MATH 227A: Mathematical Biology
Homework 2

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Problem 1

Part (i)

Study the steady states of the following ODE $y' = f(y)$, and determine their stability. Draw on a separate diagram the graphs of two sample solutions for each system.

a) $y' = 2y - 2y^3$

Given the ODE

$$y' = 2y - 2y^3 \tag{1}$$

To find the steady states we set $y' = 0$,

$$\begin{aligned} 2y - 2y^3 &= 0 \\ 2y \cdot (1 - y^2) &= 0 \\ 2y \cdot (1 - y) \cdot (1 + y) &= 0 \end{aligned} \tag{2}$$

The steady states are at $y = 0$, $y = -1$, and $y = 1$, and are shown in Figure 1. B ($y = 1$) and C ($y = -1$) are stable steady states as y' is positive on their left and negative on their right. A ($y = 0$) is an unstable steady state as the gradients are directed away on either side.

b) $y' = e^{-y} \cdot \sin y$

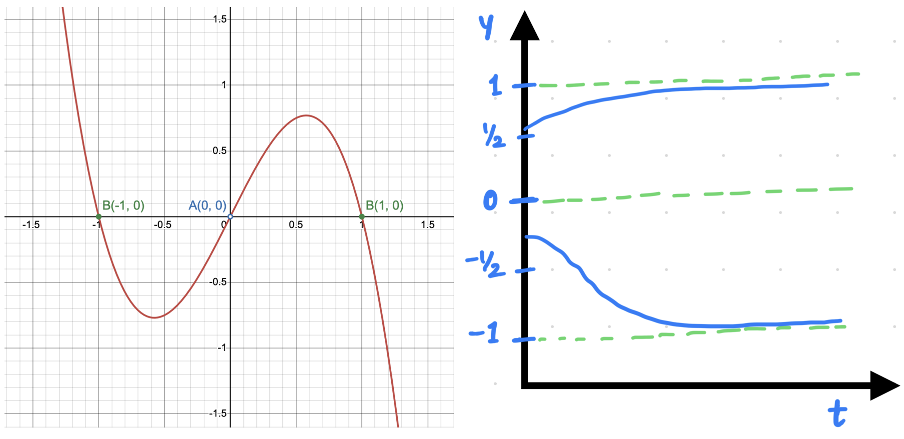

Figure 1: A: Phase plot of the system $y' = 2y - 2y^3$ with y' on the x-axis and y on the y-axis; B: Sample solutions of the system.

To find the steady states we set $y' = 0$,

$$\begin{aligned} y' &= e^{-y} \cdot \sin y \\ 0 &= e^{-y} \cdot \sin y \end{aligned} \tag{3}$$

The solutions are then,

$$\begin{array}{ll} e^{-y} = 0 & \sin y = 0 \\ \text{No solutions} & y = n\pi, \text{ where } n \in Z \end{array}$$

So, the system has steady states at $y = n\pi$, where $n \in Z$. The steady state is unstable at $y = 2n\pi$ for $n \in Z$ and stable at $y = (2n+1) \cdot \pi$ for $n \in Z$. The steady states are shown in Figure 2. B and C are examples of stable steady states, while A is an example of an unstable one.

Problem 2

Working backwards, find an equation for a system $y' = f(y)$, with exactly four steady states at y = 0, 1, 2, 3, and which are stable, unstable, stable, and unstable, respectively. Is there more than one system with the same steady states and stability? Explain.

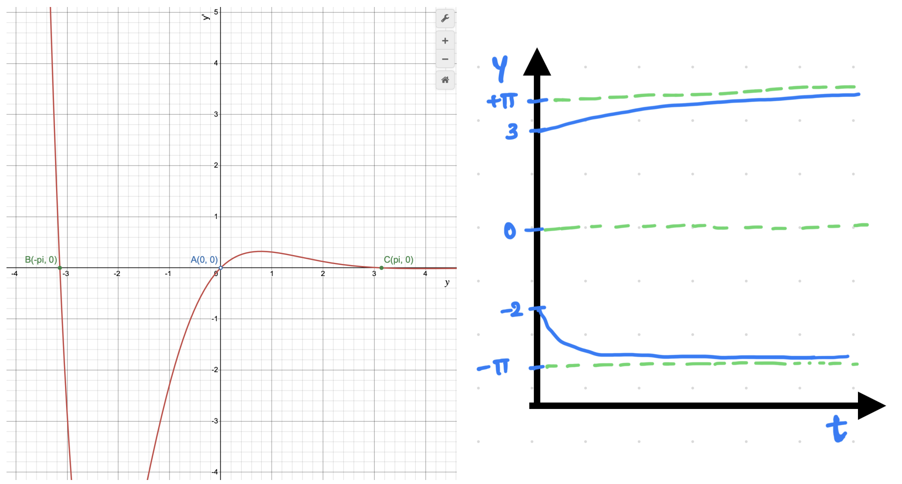

Figure 2: A: Phase plot of the system $y' = e^{-y} \cdot \sin y$ with y' on the x-axis and y on the y-axis; B: Sample solutions of the system.

