Exercise #3
MB&B 361/562.
Due: before class on Tuesday, February 6, 2024
Please upload it to the Canvas Box (title: 'LastnameFirstname_Exercise2'). You can scan handwritten parts.

1. Use a variable substitution to change the following second order (non-linear) differential equation, called the van der Pol oscillator, into a pair of first order (nonlinear) differential equations:

$$\ddot{x} - \mu\left(1 - x^2\right)\dot{x} + x = 0$$

2. Use MATLAB code from class 4 can solve the harmonic oscillator

$$\ddot{x} + \gamma\dot{x} + x = 0$$

for various values of the (positive) damping coefficient (γ) ranging from small (underdamped) to large (overdamped). The mass-damped-spring equation is non-dimensionalized with $m=1$ and $\kappa=1$.
(i) What is the critical value of γ for the transition between being underdamped and overdamped.
(ii) Change the code to plot out the solutions when $\gamma<0$. Hand in the published pdf.

3. (a) Consider the 2D linear dynamical system

$$\dot{x} = Ax \qquad A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \qquad a \neq d$$

(i) Find the eigenvalues λ_1 and λ_2.
(ii) Find the eigenvectors x_1 and x_2 ($A x_1 = \lambda_1 x_1$) and ($A x_2 = \lambda_2 x_2$). Note that these vectors are defined up to a constant.
(iii) Suppose both a and d are both positive. Is the dynamical system stable or unstable?
(iv) Describe in words how the matrix transforms the plane. I.e., how does the matrix transform a unit square centered on the origin and aligned with the x and y axes?
(iv) Suppose both a and d are negative. Is the dynamical system stable or unstable? Describe in words how the matrix transforms the plane.
(v) Suppose a is positive and d are negative. Is the system stable?
(vi) Suppose that $d = 1/a$. Describe in words how the matrix transforms the plane.

(b) Consider the 2D linear dynamical system

$$\dot{x} = Ax \qquad A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$$

(i) Find the eigenvalues λ_1 and λ_2.
(ii) What are the forms of solutions for $x(t)$ and $y(t)$ (i.e., what kind of functions are they)?