Gierer-Meihhardt equations

PS5_GM_equation.mlx

Joe Howard, modified from Hugo Bowne-Anderson 2014

April 8, 2024

In this script, we solve the Gierer-Meinhardt equations 12a/b or 15a/b from (Kybernetik, 12:30-39, 1972) for a variety of parameters.

The partial differential equations are the dimensionless G-M system:

$$\frac{\partial}{\partial t}a(x,t) = \frac{a^2}{h+h_0} - a + \sigma + D\frac{\partial^2}{\partial x^2}a$$

$$\frac{\partial}{\partial t}h(x,t) = \omega(a^2 - h) + \frac{\partial^2}{\partial x^2}h$$

%Notes:

(1) the derivates are partial derivatives (t is time, x is space);

(2) t, x, a and h are all dimensionless;

(4) D is ratio of diffusion constants: activator/inhibitor and is < 1

(5) ω is a reaction term

(6) σ is a baseline synthesis term for a (assumed small, 0.01).

7) h_0 is a small positive term (0.1); it keeps the first term in Eqn 1 finite

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clear;
close all;
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GM_solve_record is the PDE solver. It takes 3 main parameters:

P(1) = D

P(2) = ω

P(3) = σ (chosen to be close to zero);

There are also three other constants specified below:

tmax = amount of time that we run the system;

delt = time increment;

L = length (spatial) of the system;

and there is h0, which is set at 0.1 in the function

GM_solve_record outputs the concentrations of activator and inhibitor over the time course and plots the solution as surfaces for $a(x, t)$ and $h(x, t)$ and as an animation.

We consider the phase plane of the G-M system with x-axis given by D (ratio of diffusion constants: activator/inhibitor) and y-axis ω (the reaction

% term) while keeping everything else constant.

Question 1.

Q1) $$\frac{\partial A}{\partial t} = k_a \frac{A^2}{H+H_0} - \omega_\alpha A + \Sigma + D_a \frac{\partial^2 A}{\partial X^2} \quad \text{—①}$$

$$\frac{\partial H}{\partial T} = k_h A^2 - \omega_h H + D_h \frac{\partial^2 H}{\partial X^2}$$

[using: $T = \alpha t \Rightarrow dT = \alpha dt$

$X = \beta x \Rightarrow dX = \beta x$

$A = \chi a \Rightarrow dA = \chi \cdot da$

$H = \delta h \Rightarrow dH = \delta \cdot dh$

$$\frac{\partial a}{\partial t} = \frac{\alpha}{\chi}\left(k_a \cdot \frac{\chi^2 a^2}{\delta(h+h_0)} - \omega_\alpha \chi a + \Sigma + D_a \frac{\chi^2}{\beta^2} \cdot \frac{\partial a^2}{\partial x^2} \right)$$

$$= \frac{k_a \chi \alpha}{\delta} \cdot \frac{a^2}{h+h_0} - \omega_\alpha \chi a + \Sigma + \frac{D_a \chi \alpha}{\beta^2} \cdot \frac{\partial a^2}{\partial x^2}$$

$$\frac{\partial h}{\partial t} = \frac{\alpha}{\delta}\left(k_h \chi^2 a^2 - \omega_h \delta h + \frac{D_h \delta^2}{\beta^2} \cdot \frac{\partial h^2}{\partial x^2} \right)$$

$$= \frac{k_h \alpha \chi^2}{\delta} a^2 - \omega_h \alpha h + \frac{D_h \alpha \delta}{\beta^2} \cdot \frac{\partial h^2}{\partial x^2}$$

Let $\chi = 1/\omega_a \longrightarrow \omega_h = \omega \omega_a = \frac{\omega}{\chi}$

$\frac{\beta^2}{\alpha\delta} = D_h \longrightarrow D_a = D D_h = \frac{D\beta^2}{\alpha\delta}$

$\frac{\delta}{\chi\alpha} = k_a$

$$\frac{\partial a}{\partial t} = \frac{a^2}{h+h_0} - a + \Sigma + \frac{D\chi}{\delta} \cdot \frac{\partial a^2}{\partial x^2}$$

$$\frac{\partial h}{\partial t} = \frac{k_h \chi}{k_a} a^2 - \omega \cdot \frac{\alpha}{\chi} h + \frac{\partial h^2}{\partial x^2}$$

Question 2.

Q2) $h_0 \ll 1$, $\sigma = 0$

Also, $\frac{\partial^2 a}{\partial x^2} = 0$; $\frac{\partial^2 h}{\partial x^2} = ($

Setting $\frac{\partial a}{\partial t} = \frac{\partial h}{\partial t} = 0$,

$\omega(a^2 - h) = 0$

So, $a^2 = h$

Now, $\frac{a^2}{h} - a + \sigma = 0$

$\Rightarrow \boxed{a = 1}$ $\quad \because \sigma = 0$

and $\boxed{h = 1}$

Question 3.

Q3) $a = a_0 \sin kx$
$h = h_0 \sin kx$

So, $\dfrac{\partial^2 a}{\partial x^2} = -a_0 k^2 \sin kx = -k^2 a$

$\dfrac{\partial^2 h}{\partial x^2} = -h_0 k^2 \sin kx = -k^2 h$

The dynamical system is then:

$$\frac{\partial a}{\partial t} = \frac{a^2}{h+h_0} - a(1 + Dk^2)$$

$$\frac{\partial h}{\partial t} = \omega(a^2 - h) - k^2 h$$

Now, the Jacobian,

$$J = \begin{pmatrix} \frac{2a}{h} - 1 - Dk^2 & \frac{-a^2}{h^2} \\ 2a\omega & -\omega - k^2 \end{pmatrix}$$

$$J(1,1) = \begin{pmatrix} 1 - Dk^2 & -1 \\ 2\omega & -\omega - k^2 \end{pmatrix}$$

(ii) When spatial gradient is 0,

$$J(1,1) = \begin{pmatrix} 1 & -1 \\ 2\omega & -\omega \end{pmatrix}$$

$tr_J = 1 - \omega$

$det_J = \omega$

For a stable fixed pt.,

$\underline{tr < 0}$ and $\underline{4det > 0}$

$1 - \omega < 0$ $\quad\quad$ $4\omega > 0$

$\omega > 1$

So we need: $\boxed{\omega > 1}$

(iii) $\omega = \dfrac{\omega_h}{\omega_a} > 1$

This means that degradation rate of the inhibitor (H) must be higher than that of the activator (A) for the fixed point to be stable.

Question 4.

Q4) From 3(i),

$$J = \begin{pmatrix} \frac{2a}{h} - 1 - Dk^2 & \frac{-a^2}{h^2} \\ 2a\omega & -\omega - k^2 \end{pmatrix}$$

So, $\det J = \left(\frac{2a}{h} - 1 - Dk^2\right)(-\omega - k^2) + \frac{2a^3\omega}{h^2}$

$$= Dk^4 + \left(\omega D - \frac{2a}{h} + 1\right)k^2 + \left(1 - \frac{2a}{h}\right)\omega$$

$$\frac{\partial(\det J)}{\partial(k^2)} = \frac{\partial(\det J)}{\partial k} \cdot \frac{1}{2k}.$$

$$= 2Dk^2 + \omega D - \frac{2a}{h} + 1$$

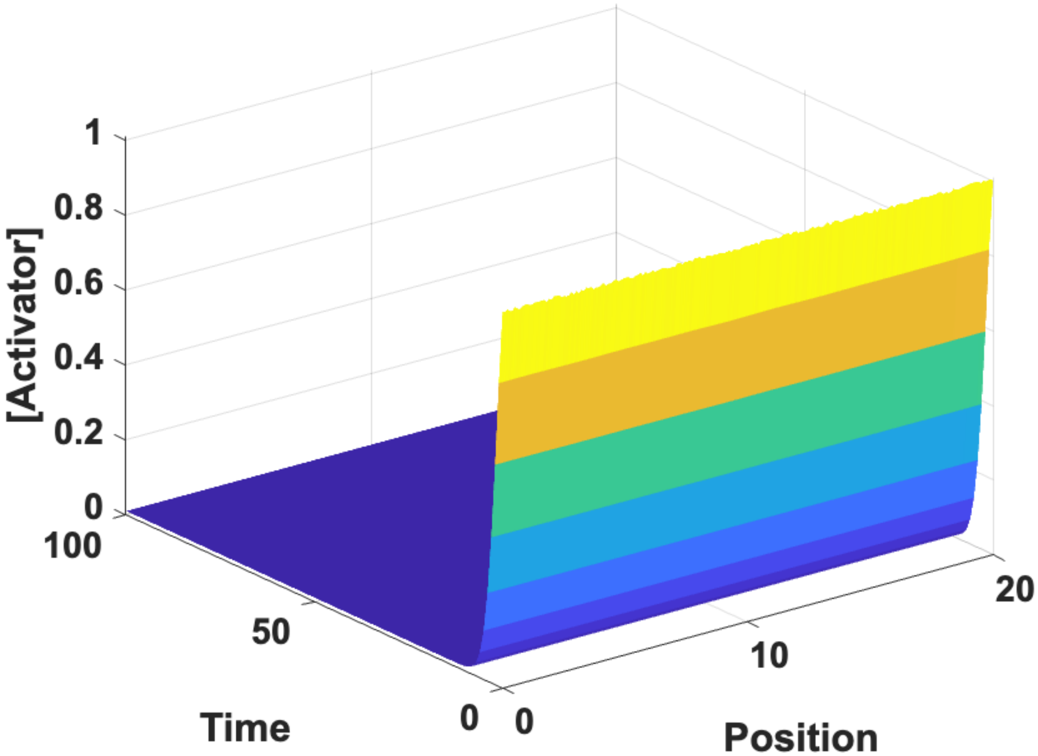
Setting $\det' J = 0$,

$$k^2 = \frac{a}{Dh} - \frac{\omega}{2} - \frac{1}{2D}$$

At the fixed pt. $(1,1)$,

$$k^2 = \frac{1}{2D} - \frac{\omega}{2}$$

```
tmax = 100;     %maximum time (can be changed from 50 to 500 if necessary)
delt = 1;      %time interval
L = 20;        %length
% P = [0.05 ; 2 ; 0.01 ];     %Spatial pattern
% P = [0.4 ; 2 ; 0.01 ];      %Homogenous in time and space
% P = [0.4 ; 1; 0.01];       %temporal oscillation
P = [0.4 ; 0.5; 0.01];     %trivial solution
soln = GM_solve_record_JH(P , tmax , delt, L, 'test_pattern1.avi');
```



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0.6
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0.2
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100
50
Time
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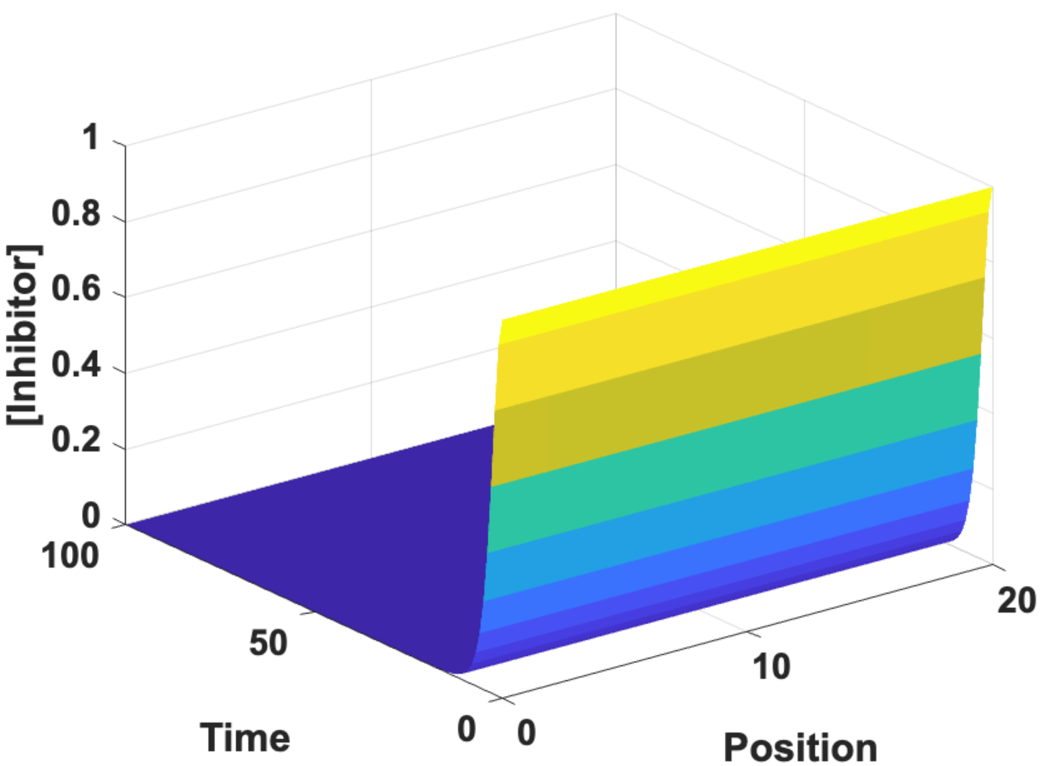
20
Position



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1
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0.2
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100
50
Time
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0
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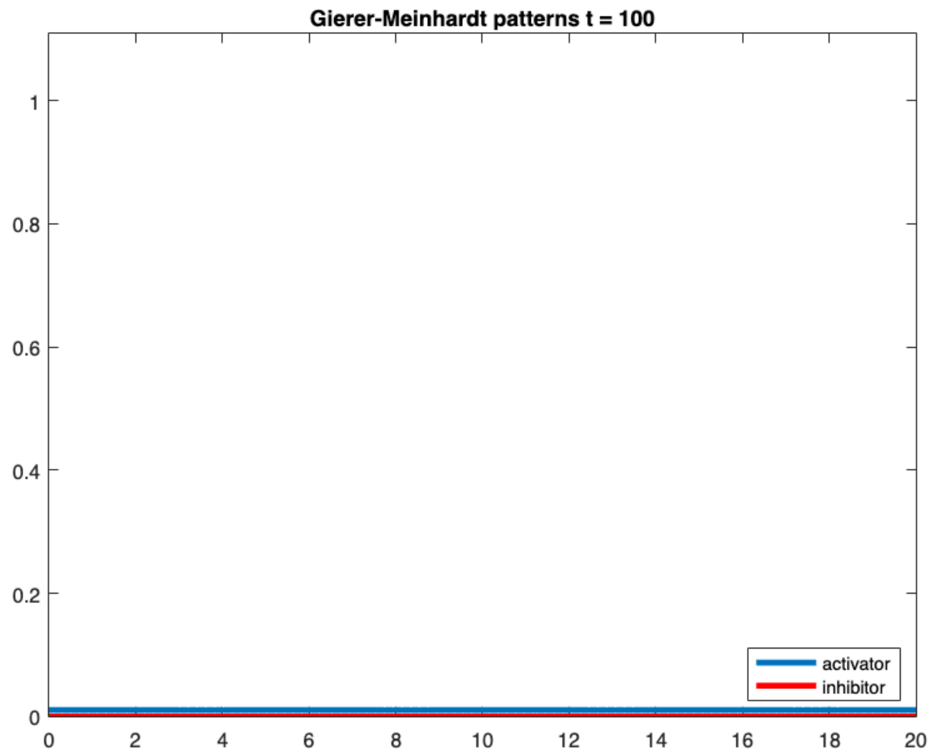
20
Position



(i) The wavelength from the plot is ~3.25 units. Based on the prediction from question 4, it is ~2.09 units.

(ii) The steady fixed point occurs at (a, h) = (0.75, 0.575). The prediction from question 2 was (1, 1).

(iii)

(iv) The fixed point occurs at (0, 0). The Jacobian evaluated at (0, 0) shows that the fixed point is stable (see below).

Q5) $J = \begin{pmatrix} \frac{2a}{h} - 1 - Dk^2 & \frac{-a^2}{h^2} \\ 2a\omega & -\omega - k^2 \end{pmatrix}$

$J(0,0) = \begin{pmatrix} -1-Dk^2 & 0 \\ 0 & -\omega-k^2 \end{pmatrix}$

$$\begin{aligned} tr_J &= -1 - Dk^2 - \omega - k^2 \\ &= -1 - (0.4)k^2 - 0.5 - k^2 \\ &= -(1.5 + 1.4k^2) \\ &< 0 \end{aligned}$$

So the fixed pt is stable.