If the steady states occur at $y = 0, 1, 2, 3$, then y, $(y-1)$, $(y-2)$, and $(y-3)$ could be their corresponding factors of $y' = f'(y)$ among several other possibilities. Multiplying the factors, we get a candidate function for $f(y)$. Since $y = 0$ is a stable point, we need the function to be concave upward; so, the coefficient of y^4 must positive.

$$\begin{aligned} y' &= y \cdot (y-1) \cdot (y-2) \cdot (y-3) \\ &= y^4 - 6y^3 + 11y^2 - 6y \end{aligned} \tag{4}$$

The plot for the function in (4) is shown in Figure 3. As required, the steady states are stable, unstable, stable, and unstable, respectively.

Sketched on the same curve in Figure 3 are other candidate functions for $f(y)$ that still only pass through $y = 0, 1, 2, 3$ for $y' = 0$, and thus have their steady states at those same points—these were generated by multiplying the RHS of (4) by a constant c. Specifically, the dashed blue curve is obtained using $c = 0.5$ and the dotted blue line using $c = 2$. Clearly, there are infinitely many curves that could be drawn following the same logic; thus, **there is more than one system** that can have the same set of steady states and stability. Furthermore, there are also other completely unrelated functions that could be drawn through the same four steady states.

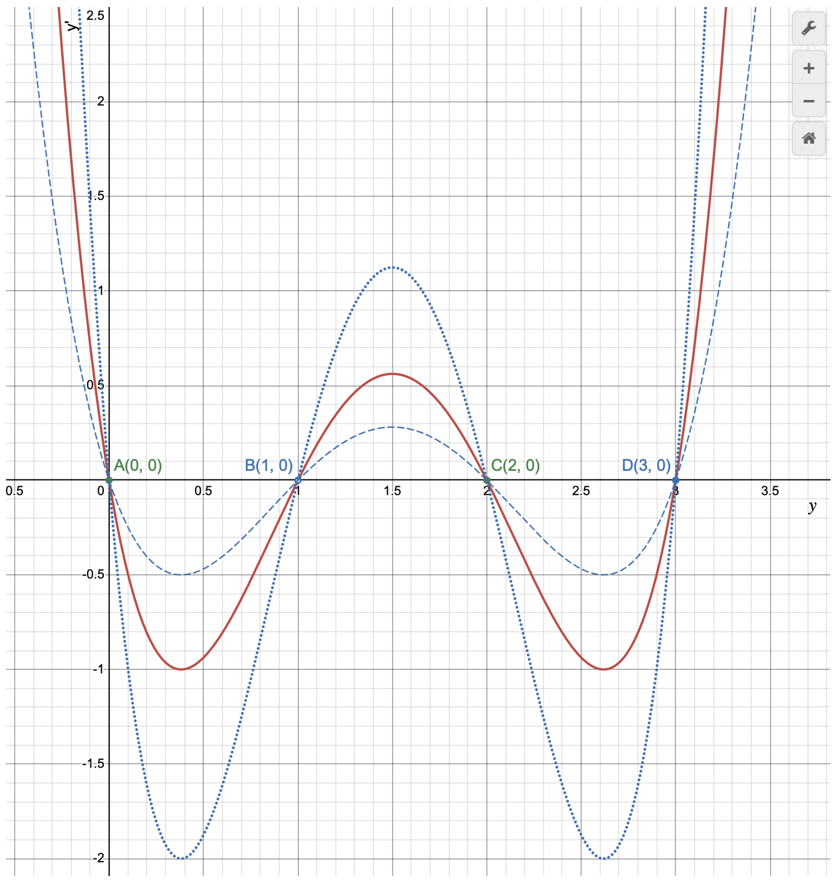

Figure 3: Phase plot of the system with steady states at $y = 0, 1, 2, 3$. The red plot shows the function in (4). The blue plots show other candidate functions. Specifically, the dashed plot is obtained by multiplying the RHS of (4) by 0.5 and the dotted plot by multiplying by 2. They all have the same steady states.

Problem 3

Consider the ODE $y' = y^2 - 1 + r$.

a) Draw a phase plot for two different values of the bifurcation parameter.

The phase plots are shown in Figure 4 for $r = 0$ and $r = 1.5$. The system has one stable and one unstable steady state when $r < 1$, no steady states when $r > 1$, and one saddle node (stable on one side and unstable on the other) when $r = 0$ (not shown).

b) Draw the bifurcation diagram for this system, and determine what type

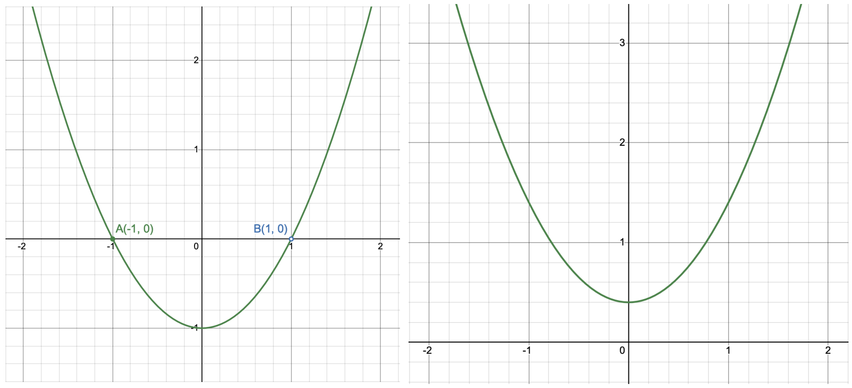

Figure 4: Phase plots of the system $y' = y^2 - 1 + r$ with y' on the y-axis and y on the x-axis for $r = 0$ (left) and $r = 1.5$ (right). On the left, A is a stable steady state while B is unstable.

of bifurcation it is.

To arrive at the bifurcation diagram, we set $y' = 0$,

$$\begin{aligned} y' &= y^2 - 1 + r \\ 0 &= y_{ss}^2 - (1 - r) \\ y_{ss}^2 &= (1 - r) \end{aligned} \tag{5}$$

So, the steady state of the system is described by (5). This is sketched as a bifurcation plot in 5. As expected based on the phase plot, the system does not have steady states for $r > 1$, and has two steady states for $r < 1$. The steady states with $y_{ss} < 0$ are stable, while those with $y_{ss} > 0$ are unstable. At $r = 0$ there is one steady state that is unstable upward and stable downward. Hence, this is a **saddle node bifurcation**.

Problem 4

Consider the ODE $y' = -r \cdot \ln y + y - 1$.

a) In order to study the steady states for different values of r, set the ODE equal to zero, that is, $r \cdot \ln y = y - 1$. Plot both functions (right and left hand side) on the same graph, and show how this plot can be used to graphically display the steady states for every value of r. Show that $y = 1$ is a steady state for every value of r.

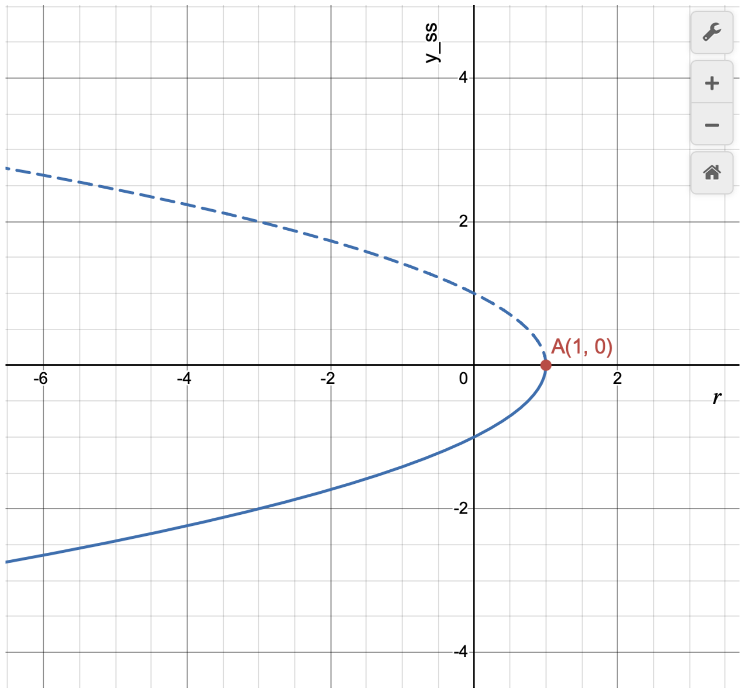

Figure 5: Bifurcation plot of the system $y' = y^2 - 1 + r$ with y_{ss} on the y-axis and r on the x-axis. The system has stable steady states for $y < 0$ and unstable ones for $y > 0$. A(1, 0) is the bifurcation point.

Setting $y' = 0$,

$$y' = -r \cdot \ln y + y - 1 \tag{6}$$

$$0 = -r \cdot \ln y_{ss} + y_{ss} - 1 \tag{7}$$

$$= -g(y_{ss}) + h(y_{ss})$$

$$g(y_{ss}) = h(y_{ss}) \tag{8}$$

where $g(y_{ss}) = r \cdot \ln y_{ss}$ and $g(y_{ss}) = y_{ss} - 1$.

The plot of $g(y_{ss})$ and $h(y_{ss})$ from (8) is shown on a phase plot in Figure 6 for two choices of r. Since $y' = -g(y_{ss}) + h(y_{ss})$, the point(s) at which the two functions intersect on the phase plot are the points at which $y' = 0$, i.e., the steady states. The plot of $g(y)$ shifts on the phase plot for different choices of r; this in turn moves the points of intersection between $g(y)$ and $h(y)$ and determines the steady states for that choice of r.

