MATH 227A: Mathematical Biology
Homework 3

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Problem 1

Consider the system $y' = y^{1/3}$, $y(0) = 0$. Show that there are at least two different solutions of this system. Does this violate the Existence and Uniqueness Theorem? Why?

Given the ODE

$$\begin{aligned} y' &= y^{1/3} \\ y' &= y^{1/3} \cdot 1 \end{aligned} \tag{1}$$

Comparing (1) with the standard separable ODE form, $y'(t) = g(y) \cdot h(t)$, we have $g(y) = y^{1/3}$ and $h(t) = 1$.

Rewriting the separable ODE using Leibnitz's notation,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= g(y) \cdot h(t) \\ \frac{1}{g(y)} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} &= h(t) \end{aligned}$$

Integrating both sides wrt t,

$$\begin{aligned}\int \frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}t}\,dt &= \int h(t)\,dt\\ \int \frac{1}{y^{1/3}}\,dy &= \int 1\cdot dt\\ \int y^{-2/3}\,dy &= t + c_1\\ 3\cdot y^{1/3} &= t + c_2\\ y &= \frac{(t+c_2)^3}{27}\end{aligned}$$

However, $y(t) = 0$ is also a trivial solution of the system in the neighborhood of $y = 0$ as shown below.

$$y'(t) = \frac{\mathrm{d}}{\mathrm{d}t}(0) = 0 = 0^{1/3} = y^{1/3}$$

Clearly, both $y(t) = 0$ and $y(t) = \frac{(t+c_2)^3}{27}$ are solutions of the system, with the latter solution satisfying $y(0) = 0$ when $c_2 = 0$. However, the existence of two distinct solutions is not a violation of the existence and uniqueness theorem since f in $y' = f(y)$ does not satisfy the requirements of the theorem: the derivative of $f(y) = y^{1/3}$ is not defined at $y = 0$ as shown below.

$$\begin{aligned}f(y) &= y^{1/3}\\ \frac{\partial f(y)}{\partial y} &= \frac{1}{3y^{2/3}}\end{aligned}$$

which is not defined at $y = 0$.

Problem 2

a) Consider the ODE $y' = \sqrt{1-y^2}$, defined for $-1 \le y \le 1$. Find the solution $y(t)$ of this system through mathematical analysis, and in particular a nontrivial solution with $y(0) = -1$.

Given the ODE,

$$\begin{aligned}y' &= \sqrt{1-y^2}\\ y' &= \sqrt{1-y^2}\cdot 1 \end{aligned} \tag{2}$$

Comparing (2) with the standard separable ODE form, $y'(t) = g(y) \cdot h(t)$, we have $g(y) = \sqrt{1-y^2}$ and $h(t) = 1$.

Rewriting the separable ODE using Leibnitz's notation,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= g(y) \cdot h(t) \\ \frac{1}{g(y)} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} &= h(t)\end{aligned}$$

Integrating both sides wrt t,

$$\begin{aligned}\int \frac{1}{g(y)} \frac{\mathrm{d}y}{\mathrm{d}t}\, dt &= \int h(t)\, dt \\ \int \frac{1}{\sqrt{1-y^2}}\, dy &= \int 1 \cdot dt \\ \arcsin y &= t + c_1 \\ y &= \sin(t + c_1) \qquad (3)\end{aligned}$$

Applying the given initial condition $y(0) = -1$,

$$\begin{aligned}-1 &= \sin(c_1) \\ c_1 &= (4n-1) \cdot \frac{\pi}{2} \qquad [\text{ where } n \in Z\ ]\end{aligned}$$

So, a possible non-trivial, specific solution is $y = -\cos t$. More generally, the solution is $y = -\cos(t + 2n\pi)$ for $n \in Z$.

b) Can you find another solution of this ODE with y(0) = -1? Explain why this doesn't violate the Existence and Uniqueness Theorem.

Yes, another solution can be found. For instance, $y(t) = -1$ is a trivial solution in the neighborhood of $y = -1$ as shown below by plugging in.

$$y'(t) = \frac{\mathrm{d}}{\mathrm{d}t}(-1) = 0 = \sqrt{1-(-1)^2} = \sqrt{1-y^2}$$

However, the existence of two distinct solutions is not a violation of the existence and uniqueness theorem since f in $y' = f(y)$ does not satisfy the requirements of the theorem: the derivative of $f(y) = \sqrt{1-y^2}$ is not defined at $y = -1$ as shown below.

$$\begin{aligned} f(y) &= \sqrt{1-y^2} \\ \frac{\partial f(y)}{\partial y} &= \frac{-1}{\sqrt{1-y^2}} \end{aligned}$$

which is not defined at $y = -1$ or anywhere outside of the interval $(-1, 1)$.

c) Consider the function $f(y)$ in the attached figure, made up of many copies of the function in (a). Show in a timecourse graph different solutions of the system with initial condition y(0) = 0, with the property that over time one solution converges towards 2, another towards 4, and another towards 6.

