$, $P(1<X<6)$

iii) The distribution function of X

iv) Mean and variance of X

i) Getting X is discrete distribution,

$$\sum P_i = 1$$

$$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$10k^2 + 9k = 1$$

$$10k^2 + 9k - 1 = 0$$

$$k = \frac{-9 \pm \sqrt{81 - 4(10)(-1)}}{2(10)}$$

$$= \frac{-9 \pm \sqrt{81+40}}{20}$$

$$= \frac{-9 \pm \sqrt{121}}{20}$$

$$= \frac{-9 \pm 11}{20}$$

$$k = \frac{-9-11}{20}, \quad \frac{-9+11}{20}$$

$$= \frac{-20}{20}, \quad \frac{2}{20}$$

$$k = -1, \ 1/10$$

$$k = -1, \ 0.1$$

$$k \neq -1 \qquad 0 \leq p \leq 1 \quad , k = 0.1$$

ii)

$X=x$:	0	1	2	3	4	5	6	7
$P(X=x)$:	0	$1/10$	$2/10$	$2/10$	$3/10$	$1/10^2$	$2/10^2$	$7/10^2 + \frac{1}{10}$

$\Rightarrow P(X<5) = 0 + \frac{1}{10} + \frac{2}{10} + 2/10 + 3/10$

$= 8/10$

$P(X<5) = 0.8$

$P(1<X<6) = 2/10 + 2/10 + 3/10 + \frac{1}{10^2}$

$= \frac{20 + 20 + 30 + 1}{100}$

$= \frac{71}{100}$

$P(1<X<6) = 0.71$

iii) cummulative distribution of x

$X=x$:	0	1	2	3	4	5	6	7
$P(X=x)$:	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17

$X=x$:	0	1	2	3	4	5	6	7
$P(X=x)$:	0	0.1	0.3	0.5	0.8	0.81	0.83	1.00

iv) Mean and variance of x

Mean $E(x) = \sum x_i p_i$

$= 0\times 0 + 1\times 0.1 + 2\times 0.2 + 3\times 0.2 +$
$4\times 0.3 + 5\times 0.01 + 6\times 0.02 + 7\times 0.17$

$= 0 + 0.1 + 0.4 + 0.6 + 1.2 + 0.05 +$
$0.12 + 1.19$

mean $E(x) = 3.66$

variance $= E(x^2) - E(x)^2$

$E(x^2) = 0\times 0 + 0.1\times 1^2 + 2^2\times 0.2 + 3^2\times 0.2 +$
$4^2\times 0.3 + 5^2\times 0.01 + 6^2\times 0.02 +$
$7^2\times 0.17$

$= 0 + 0.1 + 0.8 + 1.8 + 4.8 + 0.25$
$+ 0.72 + 8.33$

$= 16.8$

$$var(x) = E(x)^2 - E(x^2)$$

$$= 16.8 - (3.66)^2$$

$$= 3.4044$$

$$\boxed{var(x) = 3.404}$$

6 B) The density function of r.v x is $f(x) = kx(2-x)$, $0 \leq x \leq 2$ find the k, mean and variance of x.

soln :-

Getting x is a constant . r.v

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\int_{0}^{2} k\, x\,(2-x)\, dx = 1$$

$$k \int_{0}^{2} 2x - x^2\, dx = 1$$

$$k \left[\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^3}{3} \right]_0^2 = 1$$

$$K\left[4-\frac{8}{3}\right]=1$$

$$K\left(\frac{12-8}{3}\right)=1$$

$$K\left(\frac{4}{3}\right)=1$$

$$\boxed{K=3/4}$$

$f(x)= 3/4\ x(2-x),\ 0\le x\le 2$

mean $E(x)=\int_{-\infty}^{\infty} x\, f(x)\,dx$

$$=\int_0^2 x\,\frac{3}{4}x(2-x)\,dx$$

$$=3/4\int_0^2 x^2(2-x)\,dx$$

$$=3/4\int_0^2 2x^2-x^3\,dx$$

$$= \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2$$

$$= \frac{3}{4}\left[\frac{16}{3} - \frac{16}{4}\right]$$

$$= \frac{3}{4}\left[\frac{64-48}{12}\right]$$

$$= \frac{\cancel{3}}{\cancel{4}}\left[\frac{\cancel{16}}{\cancel{12}}\right]^{\cancel{4}}_{\cancel{3}}$$

$$\boxed{E(X) = 1}$$

$$\text{variance} = E(X^2) - E(X)^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_0^2 x^2 \, \frac{3}{4} x(2-x)\,dx$$

