Exercise #7
MB&B 361/562.
Due: before class on Tuesday, March 5, 2024
Please upload it to the Canvas Box (title: 'LastnameFirstname_ExerciseXX').

1) Background

The mechanical properties of auditory hair cells were discussed last week. In this exercise, we will work through a study that used a vibrating glass capillary to entrain the motion of an oscillating hair bundle (Figure 1).

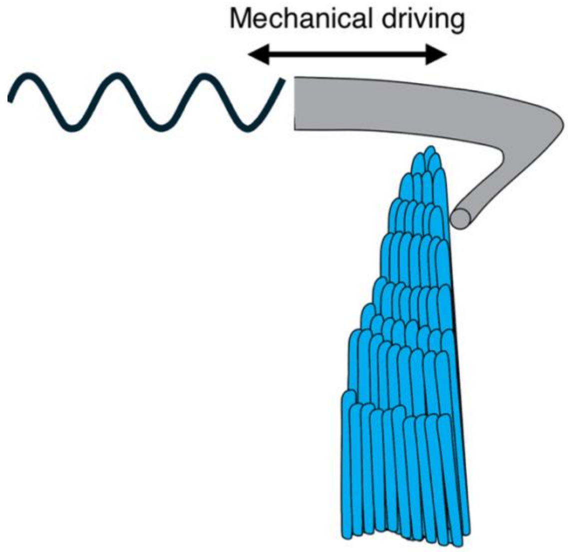

Figure 1. Schematic of a hair bundle stimulated with a bent glass capillary.

It was found that, when the capillary is vibrated with a frequency near the intrinsic frequency of the hair bundle (20 Hz in the cell in Figure 2), the hair bundle synchronized with the driving capillary, with occasional slips when the driving frequency deviated too much from the intrinsic frequency of the hair bundle.

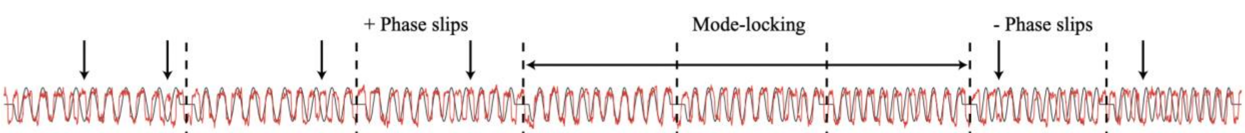

Figure 2. The oscillation frequency of the capillary was increased stepwise from 16 Hz to 23 Hz. The hair bundle phase-locked with the capillary at frequencies near the hair bundle's intrinsic frequency of 20 Hz. The hair bundle motion is in red and the capillary motion is in black. Arrows indicate slips in relative phase. Figure 1 from Fredrickson-Hemsing et al. (2012).

Question 1

Q1.1 (2 point). Consider a hair cell which is undergoing noisy oscillations with angular frequency ω_1 and noise $\xi(t)$. Suppose that it is being entrained by a sinusoidal driving force with a frequency ω_2: that is, $\frac{d\theta_2}{dt} = \omega_2$. Write an equation to describe the rate of change of the phase of the hair cell, $\frac{d\theta_1}{dt}$. (Hint: this is just Example 1 in the Class 13 (Adler) notes with a noise term added).

Q1.2 (2 point). Write the equation from Q1 as a phase difference between the hair cell and the driver: i.e., in $\Delta\theta = \theta_1 - \theta_2$.

Q1.3 (2 point). In Figure 3 below, the time evolution of the phase difference between a hair bundle with intrinsic frequency $\omega_1 = 90\frac{rad}{s}$ and the capillary with driving frequency $\omega_2 = 60\frac{rad}{s}$ is plotted for different driving strengths. The hair bundle begins to phase lock at a driving force of f_0 0.35 to 0.5 pN where there are horizontal regions (plateaus) indicating that the relative phase is not changing.