Setting $y = 1$ in (6),

$$\begin{aligned} y' &= -r \cdot \ln(1) + 1 - 1 \\ &= -r \cdot 0 + 0 \\ &= 0 \end{aligned}$$

Thus, regardless of the value of r, the system has $y' = 0$ at $y = 1$ and thus has a steady state at that point.

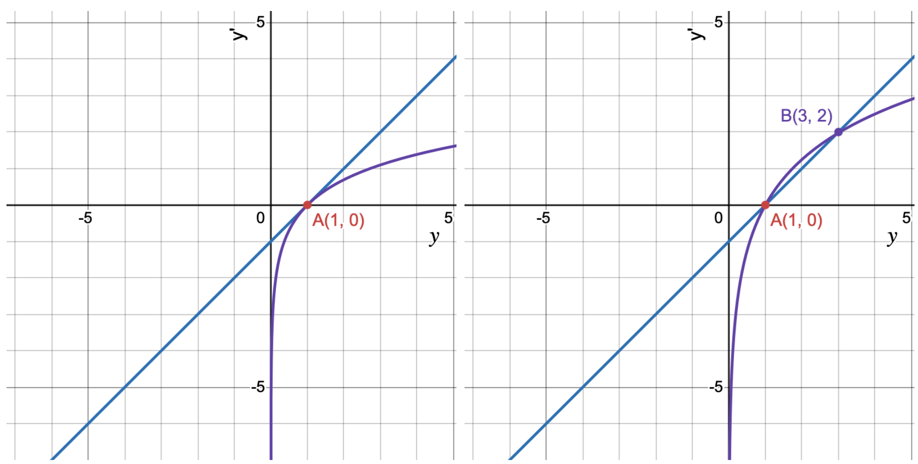

Figure 6: Plot of the functions comprising the system $y' = -r \cdot \ln y + y - 1$ for $r = 1$ (left) and $r = 1.8$ (right). The blue curve shows $h(y)$ and purple shows $g(y)$ as defined in (8).

c) Sketch a phase plot for the system when $0 < r < 1$ and $r > 1$. How many steady states are there in each case, and what is their stability?

Figure 8 shows the phase plot for the system (specifically, relation (7) is plotted) for $r = 0.5$ (left) and $r = 1.3$ (right). In both cases, there are two steady states and one of them is always at $y = 1$ as we showed previously. When $r = 0.5$, the steady state at $y = 1$ is unstable and the other one to its left is stable. When $r = 1.3$, the steady state at $y = 1$ is stable whole the one to its right is unstable.

b) Show that $\bar{r} = 1$ is a bifurcation point for this system, with the associated critical point $\bar{y} = 1$. What information do you need to know to show this?

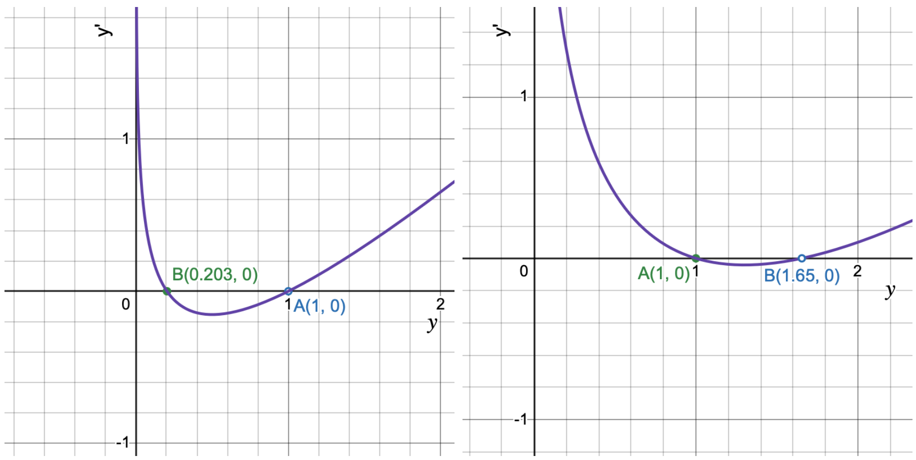

Figure 7: Phase plot of the system $y' = -r \cdot \ln y + y - 1$ for $r = 0.5$ (left) and $r = 1.3$ (right). On the left, A is an unstable steady state while B is stable. On the right, A is a stable steady state while B is unstable.