Rewriting the function from (a) in (2) defined for $y \in [-1, 1]$,

$$y' = \sqrt{1-y^2}$$

$f(x)$ can be defined as many copies of (2), each operating in a specific interval.

$$y' = f(x) = \sqrt{1-(y-2k-1)^2} \qquad \text{(where } y \in [2k, 2k+2] \text{ and } k \in \{0,1,2\}\text{)}$$

Or equivalently,

$$y' = f(x) = \begin{cases} \sqrt{1-(y-1)^2}, & y \in [0,2] \\ \sqrt{1-(y-3)^2}, & y \in [2,4] \\ \sqrt{1-(y-5)^2}, & y \in [4,6] \end{cases} \tag{4}$$

which has the same form as the function in (a) in each interval, only transformed along the y-dimension. At the boundaries $y = 2$ and $y = 4$, $f_{left}(y) = f_{right}(y) = 0$; thus, $f(y)$ is **continuous**.

$$f'(y) = \frac{-(y-2k-1)}{\sqrt{1-(y-2k-1)^2}}$$

However, the derivatives at the boundaries $y = 2$ and $y = 4$, $f_{left}(y) \to -\infty$ and $f_{right}(y) \to \infty$; so, the function is **not differentiable** at the boundaries which is why the system does not follow the existence and uniqueness theorem—specifically, the function is **not Lipschitz**. Since the system can have extremely high slopes at the right bound of its interval, an infinitesimal nudge can push the function over to the subsequent interval in the piecewise definition.

Since we need the function to begin at $y(0) = 0$, we find the specific solution for its specification in the interval $y \in [0, 2]$.

$$\begin{aligned}
y' &= \sqrt{1 - (y-1)^2} \\
\implies y &= 1 + \sin(t + c_2) && (5) \\
0 &= 1 + \sin(c_2) && [\text{ solving for } y(0) = 0 \text{ }] \\
\implies c_2 &= (4n - 1) \cdot \frac{\pi}{2} && [\text{ where } n \in Z \text{ }] \\
\implies y &= 1 - \cos t && [\text{ one possible specific solution }]
\end{aligned}$$

Now, we require three different solutions that converge toward 2, 4, and 6, respectively, while still being continuous but not Lipschitz. Since, for continuity, we need the subsequent piecewise definitions to meet at the right bound of the previous interval, we define the time intervals based on the time it takes for the function to reach its right bound. Since the first function is defined for the interval $[0, 2]$, this works out to be the time for the function go from its minima to its maxima. We choose the operating range $[0, \pi]$ for the first piecewise definition in (5). Now, we require the second piece in (4) to begin at its minima at $t = \pi$. Solving the general form again, we have $c_2 = \pi$, which evaluates to $y = 3 + \sin(t + \pi) = 3 + \cos \pi$. We can similarly solve for the third piece to be at its minima at $t = 2\pi$ to get the solution $y = 5 - \cos \pi$. Concretely, the three piecewise definitions converging to 2, 4, and 6, are shown in (6), (7), and (8), and graphically in Figure 1.

$$y_1 = \begin{cases} 0, & t \leq 0 \\ 1 - \cos t, & t \in [0, \pi] \\ 2, & t \geq \pi \end{cases} \tag{6}$$

$$y_2 = \begin{cases} 0, & t \leq 0 \\ 1 - \cos t, & t \in [0, \pi] \\ 3 + \cos t, & t \in [\pi, 2\pi] \\ 4, & t \geq 2\pi \end{cases} \tag{7}$$

$$y_3 = \begin{cases} 0, & t \leq 0 \\ 1 - \cos t, & t \in [0, \pi] \\ 3 + \cos t, & t \in [\pi, 2\pi] \\ 5 - \cos t, & t \in [2\pi, 3\pi] \\ 6, & t \geq 3\pi \end{cases} \tag{8}$$

Since the steady states of the system between any piecewise two definitions is a saddle point that the system approaches from the left and leaves from the right, it is able to transition from one to the next with an infinitesimal nudge.