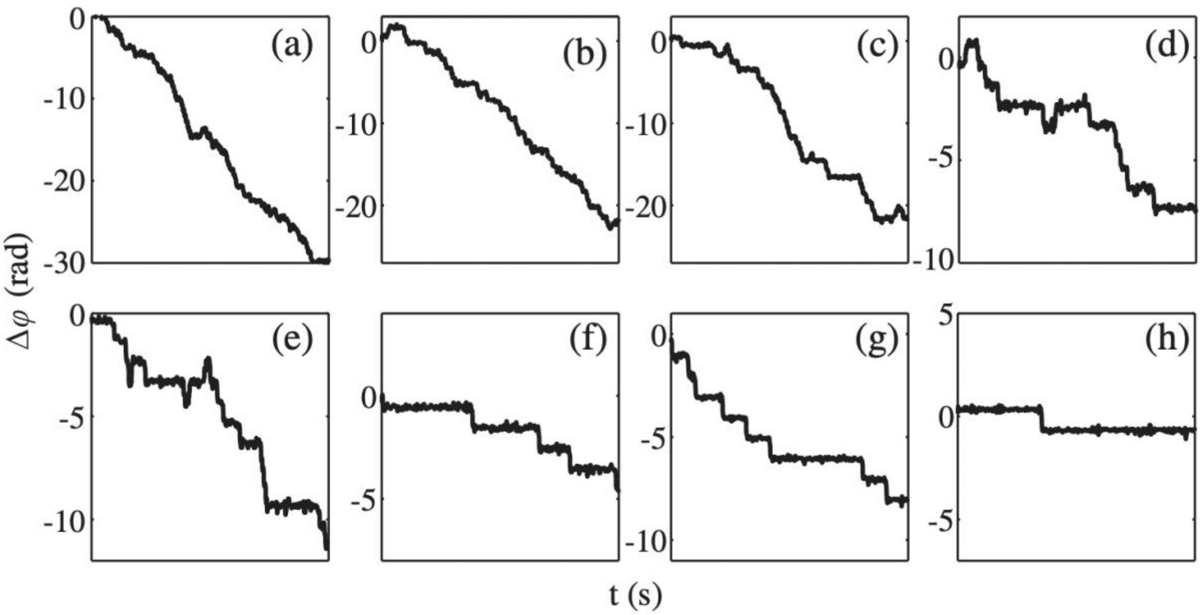

Figure 3. The time evolution of the phase difference between the hair cell ($\omega_1 = 90\frac{rad}{s}$) and the driving signal ($\omega_2 = 60\frac{rad}{s}$) at different driving strengths of 0.2, 0.35, 0.5, 0.6, 0.7, 0.8, 1.0 and 1.2 pN. The hair bundle begins to phase lock at a driving force of f_0 of 0.35 pN to 0.5 pN. Figure 3 from Roongthumskul et al. (2013).

Assume that the coupling is linear in the force $\epsilon = \alpha f_0$, where α is the constant of proportionality. Calculate α. [Hint: at the onset of coupling $\epsilon \approx \Delta\omega$.]

Question 2 (4 points)

To complete the characterization of the system, we would need to estimate the strength of the noise. We will do this by simulating the phase-locking.

Open the included MATLAB script. In the code, you have the enter α from Q1.3. This code simulates the phase difference over 20 seconds for coupling strengths between 0.2 and 1.2 pN (like in Figure 3 above). Choose the following values for the amplitude A of the Gaussian noise: (1) 250, (ii) 1,000, (iii) 2,500, (iv) 10,000. Qualitative describe the how phase locing changes when the coupling strength is increased from 0.2 pN to 1.2 pN. Which noise best matched Figure 3?

Question 3 (For graduate students or bonus for undergraduates; 2 points)

Although we only looked at the steady state solution to the Adler equation in class we can calculate the transient solution of the noise-free equation. Solve the equation $\frac{d}{dt}\Delta\theta = \Delta\omega - 2\epsilon\sin(\Delta\theta)$ numerically with $\Delta\omega = 1$, $\epsilon = 5$ and $\Delta\theta(t=0) = \pi/2$ and plot it $\Delta\theta(t)$.

References

[1] Roongthumskul, Y., et. al. (2013). *PRL*, *110*(14), 148103.

[2] Martin, Pascal, A. J. Hudspeth, and F. Jülicher. *PNAS* 98, no. 25 (2001): 14380-14385.

[3] Fredrickson-Hemsing, et. al. *Physical Review E* 86, no. 2 (2012): 021915.