MB&B 562: Exercise 3

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Exercise 1

1) $\ddot{x} - \mu(1-x^2)\dot{x} + x = 0$ — (A)

let $y = \dot{x}$,

then, $\dot{x} = 0.x + 1.y$

$\equiv f(x,y)$. $\longrightarrow$ linear

and $\dot{y} = \ddot{x}$

$= \mu(1-x^2)\dot{x} - x$ (from A)

$= \mu(1-x^2)y - x$

$\equiv g(x,y)$. $\longrightarrow$ non-linear

Where, $\dot{x}$ and $\dot{y}$ form a pair of first-order equations.

Exercise 2

Livescript PDF attached.

The **critical value of gamma is 2**.

Detailed responses to (i) and (ii) included as inline text in the livescript.

Exercise 3

3) (a) $\dot{x} = Ax$, $A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$

Trial Soln. (general)

$$x(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\lambda t} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

For $x(t)$ to be a soln,

$$\lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a\alpha + b\beta \\ c\alpha + d\beta \end{pmatrix}$$

This can be written as,

$$(a-\lambda)\alpha + b\beta = 0 \quad \text{——} \ 1$$
$$c\alpha + (d-\lambda)\beta = 0 \quad \text{——} \ 2$$

For non-trivial soln, let $\beta \neq 0$,

Multiplying 1 by c , 2 by $(\lambda - a)$, and adding,

$$bc\beta + (d-\lambda)(\lambda-a)\beta = 0$$

Dividing by $\beta \neq 0$,

$$bc + (d-\lambda)(\lambda-a) = 0$$

$$\lambda^2 + (a-d)\lambda + (ad-bc) = 0$$

→ For the given system,

$$A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

Characteristic eqn. is now,

$$\lambda^2 - (a+d)\lambda + ad = 0$$

$$\lambda_1 = \frac{(a+d) + \sqrt{(a+d)^2 - 4ad}}{2}$$

$$= ((a+d) + (a-d))/2$$

$$= \underline{a}$$

$$\lambda_2 = \frac{(a+d) - \sqrt{(a+d)^2 - 4ad}}{2}$$

$$= ((a+d) - (a-d))/2$$

$$= \underline{d}$$

The eigen values are

$$\boxed{\begin{array}{l} \lambda_1 = a \\ \lambda_2 = d \end{array}}$$

(ii) For eigenvectors $\bar{x}_1, \bar{x}_2$:

where $\bar{x}$ denotes that x is a vector

$$A\bar{x}_1 = \lambda_1 \bar{x}_1 = \lambda_1 I \bar{x}_1$$

$$(A - \lambda_1 I) . \bar{x}_1 = 0$$

$$\begin{pmatrix} a-\lambda_1 & 0 \\ 0 & d-\lambda_1 \end{pmatrix} \bar{x}_1 = 0$$

$$\begin{pmatrix} 0 & 0 \\ 0 & d-a \end{pmatrix} \bar{x}_1 = 0$$

So, $\boxed{\bar{x}_1 = k_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$ for some const. k_1

Similarly, for $\bar{x}_2$,

$$\begin{pmatrix} a-\lambda_2 & 0 \\ 0 & d-\lambda_2 \end{pmatrix} \bar{x}_2 = 0$$

$$\begin{pmatrix} a-d & 0 \\ 0 & 0 \end{pmatrix} \bar{x}_2 = 0$$

So, $\boxed{\bar{x}_2 = k_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$ for some const. k_2

iii) If both a, d are positive

Then, $\left.\begin{aligned} \lambda_1 &= a > 0 \\ \lambda_2 &= d > 0 \end{aligned}\right\}$ +ve eigenvalues

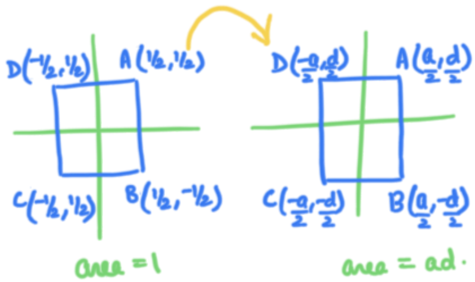
Hence, the system is <u>unstable</u>.

(iv) If both $a > 0$, $d > 0$,

then, a unit square expands

$\left.\begin{array}{l} \text{'a' times along } x\text{-axis}, \\ \text{'d' times along } y\text{-axis}, \end{array}\right\}$ The area in from 1 to a

and the vertices remain in the same quads



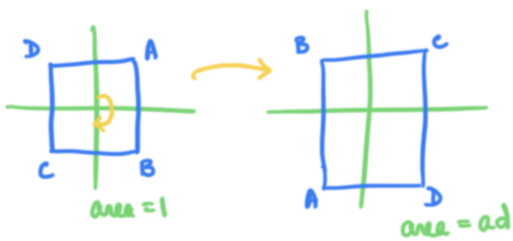
(v) If both $a < 0$ and $d < 0$

Then, $\left.\begin{array}{l} \lambda_1 = a < 0 \\ \lambda_2 = d < 0 \end{array}\right\}$ -ve eigenvalues

Hence, the system is stable.

In this case, the unit square still expands 'a' times along x, and 'd' times along y-axis. The area increases from 1 to ad.

However, it rotates by $180°$ along its center.



(vi) If $a > 0$, $d < 0$

Then, $\left.\begin{array}{l} \lambda_1 = a > 0 \\ \lambda_2 = d < 0 \end{array}\right\}$ one +ve eigenvalue

Hence, the system is unstable.

(vii) If $d = 1/a$,

the unit square expands 'a' — or, contracts if $a \in (0,1)$

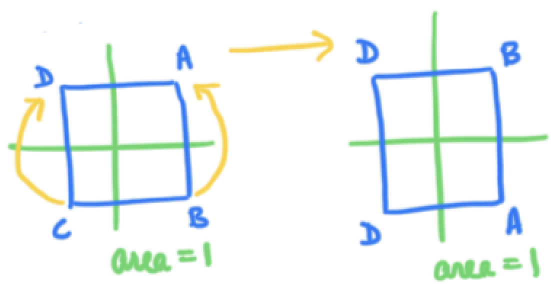
times along x-axis and

contracts 'a' times along y-axis. — expands if $a \in (0,1)$

The area is conserved at 1 sq. unit.

Further, if $a>0$, $b<0$

Then, the square also flips about the x-axis,



Question 3(b) on the next page.

3) b) i) $\dot{x} = A\bar{x}$; $A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$

The characteristic eqn. is :

$\lambda^2 + \omega^2 = 0$

Now,

$\lambda_1 = \dfrac{+\sqrt{-4\omega^2}}{2}$

$= \omega i$

$\lambda_2 = \dfrac{-\sqrt{-4\omega^2}}{2}$

$= -\omega i$

The eigenvalues are :

$$\boxed{\begin{aligned} \lambda_1 &= \omega i \\ \lambda_2 &= -\omega i \end{aligned}}$$

(ii) From the eigenvalues, the trial soln. is

$x(t) = \alpha e^{\lambda t}$

$= \alpha e^{i\omega t}$ (for λ_1)

$= \alpha \cos \omega t + i\alpha \sin \omega t$

and, $y(t) = \beta e^{\lambda t}$

$= \beta e^{i\omega t}$ (for λ_1)

$= \beta \cos \omega t + i\beta \sin \omega t$

The solns. w/ λ_2 will have similar form.

Hence, x(t) and y(t) are complex functions.

The values lie on a circle of radius α for $x(t)$, and β for $y(t)$ on the complex coordinate plane.