Phase diagram

Alter P(1) = D and P(2) = ω to move around the phase plane & draw (by hand or by comouter) a phase diagram for this Gierer-Meinhardt system with D on the x-axis and ω on the y-axis.

```
tmax = 200;
ctr = 1;
for x1 = 0.05:0.1:0.5
    for x2 = 0.05:0.5:5
        P = [x1; x2; 0.01];     %trivial solution
        disp(P);
        sol = GM_solve_record_JH(P, tmax, delt, L, 'test_pattern2.avi');

        t = linspace(0,tmax,tmax/delt);
        x = linspace(0,L,200); %the mesh on which we solve (200 space
points)
        figure(100+ctr);
        for n = 2:2:length(t)
            set(gca, 'FontSize', 18, 'LineWidth', 1); %<- Set properties
            plot( x , sol(n,:,1), 'LineWidth',3);
            hold on
            plot( x , sol(n,:,2), 'r', 'LineWidth',3);
            hold off
            legend('activator', 'inhibitor', 'Location', 'SouthEast');
            title(strcat('Gierer-Meinhardt patterns t =' , sprintf(' %d ',
ceil(t(n)))));
            axis([0 L 0 max(max(max(sol(:,:,:))))+0.1])
        end
        ctr = ctr + 1;
    end
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end
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    0.0500
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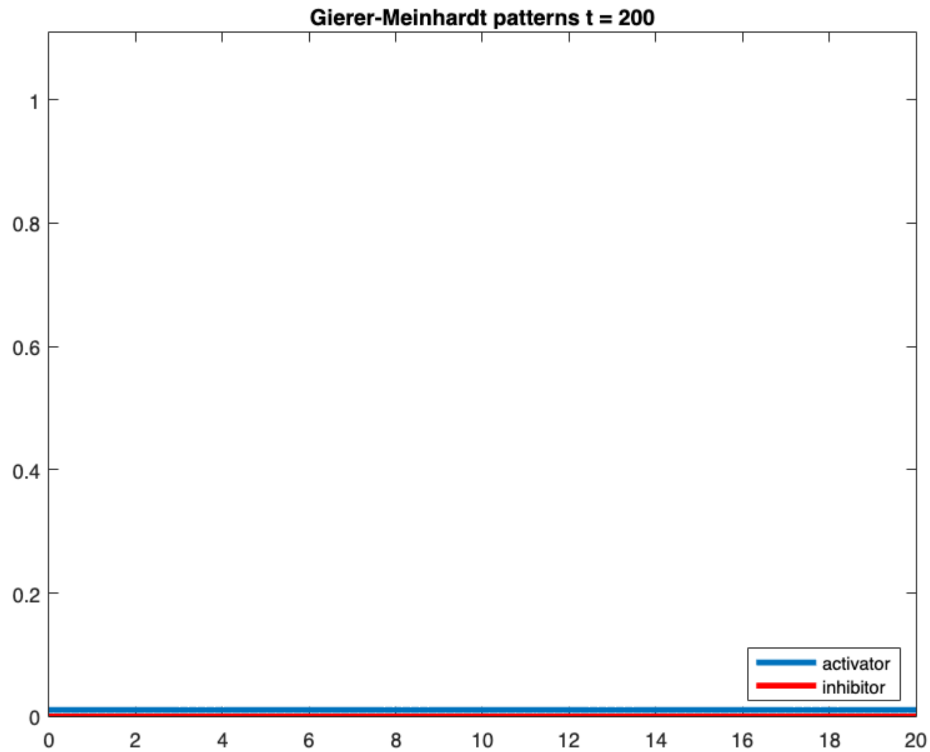


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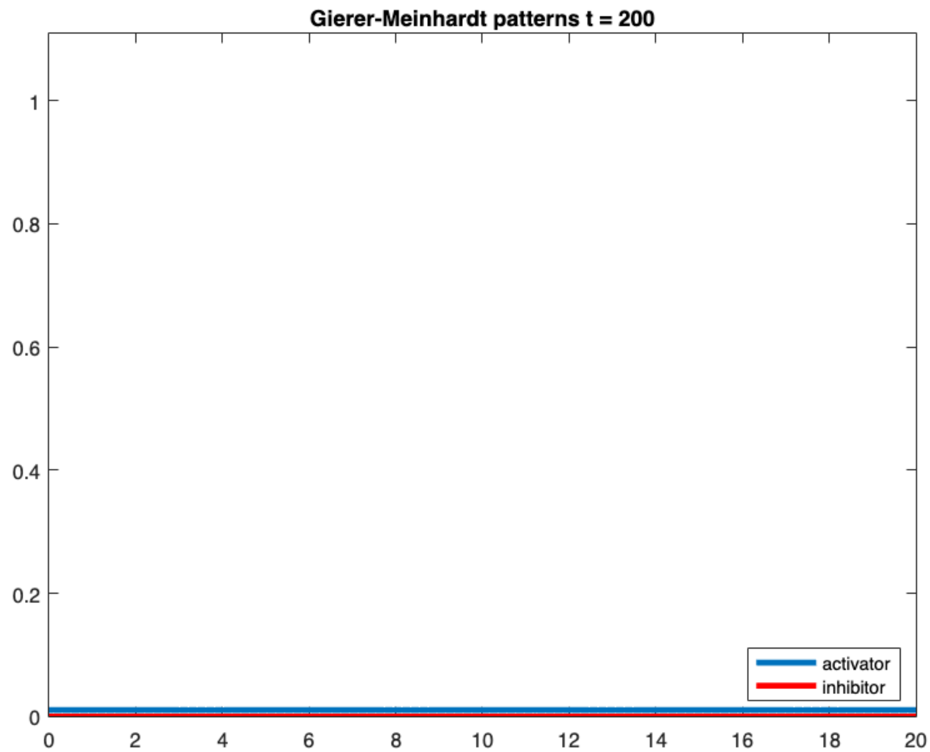


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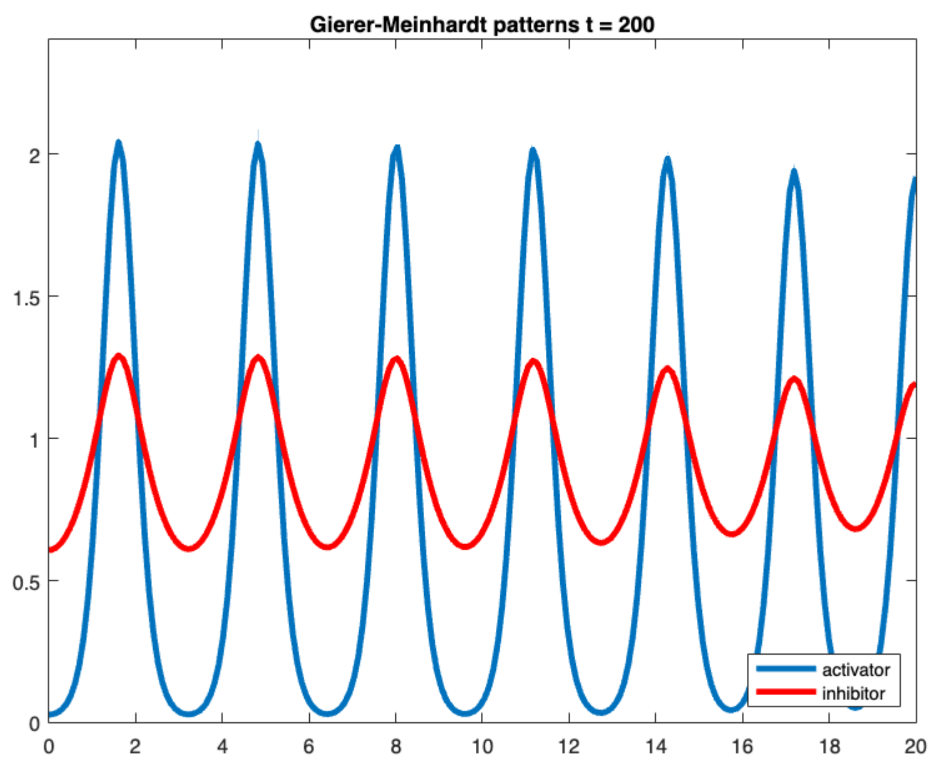

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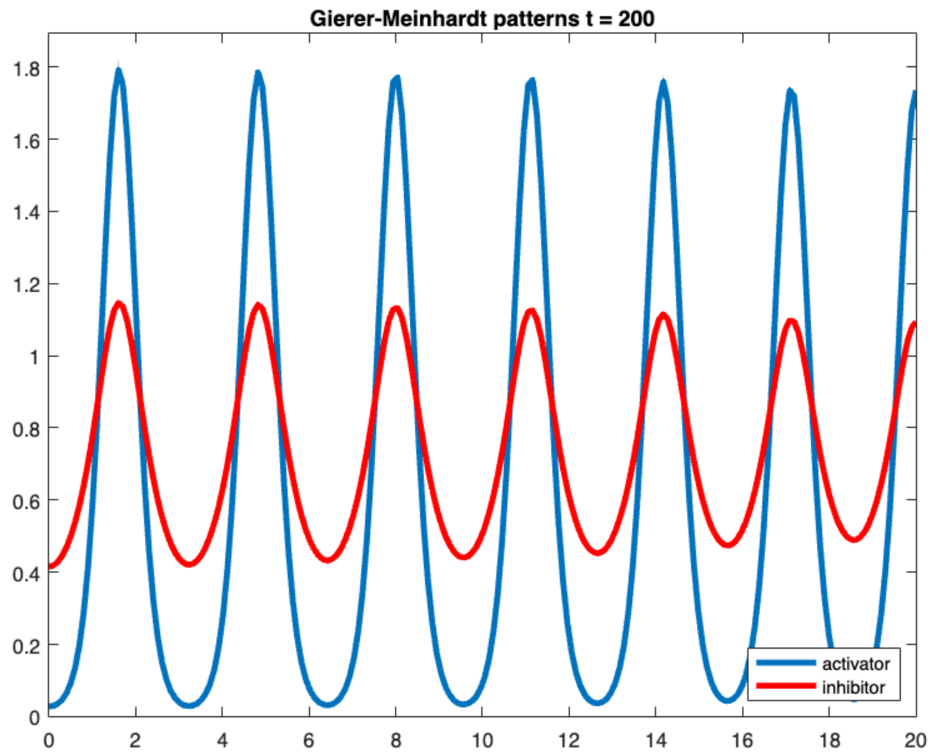


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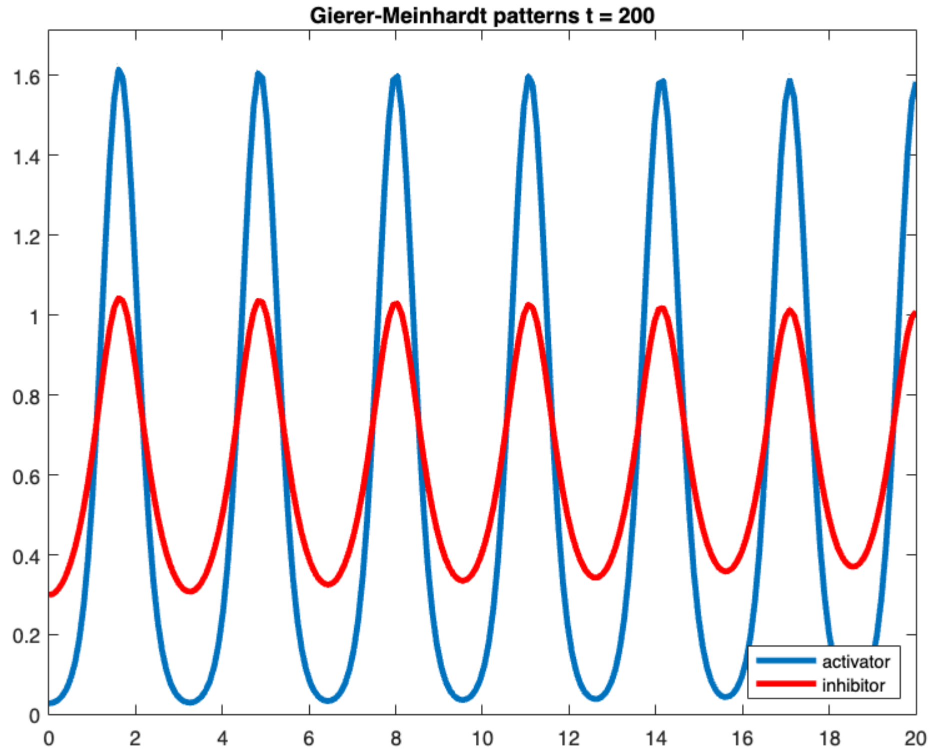


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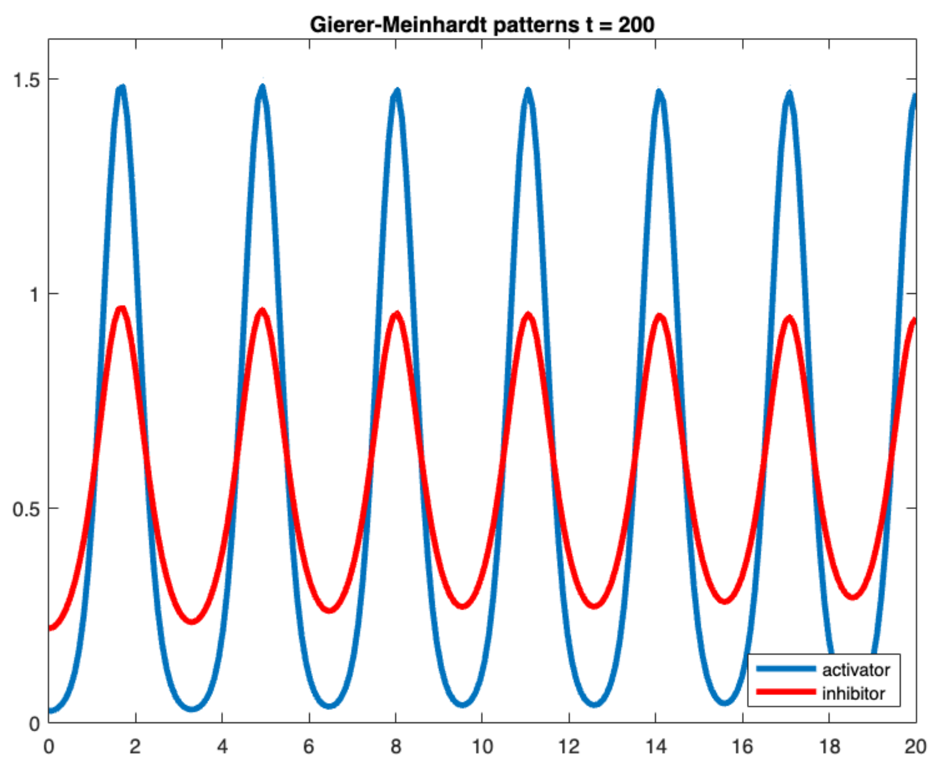

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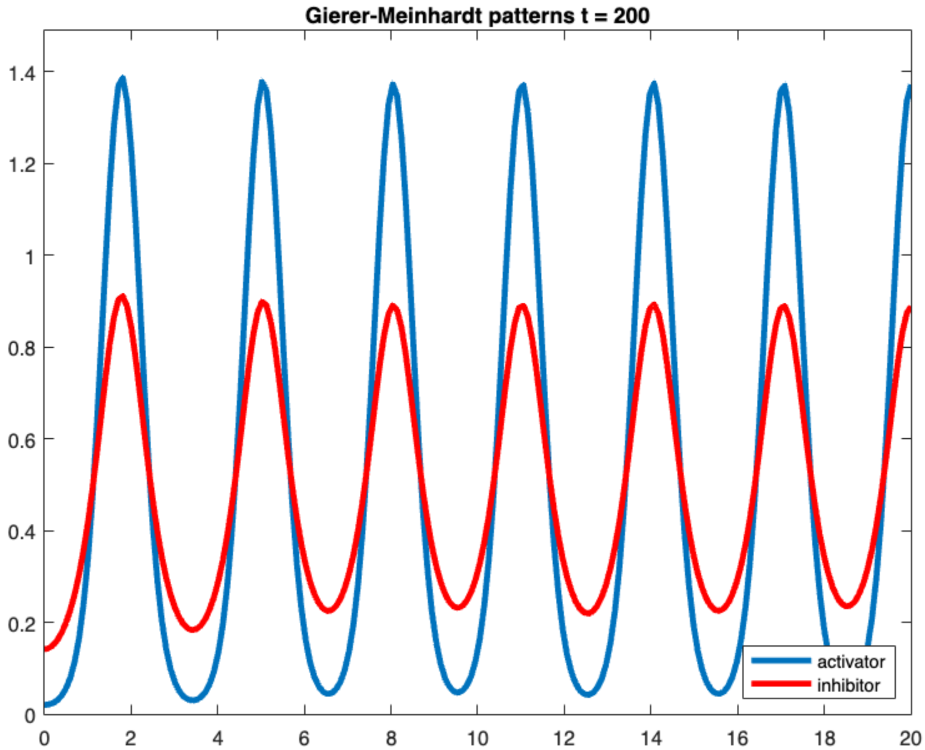


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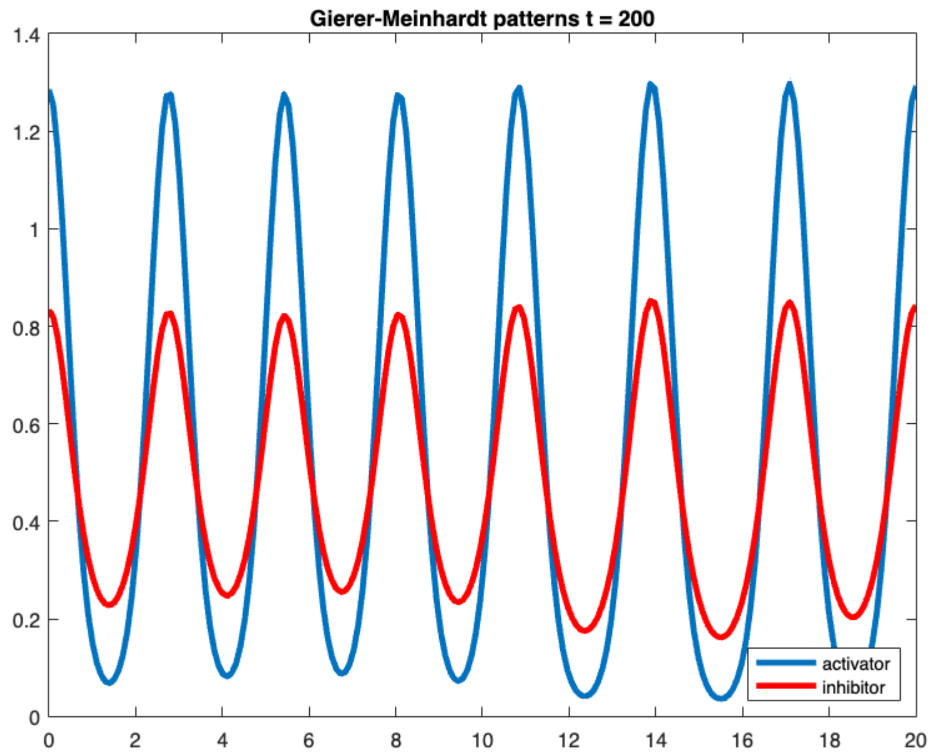