The given ODE in (6) can be expressed, as we did above, as a combination of two functions. Concretely,

$$y' = -g(y) + h(y)$$

where $g(y) = r \cdot \ln y$ and $h(y) = y - 1$. Thus, $y' > 0$ when $h(y) > g(y)$ and $y' < 0$ when $h(y) < g(y)$. So, by comparing the magnitudes of the two functions we can determine the direction of the system at any given state. To examine the behavior of the system in the neighborhood of $\bar{y} = 1$, we estimate the the ratio of $g(y)$ to $h(y)$. Specifically,

$$\begin{aligned}
\frac{g(y)}{h(y)} &= \lim_{y \to 1} \left(\frac{r \cdot \ln y}{y - 1} \right) \\
&= r \cdot \lim_{y \to 1} \left(\frac{\ln y}{y - 1} \right) \\
&= r \cdot \lim_{y \to 1} \left(\frac{1/y}{1} \right) && \text{[using L'Hopital's rule]} \\
&= r
\end{aligned}$$

In the neighborhood of the critical point $\bar{y} = 1$, since $r = 1$ and the slopes of $h(y)$ and $g(y)$ are equal, the two functions are tangential to one another at $y = 1$.

Now, we consider points at $r \neq 1$. When $r > 1$, we have $g(y) > h(y)$ making

$y' < 0$ and *vice versa*. Since their slopes are unequal, the curves cross each other and are not tangential. Thus, there must at least be one more point where they cross each other as a log curve must cross a linear curve at least twice if not tangential. The three scenarios for $r < 1$, $r > 1$, and $r = 1$ with $h(y)$ and $g(y)$ are depicted in Figure {. It is clear from these figures that $\bar{r} = 1, \bar{y} = 1$ is a bifurcation point in the system as it transitions from having two steady states to one at this point.

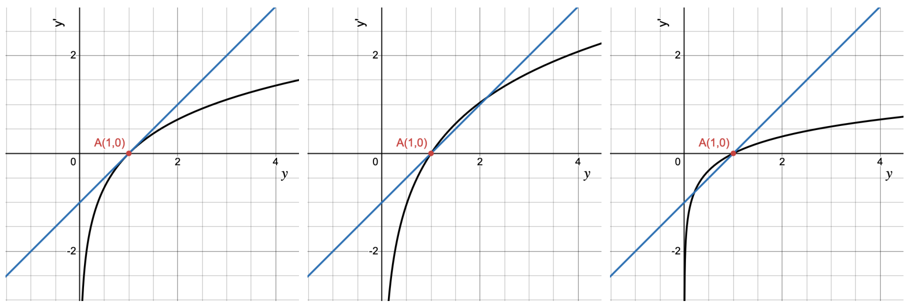

Figure 8: Plots of $h(y)$ and $g(y)$ for the system $y' = -r \cdot \ln y + y - 1$ for $r = 1$ (left), $r = 0.1.3$ (center), and $r = 0.6$ (right). A(1, 0) is a bifurcation point.

d) Sketch the bifurcation diagram for this system and determine the type of bifurcation (eg: saddle-node, transcritical).

To arrive at the bifurcation diagram, we set $y' = 0$,

$$\begin{aligned} 0 &= -r \cdot \ln y_{ss} + y_{ss} - 1 \\ r &= \frac{y_{ss} - 1}{\ln y_{ss}} \end{aligned} \tag{9}$$

The bifurcation plots is sketched in Figure 9 using the relation in (9). This is a **transcritical bifurcation** since the steady states exchange stabilities at the bifurcation point A(1,1). Specifically, the steady state at $y = 1$ changes from stable to unstable at A as r is increases, while the other steady state changes from unstable to stable at A.

Problem 5

Consider the ODE $y' = ay \cdot (y - b) \cdot \left(1 - \dfrac{y}{K}\right)$ encountered in ecological dynamics. Draw a bifurcation diagram for this system using the parameter K. What type of bifurcation is present?

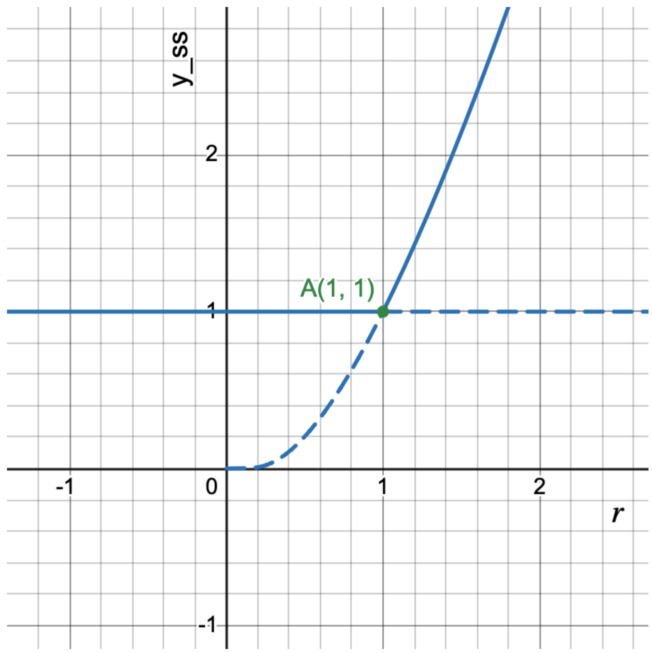

Figure 9: Bifurcation plot of the the system $y' = -r \cdot \ln y + y - 1$. A(1, 1) is the bifurcation point.

