NAME:____

Math 227A, Computational Biology

Fall 2025, German Enciso Due: Friday, October 24th, at the beginning of class

1. Consider the system $y' = y^{1/3}$, $y(0) = 0$. Show that there are at least two different solutions of this system. Does this violate the Existence and Uniqueness Theorem? Why?

2. a) Consider the ODE $y' = \sqrt{1 - y^2}$, defined for $-1 \leq y \leq 1$. Find the solution $y(t)$ of this system through mathematical analysis, and in particular a nontrivial solution with $y(0) = -1$.
b) Can you find another solution of this ODE with $y(0) = -1$? Explain why this doesn't violate the Existence and Uniqueness Theorem.
c) Consider the function $f(y)$ in the attached figure, made up of many copies of the function in a). Show in a timecourse graph different solutions of the system with initial condition $y(0) = 0$, with the property that over time one solution converges towards 2, another towards 4, and another towards 6.

3. Insect Outbreak (from Strogatz textbook) Consider the population $y(t)$ of insects in a forest, with the following equation
$$y'(t) = ry(1 - y/K) - y^2/(1 + y^2)$$
The last term is a predation term corresponding to the amount of insects eaten by birds.
a) Notice that the predation term is a sigmoidal function, and describe what might be the biological reason for such a shape, as opposed to a nonsigmoidal function $y/(1 + y)$.
b) Find the steady states of the system by setting $y' = 0$ and dividing by y on both sides. Plot each of the functions $r(1-y/K)$ and $y/(1+y^2)$ on their own and study where they intersect, for different values of r and K.
c) Show that for a fixed large value of K, as r grows the system transitions from having one steady state, to bistability, and then one steady state again. What kind of behavior is this? Explain.

4. Consider a general model of insect outbreak, where $y(t)$ is the insect population:

$$\frac{dy}{dt} = ay(1 - y/K) - b\frac{y^2}{r^2 + y^2}$$

Show that by carrying out a nondimensionalization of this system, it can be reduced to the model

$$\frac{dz}{d\tau} = \rho z(1 - z/\kappa) - \frac{z^2}{1 + z^2},$$

where z is a nondimensional variable and ρ, κ are nondimensional parameters.

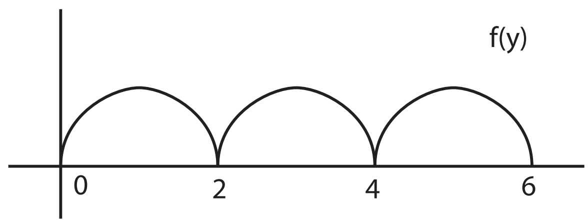

Figure 1:

5. Consider the linear system $y_1' = y_2$, $y_2' = -2y_1 - 3y_2$.
a) Find the eigenvalues of the system.
b) For each eigenvalue, find one corresponding eigenvector.
c) Find the general solution of the ODE, and the solution for $y_1(0) = 4$, $y_2(0) = 3$.
d) Draw a rough phase plot of the system.

6. Consider the linear system $y_1' = 3y_1 + 9y_2$, $y_2' = -4y_1 - 3y_2$.
a) Find the eigenvalues of the system.
b) For each eigenvalue, find one corresponding eigenvector.
c) Find the general solution of the ODE, by finding two real valued solutions and multiplying each by a general contstant.
d) Draw a rough phase plot of the system.

7. Suppose that a linear $y' = Ay$ system with four variables has eigenvalues $\lambda_1 = -2$, $\lambda_2 = 1$, $\lambda_3 = -1 - i$, $\lambda_1 = -1 + i$, and respective eigenvectors $w_1 = [1\ 0\ 2\ 0]^t$, $w_2 = [3\ 1\ 0\ 1]^t$, $w_3 = [2i\ 0\ 1\ \ -i]^t$, and $w_4 = [-2i\ 0\ 1\ i]^t$. Here the t stands for the transposition of the vector, so that each of these is a column vector.

a) Find a solution to the system associated to each eigenvalue and eigenvector pair. Notice that some of these solutions might have complex values.
b) Find four different, real valued solutions of the system.
c) Using the information above, find the general solution of the linear system $y' = Ay$.
d) What can you conclude about the stability of the solutions of this system around the steady state $y = [0\ 0\ 0\ 0]^t$? Explain