NAME:____

Math 227A, Computational Biology Homework 6

Fall 2025, German Enciso
Due Friday, November 21st, at the beginning of class

1. For the system $x' = y + \mu x,\ y' = -x + \mu y - x^2 y$, show that a Hopf bifurcation occurs. To verify the asymptotic stability at the critical point, use a computer simulation. Also, plot the limit cycle using a computer system for a relevant value of μ.

2. Suppose that $x' = f(x, y),\ y' = g(x, y)$ is a 2D ODE and that there is a periodic solution P.
a) Calculate the index of the system using the periodic solution itself as a closed path.
b) Suppose that there are two stable spirals inside of the periodic solution. Show that there must also exist at least one saddle in the system.
c) sketch a possible phase plot of this ODE.

3. One of the best known excitable systems is the so-called Fitzhugh-Nagumo model, representing the voltage of a neuron over time. This system can be written as

$$v' = f(v) - w + I(t), \qquad w' = \epsilon(v + d - cw),$$

where $I(t)$ is a time-varying function chosen independently. Set $f(v) = v - v^3/3$, $\epsilon = 0.08$, $d = 0.7$, $c = 0.8$.
a) Set first $I(t) = 0$ for all t. Draw the nullclines of this system on the paper (possibly using Matlab), and plot sample solutions. What behavior do you observe?
b) Now set $I(t)$ to be a constant larger than zero, e.g. $I = 0.3$ or $I = 0.5$. How do the nullclines change, and what behavior do you observe?
c) Interpret this behavior in terms of relaxation oscillations and timescale decomposition.
d) Explain why this system can be considered excitable with respect to the input function $I(t)$.

4. Carry out the following problem from Strogatz:

8.2.9 Consider the predator-prey model

$$\dot{x} = x\left(b - x - \frac{y}{1+x}\right), \qquad \dot{y} = y\left(\frac{x}{1+x} - ay\right),$$

where $x, y \geq 0$ are the populations and $a, b > 0$ are parameters.

a) Sketch the nullclines and discuss the bifurcations that occur as b varies.

b) Show that a positive fixed point $x^* > 0$, $y^* > 0$ exists for all $a, b > 0$. (Don't try to find the fixed point explicitly; use a graphical argument instead.)

c) Show that a Hopf bifurcation occurs at the positive fixed point if

$$a = a_c = \frac{4(b-2)}{b^2(b+2)}$$

and $b > 2$. (Hint: A necessary condition for a Hopf bifurcation to occur is $\tau = 0$, where τ is the trace of the Jacobian matrix at the fixed point. Show that $\tau = 0$ if and only if $2x^* = b - 2$. Then use the fixed point conditions to express a_c in terms of x^*. Finally, substitute $x^* = (b-2)/2$ into the expression for a_c and you're done.)

d) Using a computer, check the validity of the expression in (c) and determine whether the bifurcation is subcritical or supercritical. Plot typical phase portraits above and below the Hopf bifurcation.

Figure 1: