MATH 227A: Mathematical Biology
Homework 4

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Problem 1

Use Matlab command $[P, D] = \text{eig}(A)$ (or a similar command in another programming language) to calculate the eigenvalues and eigenvectors of the following systems. The output of this function is such that $A = PDP^{-1}$. Write the general solution formula and a phase diagram including the relevant eigenvectors. *Note:* Matlab normalizes the eigenvectors, you might want to rescale them to obtain a simpler form e.g. $(1, 2)$ instead of $(0.44, 0.88)$.

a) $x' = -3x + 4y$; $y' = -2x + 3y$

For the given ODE system, we have,

$$A = \begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix}$$

Matlab produces the following diagonal matrix D.

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1}$$

So, the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 1$. We now compute the eigenvectors by solving for $(A - \lambda I) \cdot v_1 = 0$. For $\lambda_1 = -1$,

$$\begin{pmatrix} -3 - \lambda_1 & 4 \\ -2 & 3 - \lambda_1 \end{pmatrix} \cdot v_1 = 0$$

$$\begin{pmatrix} -2 & 4 \\ -2 & 4 \end{pmatrix} \cdot v_1 = 0$$

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

is an eigenvector. Similarly, for $\lambda_2 = 1$,

$$\begin{pmatrix} -3-\lambda_2 & 4 \\ -2 & 3-\lambda_2 \end{pmatrix} \cdot v_2 = 0$$
$$\begin{pmatrix} -4 & 4 \\ -2 & 2 \end{pmatrix} \cdot v_2 = 0$$
$$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is the second eigenvector. Given the eigenvalues $\lambda_1 = -1$, $\lambda_2 = 1$ and their corresponding eigenvectors v_1 and v_2, we have the following two solution components.

$$\begin{aligned} z(t) &= v_1 e^{\lambda_1 t} \\ &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} \qquad (2) \\ w(t) &= v_2 e^{\lambda_2 t} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} \qquad (3) \end{aligned}$$

Combining the solutions, we have the general solution

$$\begin{aligned} y(t) &= c_1 z(t) + c_2 w(t) \\ &= c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} \qquad (4) \end{aligned}$$

The phase plot of the system is shown in Figure 1. Since there is one positive real and one negative real eigenvalue, the system is an **unstable saddle**.

b) $x' = 5x + 2y$; $y' = -17x - 5y$

For the given ODE system, we have,

$$A = \begin{pmatrix} 5 & 2 \\ -17 & -5 \end{pmatrix}$$

Matlab produces the following diagonal matrix D.

$$D = \begin{pmatrix} 3i & 0 \\ 0 & -3i \end{pmatrix} \qquad (5)$$

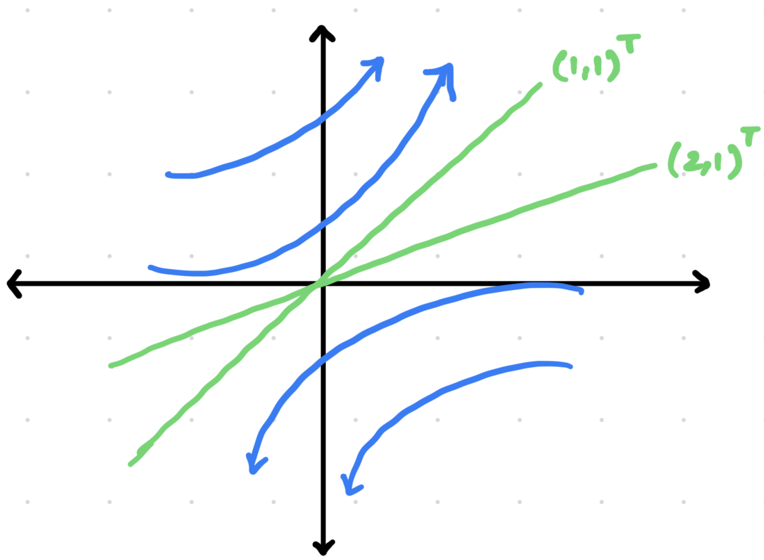

Figure 1: Phase plot of the ODE system in 1a. Eigenvectors are shown in green. Trajectories are shown in blue. This is a saddle node. x and y axes are horizontal and vertical, respectively.