$$= \frac{3}{4}\int_0^2 x^3(2-x)\,dx$$

$$= \frac{3}{4}\int_0^2 (2x^3 - x^4)\,dx$$

$$= \frac{3}{4}\left[\frac{2x^4}{4} - \frac{x^5}{5}\right]_0^2$$

$$= \frac{3}{4}\left[\frac{2\times 16^4}{4} - \frac{32}{5}\right]$$

$$= \frac{3}{4}\left[8 - \frac{32}{5}\right]$$

$$= \frac{3}{4}\left(\frac{8}{5}\right)^2$$

$$E(X^2) = \frac{6}{5}$$

$$\text{variance} = \frac{6}{5} - (1)^2$$

$$= \frac{6}{5} - 1$$

$$\boxed{\text{variance} = \frac{1}{5}}$$

Standard distributions

- Discrete
  - Binomial Distribution
  - poisson "
  - Geometric "
- continuous
  - Uniform Distribution
  - Normal "
  - Exponential "

Binomial distribution :-

A r.v X is said to follow Binomial Distribution if it has the following probability distribution

$$P(X=x) = nc_x\, p^x\, q^{n-x}$$

$x = 0, 1, 2, \ldots n$

$n \rightarrow$ no. of. trials

$p \rightarrow$ probability of success

$q \rightarrow$ prob of failure $(q = 1-p)$

mean $= np$, Variance $= npq$

7 A) The mean of binomial distributions is 20 and standard deviation 4. find the parameter of distribution.

soln:-

mean $= 20$

$np = 20 \rightarrow$ ①

$S.D = 4$

$\sqrt{var} = 4$

$npq = 16 \rightarrow$ ②

$\frac{②}{①} \Rightarrow \frac{\cancel{n}\cancel{p}q}{\cancel{n}\cancel{p}} = \frac{16}{20}$

$\boxed{q = 4/5}$

$p = 1 - q$

$= 1 - 4/5$

$\boxed{p = 1/5}$

Using ①

$np = 20$

$n(1/5) = 20$

$\boxed{n = 100}$

A r.v is x said to follow poisson distribution if it has the following probability distribution.

$$p(X=x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$x = 0, 1, 2, \ldots . n$

$\lambda \to$ mean

variance $= \lambda$

Now:-

If x is larger than Binomial $\approx$ poisson

Geometric distribution :-

A r.v x is said to follow geometric distribution if it has the following probability distribution.

$$P(X = x) = pq^{x-1}$$

$p \to$ prob success

$q \to$ failure

$\boxed{p+q}$

8) a) State memoryless property of geometric distribution?

Soln :-

Let 's' and 't' two positive integers then for $s > t$

$$P[X > s+t / X > t] = P[X > s]$$

9) A) State any two properties of Normal distributions.

1. The mean, mode and median are same.

2. The distribution is symmetric about the mean (half the value fall below and half above mean).

Uniform distribution :-

Let 'a' and 'b' are any two parameters then the p.d.f of uniform distribution is

$$f(x) = \begin{cases} \frac{1}{b-a} , & a \le x \le b \\ 0 , & \text{otherwise} \end{cases}$$

10) find the mean variance and moment
5B) generating function for uniform distribution ?

Soln :-

Let 'a' and 'b' are parameters of uniform distribution then its p.d.f

$$f(x) = \begin{cases} \frac{1}{b-a} , & a \le x \le b \\ 0 , & \text{otherwise} \end{cases}$$

$$\text{mean } E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_{a}^{b} x \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \int_{a}^{b} x \, dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a$$

$$= \frac{1}{b-a}\left[\frac{b^2}{2} - \frac{a^2}{2}\right]$$

$$= \frac{1}{2}\frac{1}{b-a}(b^2-a^2)$$

$$= \frac{1}{2}\frac{1}{\cancel{b-a}}(b+a)\cancel{(b-a)}$$

$$E(x) = \frac{a+b}{2}$$

$$E(x^2) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_a^b x^2 \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a}\int_a^b x^2\, dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{1}{b-a}\left[\frac{b^3}{3} - \frac{a^3}{3}\right]$$

$$= \frac{1}{3}\,\frac{1}{b-a}\,(b^3 - a^3)$$

$$= \frac{1}{3}\,\frac{1}{b-a}\,(b-a)(b^2+ab+a^2)$$

$$= \frac{b^2+ab+a^2}{3}$$

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

$$\text{variance} = E(x^2) - E(x)^2$$

$$= \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{b^2+ab+a^2}{3} - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{4b^2+4ab+4a^2-3a^2-6ab-3b^2}{12}$$