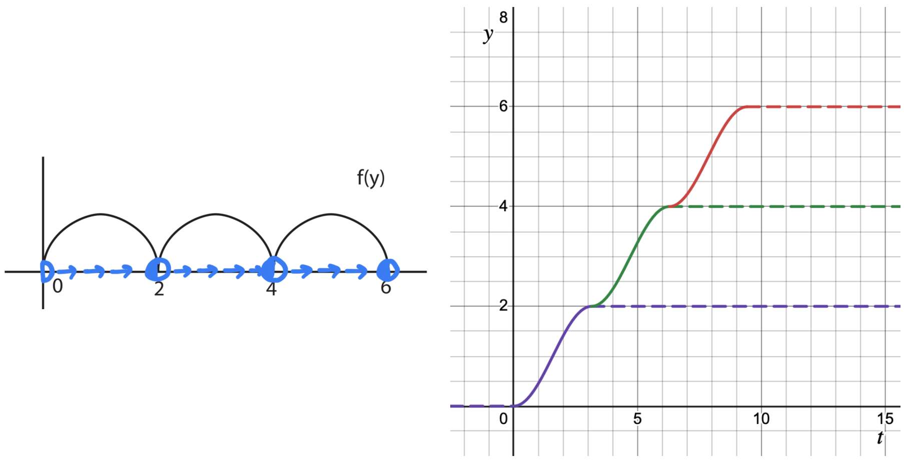

Figure 1: The complete phase plot of the system (left) showing the saddle points, and the three different solutions for the system (right) to converge to $y = 2$, 4, and 6 in purple, purple $\rightarrow$ green, and purple $\rightarrow$ green $\rightarrow$ red, respectively.

Problem 3

[**Insect Outbreak**] Consider the population $y(t)$ of insects in a forest, with the following equation $y'(t) = r \cdot y(1 - y/K) - y^2/(1 + y^2)$. The last term is a predation term corresponding to the amount of insects eaten by birds.

a) Notice that the predation term is a sigmoidal function, and describe what might be the biological reason for such a shape, as opposed to a non-sigmoidal function y/(1 + y).

Figure 2 shows the predation term as a sigmoidal function $y^2/(1 + y^2)$ in red, and as a non-sigmoidal function $y/(1 + y)$. The sigmoidal plot is more biologically accurate since initially, when the insect population is very low, the likelihood of a bird finding an insect is also very low. Once the amount of insects reaches a substantial number, they are easier to spot/find for birds and they get predated more easily. This phenomenon of gradual increase in rate followed by exponential increase in the rate is captured by the change in concavity at $y \approx 0.5$ by the sigmoidal function, while the non-sigmoidal function just keeps growing exponentially in this region.

Both functions, however, do approach a saturating value at high values of y because there is a finite rate at which they can be consumed.

b) Find the steady states of the system by setting $y' = 0$ and dividing by y on both sides. Plot each of the functions $r \cdot (1 - y/K)$ and $y/(1 + y^2)$ on their own and study where they intersect, for different values of r and K.

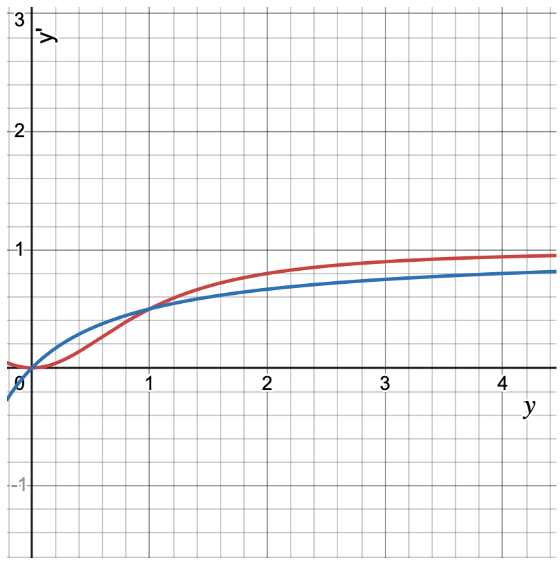

Figure 2: Comparison of sigmoidal (red) and non-sigmoidal (blue) functions for the predation term.

Setting $y' = 0$ and dividing by y,

$$y' = r \cdot y(1 - y/K) - y^2/(1 + y^2) \tag{9}$$

$$0 = r \cdot (1 - y_{ss}/K) - y_{ss}/(1 + y_{ss}^2) \tag{10}$$

$$y' = g(y) - h(y) \tag{11}$$

where $g(y) = r \cdot (1 - y/K)$ and $h(y) = y/(1+y^2)$. In $g(y)$, K and r are essentially the x- and y-intercepts, respectively.

Figure 4 plots the functions $h(y)$ and $g(y)$. For a fixed value of low K, say $K = 4$, the system has exactly one steady state for any r. However, for high K, say $K = 10$, as r increases from 0 to 1, the system goes from having one steady state to three back to just one. Similarly, for a fixed value of $r > 0.5$ (since $y' = 0.5$ is the point of maxima for $h(y)$), say $r = 0.55$, the system transitions from a regime of one steady state to three back to one as K increases. However, for $r < 0.5$, say $r = 0.45$, the system has only one continuous regime where it has exactly one fixed point when K is small. As K increases the system moves into the regime of three fixed points and remains there.

c) Show that for a fixed large value of K, as r grows the system transitions from having one steady state, to bistability, and then one steady state again. What kind of behavior is this? Explain.

