MB&B 562: Exercise 4

Karthik Desingu

Question 1

Q1) $\dot{u} = f(u,v) = \frac{2}{1+v^4} - u.$

$\dot{v} = g(u,v) = \frac{2}{1+u^4} - v$

Setting $f(u,v)$ and $g(u,v) = 0$ to find the nullclines, we have:

$u = \frac{2}{1+v^4}$ —— ①

$v = \frac{2}{1+u^4}$ —— ②

Given that $u = u^* = \varepsilon$ is a fixed pt.

Since ① and ② intersect at ε,

$\boxed{v^* = \frac{2}{1+\varepsilon^4}}$

If $\varepsilon \approx 0.2$,

$u^* = 0.2$

$v^* \approx 2$

(ii) $J(u^*, v^*)$

$= \begin{pmatrix} \partial f_u & \partial f_v \\ \partial g_u & \partial g_v \end{pmatrix} (u^*, v^*)$

$\partial f_u = -1$

$\partial f_v = \frac{-8v^3}{(1+v^4)^2}$

$\partial g_u = \frac{-8u^3}{(1+u^4)^2}$

$\partial g_v = -1$

$\approx \begin{pmatrix} -1 & 0.22 \\ 0.06 & -1 \end{pmatrix}$

For this linear approximation, characteristic eqn. of eigenvalues,

$\lambda^2 + 2\lambda + 0.9868 = 0$

$\lambda_+ = \frac{-2 + \sqrt{4 - 4(0.99)}}{2}$

(-ve, real)

$\lambda_- = \frac{-2 - \sqrt{4-4(0.99)}}{2}$

(-ve, real)

Since both eigenvalues are -ve at the linear approximation near (u^*, v^*), this fixed point is <u>STABLE</u>.

Question 2

Q2) $A = \begin{pmatrix} -1 & 0.5 \\ 0.5 & -1 \end{pmatrix}$

(i) The characteristic eqn is :

$$\lambda^2 + 2\lambda + 0.75 = 0$$

Solving for λ,

$$\lambda_+ = \frac{-2 + \sqrt{4-3}}{2} = \frac{-1}{2}$$

$$\lambda_- = \frac{-2 - \sqrt{4-3}}{2} = \frac{-3}{2}$$

Since $\lambda_1 > \lambda_2$,

$$\boxed{\begin{array}{l} \lambda_1 = -1/2 \\ \lambda_2 = -3/2 \end{array}}$$

(ii) $A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 0.5 \\ 0.5 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} -0.5 \\ -0.5 \end{pmatrix}$

$= -\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$= \lambda_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ So, $\bar{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(eigenvector)

and $A\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-1 & 0.5\\ 0.5 & -1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}$

$= \begin{pmatrix}-1.5\\-1.5\end{pmatrix}$

$= \frac{-3}{2}\begin{pmatrix}1\\1\end{pmatrix}$

$= \underline{\underline{\lambda_2}}\begin{pmatrix}1\\1\end{pmatrix}$ So, $\bar{x}_2 = \begin{pmatrix}1\\-1\end{pmatrix}$
(eigenvector)

(iii) $\dot{x} = Ax$

at $x = x_0 = \begin{pmatrix}1\\1\end{pmatrix}$

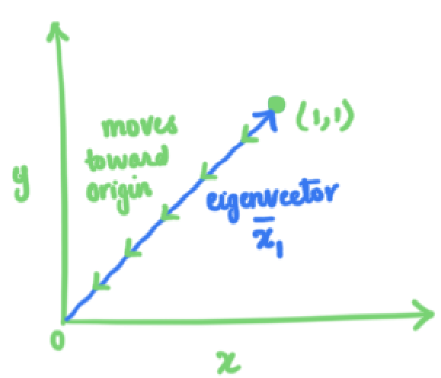
Since x_0 is on the eigenvector, $\bar{x}_1$,

$\dot{x}(x_0) = \lambda_1 x_0$

(ie) $\dot{x}$ is also along $\bar{x}_1$.

And since $\lambda_1 = \frac{-1}{2} < 0$,

x moves along $\bar{x}_1$
and toward the origin.



Further, the rate of movement toward origin reduces as it gets closer to the origin.