So, the eigenvalues are $\lambda_1 = 3i$ and $\lambda_2 = -3i$. We now compute the eigenvectors by solving for $(A - \lambda I) \cdot v_1 = 0$. For $\lambda_1 = 3i$,

$$\begin{pmatrix} 5-\lambda_1 & 2 \\ -17 & -5-\lambda_1 \end{pmatrix} \cdot v_1 = 0$$

$$\begin{pmatrix} 5-3i & 2 \\ -17 & -5-3i \end{pmatrix} \cdot v_1 = 0$$

$$\begin{aligned} v_1 &= \begin{pmatrix} -5-3i \\ 17 \end{pmatrix} \\ &= \begin{pmatrix} -5 \\ 17 \end{pmatrix} - i \begin{pmatrix} 3 \\ 0 \end{pmatrix} \end{aligned}$$

is an eigenvector. Similarly, for $\lambda_2 = -3i$,

$$\begin{pmatrix} 5-\lambda_2 & 2 \\ -17 & -5-\lambda_2 \end{pmatrix} \cdot v_1 = 0$$

$$\begin{aligned}
\begin{pmatrix} 5+3i & 2 \\ -17 & -5+3i \end{pmatrix} \cdot v_2 = 0 \\
v_2 &= \begin{pmatrix} -5+3i \\ 17 \end{pmatrix} \\
&= \begin{pmatrix} -5 \\ 17 \end{pmatrix} + i \begin{pmatrix} 3 \\ 0 \end{pmatrix}
\end{aligned}$$

is the second eigenvector. Using the eigenvalue $\lambda_2 = -i3$ and its corresponding eigenvector v_2, we have the following two real-valued solutions arise from the real and imaginary parts of v_2,

$$\begin{aligned}
y_{Re}(t) &= \begin{pmatrix} -5 \\ 17 \end{pmatrix} e^{0\cdot t} \sin(-3t) + \begin{pmatrix} -5 \\ 17 \end{pmatrix} e^{0\cdot t} \cos(-3t) \\
&= -\begin{pmatrix} -5 \\ 17 \end{pmatrix} \sin(3t) + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \cos(3t) \qquad (6) \\
y_{Im}(t) &= \begin{pmatrix} -5 \\ 17 \end{pmatrix} e^{0\cdot t} \cos(-3t) - \begin{pmatrix} 3 \\ 0 \end{pmatrix} e^{0\cdot t} \sin(-3t) \\
&= \begin{pmatrix} -5 \\ 17 \end{pmatrix} \cos(3t) + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \sin(3t) \qquad (7)
\end{aligned}$$

Combining the solutions by multiplying (6) by c_1 and (7) by c_2, we have

$$y(t) = \begin{pmatrix} -5 \\ 17 \end{pmatrix} (-c_1 \sin 3t + c_2 \cos 3t) + \begin{pmatrix} 3 \\ 0 \end{pmatrix} (c_1 \cos 3t + c_2 \sin 3t) \qquad (8)$$

The phase plot of the system is shown in Figure 2. Since the real parts of the both complex eigenvalues are zero, the system has closed elliptical orbits about the origin.

Problem 2

Similarly as in the previous problem, except for the system $x' = 3x - 4y$, $y' = x - y$.

a) Use the command eig(A) as in the previous problem, and realize that this matrix is not diagonalizable (Why?).

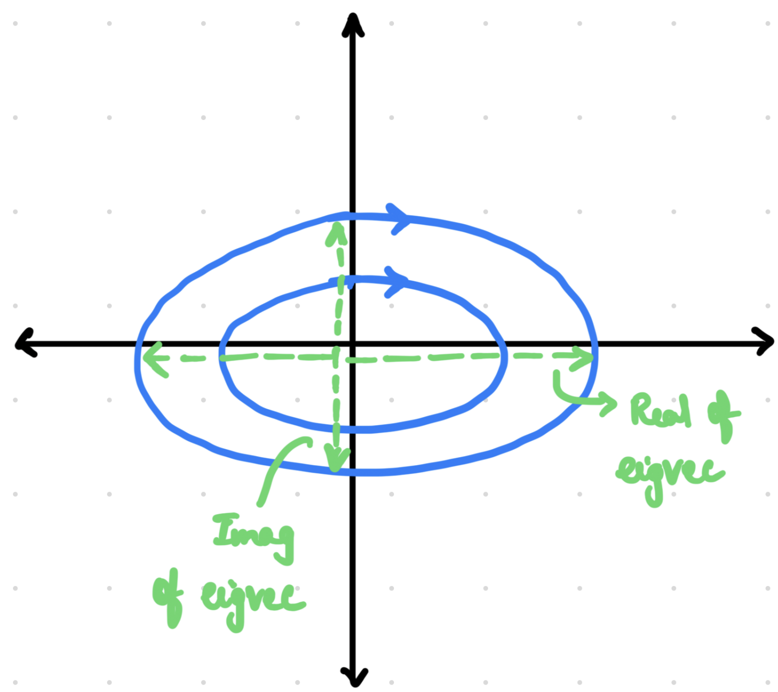

Figure 2: Phase plot of the ODE system in 1b. Eigenvectors are shown in green. Trajectories are shown in blue. This is a closed orbit. x and y axes are horizontal and vertical, respectively.