To arrive at the bifurcation diagram, we set $y' = 0$,

$$\begin{aligned}
y' &= ay \cdot (y - b) \cdot \left(1 - \frac{y}{K}\right) \\
0 &= ay_{ss} \cdot (y_{ss} - b) \cdot \left(1 - \frac{y_{ss}}{K}\right) \qquad (10) \\
\left(\frac{y_{ss}}{K}\right) \cdot ay_{ss} \cdot (y_{ss} - b) &= ay_{ss} \cdot (y_{ss} - b) \\
K &= \frac{y_{ss} \cdot ay_{ss} \cdot (y_{ss} - b)}{ay \cdot (y_{ss} - b)}
\end{aligned}$$

Based on (10), the system is at steady state at $y = 0$, $y = b$, and $y = K$. The phase plot and bifurcation diagrams are sketched in Figure 10.

The phase plots in Figure 10 A, B, and C show three different choices of parameters for a, b, and c. The system always has either two or three steady states. When $b = K$, two steady states (at $y = b$ and $y = K$) coincide and it becomes a saddle point. Likewise, when $b = 0$, two steady states (at $y = 0$ and

$y = b$) coincide and it becomes a saddle point. While such coincidence can also occur between $y = 0$ and $y = K$, the finite system is not defined for $K = 0$. In all other situations, the system has three steady states.

For the scenario $b < K$ shown in Figure 10C, which is $a = 1$, $b = 1$, $K = 2$, there are exactly three steady states of which A (at $y = 0$) and B (at $y = 2$) are stable while C (at $y = 1$) is unstable. However, when $b > K$ the steady states at $y = b$ and $y = K$ exchange stabilities. This is shown in Figure 10D, where $a = 1$, $b = 4$, $K = 2$. Here, the steady state corresponding to $b = 4$ is now stable while that associated with $K = 2$ is unstable.

The value of a determines the concavity and scale of the function. Furthermore, the corresponding stability of the steady states at $y = 0$, $y = b$, and $y = K$ are reversed between when $a > 0$ and $a < 0$. This is illustrated in Figure 10E, where $a = -1$, $b = 4$, $K = 2$. Compared to 10D, this system has its stabilities inverted—steady states that were stable are now unstable and *vice versa*.

Finally, Figure 10F shows the bifurcation plot of the system at $a = 1$, $b = 2$. The observations from earlier can be confirmed here. The system has a **transcritical bifurcation point** at $A(k = 2, y_{ss} = 2)$. Concretely, the system can exist in the following regimes for $a > 0$. When $a < 0$ all stabilities reversed.

- $0 < b < K$: Stable at $y = 0$ and $y = K$; unstable at $y = b$.
- $0 < K < b$: Stable at $y = 0$ and $y = b$; unstable at $y = K$.
- $b < K < 0$: Unstable at $y = b$ and $y = 0$; unstable at $y = K$.
- $K < b < 0$: Unstable at $y = K$ and $y = 0$; stable at $y = b$.
- $b < 0 < K$: Stable at $y = b$ and $y = K$; unstable at $y = 0$.
- $K < 0 < b$: Unstable at $y = K$ and $y = b$; unstable at $y = 0$.

Problem 6

Consider the system $y' = y^2 \cdot (5 - y) + 1 - ry$.

a) Carry out a mathematical analysis to show that this system presents hysteresis.

To arrive at the phase plot and bifurcation diagram, we set $y' = 0$,

$$\begin{aligned} y' &= y^2 \cdot (5 - y) + 1 - r \cdot y \\ &= -y^3 + 5y^2 - r \cdot y + 1 \end{aligned} \tag{11}$$

$$\begin{aligned} -y_{ss}^3 + 5y_{ss}^2 + 1 - r \cdot y_{ss} &= 0 \\ -y_{ss}^3 + 5y_{ss}^2 + 1 &= r \cdot y_{ss} \\ \frac{-1}{y_{ss}} \cdot \left(y_{ss}^3 - 5y_{ss}^2 - 1\right) &= r \end{aligned} \tag{12}$$

The relations in (11) and (12) are sketched as phase plot and bifurcation plot, respectively, in Figure 12.

From the phase plots we see that the system transitions form having one steady state to three steady states back to having just one steady state. Specifically, in Figure 12A with $r = 7$, there is one stable steady state. 12B shows $r = 6$ with three steady states where B is unstable while A and B are stable steady states. Finally, 12C shows the system with $r = 3$ that has exactly one stable steady state.