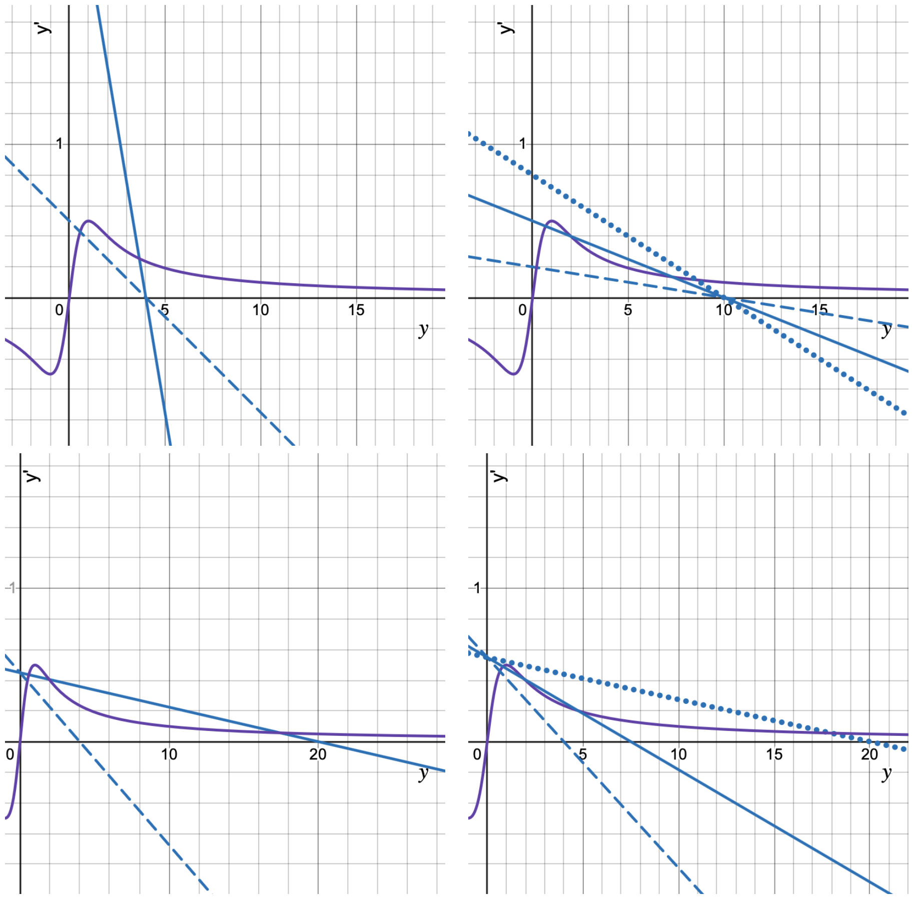

Figure 3: Phase plots with $h(y)$ and $g(y)$ plotted separately. Upper panels show different r values at low $K = 4$ (left) and high $K = 10$ (right). Lower panels show different K values at low $r = 0.45$ (left) and high $r = 0.55$ (right). Dotted, dashed, and solid lines within a plot show different parameter settings for $h(y)$.

Figure 3 plots the functions $h(y)$ and $g(y)$. Specifically, parts G-I show the described condition where for fixed high K, the system transitions from having one steady state to three back to one a r increases. To plot the bifurcation diagram, (10) can be rewritten to express r in terms of y as follows,

$$r = \frac{y_{ss}}{(1 - y_{ss})(1 + y_{ss}^2)} \tag{12}$$

Figure 4 plots the bifurcation plot using (12). It is evident that the **system**

shows hysterisis. Specifically for $K = 25$, as r increases from 0 the system has exactly one steady state value of y. For $r \in [\sim 0.15, \sim 0.5]$, the system has three steady states—one unstable and two stable.

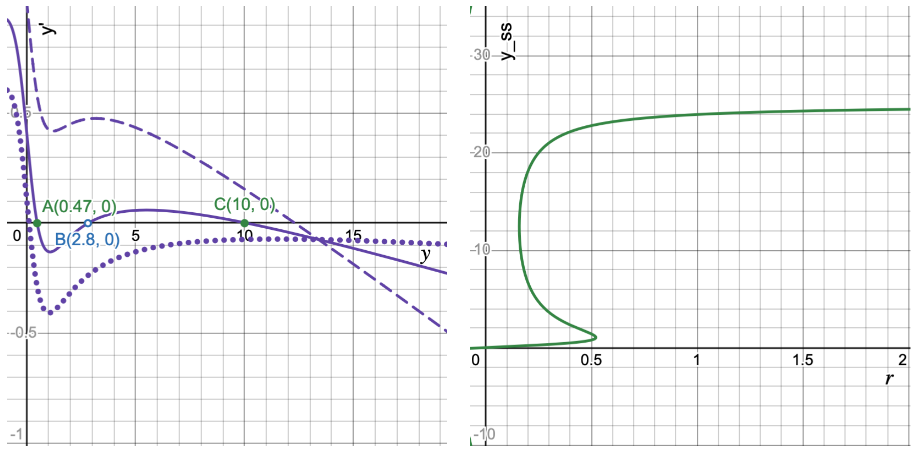

Figure 4: Phase plots (left) of the complete ODE in (9) for different settings of r between 0 and 1 (dotted: lowest, solid: intermediate, dashed: highest) with fixed high $K = 12$ and a bifurcation plot (right) of y_{ss} as a function of r for fixed high $K = 25$. In the phase plot, A and C are stable steady states while B is unstable.