The system can be written in the form $y' = Ay$ where,

$$A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$$

Running eig(A) in Matlab returns a matrix with two identical eigenvalues, $\lambda_1 = \lambda_2 = 1$; i.e., the eigenvectors matrix is not full-rank. This means that A is defective, and not diagonalizable. Specifically, it cannot be decomposed into a full linearly independent set of eigenvectors.

b) Use instead the command $[P, J] = \text{jordan}(A)$ to find the associated Jordan matrix J such that $A = PJP^{-1}$. Write the general solution of the system algebraically.

Matlab finds the Jordan matrix J and model matrix P as follows,

$$J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
$$P = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

We now solve the system $z' = Jz$ for each Jordan block. J has exactly one Jordan block with $\lambda = 1$. Computing the matrix exponential and the solution,

$$\begin{aligned} e^{tJ} &= e^t \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \\ z(t) &= e^{tJ} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= e^t \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \end{aligned}$$

So, the general solution can be written as

$$\begin{aligned} y(t) &= Pe^{tJ} \\ &= \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \begin{pmatrix} 2e^t & 2te^t + e^t \\ e^t & te^t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \end{aligned}$$

Problem 3

Describe the qualitative behavior of a system $y' = Ay$, where A is a diagonalizable matrix with the following eigenvalues. Do so based on the form of the general solution of the system.

Let the eigenvalues be λ_i and their corresponding eigenvectors be $w_i = u_i + iv_i$ for the given ODE systems.

a) $\lambda =$ -1, -3, 4, 2

The general solution is

$$y(t) = c_1 e^{-t} w_1 + c_2 e^{-3t} w_2 + c_3 e^{4t} w_3 + c_4 e^{2t} w_4$$

Since we have only real eigenvalues and some of them are positive, some exponential terms can go to infinity as $t \to \infty$ and others can go to zero. So, the system is an **unstable saddle**.

b) λ = -2, $-1 \pm 3i$, -3

The general solution is

$$y(t) = c_1 e^{-2t} w_1 + c_3 e^{-3t} w_3 + c_4 e^{-t}(\cos 3t + i \sin 3t) w_4 + c_2 e^{-t}(\cos 3t - i \sin 3t) w_2$$

Since all real eigenvalues and real components of imaginary eigenvalues are negative, all exponential terms go to zero at $t \to \infty$. The complex eigenvalues give rise to oscillations. So, the system is a **stable inward spiral**.

c) λ = -2, 0, 1, $2 \pm i$

The general solution is

$$y(t) = c_1 e^{-2t} w_1 + c_2 w_2 + c_3 e^{t} w_3 + c_4 e^{2t}(\cos t + i \sin t) w_4 + c_5 e^{2t}(\cos t - i \sin t) w_5$$

Since there are real positive eigenvalues, some exponential terms go to infinity as $t \to \infty$. The complex eigenvalues give rise to oscillations. So, the system is an **unstable outward spiral**.

d) $\lambda = -3 \pm 4i$, -2, -1

The general solution is

$$y(t) = c_1 e^{-3t}(\cos 4t + i \sin 4t) w_1 + c_2 e^{-3t}(\cos 4t - i \sin 4t) w_2 + c_3 e^{-2t} w_3 + c_4 e^{-t} w_4$$

Since all real eigenvalues and real components of complex eigenvalues are negative, all exponential terms go to zero as $t \to \infty$. The complex eigenvalues give rise to oscillations. So, the system is an **unstable outward spiral**.

Problem 4

Consider the system $y_1' = y_1 - y_2$, $y_2' = y_1 + y_2$.

a) Write the ODE in format $y' = Ay$, and show that it has eigenvalues $\lambda_1 = 1 + i$, $\lambda_2 = 1 - i$, with eigenvectors $w^{(1)} = (i, 1)^T$, $w^{(2)} = (-i, 1)^T$, where T stands for the transpose operation that transforms row vectors into column vectors.