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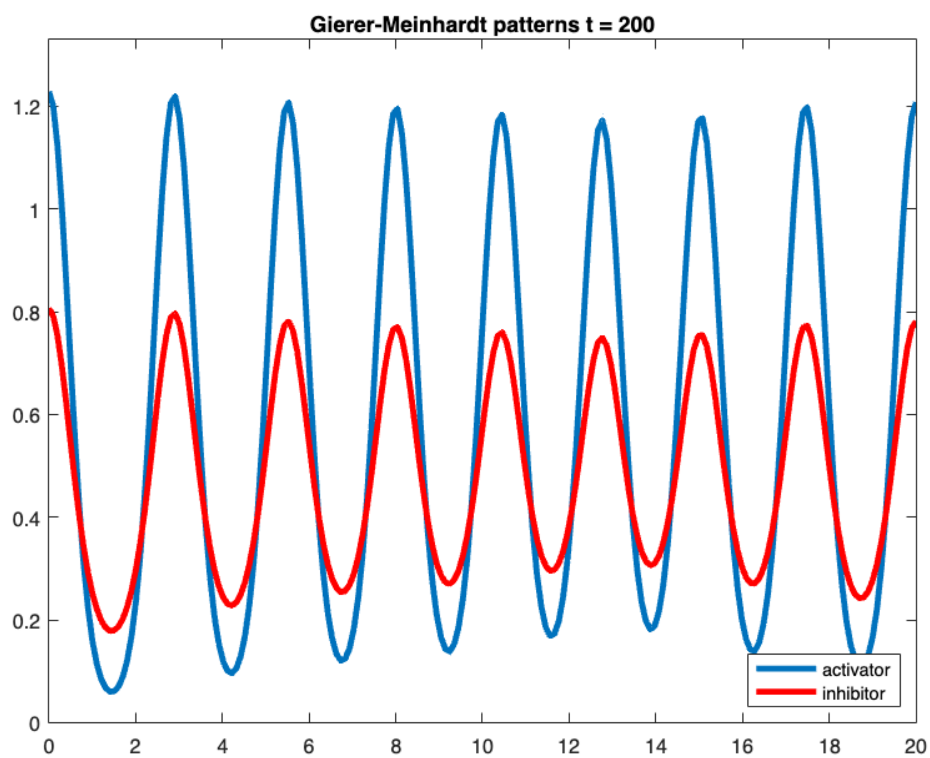

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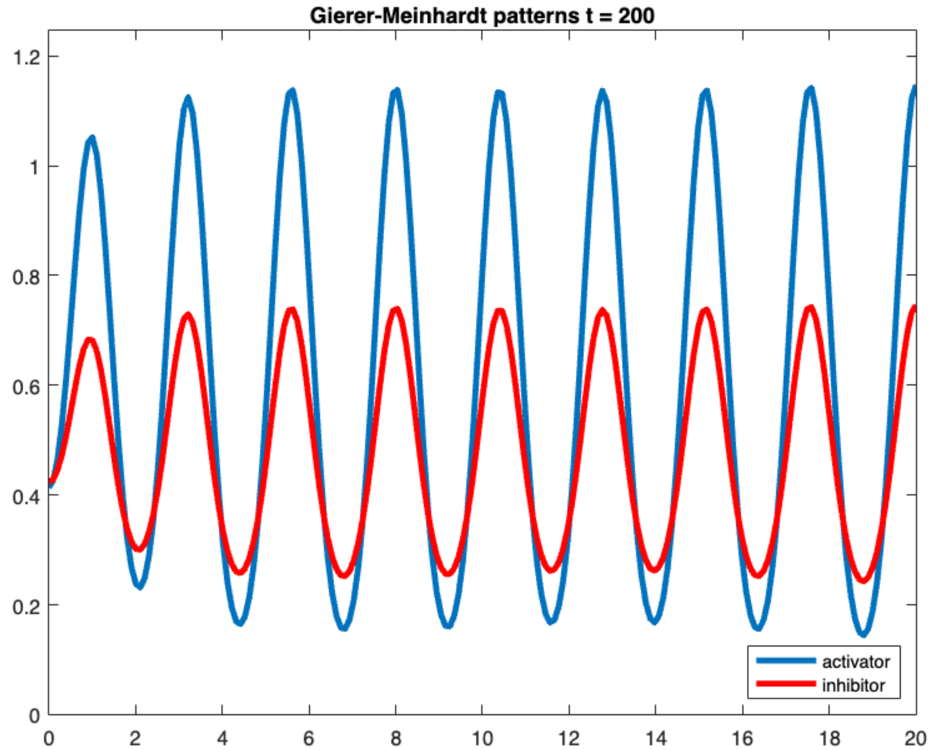


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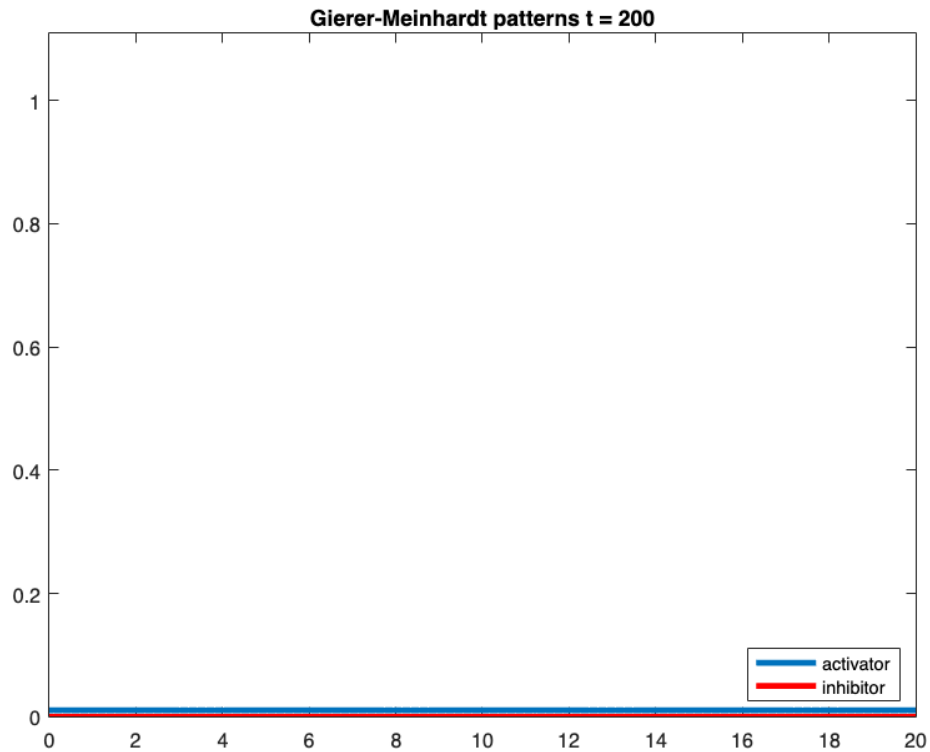


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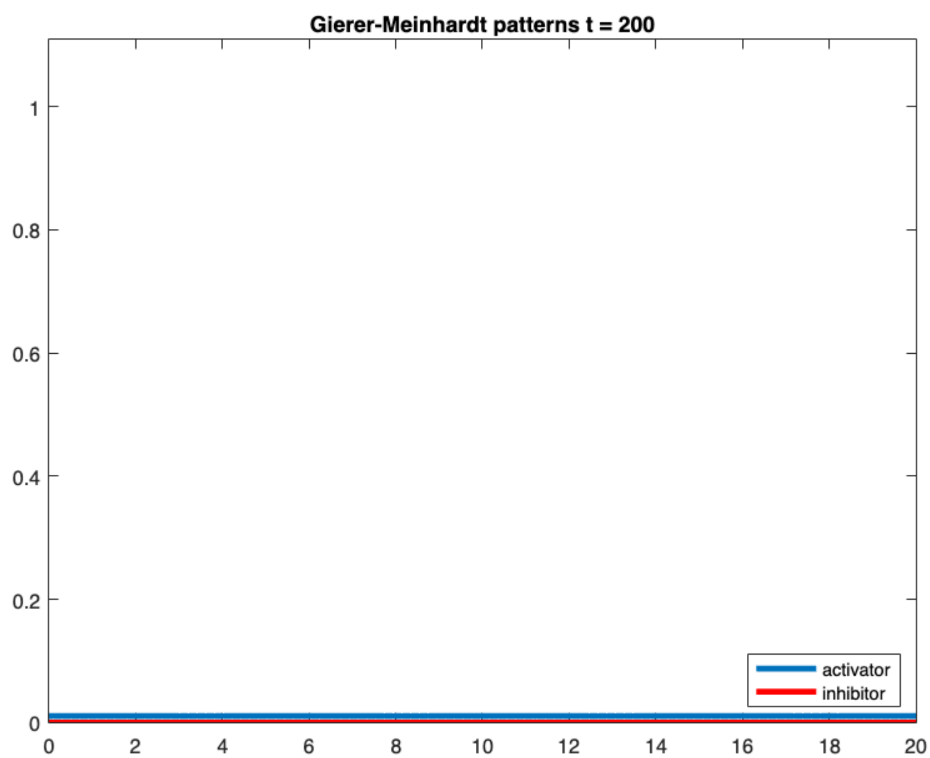

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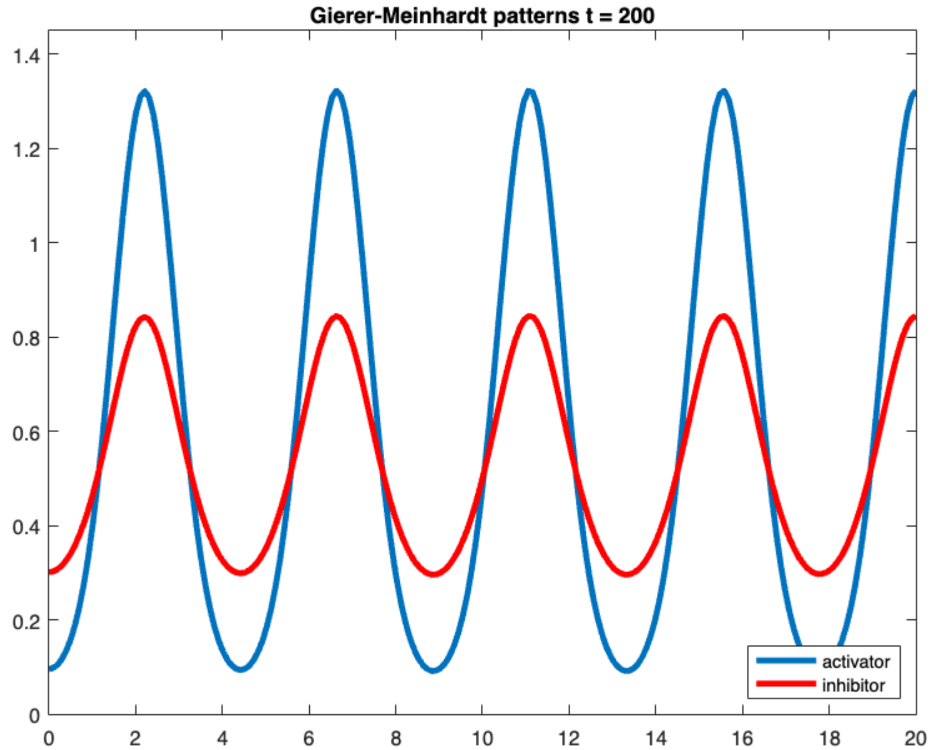


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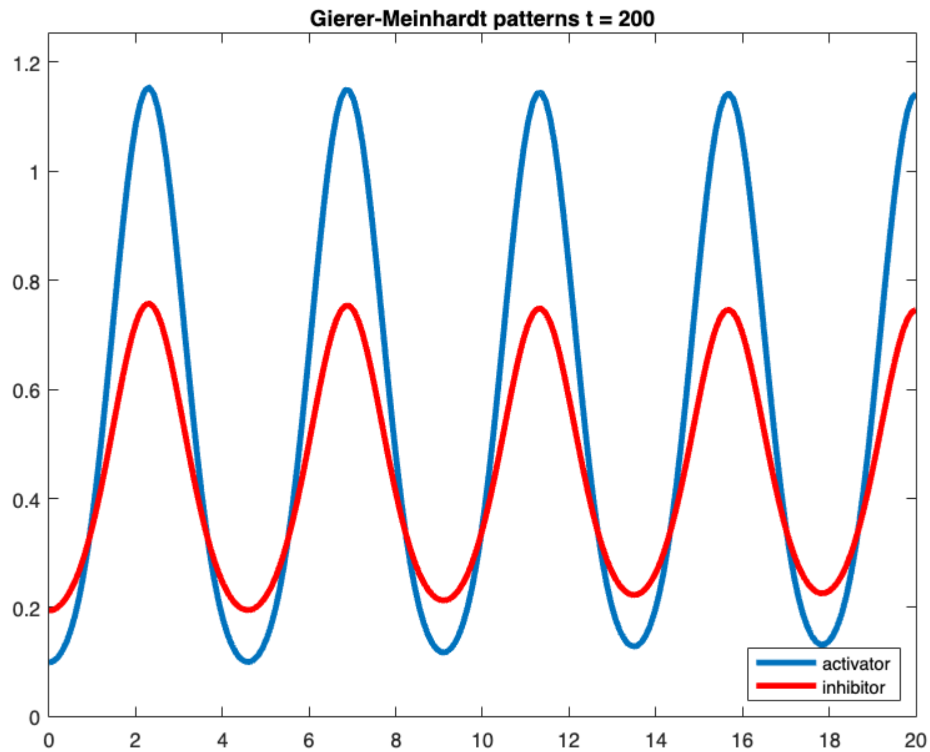


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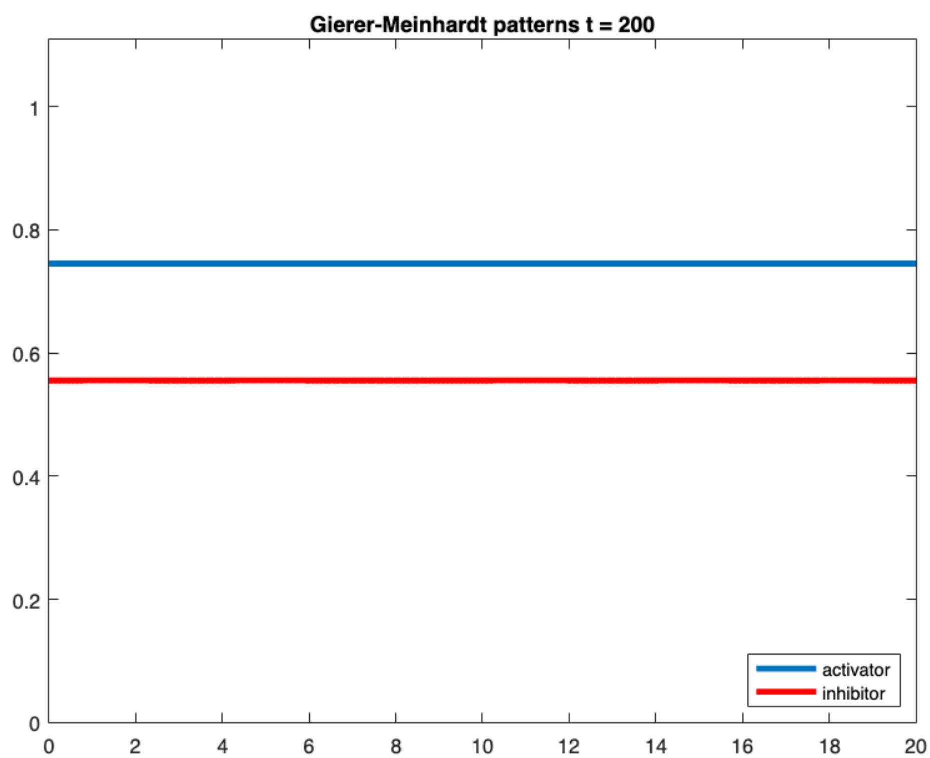

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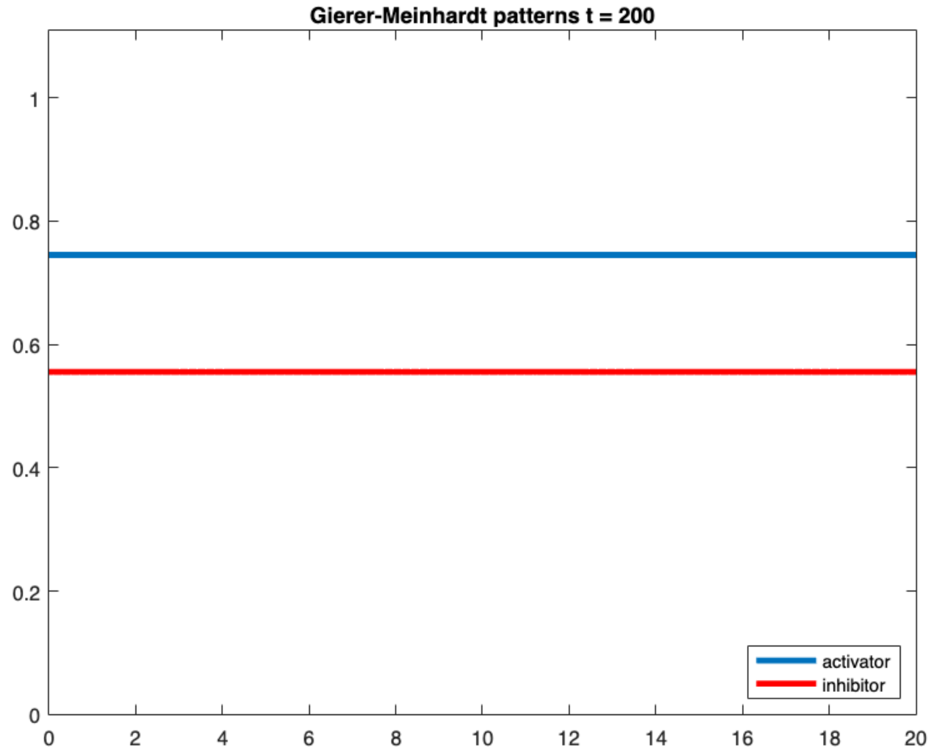


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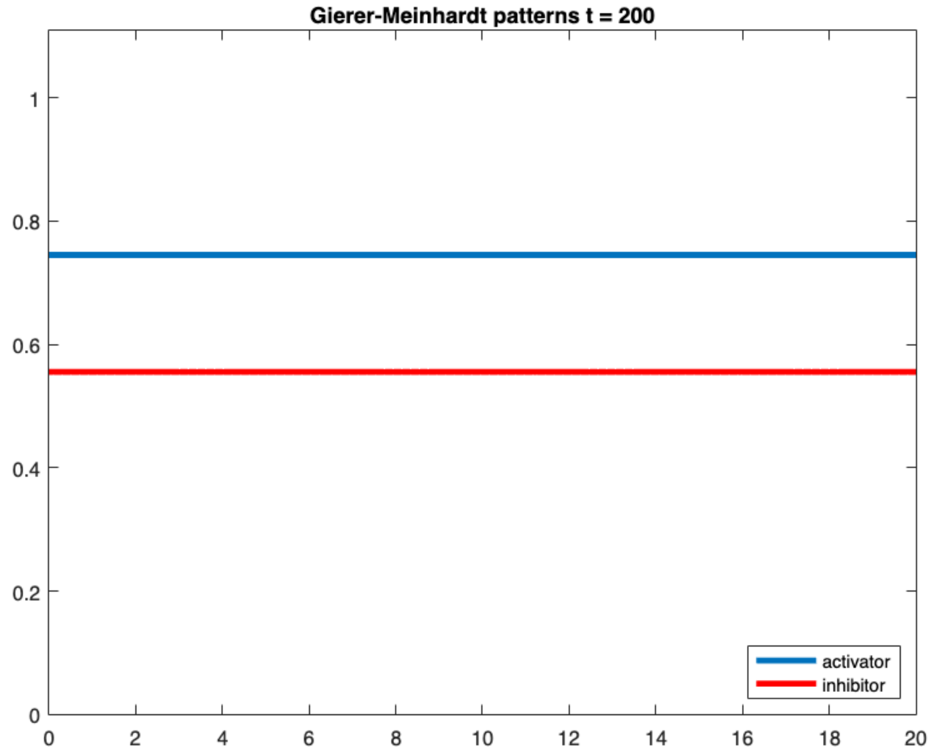


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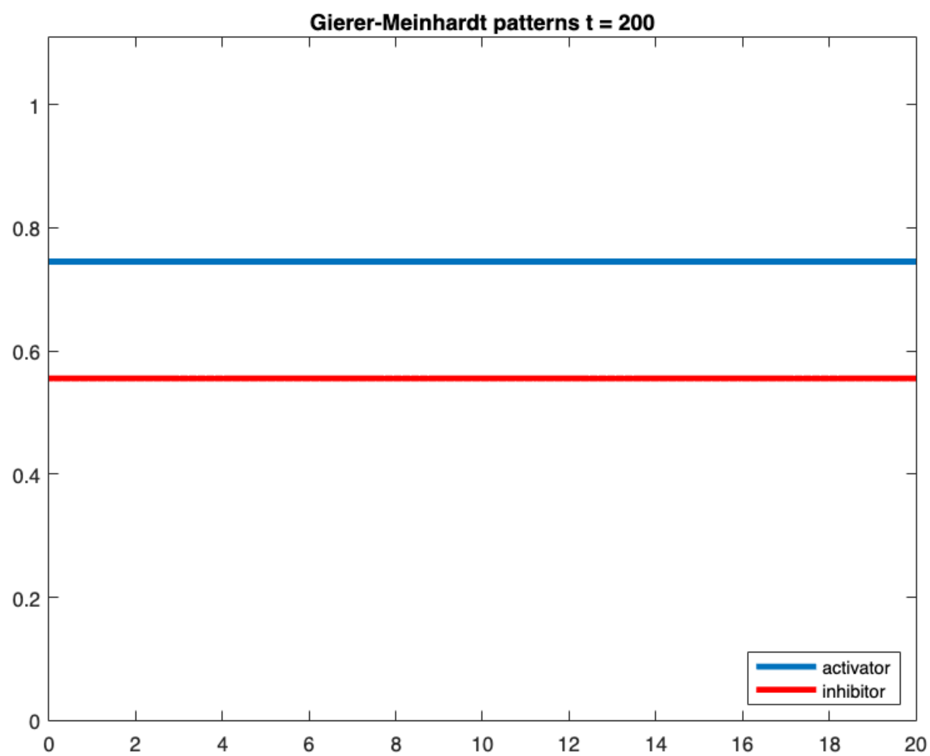