Problem 4

Consider a general model of insect outbreak, where y(t) is the insect population.:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay\left(1 - \frac{y}{K}\right) - b\frac{y^2}{r^2 + y^2}$$

Show that by carrying out a non-dimensionalization of this system, it can be reduced to the model

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = \rho z\left(1 - \frac{z}{\kappa}\right) - \frac{z^2}{1 + z^2}$$

where z is a non-dimensional variable and ρ, κ are non-dimensional parameters.

Consider the dimensional model,

$$\frac{dy}{dt} = a\,y\Big(1 - \frac{y}{K}\Big) - b\,\frac{y^2}{r^2 + y^2},$$

where $y(t)$ is the insect population, a is an intrinsic growth rate, K is the carrying capacity, b scales the strength of the saturating loss term, and r is the half-saturation population for the loss term.

We choose scales for the dependent and independent variables so that the saturating term becomes simple. Defining the non-dimensional variables,

$$y = r\,z, \qquad t = \frac{r}{b}\,\tau,$$

so that $z(\tau)$ is the nondimensional population and τ is nondimensional time. The time scaling $t = (r/b)\tau$ is chosen so that the coefficient of the saturating term on the left-hand side becomes 1 after rescaling.

Differentiating $y = rz$ with respect to t using the chain rule,

$$\frac{dy}{dt} = \frac{d(rz)}{d\left(\frac{r}{b}\tau\right)} = \frac{dz}{d\tau} \cdot \frac{d\tau}{dt} \cdot r = \frac{dz}{d\tau} \cdot \frac{b}{r} \cdot r = b\,\frac{dz}{d\tau}.$$

Now substituting $y = rz$ into the right-hand side of the original equation,

$$\begin{aligned} a\,y\Big(1 - \frac{y}{K}\Big) - b\frac{y^2}{r^2 + y^2} &= a(rz)\Big(1 - \frac{rz}{K}\Big) - b\frac{r^2 z^2}{r^2(1 + z^2)} \\ &= ar\,z\Big(1 - \frac{r}{K}z\Big) - b\frac{z^2}{1 + z^2}. \end{aligned}$$

Equate the left and right sides $b\,\dfrac{dz}{d\tau} =$ (right-hand side) and divide through by b:

$$\frac{dz}{d\tau} = \frac{ar}{b}\,z\Big(1 - \frac{r}{K}z\Big) - \frac{z^2}{1 + z^2}.$$

Introducing the non-dimensional parameters,

$$\rho := \frac{ar}{b}, \qquad \kappa := \frac{K}{r}.$$

Observe that $1 - \dfrac{r}{K}z = 1 - \dfrac{z}{\kappa}$. Substituting these definitions yields the nondimensional model

$$\frac{dz}{d\tau} = \rho\, z\Big(1 - \frac{z}{\kappa}\Big) - \frac{z^2}{1+z^2}$$

with the definitions

$$z = \frac{y}{r}, \qquad \tau = \frac{b}{r}\,t, \qquad \rho = \frac{ar}{b}, \qquad \kappa = \frac{K}{r}.$$

Problem 5

Consider the linear system $y_1' = y_2$, $y_2' = -2y_1 - 3y_2$.

a) Find the eigenvalues of the system.

The system can be written in matrix form, $y' = Ay$, as follows:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{13}$$

To find the eigen values of A, we solve $det(A - \lambda I) = 0$,

$$\begin{aligned} det\left(\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) &= 0 \\ det\begin{pmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{pmatrix} &= 0 \\ \lambda^2 + 3\lambda + 2 &= 0 \\ \lambda^2 + 2\lambda + \lambda + 2 &= 0 \\ (\lambda+2)\cdot(\lambda+1) &= 0 \\ \lambda_1 = -2, \lambda_2 &= -1 \end{aligned} \tag{14}$$

are the eigen values of A.

b) For each eigenvalue, find one corresponding eigenvector.

To find the eigenvectors, we solve $(A - \lambda I)\cdot v = 0$. For $\lambda_1 = -2$,

$$\begin{aligned} \begin{pmatrix} -\lambda_1 & 1 \\ -2 & -3-\lambda_1 \end{pmatrix} \cdot v_1 &= 0 \\ \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \cdot v_1 &= 0 \\ v_1 &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \end{aligned} \tag{15}$$

is an eigenvector. Similarly, solving fo $\lambda_2 = -1$,

$$\begin{pmatrix} -\lambda_2 & 1 \\ -2 & -3-\lambda_2 \end{pmatrix} \cdot v_2 = 0$$
$$\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \cdot v_2 = 0$$
$$v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{16}$$

is the second eigenvector.

c) Find the general solution of the ODE, and the solution for $y_1(0) = 4$, $y_2(0) = 3$.