From the bifurcation plot in Figure 12D it is clear that the system presents hysteresis. As r increases from 0, the system has exactly one steady state until r increases just past 4. Then, system has three steady states as r gets closer to 7. Finally, the system switches back to having just one steady state. Since in some interval between 4 and 7 the system can exist at one of two steady states depending on the history of the system, (i.e., depending on where the system was just before reaching its current state), the system exhibits hysterisis.

b) Use Matlab to numerically plot the bifurcation graph of this system. You can use a Matlab function to find the roots of algebraic equations.

```
rvals = linspace(0, 9, 1000);
eps = 1e-2;
figure(); hold on;
for r=rvals
    % A.y^3 + B.y^2 + C.y + D = 0.
    ode_coeffs = [-1 5 -r 1];
    y_ss_vec = roots(ode_coeffs);
    for j=1:length(y_ss_vec)
        % plot only real solutions.
        if (abs(imag(y_ss_vec(j))) < eps)
            plot(r, y_ss_vec(j), 'b.');
        end
    end
end
hold off;
title('Bifurcation Plot')
xlabel('r');
ylabel('y at steady state');
```

```
prettyfig;
```

Listing 1: Matlab script to numerically generate the bifurcation graph of the system y'

The code used to generate the bifurcation graph of this system is shown in Listing 1, and the corresponding bifurcation plot is shown in Figure **??**. We obtain the same system as we did analytically in Figure 12D. It presents hysteresis roughly in the range $r \in [4, 6.75]$.

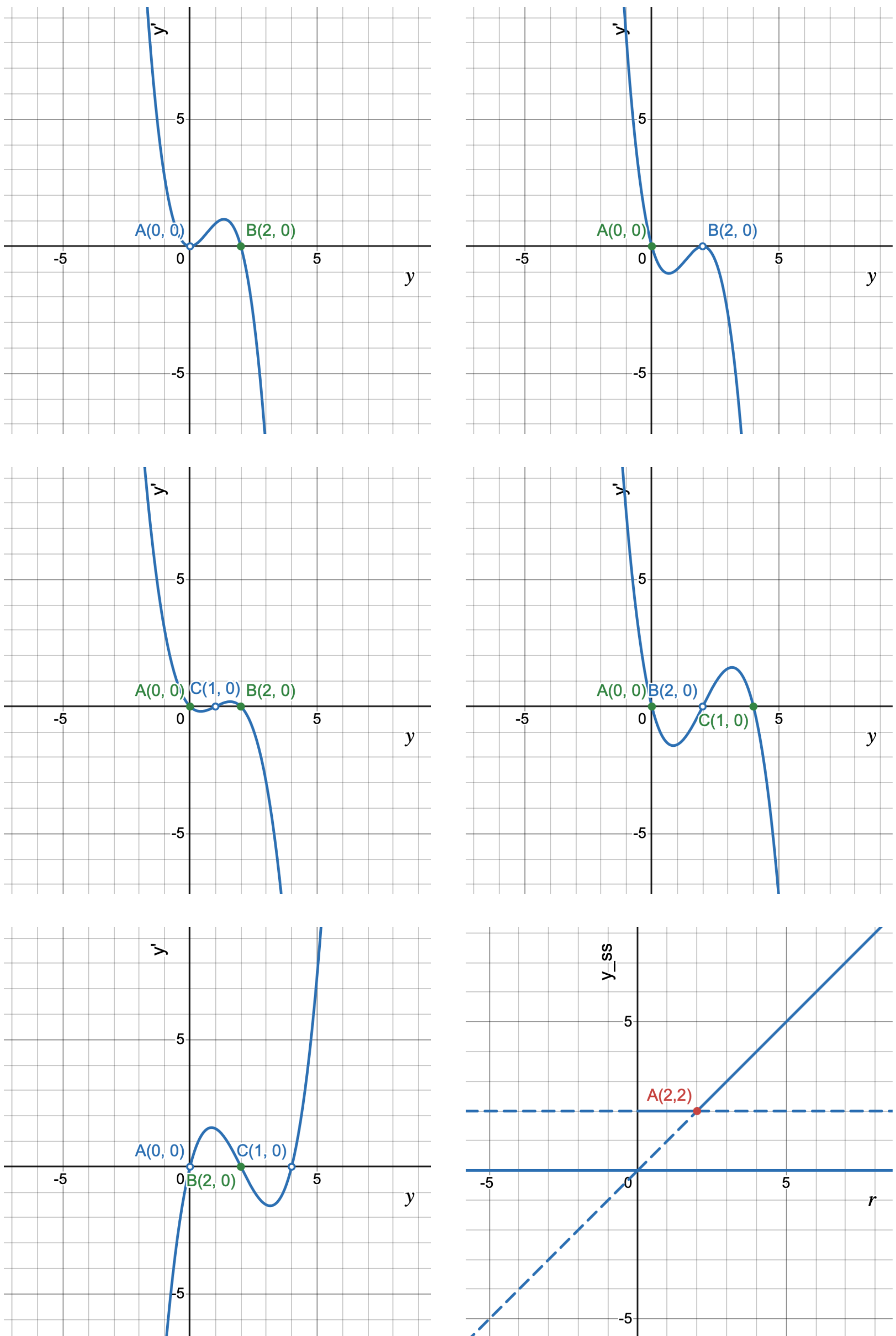

Figure 10: Analysis of the system $y' = ay \cdot (y-b) \cdot \left(1 - \dfrac{y}{K}\right)$. Subfigure F shows a transcritical bifurcation point at A(2,2).