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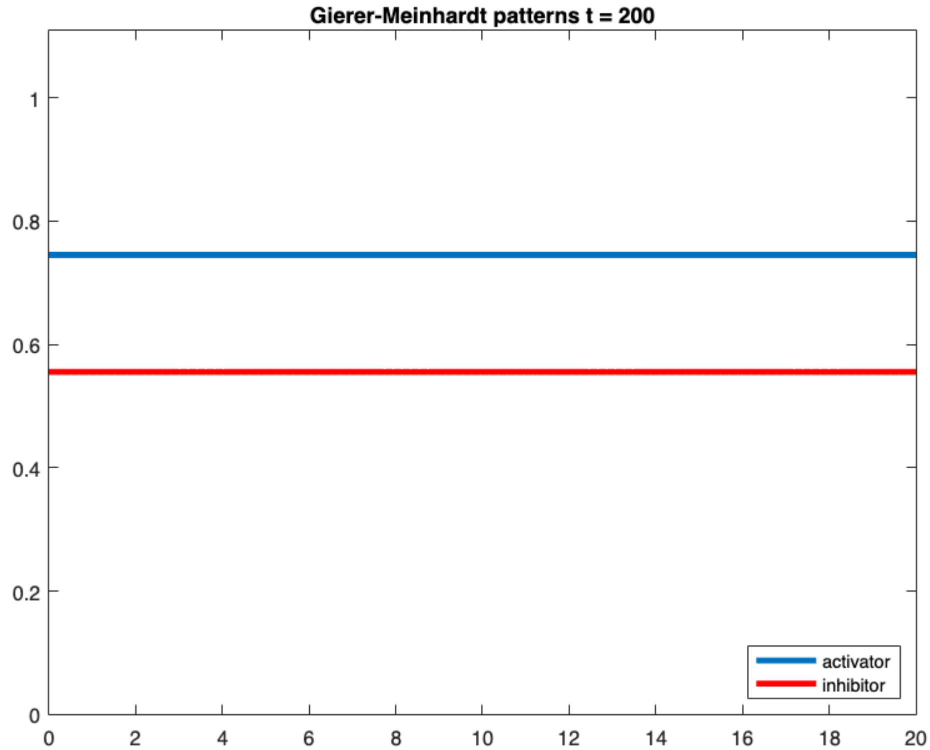


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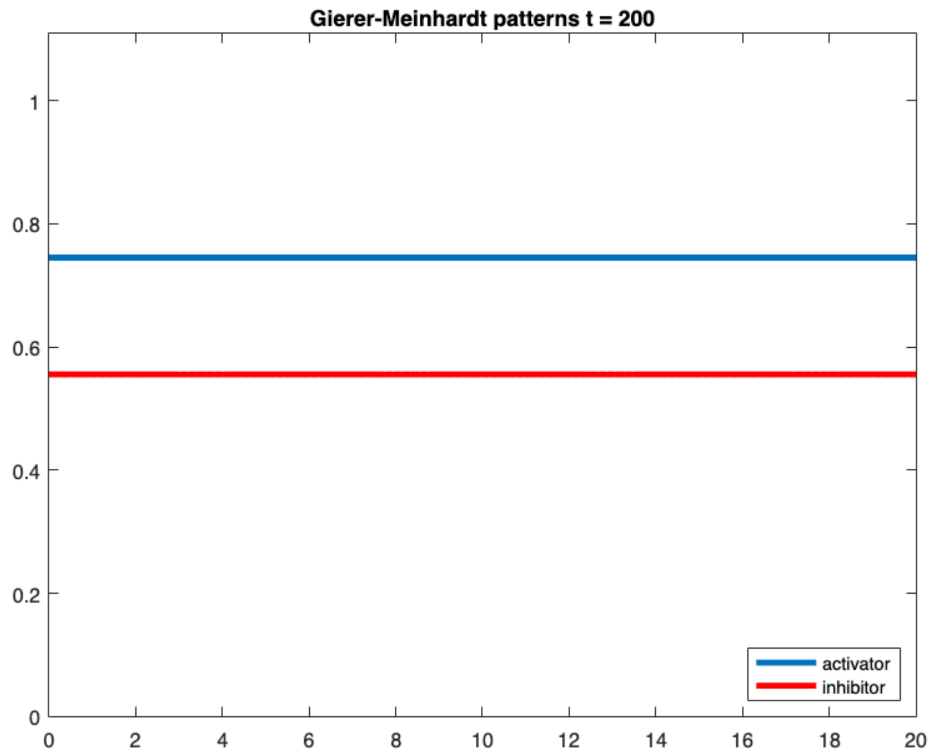


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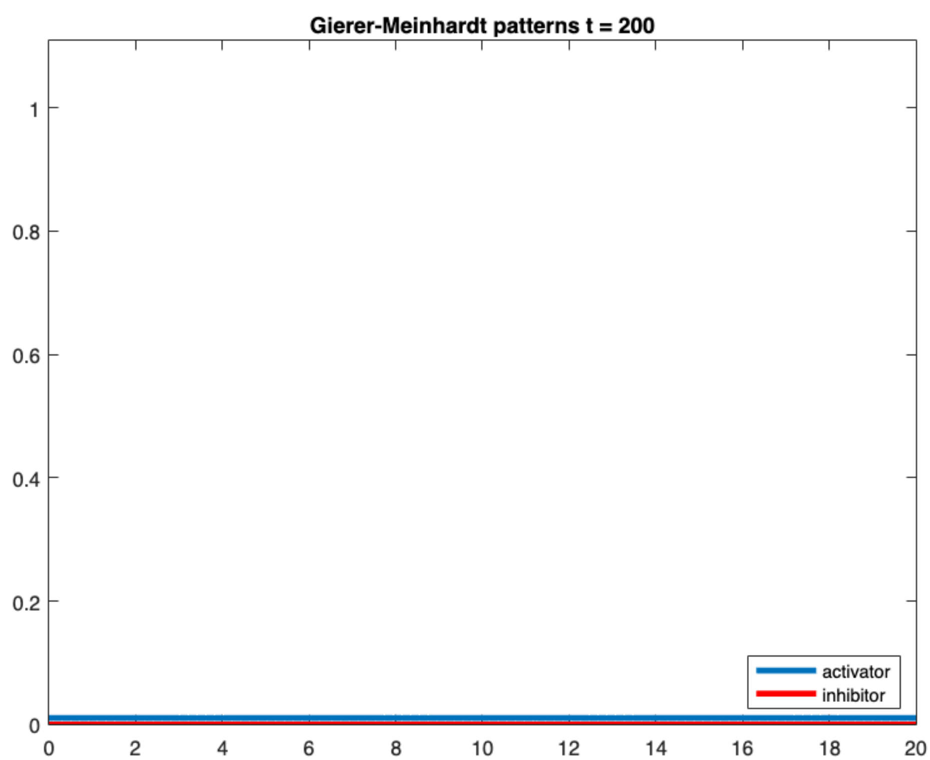

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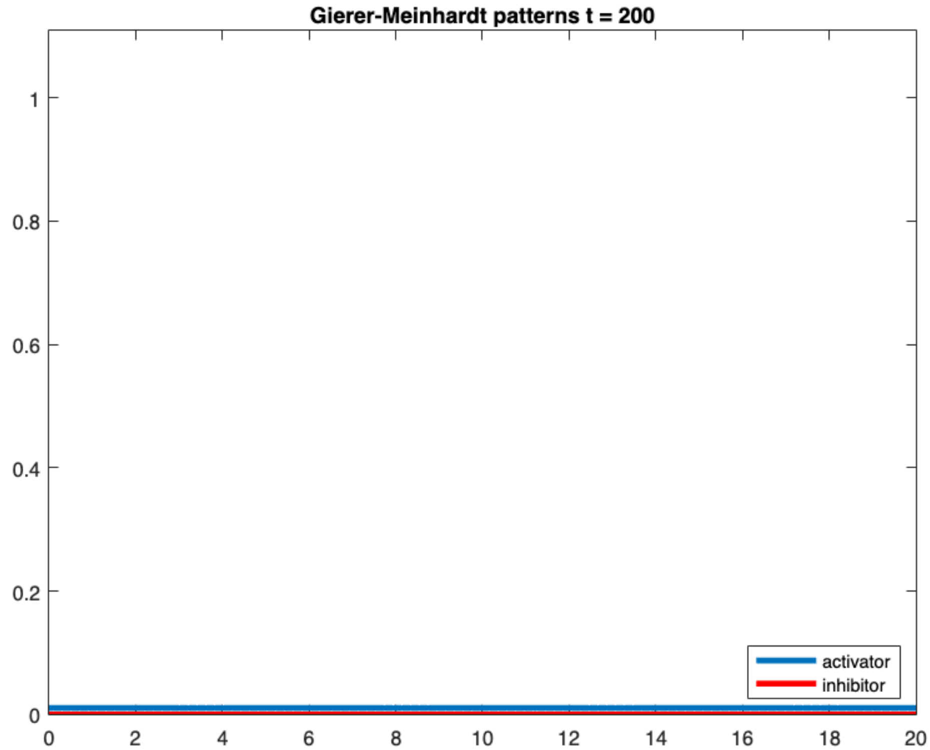


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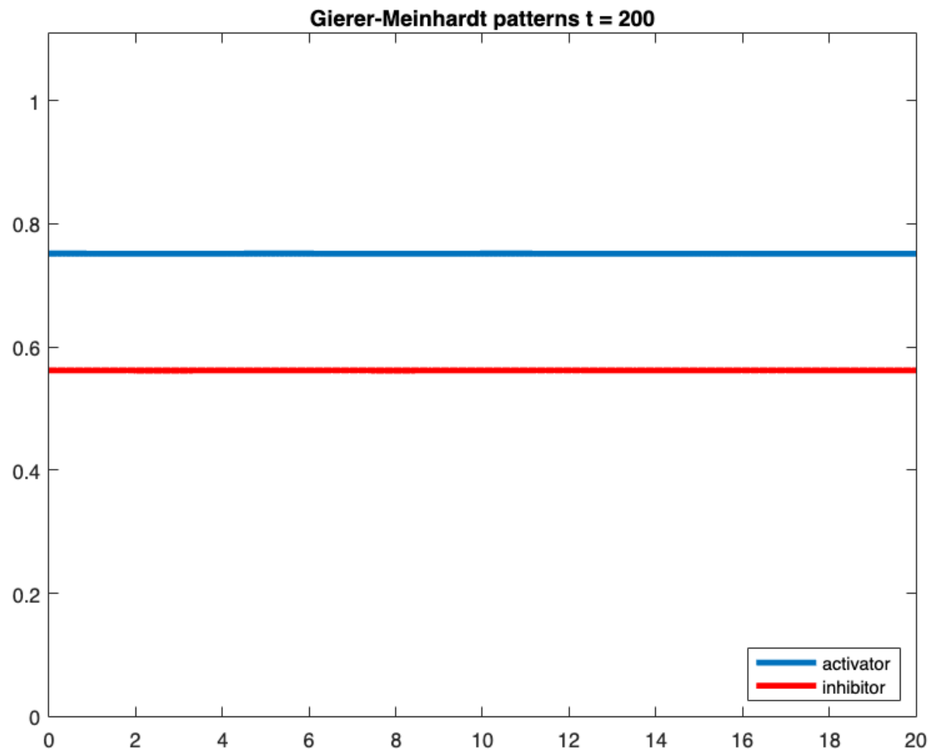


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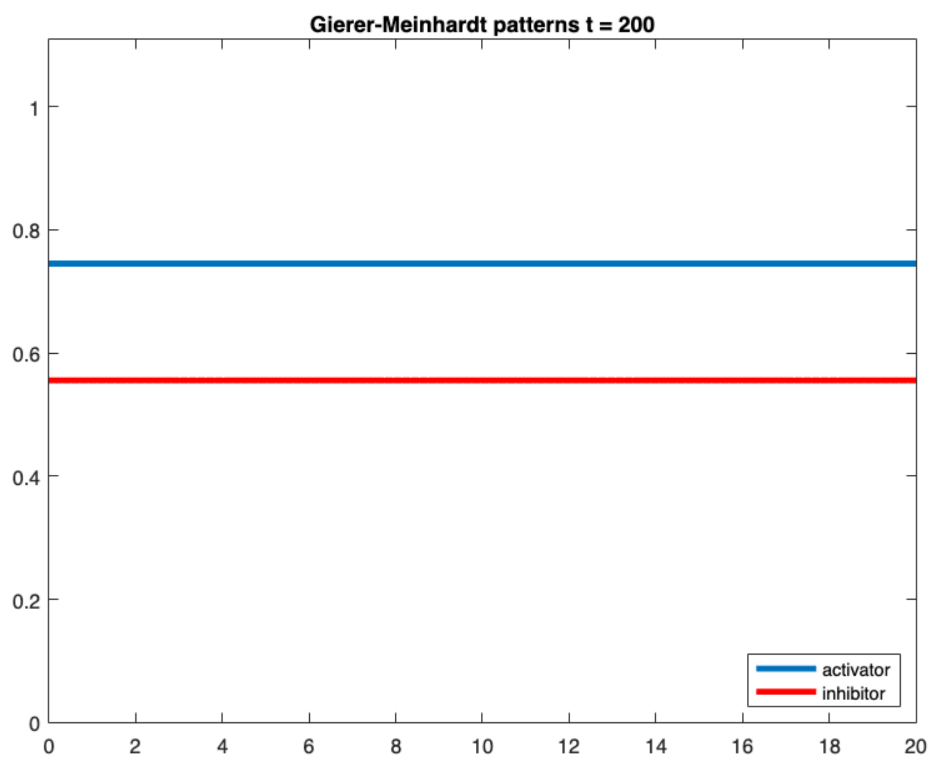

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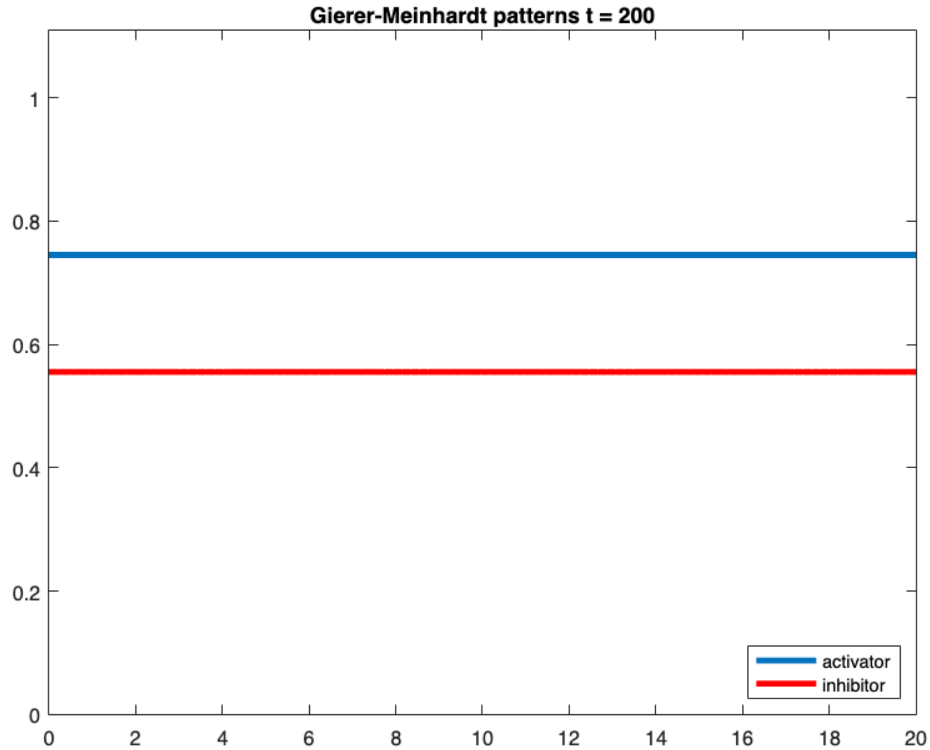


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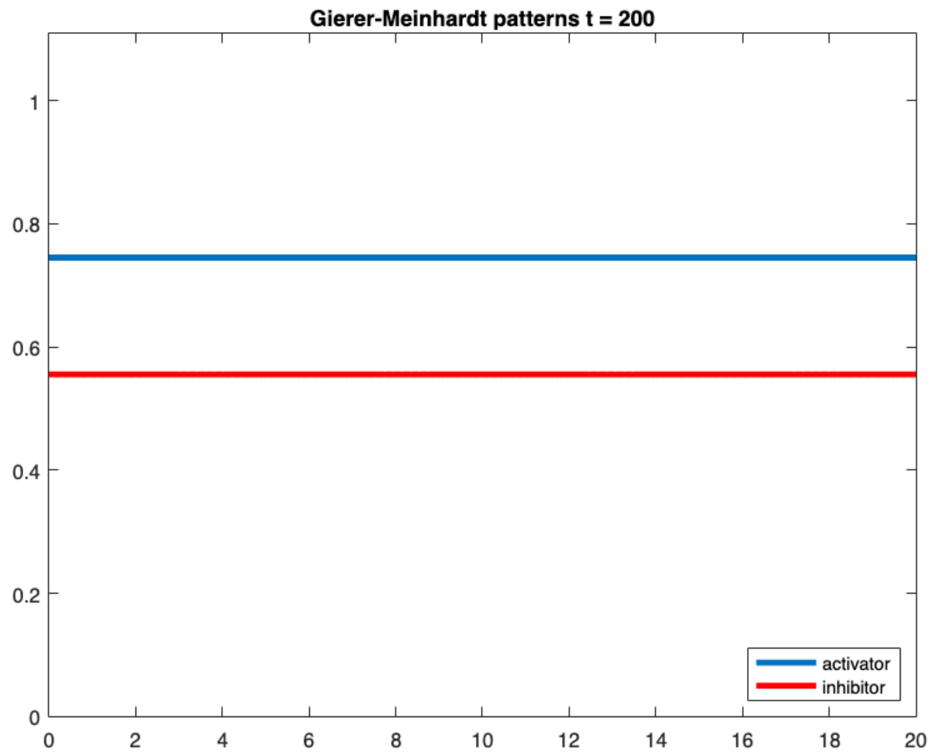


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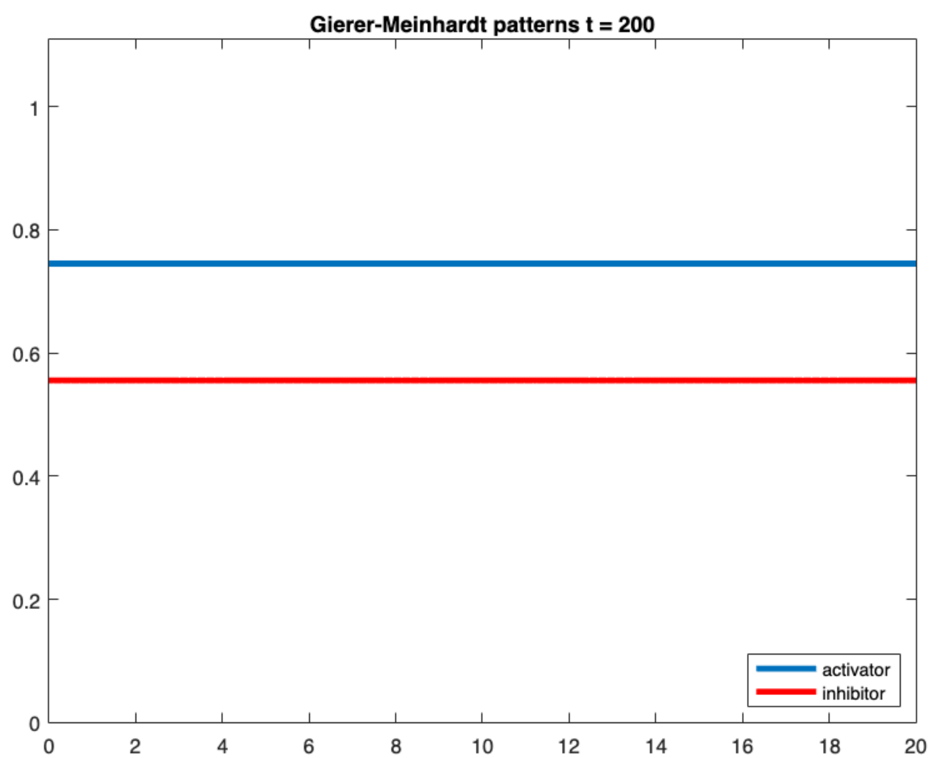

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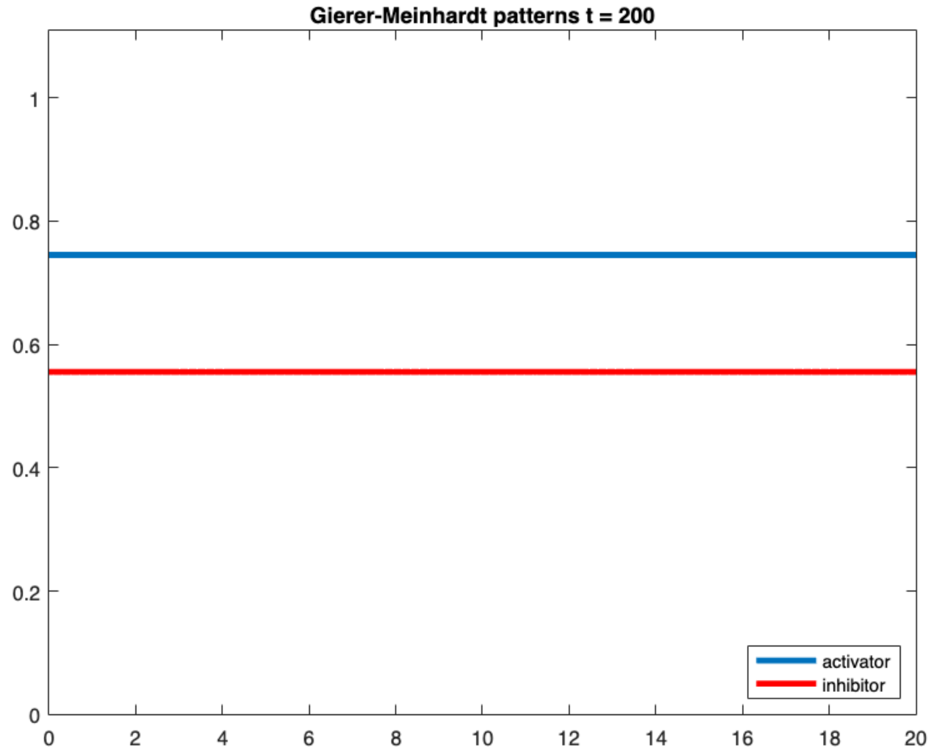


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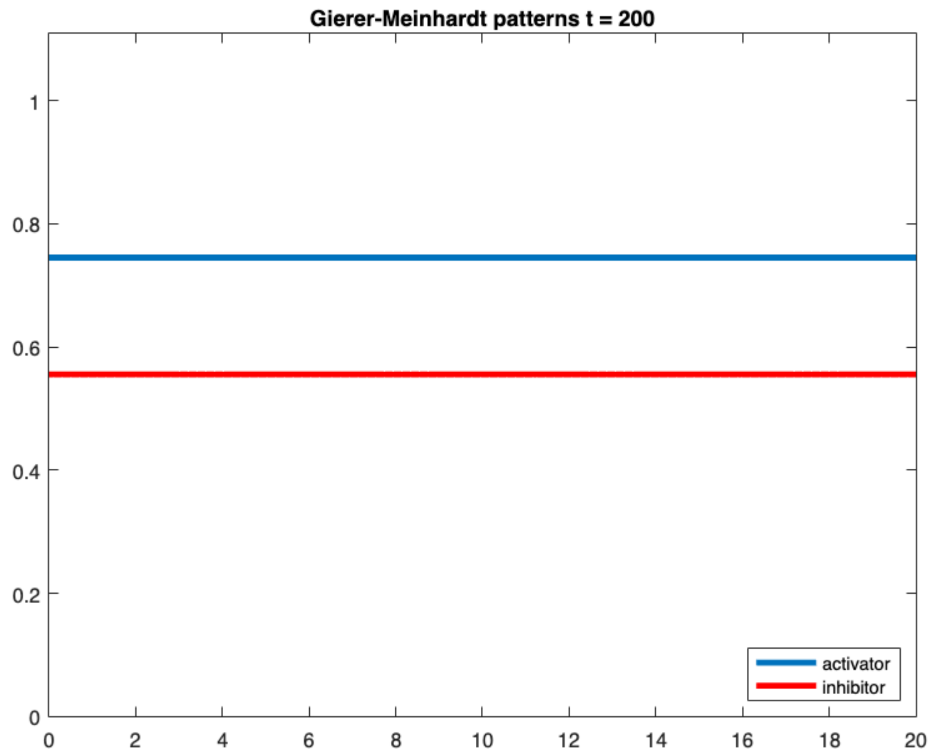