$$= \frac{b^2-2ab+a^2}{12}$$

$$= \frac{(b-a)^2}{12}$$

$$\text{variance} = \frac{(a-b)^2}{12}$$

Moment generating function :-

$$M_X(t) = E\left[e^{tX}\right]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{a}^{b} e^{tx} \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \int_{a}^{b} e^{tx}\, dx$$

$$= \frac{1}{b-a} \left[\frac{e^{tx}}{t}\right]_a^b$$

$$= \frac{1}{b-a} \cdot \frac{e^{bt} - e^{at}}{t}$$

Note :-

$$\text{Mean} = \frac{a+b}{2}$$

$$\text{variance} = \frac{(b-a)^2}{12}$$

11) 3B) six dice are thrown 729 times. How many times do you expect at least three dice to show a five or six?

Soln :-

prob success $p = \frac{2}{6} = 1/3$

$q = 1 - p$

$q = 1 - 1/3 = 2/3$

$$\boxed{q = 2/3}$$

no. of. trials $n = 6$

$\therefore$ by using binomial distribution

$$P(X = x) = {}^nC_x \, p^x \, q^{n-x}$$

$$x = 0, 1, 2, \ldots\ldots n$$

$$P(X=x) = {}^6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}$$

$$= {}^6C_x \frac{1^x}{3^x} \frac{(2/3)^6}{(2/3)^x}$$

$$= {}^6C_x \frac{1}{\cancel{3^x}} \frac{(2/3)^6}{\frac{2^x}{\cancel{3^x}}} \Rightarrow \left[{}^nC_r = \frac{n!}{(n-r)! \times r!} \right]$$

$$P(X=x) = {}^6C_x \left(\frac{2}{3}\right)^6 \frac{1}{2^x}$$

$$\frac{\cancel{6} \times 5 \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{1 \times \cancel{2} \times \cancel{3} \times 1 \times 2 \times 3}$$

$$x = 0, 1, 2, \ldots\ldots 6$$

$$P(X \geq 3) = P(X=3) + P(X=4) + P(X=5) + P(X=6)$$

$$= {}^6C_3 \left(\frac{2}{3}\right)^6 \frac{1}{2^3} + {}^6C_4 \left(\frac{2}{3}\right)^6 \frac{1}{2^4} + {}^6C_5 \left(\frac{2}{3}\right)^6 \frac{1}{2^5}$$

$$+ {}^6C_6 \left(\frac{2}{3}\right)^6 \frac{1}{2^6}$$

$$= 20 \frac{(2^{\cancel{6}\,3})}{3^6} \frac{1}{\cancel{2^3}} + (15) \frac{2^{\cancel{6}\,2}}{3^6} \frac{1}{\cancel{2^4}} + 6 \frac{2^{\cancel{6}\,1}}{3^6} \frac{1}{\cancel{2^5}}$$

$$+ \frac{\cancel{2^6}}{3^6} \cdot \frac{1}{\cancel{2^6}}$$

$= \frac{160}{729} + \frac{60}{729} + \frac{12}{729} + \frac{1}{729}$

$= \frac{233}{729}$

No. of times getting at least three dice to show a 5 or 6 $= \frac{233}{\cancel{729}} \times \cancel{729}$

$= 233$ times //.

12) 4B) The no. of monthly breakdown of a computer is a random variable having a poisson distribution with mean equal to 1.8. Find the probability that this computer will function for a month i) without a breakdown ii) with two breakdown and iii) with at least one breakdown.

Soln;

The rv x denote "no. of monthly breakdown of a computer

mean $= 1.8$

$\lambda = 1.8$

$\therefore$ using poisson distribution

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$x = 0, 1, 2 \ldots\ldots$

$$P(X=x) = \frac{e^{-1.8}(1.8^{x})}{x!}$$

$$P(x=x) = (0.1652)\frac{1.8^{x}}{x!}$$

i) P[without a breakdown] $= P[X=0]$

$$= 0.1652 \frac{1.8^{0}}{0!}$$

$$= 0.1652$$

ii) P[with 2 breakdown]

$= P[x=2]$

$= 0.1652 \frac{(1.8)^2}{2!}$

$= 0.82676$

iii) P[At least one break down]

$= P[x \geq 1]$

$= 1 - P[x < 1]$

$= 1 - P(x=0)$

$= 1 - 0.1652$

$= 0.8348$ //