The ODE system can be written in the form $y' = Ay$ where,

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

To find the eigenvalues, we solve for λ in $det(A - \lambda I) = 0$.

$$\begin{aligned} det(A - \lambda I) &= 0 \\ det \begin{pmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix} &= 0 \\ \lambda^2 - 2\lambda + 2 &= 0 \\ &\lambda_1 = 1+i, \lambda_2 = 1-i \end{aligned}$$

We now solve $(A - \lambda I) \cdot v = 0$ to obtain the eigenvectors. For the eigenvalue $\lambda_1 = 1 + i$,

$$\begin{aligned} \begin{pmatrix} 1-\lambda_1 & -1 \\ 1 & 1-\lambda_1 \end{pmatrix} \cdot w_1 &= 0 \\ \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \cdot w_1 &= 0 \\ w_1 &= \begin{pmatrix} i \\ 1 \end{pmatrix} \end{aligned}$$

Now solving for $\lambda_2 = 1 - i$,

$$\begin{aligned} \begin{pmatrix} 1-\lambda_2 & -1 \\ 1 & 1-\lambda_2 \end{pmatrix} \cdot w_2 &= 0 \\ \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \cdot w_2 &= 0 \\ w_2 &= \begin{pmatrix} -i \\ 1 \end{pmatrix} \end{aligned}$$

b) The general solution of this system is $y(t) = c_1 e^{\lambda_1 t} w^{(1)} + c_2 e^{\lambda_2 t} w^{(2)}$. However, this is in general a complex solution. By properly choosing the value of c_1, c_2, show by direct calculation (i.e. do not use the theorem for solutions with complex eigenvalues) that the functions $x(t) = e^t \sin tu + e^t \cos tv$, $z(t) = e^t \cos tu - e^t \sin tv$ are real valued cases of the general solution of the ODE, where u, v are such that $w^{(1)} = u + iv$, $w^{(2)} = u - iv$. *Hint: use the fact that* $e^{ix} = \cos x + i \sin x$, *and try eg* $c_1 = c_2 = 1$, *or* $c_1 = -i$, $c_2 = i$

Given the general solution,

$$y(t) = c_1 e^{\lambda_1 t} w^{(1)} + c_2 e^{\lambda_2 t} w^{(2)} \tag{9}$$

Using the values $c_1 = 1$ and $c_2 = 1$; and $w^{(1)} = u + iv$ and $w^{(2)} = u - iv$,

$$\begin{aligned}
y(t) &= e^{(1+i)t} w^{(1)} + e^{(1-i)t} w^{(2)} \\
&= e^t\, e^{it}\, (u + iv) + e^t\, e^{-it}\, (u - iv) \\
&= e^t\ (\cos t + i \sin t)\, (u + iv) + e^t\ (\cos t - i \sin t)\, (u - iv) \\
&= 2\, e^t\, (u \cdot \cos t - v \cdot \sin t)
\end{aligned}$$

which is a real solution. So, dividing the solution by 2 also produces a real solution.

$$y(t) = e^t\, (u \cdot \cos t - v \cdot \sin t) = z(t) \tag{10}$$

Now, using the values $c_1 = -i$ and $c_2 = i$; and $w^{(1)} = u + iv$ and $w^{(2)} = u - iv$ with (9),

$$\begin{aligned}
y(t) &= -i\, e^{(1+i)t} w^{(1)} + i\, e^{(1-i)t} w^{(2)} \\
&= -i\, e^t\, e^{it}\, (u + iv) + i\, e^t\, e^{-it}\, (u - iv) \\
&= e^t\ (-i \cos t + \sin t)\, (u + iv) + e^t\ (i \cos t + \sin t)\, (u - iv) \\
&= 2\, e^t\, (u \cdot \sin t + v \cdot \cos t)
\end{aligned}$$

which is a real solution. So, dividing the solution by 2 also produces a real solution.

$$y(t) = e^t\, (u \cdot \sin t + v \cdot \cos t) = x(t) \tag{11}$$

Hence, by direct calculation, we find two real solutions, (10) and (11), from the general solutions simply by plugging in values for the constants c_1 and c_2.

Problem 5

Consider the system $z' = Jz$, where J is $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$.

a) Show by solving the corresponding ODEs, starting with z_3', the general solution of this problem.