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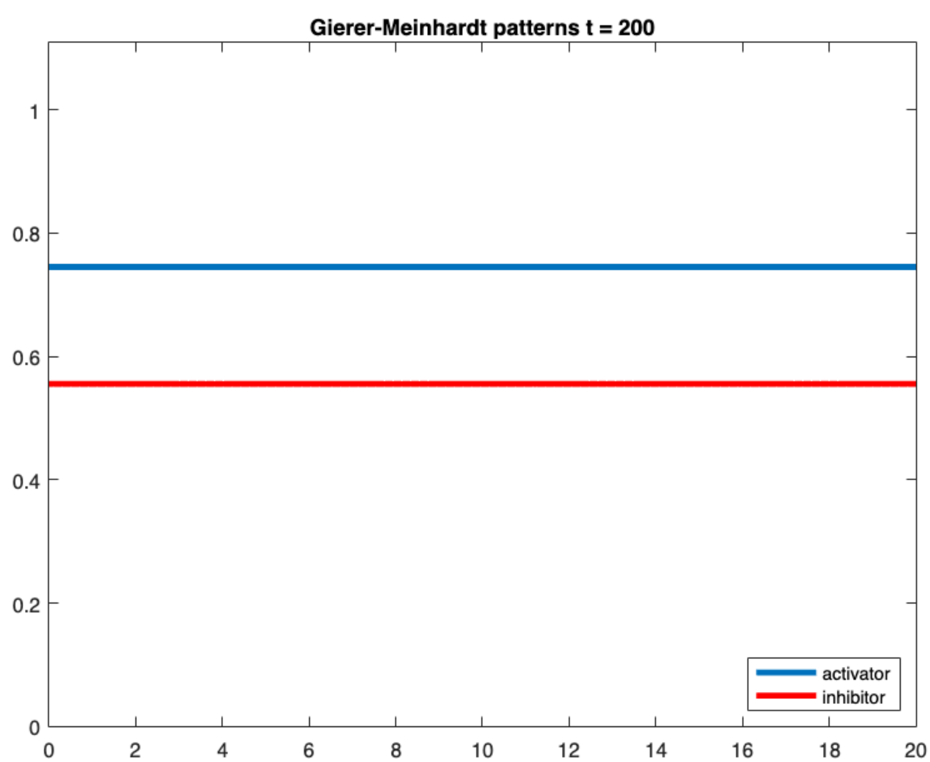

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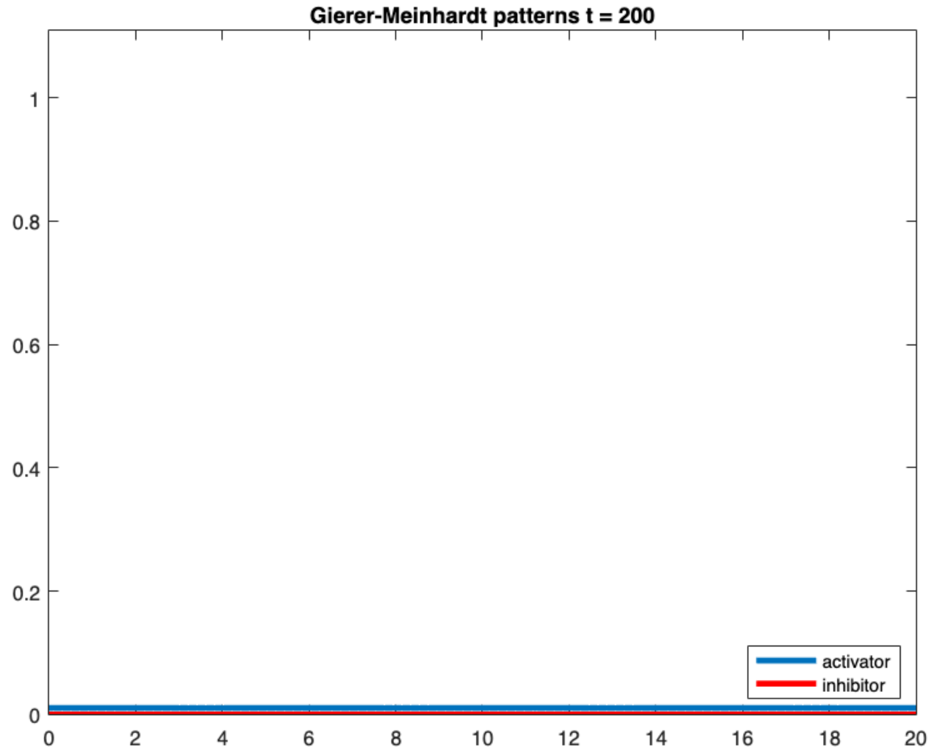


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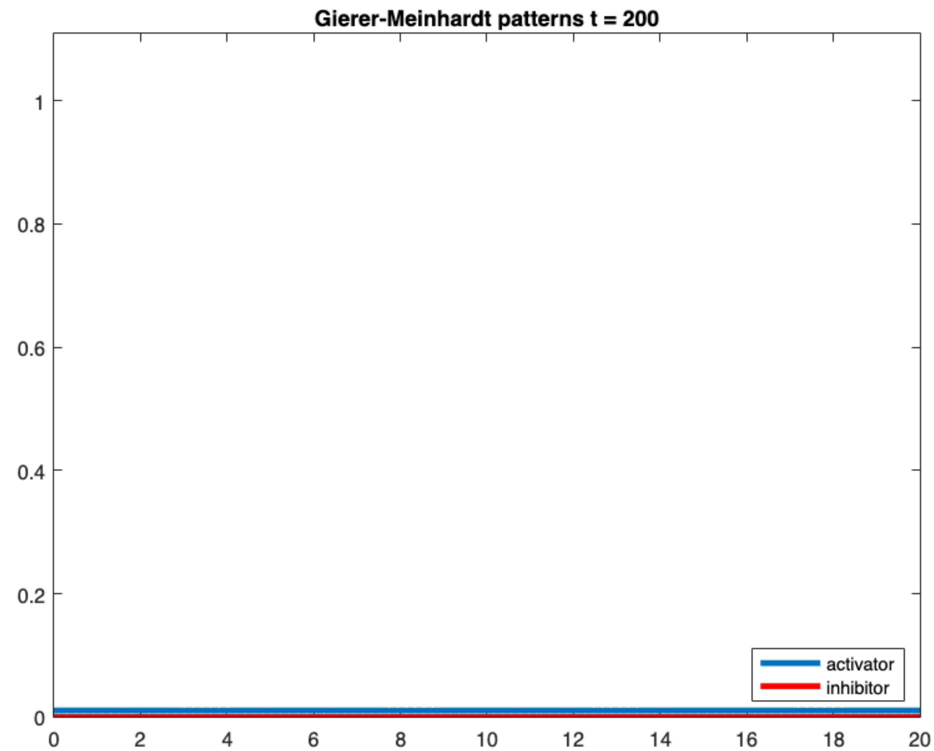


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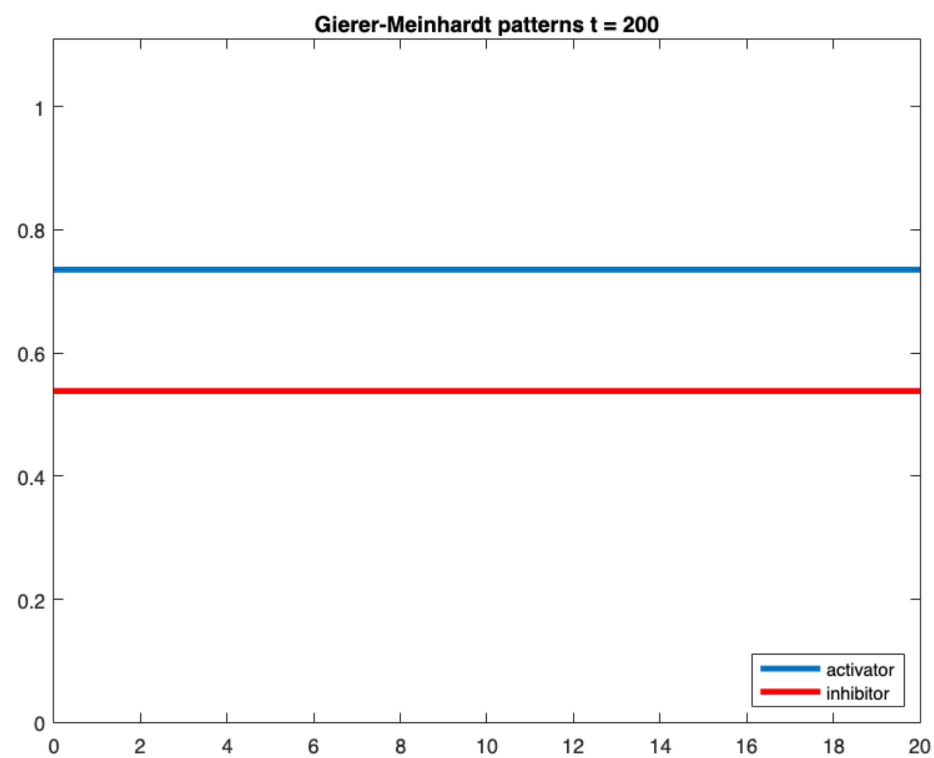

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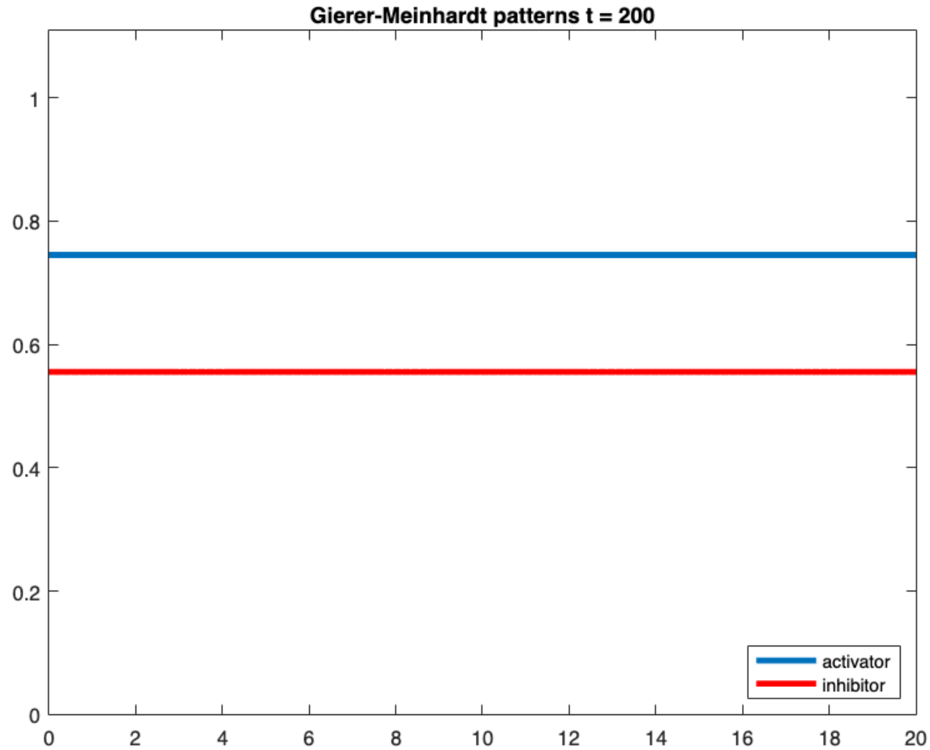


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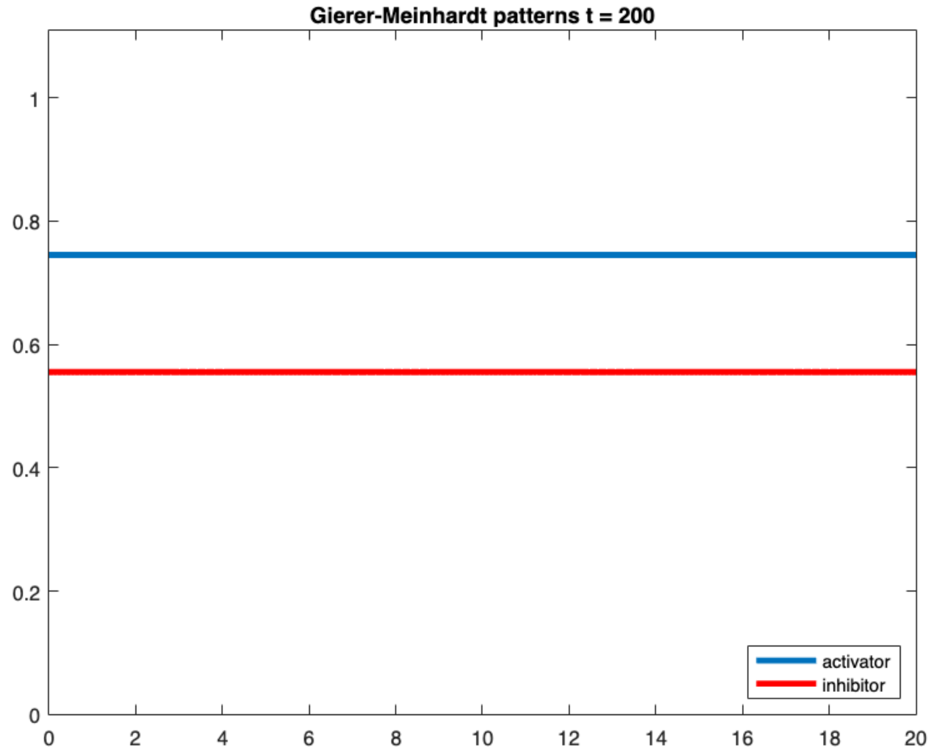


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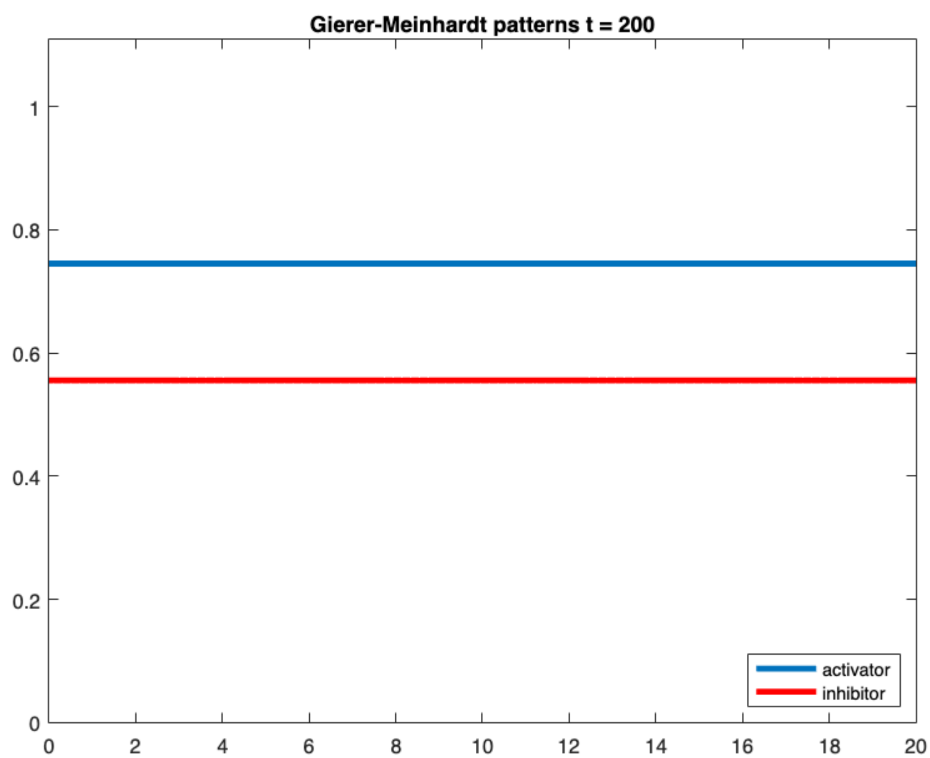

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0.0100
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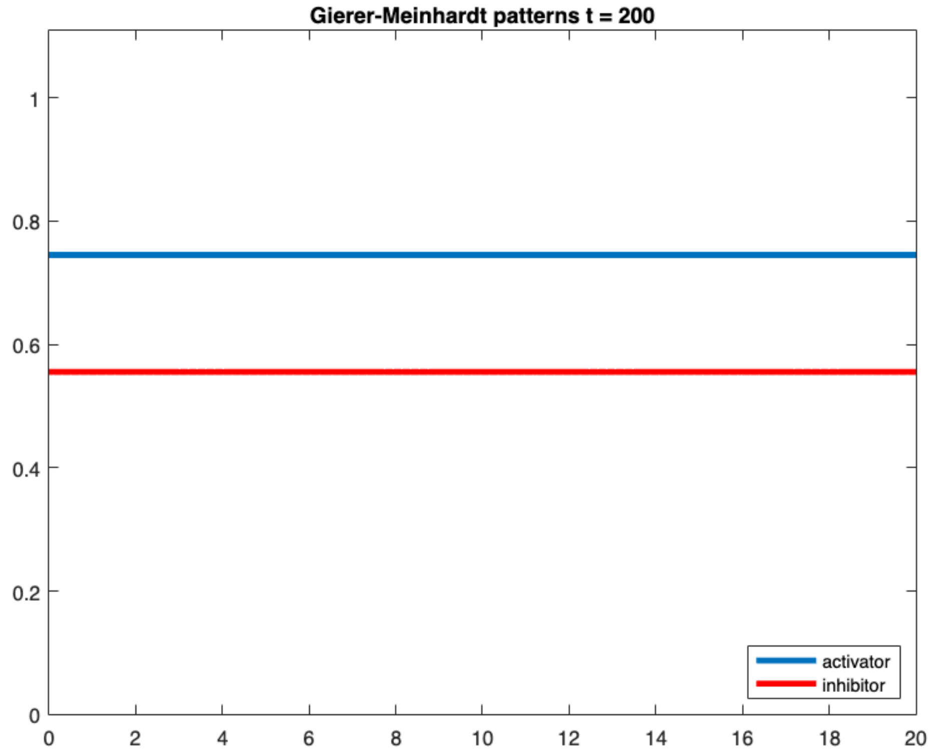
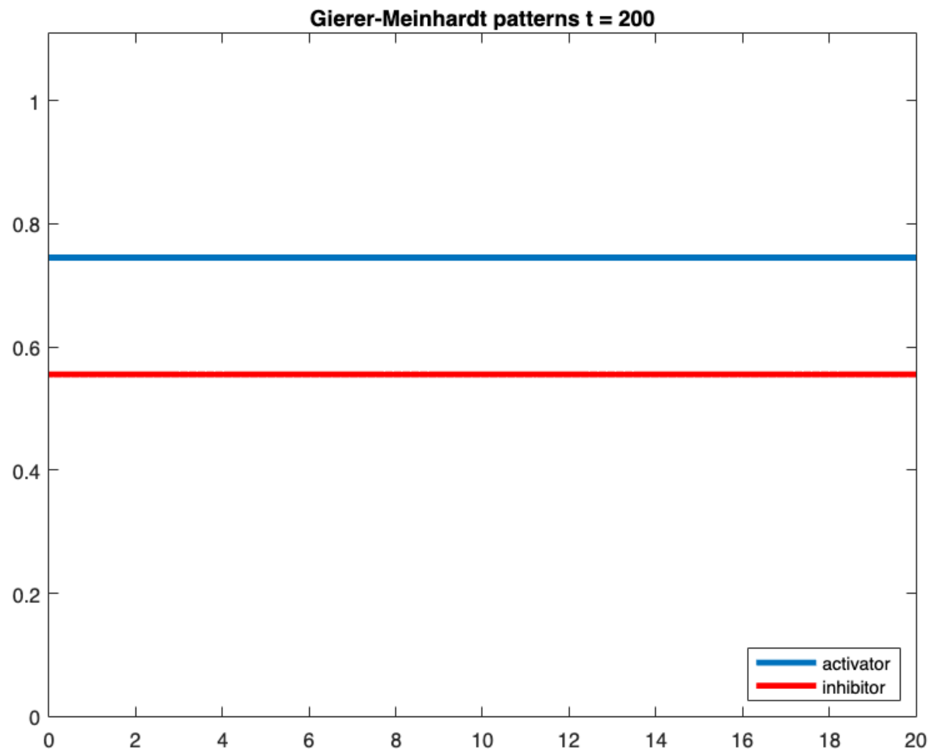
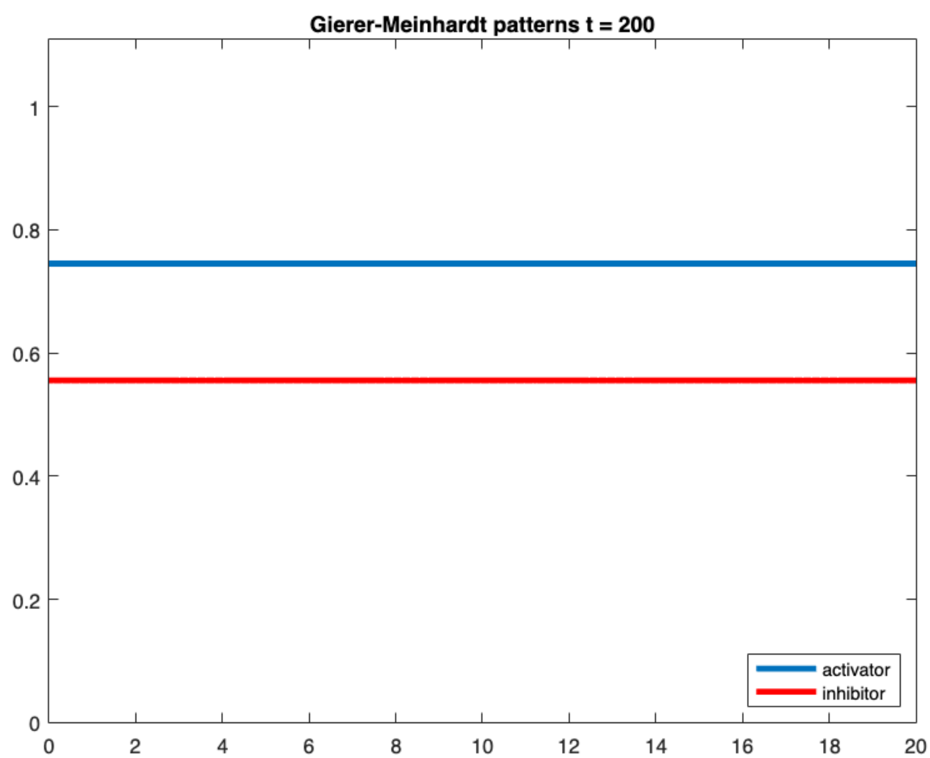


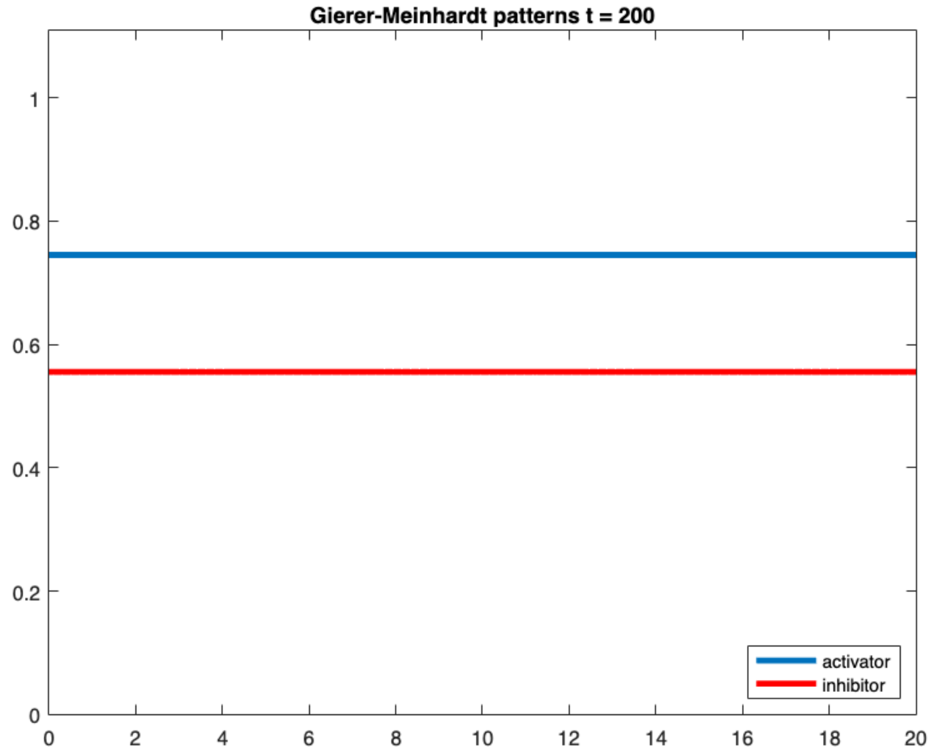

