NAME:_______________________________________

## Math 227A, Computational Biology Homework 4

Fall 2025, German Enciso
Due: Friday, October 30th, at the beginning of class

1. Use Matlab command [P,D]=eig(A) (or a similar command in another programming language) to calculate the eigenvalues and eigenvectors of the following systems. The output of this function is such that $A = PDP^{-1}$. Write the general solution formula and a phase diagram including the relevant eigenvectors. Note: Matlab normalizes the eigenvectors, you might want to rescale them to obtain a simpler form eg (1,2) instead of (0.44,0.88).

   a) $x' = -3x + 4y,\ y' = -2x + 3y$

   b) $x' = 5x + 2y,\ y' = -17x - 5y$

2. Similarly as in the previous problem, except for the system $x' = 3x - 4y,\ y' = x - y$.

   a) Use the command eig(A) as in the previous problem, and realize that this matrix is not diagonalizable (why?).

   b) Use instead the command [P,J]=jordan(A) to find the associated Jordan matrix $J$ such that $A = PJP^{-1}$. Write the general solution of the system algebraically.

3. Describe the qualitative behavior of a system $y' = Ay$, where $A$ is a diagonalizable matrix with the following eigenvalues. Do so based on the form of the general solution of the system.

   a) $\lambda = -1, -3, 4, 2$

   b) $\lambda = -2, -1 \pm 3i, -3$

   c) $\lambda = -2, 0, 1, 2 \pm i$

   d) $\lambda = -3 \pm 4i, -2, -1$

4. Consider the system $y_1' = y_1 - y_2, \quad y_2' = y_1 + y_2$.

   a) Write the ODE in format $y' = Ay$, and show that it has eigenvalues $\lambda_1 = 1+i$, $\lambda_2 = 1-i$, with eigenvectors $w^{(1)} = (i,1)^T$, $w^{(2)} = (-i,1)^T$, where $T$ stands for the transpose operation that transforms row vectors into column vectors.

b) The general solution of this system is $y(t) = c_1 e^{\lambda_1 t} w^{(1)} + c_2 e^{\lambda t} w^{(2)}$. However this is in general a complex solution. By properly choosing the value of $c_1, c_2$, show by direct calculation (i.e. do not use the theorem for solutions with complex eigenvalues) that the functions

$$x(t) = e^t \sin tu + e^t \cos tv, z(t) = e^t \cos tu - e^t \sin tv$$

are real valued cases of the general solution of the ODE, where $u, v$ are such that $w^{(1)} = u + iv$, $w^{(2)} = u - iv$. Hint: use the fact that $e^{ix} = \cos(x) + i \sin(x)$, and try eg $c_1 = c_2 = 1$, or $c_1 = -i, c_2 = i$.

5. Consider the system $z' = Jz$, where

$$J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

and $\lambda$ is an arbitrary eigenvalue.

a) Show by solving the corresponding ODEs (starting with the equation for $z_3'$) that the general solution of this problem is

$$z = e^{\lambda t} \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

b) Suppose that $y' = Ay$ is a given linear system such that $A = PJP^{-1}$, where $J$ is the matrix above. Write the general solution of this system in terms of the above system and the unknown matrix $P$.