Given the eigenvalues $\lambda_1 = -2$, $\lambda_2 = -1$ and their corresponding eigenvectors v_1 and v_2, we have the following two solution components.

$$\begin{aligned} z(t) &= v_1 e^{\lambda_1 t} \\ &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t} \end{aligned} \tag{17}$$
$$\begin{aligned} w(t) &= v_2 e^{\lambda_2 t} \\ &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \end{aligned} \tag{18}$$

Combining the solutions, we have

$$\begin{aligned} y(t) &= c_1 z(t) + c_2 w(t) \\ &= c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \end{aligned} \tag{19}$$

Using the conditions $y_1(0) = 4$ and $y_2(0) = 3$ for the solution in (19),

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{0} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{0}$$
$$\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ -2c_1 - c_2 \end{pmatrix}$$
$$c_1 = -7, c_2 = 11$$

So, the specific solution at $y_1(0) = 4$ and $y_2(0) = 3$ is

$$y(t) = \begin{pmatrix} -7 \\ 14 \end{pmatrix} e^{-2t} + \begin{pmatrix} 11 \\ -11 \end{pmatrix} e^{-t} \tag{20}$$

d) Draw a rough phase plot of the system.

The phase portrait is shown in Figure 5. Since both eigenvalues are real and negative, there is a stable node at the origin. All trajectories (shown in blue) are directed inward toward the origin, and approaches the origin parallel to one of the eigenvectors (shown in green).

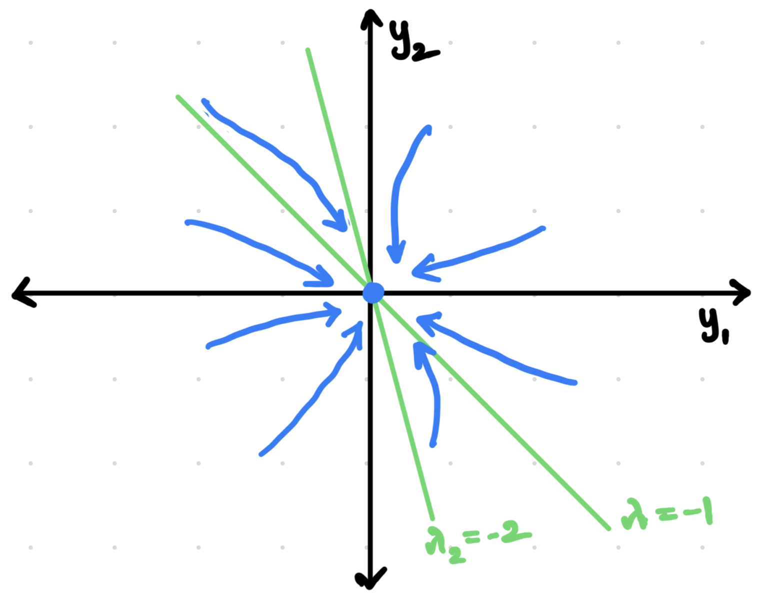

Figure 5: Rough sketch of the phase plot of the system in (21). The eigenvectors are shown in green. The trajectories of candidate systems are shown in blue.

Problem 6

Consider the linear system $y_1' = 3y_1 + 9y_2$, $y_2' = -4y_1 - 3y_2$.

a) Find the eigenvalues of the system.

The system can be written in matrix form, $y' = Ay$, as follows:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{21}$$

To find the eigen values of A, we solve $det(A - \lambda I) = 0$,

$$
\begin{aligned}
det\left(\begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) &= 0 \\
det\begin{pmatrix} 3-\lambda & 9 \\ -4 & -3-\lambda \end{pmatrix} &= 0 \\
\lambda^2 - 9 - (-36) &= 0 \\
\lambda^2 + 27 &= 0 \\
\lambda &= \pm i3\sqrt{3}
\end{aligned} \tag{22}
$$

So, the eigenvalues A are $\lambda_1 = i3\sqrt{3}i$ and $\lambda_2 = -i3\sqrt{3}$.

b) For each eigenvalue, find one corresponding eigenvector.

To find the eigenvectors, we solve $(A - \lambda I) \cdot v = 0$. For $\lambda_1 = 3\sqrt{3}i$,

$$
\begin{aligned}
\begin{pmatrix} 3-\lambda_1 & 9 \\ -4 & -3-\lambda_1 \end{pmatrix} \cdot v_1 &= 0 \\
\begin{pmatrix} 3(1-i\sqrt{3}) & 9 \\ -4 & -3(1+i\sqrt{3}) \end{pmatrix} \cdot v_1 &= 0 \\
v_1 &= \begin{pmatrix} -3 \\ 1-i\sqrt{3} \end{pmatrix} \\
&= \begin{pmatrix} -3 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -\sqrt{3} \end{pmatrix}
\end{aligned} \tag{23}
$$

is an eigenvector. Similarly, solving fo $\lambda_2 = -i3\sqrt{3}$,

$$
\begin{aligned}
\begin{pmatrix} 3-\lambda_2 & 9 \\ -4 & -3-\lambda_2 \end{pmatrix} \cdot v_2 &= 0 \\
\begin{pmatrix} 3(1+i\sqrt{3}) & 9 \\ -4 & -3(1-i\sqrt{3}) \end{pmatrix} \cdot v_2 &= 0 \\
v_2 &= \begin{pmatrix} -3 \\ 1+i\sqrt{3} \end{pmatrix} \\
&= \begin{pmatrix} -3 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}
\end{aligned} \tag{24}
$$

is the second eigenvector.