```
0.3500
3.5500
0.0100
```



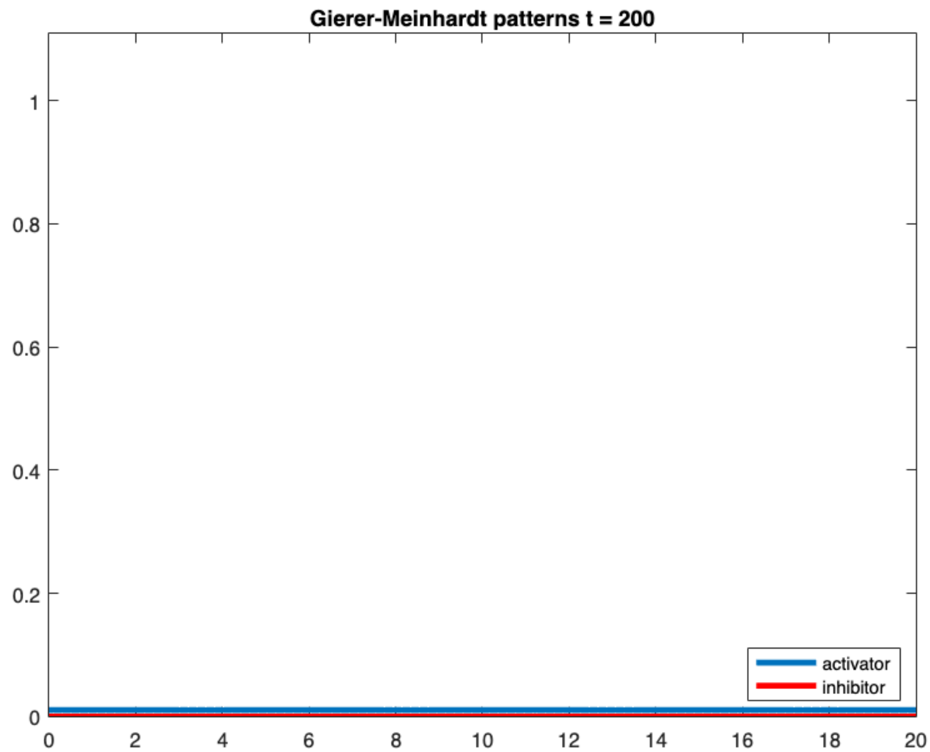

```
0.3500
4.0500
0.0100
```



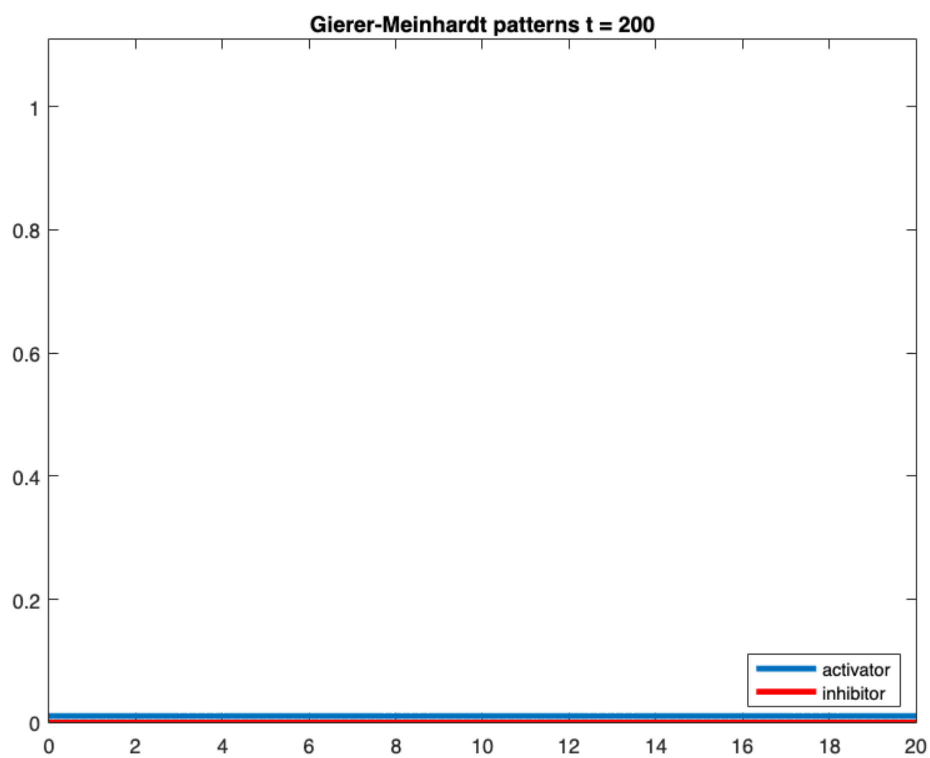
```
0.3500
4.5500
0.0100
```



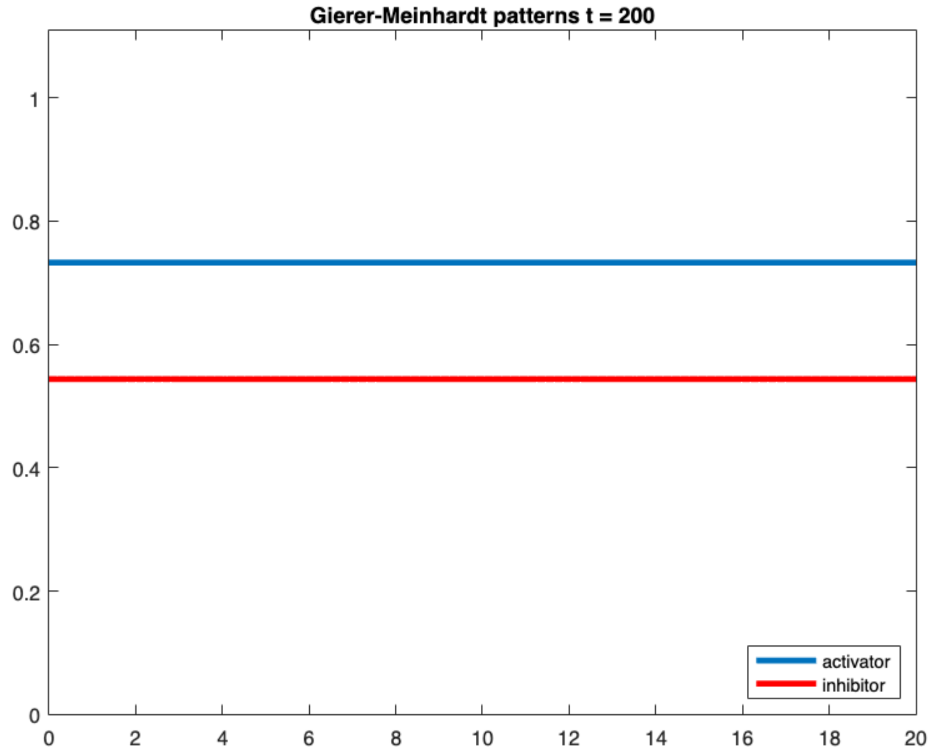

```
0.4500
0.0500
0.0100
```



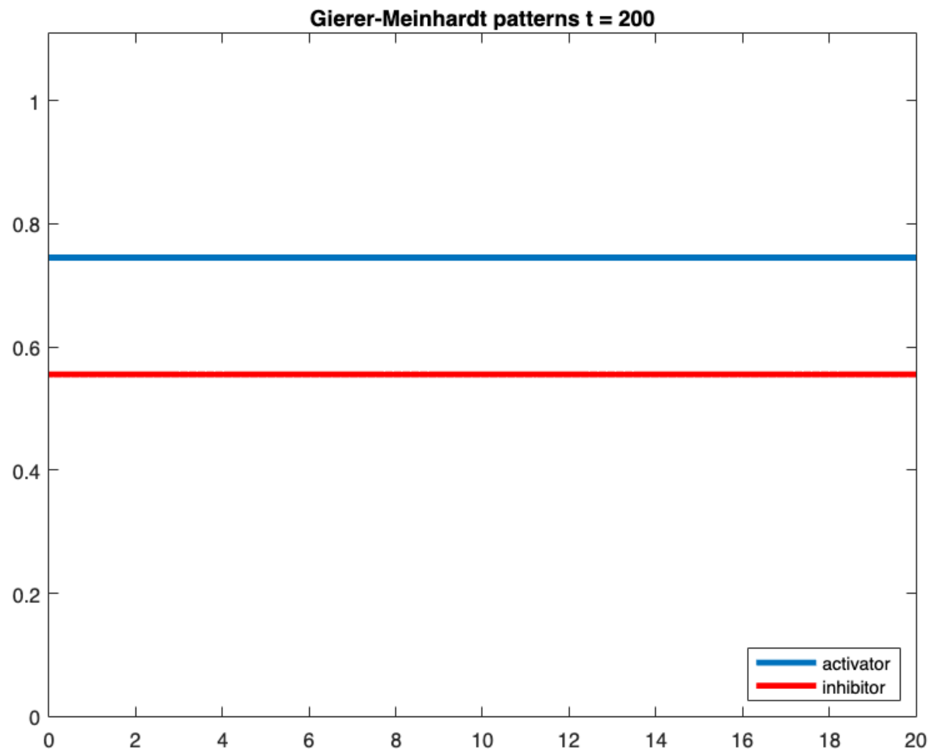

```
0.4500
0.5500
0.0100
```



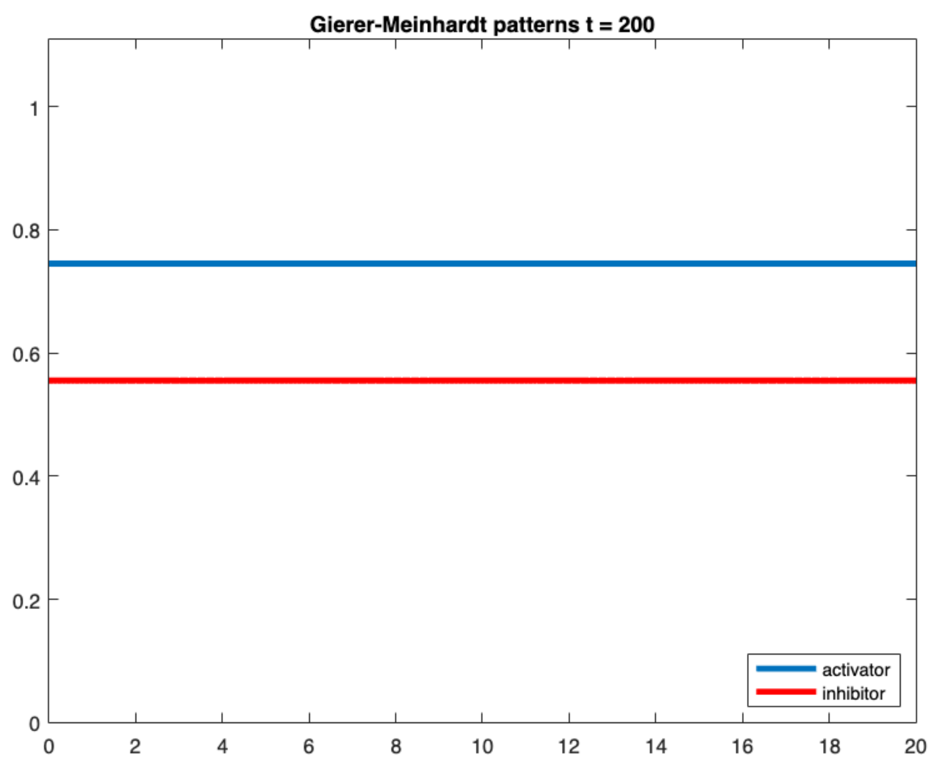
```
0.4500
1.0500
0.0100
```



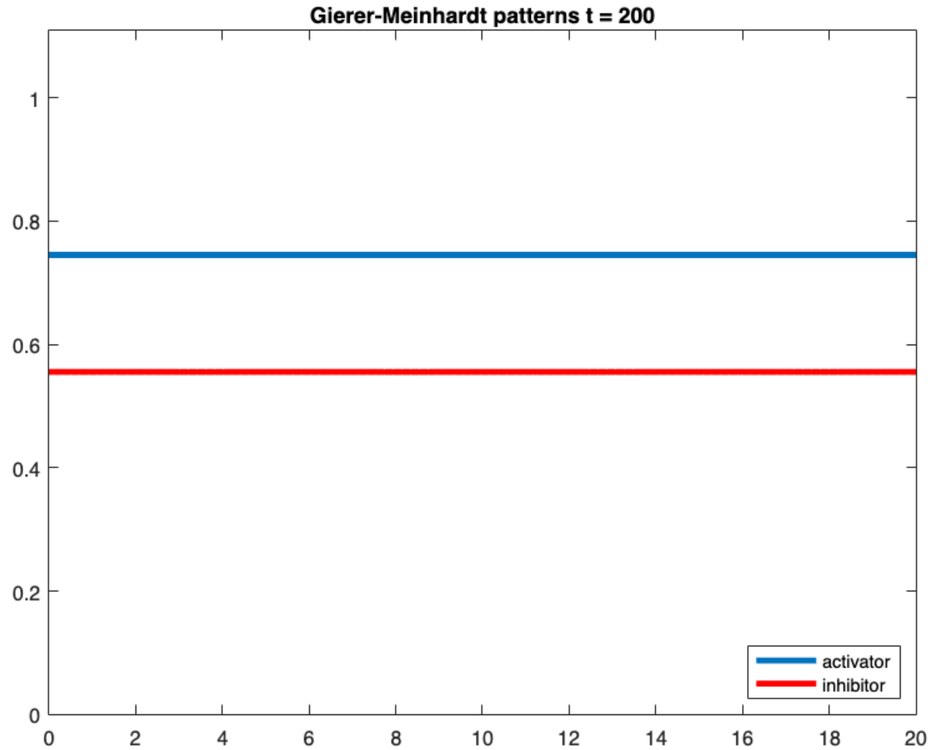