Given the Jordan matrix J, say we have fixed vectors v_1, v_2, v_3 forming a Jordan chain of length three. We note by comparing J and $z' = Jz$ that z_1 is dependent on z_2 and z_2 on z_1. Correspondingly, we infer that v_3 is the independent eigenvector for eigenvalue λ, and that v_2 and v_1 are chained starting from v_3. So, we begin by solving for z_3'.

$$\begin{aligned}
\begin{pmatrix} z_1' \\ z_2' \\ z_3' \end{pmatrix} &= \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \\
&\Longrightarrow z_3' = \lambda z_3 \\
&\Longrightarrow z_3(t) = c_3\, e^{\lambda t} \qquad (12)
\end{aligned}$$

We now expand and solve for z_2 using z_3 from (12),

$$\begin{aligned}
z_2' &= \lambda z_2 + z_3 \\
&= \lambda z_2 + c_3\, e^{\lambda t} \\
z_2' - \lambda z_2 &= c_3\, e^{\lambda t} \qquad (13)
\end{aligned}$$

Comparing (13) with the standard linear ODE form, $y'(t) + y(t) \cdot p(t) = q(t)$, we have $p(t) = -\lambda$ and $q(t) = c_3 e^{\lambda t}$.

We now define an integrating factor $E(t)$

$$\begin{aligned}
E(t) &= e^{\int_0^t p(m)\, dm} \\
&= e^{\int_0^t -\lambda\, dm} \\
&= e^{-\lambda t}
\end{aligned}$$

Using $(E(t) \cdot z(t))' = E(t) \cdot q(t)$ and integrating on both sides wrt dt

$$\begin{aligned}
E(t) \cdot z_2(t) &= \int E(t) \cdot q(t)\, dt \\
z_2(t) &= \frac{1}{E(t)} \int E(t) \cdot q(t)\, dt \\
&= e^{\lambda t} \int e^{-\lambda t} \cdot c_3 e^{\lambda t}\, dt \\
&= e^{\lambda t}\, (c_2 + c_3\, t) \qquad (14)
\end{aligned}$$

Finally, we expand and solve for z_1 using z_2 from (14),

$$\begin{aligned}
z_1' &= \lambda z_1 + z_2 \\
&= \lambda z_1 + e^{\lambda t}(c_2 + c_3\, t) \\
z_1' - \lambda z_1 &= e^{\lambda t}\, (c_2 + c_3\, t) \qquad (15)
\end{aligned}$$

Comparing (15) with the standard linear ODE form, $y'(t) + y(t) \cdot p(t) = q(t)$, we have $p(t) = -\lambda$ and $q(t) = e^{\lambda t}(c_2 + c_3\, t)$. Since we have the same $p(t)$ as before (13), we have the same integrating factor $E(t) = e^{-\lambda t}$. Using $(E(t) \cdot z(t))' = E(t) \cdot q(t)$ and integrating on both sides wrt dt

$$\begin{aligned}
E(t) \cdot z_1(t) &= \int E(t) \cdot q(t)\, dt \\
z_1(t) &= \frac{1}{E(t)} \int E(t) \cdot q(t)\, dt \\
&= e^{\lambda t} \int e^{-\lambda t} \cdot e^{\lambda t}(c_2 + c_3\, t)\, dt \\
&= e^{\lambda t} \left(c_1 + c_2\, t + c_3\, \frac{t^2}{2} \right) \qquad (16)
\end{aligned}$$

Combining the solutions for z' for (12), (14), and (16), we have,

$$z' = e^{\lambda t} \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \qquad (17)$$

which is the expected general solution.

b) Suppose that $y' = Ay$ is a given linear system such that $A = PJP^{-1}$, where J is the matrix above. Write the general solution of this system in terms of the above system and the unknown matrix P.

Since $A = PJP^{-1}$, J is the Jordan matrix of A. Defining a new coordinate vector $z(t) = P^{-1}y(t)$, we have,

$$z(t) = P^{-1}y(t) = P^{-1}Ay(t) = P^{-1}APz(t) = Jz(t)$$

To transform back to the solution $e^{Jt}c$ original coordinates, we multiply by P on either side to get,

$$y(t) = Pz(t) = Pe^{Jt}c = Pe^{Jt}P^{-1}c$$

where c is the vector of integration constants. We already have the solution of the system $z'(t) = Jz(t)$ in (17). Defining a new vector of constants $C = P^{-1}c$, we have the solution for $y' = Ay$,

$$\begin{aligned}
y(t) &= P\, e^{Jt} \\
&= P\, e^{\lambda t} \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} C
\end{aligned}$$

where $c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ is the vector of constants.