c) Find the general solution of the ODE, by finding two real valued solutions and multiplying each by a general constant.

Using the eigenvalue $\lambda_2 = -i3\sqrt{3}$ and their corresponding eigenvector v_2, we have the following two real-valued solutions arise from real and imaginary parts of the eigenvectors:

$$\begin{aligned}
y_{Re}(t) &= \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{0\cdot t} \sin 3\sqrt{3}t + \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix} e^{0\cdot t} \cos 3\sqrt{3}t \\
&= \begin{pmatrix} -3 \\ 1 \end{pmatrix} \sin 3\sqrt{3}t + \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix} \cos 3\sqrt{3}t \qquad (25) \\
y_{Im}(t) &= \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{0\cdot t} \cos 3\sqrt{3}t - \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix} e^{0\cdot t} \sin 3\sqrt{3}t \\
&= \begin{pmatrix} -3 \\ 1 \end{pmatrix} \cos 3\sqrt{3}t - \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix} \sin 3\sqrt{3}t \qquad (26)
\end{aligned}$$

Combining the solutions by multiplying (25) by c_1 and (26) by c_2, we have

$$y(t) = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \left(c_1 \sin 3\sqrt{3}t + c_2 \cos 3\sqrt{3}t \right) + \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix} \left(c_1 \cos 3\sqrt{3}t - c_2 \sin 3\sqrt{3}t \right) \qquad (27)$$

d) Draw a rough phase plot of the system.

Since $y1(t)$ and $y2(t)$ are both sines and cosines with the same frequency, the point $(y1, y2)$ moves periodically—it traces out a closed path. The relative amplitudes (set by the coefficients) stretch the motion differently along the y_1 and y_2 directions. So instead of forming a perfect circle, the path becomes an ellipse.

Due to larger magnitude of coefficients for y_1', it changes more drastically compared to y_2' for the same magnitude of its RHS; thus, the ellipse is stretched longer along y_1. Furthermore, y_1' has positive slope, and thus y_1 increases, for positive values of its RHS. y_2 does the opposite—decreases when its RHS is positive. For instance, starting in quadrant 1 moves y_1 rightward and y_2 downward. In quadrant 3, y_1 goes leftward and y_2 goes upward. This suggests that the oscillations are clockwise.

Now, since the coupling coefficients are unequal and of opposite signs, it tilts the the axis of the ellipse. Specifically, the imaginary part of the eigenvectors form the vertical axis of the ellipse—this is perfectly vertical since they are of the form $(0, \pm\sqrt{3})^t$. The real part of the eigenvectors form the horizontal axis of the ellipse—this has negative slope and is closer to the y_1 axis that it is to y_2, producing a net clockwise geometrical tilt on the elliptical trajectory. The phase plot is shown in Figure 6.

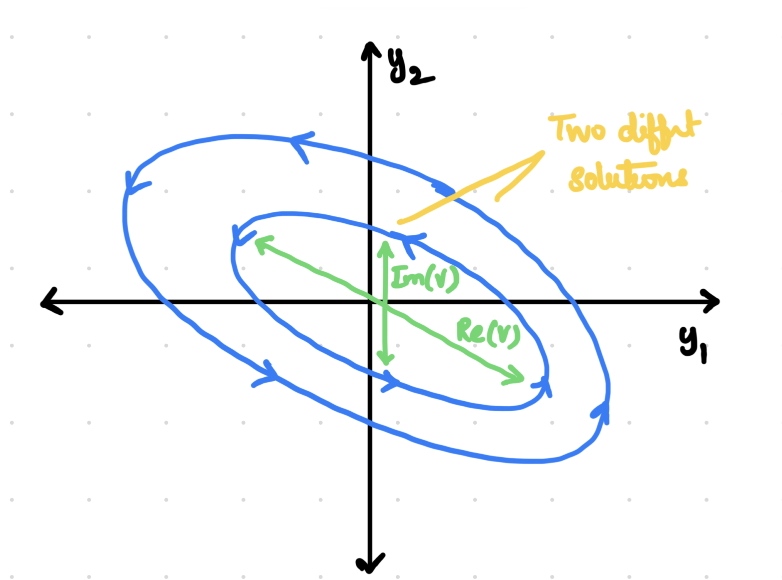

Figure 6: Rough sketch of the phase plot of the system in (27). The eigenvectors are shown in green. The trajectories of candidate systems are shown in blue.