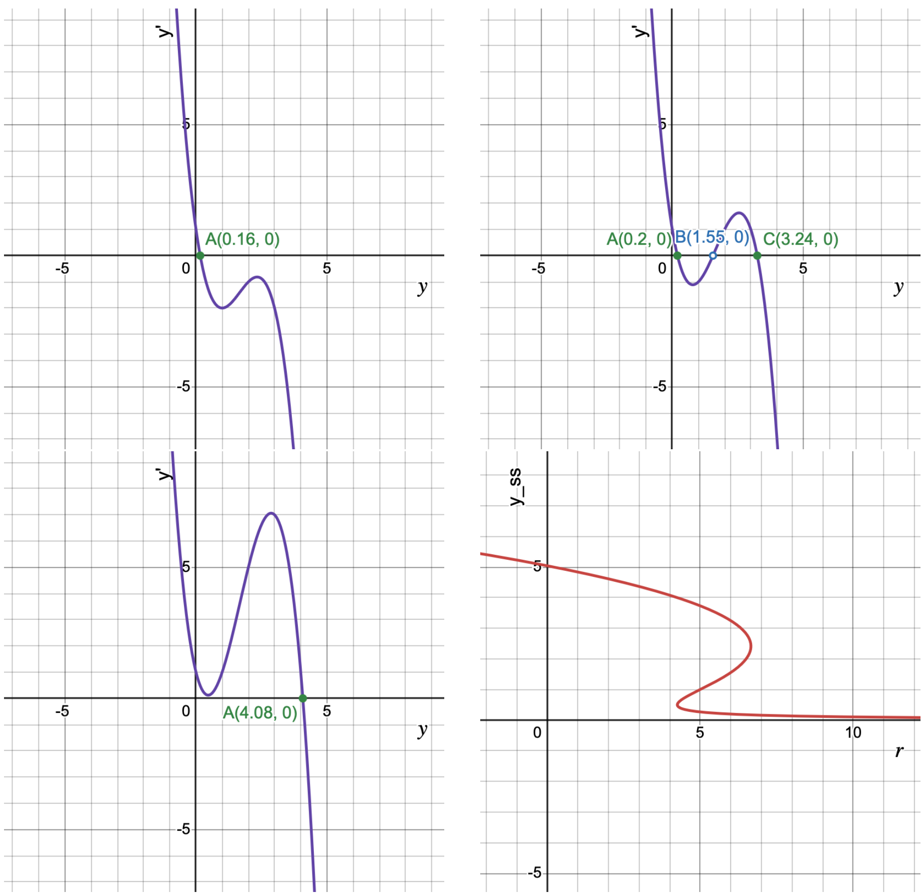

Figure 11: Analysis of the system $y' = y^2 \cdot (5 - y) + 1 - ry$. A, B, and C are phase plots for $r = 7$, $r = 6$, and $r = 3$, respectively. D is a bifurcation plot of the system exhibiting hysterisis.

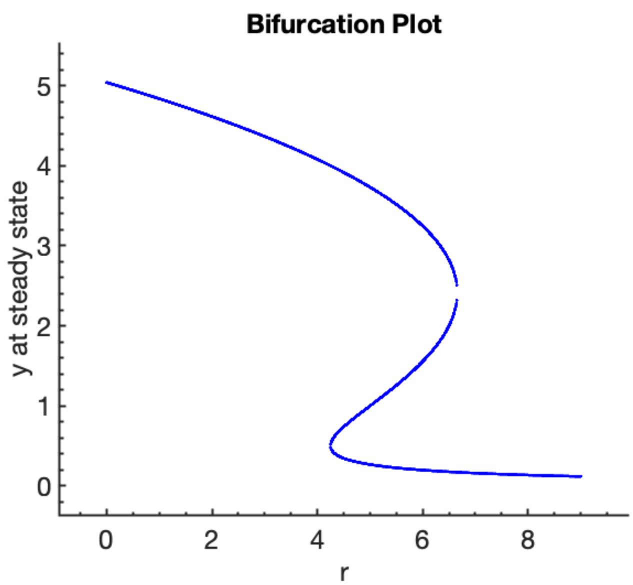

Figure 12: Analysis of the system $y' = y^2 \cdot (5 - y) + 1 - ry$. A, B, and C are phase plots for $r = 7$, $r = 6$, and $r = 3$, respectively. D is a bifurcation plot of the system exhibiting hysterisis.