```
0.4500
1.5500
0.0100
```



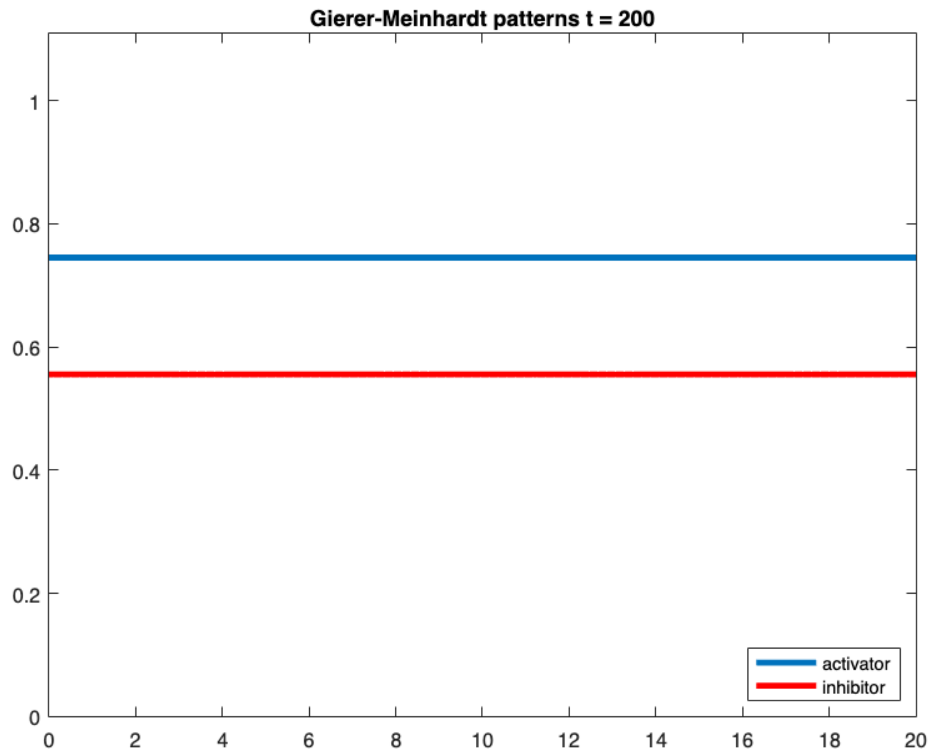

```
0.4500
2.0500
0.0100
```



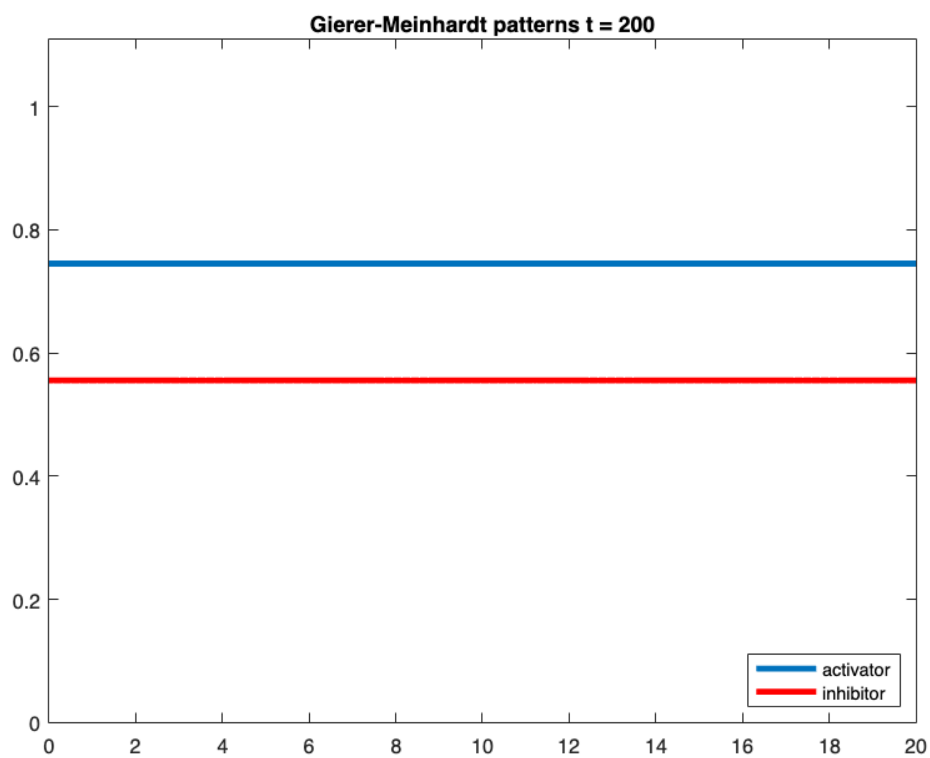
```
0.4500
2.5500
0.0100
```



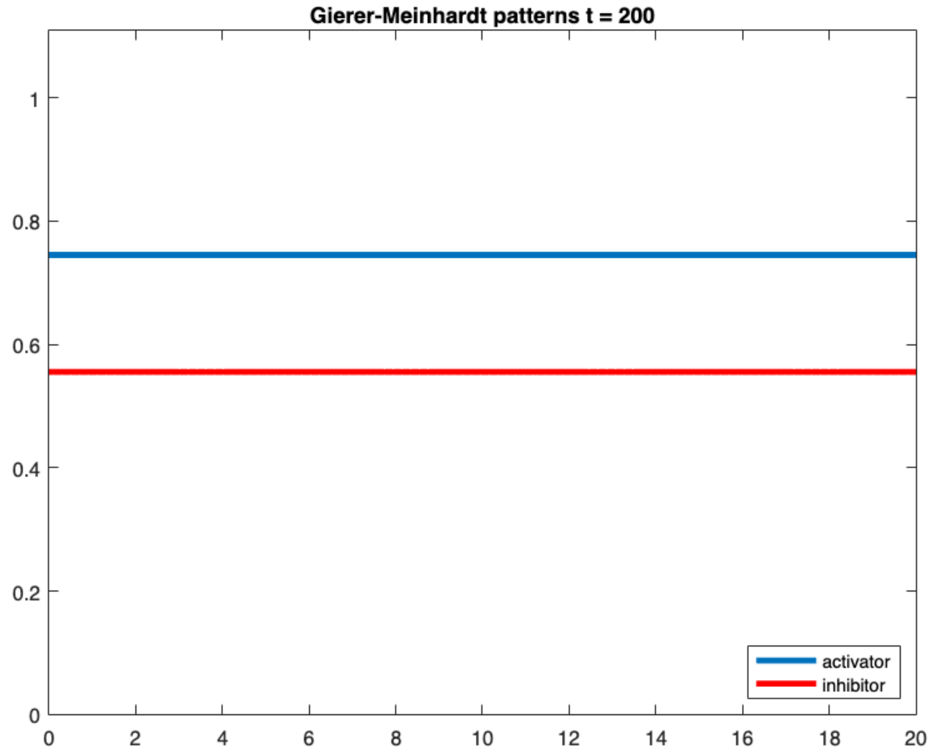

```
0.4500
3.0500
0.0100
```



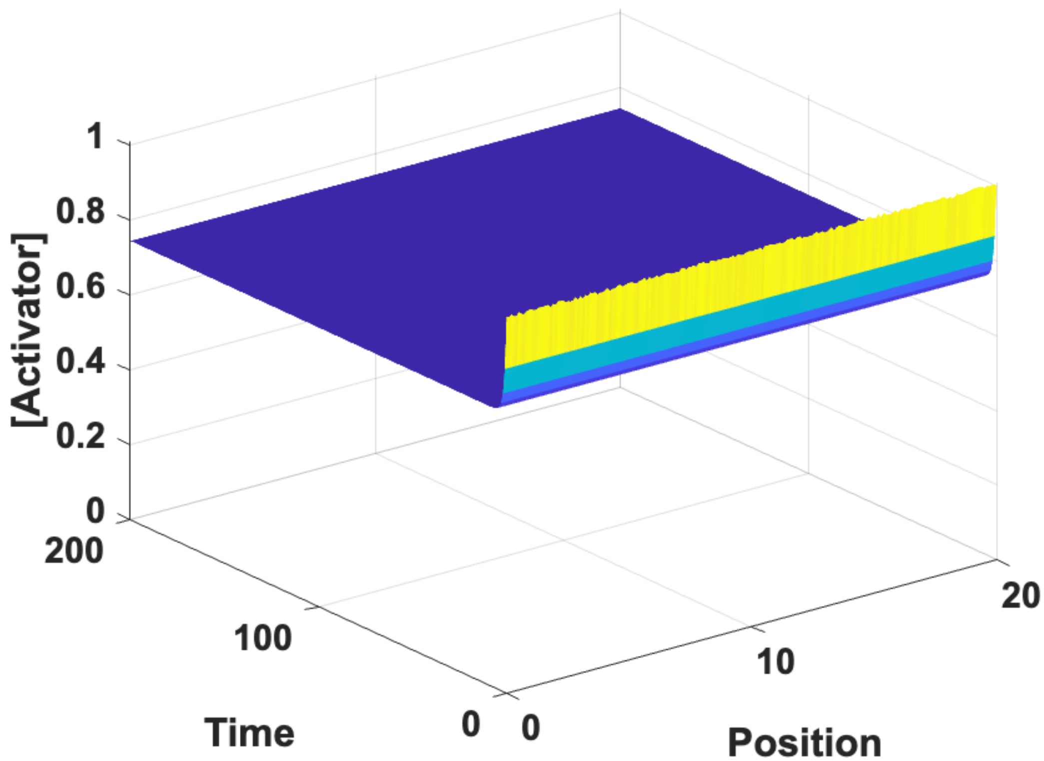

```
0.4500
3.5500
0.0100
```

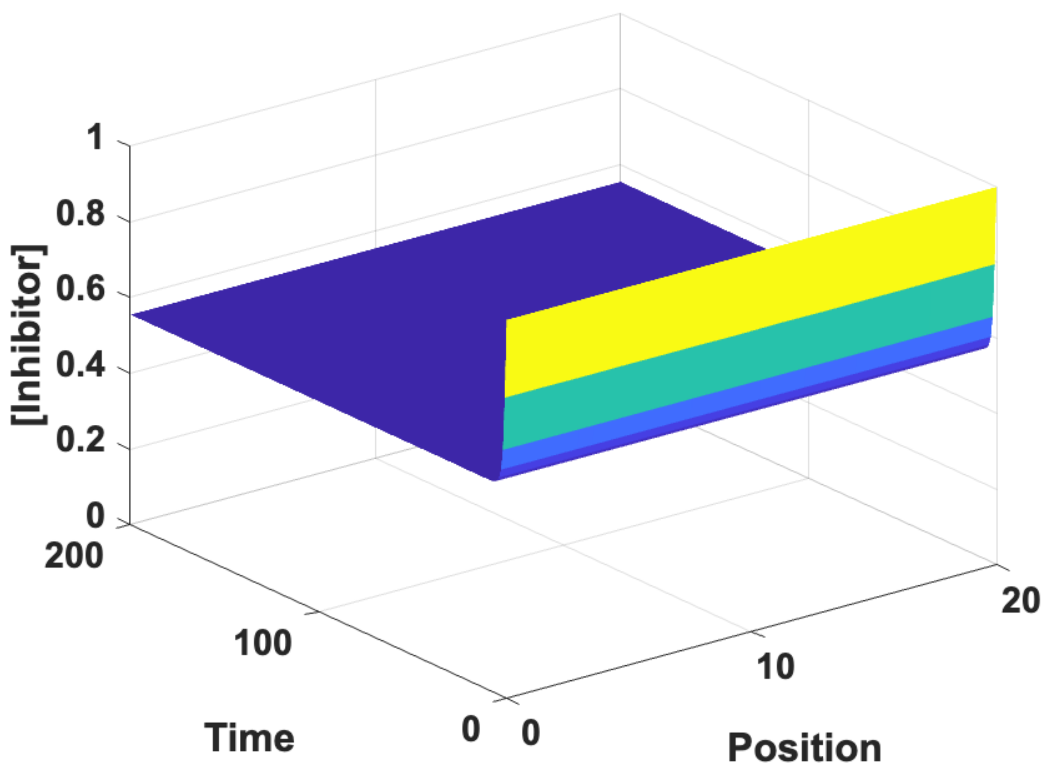


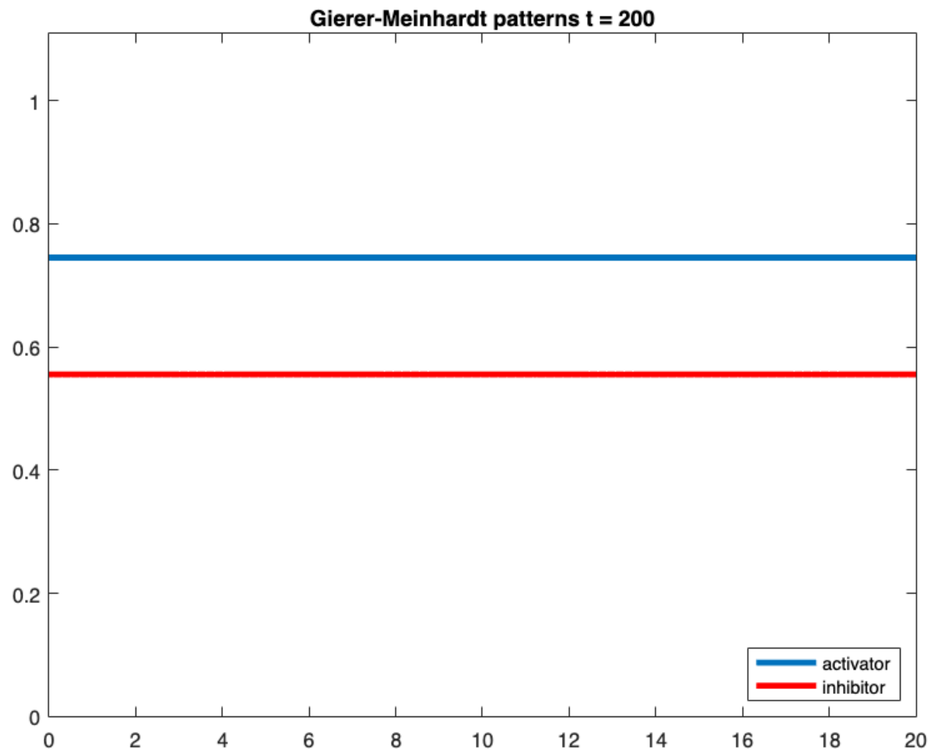
```
0.4500
4.0500
0.0100
```

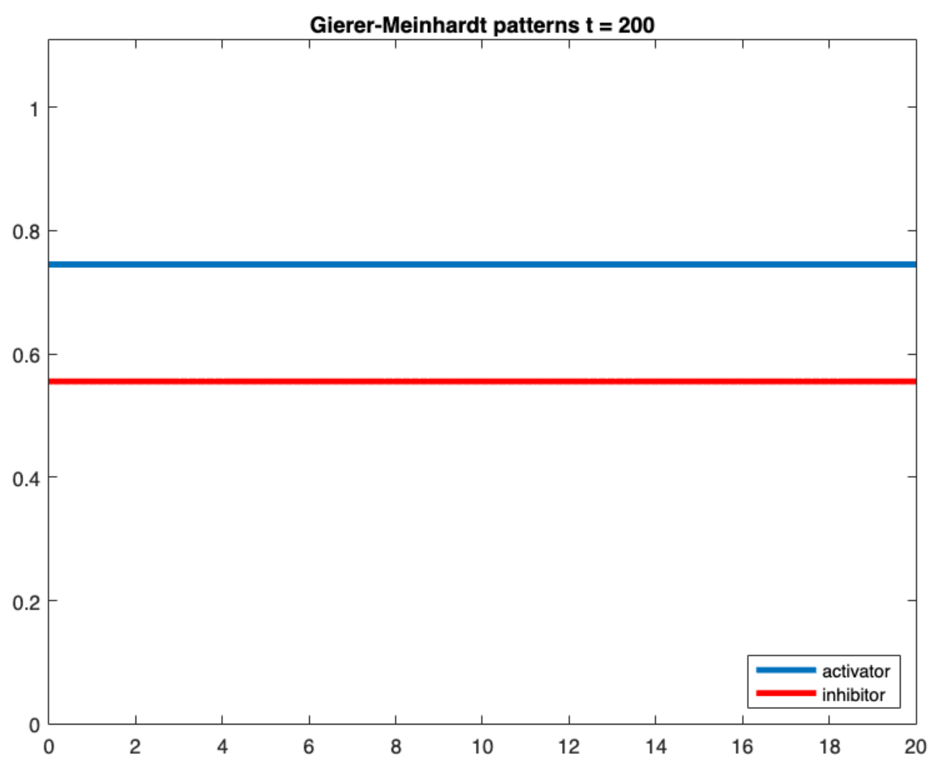



```
0.4500
4.5500
0.0100
```

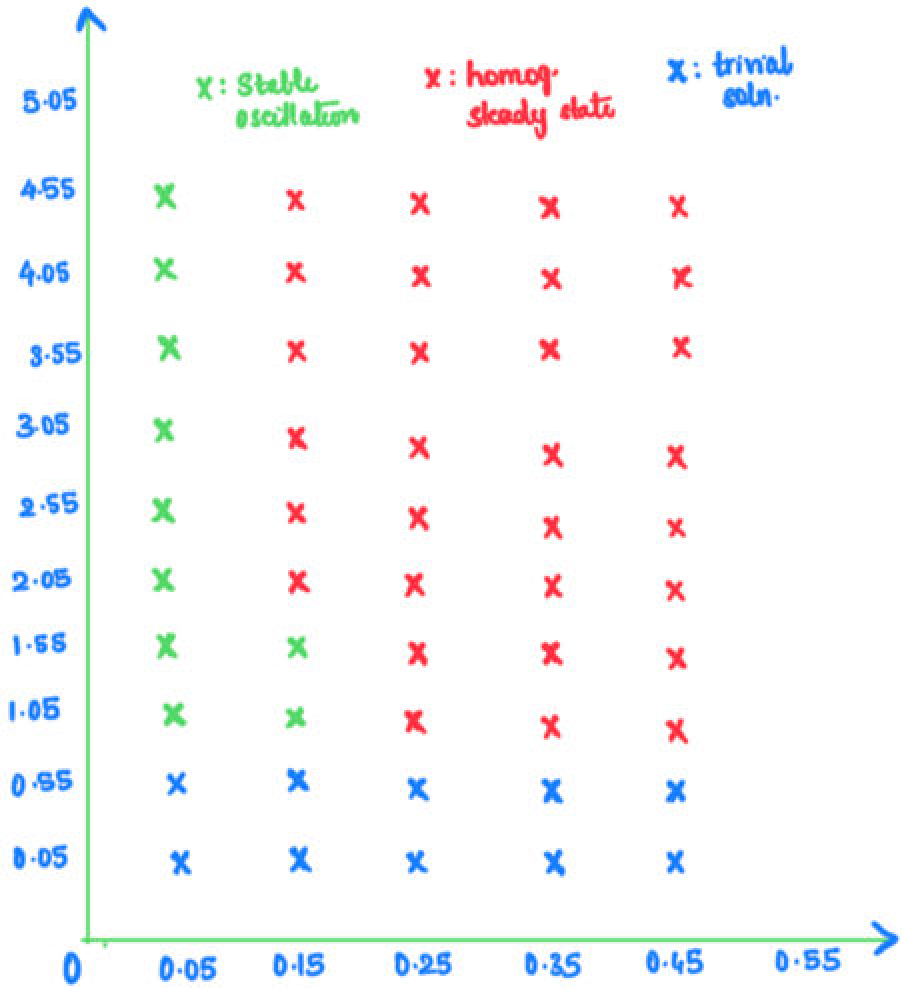

[Activator]
1
0.8
0.6
0.4
0.2
0
200
100
Time
0
0
10
20
Position


[Inhibitor]
1
0.8
0.6
0.4
0.2
0
200
100
Time
0
0
10
20
Position


Gierer-Meinhardt patterns t = 200
activator
inhibitor
0
0.2
0.4
0.6
0.8
1
0
2
4
6
8
10
12
14
16
18
20


Gierer-Meinhardt patterns t = 200
activator
inhibitor
0
0.2
0.4
0.6
0.8
1
0
2
4
6
8
10
12
14
16
18
20

Question 5.


X : Stable oscillation
X : homog. steady state
X : trivial soln.
5.05
4.55
4.05
3.55
3.05
2.55
2.05
1.55
1.05
0.55
0.05
0
0.05
0.15
0.25
0.35
0.45
0.55

Question 6.

(ii)
Q6) From Q4,

$$k^2 = \frac{a}{Dh} - \frac{\omega}{2} - \frac{1}{2D}$$

When $\det_J = 0$,

$$Dk^4 + \left(\omega D - \frac{2a}{h} + 1\right)k^2 + \left(1 - \frac{2a}{h}\right)\omega = 0$$

$$D\left(\frac{a}{Dh} - \frac{\omega}{2} - \frac{1}{2D}\right)^2 + \left(\omega D - \frac{2a}{h} + 1\right)\left(\frac{a}{Dh} - \frac{\omega}{2} - \frac{1}{2D}\right) + \left(1 - \frac{2a}{h}\right)\omega = 0.$$

(i) Setting $k^2 = 0$ in $\det_J$

$$\det_J = \left(1 - \frac{2a}{h}\right)\omega$$

For -ve $\det_J$,

$$\left(\frac{2a}{h} - 1\right)\omega > 0$$

For $\omega \neq 0$, at the boundary,

$$\boxed{2a = h}$$