Problem 7

Suppose that a linear $y' = Ay$ system with four variables has eigenvalues $\lambda_1 = -2$, $\lambda_2 = 1$, $\lambda_3 = -1 - i$, $\lambda_4 = -1 + i$, and respective eigenvectors $w_1 = [1, 0, 2, 0]^t$, $w_2 = [3, 1, 0, 1]^t$, $w_3 = [2i, 0, 1, -i]^t$, and $w4 = [-2i, 0, 1, i]^t$. Here the t stands for the transposition of the vector, so that each of these is a column vector.

a) Find a solution to the system associated with each eigenvalue and eigenvector pair. Notice that some of these solutions might have complex values.

The solutions of the ODE are of the form $y(t) = w_i e^{\lambda_i t}$. Specifically,

$$y_1(t) = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} e^{-2t} \qquad y_2(t) = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 1 \end{pmatrix} e^{t}$$

$$y_3(t) = \begin{pmatrix} i2 \\ 0 \\ 1 \\ -i \end{pmatrix} e^{-(1+i)t} \qquad y_4(t) = \begin{pmatrix} -i2 \\ 0 \\ 1 \\ i \end{pmatrix} e^{-(1-i)t}$$

b) Find four different, real valued solutions of the system.

The solutions y_1 and y_2 are already real-valued; these are solutions 1 and 2, rewritten below again.

$$y_1(t) = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} e^{-2t} \qquad y_2(t) = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 1 \end{pmatrix} e^{t} \tag{28}$$

For the imaginary solutions y_3 and y_4, we note that the corresponding eigenvalues and eigenvectors are conjugates of one another. Specifically, $\lambda_3 = \overline{\lambda_4}$ and $w_3 = \overline{w_4}$.

Thus, $y_3(t) = \overline{y_4(t)}$, and the following linear combinations of y_3 and y_4 must be real-valued since they isolate the real and imaginary parts, respectively — $Re(x) = \dfrac{x + \overline{x}}{2}$ and $Im(x) = \dfrac{x - \overline{x}}{i2}$ for complex x; and must be solutions of the ODE since they are linear combinations of two different solutions.

$$
\begin{aligned}
y_5(t) &= \frac{y_3(t)+y_4(t)}{2} \\
&= \begin{pmatrix} i2 \\ 0 \\ 1 \\ -i \end{pmatrix} \frac{e^{-t}\left(\cos t - i\sin t\right)}{2} + \begin{pmatrix} -i2 \\ 0 \\ 1 \\ i \end{pmatrix} \frac{e^{-t}\left(\cos t + i\sin t\right)}{2} \\
&= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{-t}\cos t + \begin{pmatrix} 2 \\ 0 \\ 0 \\ -1 \end{pmatrix} e^{-t}\sin t \\
&= e^{-t} \begin{pmatrix} 2\sin t \\ 0 \\ \cos t \\ -\sin t \end{pmatrix}
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
y_6(t) &= \frac{y_3(t)-y_4(t)}{i2} \\
&= \begin{pmatrix} i2 \\ 0 \\ 1 \\ -i \end{pmatrix} \frac{e^{-t}\left(\cos t - i\sin t\right)}{i2} - \begin{pmatrix} -i2 \\ 0 \\ 1 \\ i \end{pmatrix} \frac{e^{-t}\left(\cos t + i\sin t\right)}{i2} \\
&= \begin{pmatrix} 2 \\ 0 \\ 0 \\ -1 \end{pmatrix} e^{-t}\cos t + \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} e^{-t}\sin t \\
&= e^{-t} \begin{pmatrix} 2\cos t \\ 0 \\ -\sin t \\ -\cos t \end{pmatrix}
\end{aligned} \tag{30}
$$

c) Using the information above, find the general solution of the linear system $y' = Ay$.

Using the solutions y_1, y_2, y_3, and y_4 in (28)–(30), the general solution can be written down as their linear combination:

$$
y(t) = c_1 e^{-2t} \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 3 \\ 1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 2\sin t \\ 0 \\ \cos t \\ -\sin t \end{pmatrix} + c_4 e^{-t} \begin{pmatrix} 2\cos t \\ 0 \\ -\sin t \\ -\cos t \end{pmatrix}
$$

where c_1, c_2, c_3, and c_4 are arbitrary non-complex constants.

d) What can you conclude about the stability of the solutions of this system around the steady state $y = [0, 0, 0, 0]^t$? Explain.

Since at least one eigen value is positive ($\lambda_2 = 1$), the steady state is **unstable** as components in this direction will grow exponentially. Since $\lambda_1 > 0$, it is stable along this direction. And since the complex eigenvalues have negative real parts, it is a stable spiral into the origin. Thus, although stable along three directions, the steady state is unstable overall.