Vectors and Matrices

Linear Algebra

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Data

Student	Mathematics	History
A	85	75
B	89	50
C	95	100
D	56	99
E	68	98

Vectors

Student	Mathematics	History
A	85	75
B	89	50
C	95	100
D	56	99
E	68	98

$$(85, 75), \quad (89, 50), \quad (95, 100), \quad (56, 99), \quad (68, 98)$$

85 and 75 are components of the vector $(85, 75)$

Matrices

Student	Mathematics	History
A	85	75
B	89	50
C	95	100
D	56	99
E	68	98

$$\begin{bmatrix} 85 & 75 \\ 89 & 50 \\ 95 & 100 \\ 56 & 99 \\ 68 & 98 \end{bmatrix}$$

Column Vector

Student	Mathematics	History
A	85	75
B	89	50
C	95	100
D	56	99
E	68	98

$$(85, 75)$$

$$\begin{bmatrix} 85 \\ 75 \end{bmatrix}$$

$$\begin{bmatrix} \color{red}{85} & 75 \\ \color{red}{89} & 50 \\ \color{red}{95} & 100 \\ \color{red}{56} & 99 \\ \color{red}{68} & 98 \end{bmatrix}$$

Row Vector

$$(85, 75)$$

$$\begin{bmatrix} 85 & 75 \end{bmatrix}$$

Student	Mathematics	History
A	85	75
B	89	50
C	95	100
D	56	99
E	68	98

$$\begin{bmatrix} 85 & 75 \\ \color{red}{89} & \color{red}{50} \\ 95 & 100 \\ 56 & 99 \\ 68 & 98 \end{bmatrix}$$

Vector Addition

$$(1, 2, 3) \ + \ (4, 5, 6) \ = \ (5, 7, 9)$$

$$(x_1, \ \cdots, x_n) + (y_1, \ \cdots, y_n) = (x_1 + y_1, \ \cdots, x_n + y_n)$$

Components are added

Scalar Multiplication

$$3 \cdot (1, 2, 3) \;=\; (3, 6, 9)$$

$$c \cdot (x_1, \; \cdots, x_n) = (cx_1, \; \cdots, cx_n)$$

Components are scaled

Linear Combination

$$2 \cdot (1, 2) \; + \; 3 \cdot (-1, 1) \; = \; (-1, 7)$$

$$c_1 x_1 + \; \cdots \; + c_m x_m$$

$$x_i = (x_{i1}, \; \cdots \; , x_{in})$$

$\mathbb{R}^n$

$\mathbb{R}$

line

$\mathbb{R}^n$

$$\mathbb{R}$$

line

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$$

plane

$\mathbb{R}^n$

$$\mathbb{R}$$

line

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$$

plane

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

space

$\mathbb{R}^n$

$\mathbb{R}$

line

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$$

plane

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

space

$$\mathbb{R}^n = \{(x_1, \cdots, x_n) \mid x_1, \cdots, x_n \in \mathbb{R}\}$$

$$M_{m \times n}(\mathbb{R})$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{bmatrix}$$

$$3 \times 4$$

$$M_{m\times n}(\mathbb{R})$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

$$M_{3\times 4}(\mathbb{R})$$

set of all 3×4 real matrices

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{bmatrix}$$

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$M_{m\times n}(\mathbb{R})$

set of all $m \times n$ real matrices

3×4

Matrix-Vector Multiplication

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 2 \cdot \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Linear combination of the columns

Matrix-Vector Multiplication

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 2 \cdot \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Linear combination of the columns

$$\begin{bmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 c_1 + \cdots + x_n c_n$$

Matrix-Vector Multiplication

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 2 \cdot \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Linear combination of the columns

$m \times n$ $\quad n \times 1$ $\quad m \times 1$

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$M_{m \times n}(\mathbb{R})$ $\quad \mathbb{R}^n$ $\quad \mathbb{R}^m$

Vector-Matrix Multiplication

$$\begin{bmatrix} 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} + (-1) \begin{bmatrix} 2 & 4 & 6 \end{bmatrix}$$

Linear combination of the rows

Vector-Matrix Multiplication

$$\begin{bmatrix} 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} + (-1) \begin{bmatrix} 2 & 4 & 6 \end{bmatrix}$$

Linear combination of the rows

$$\begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix} \begin{bmatrix} — & r_1^T & — \\ & \vdots & \\ — & r_m^T & — \end{bmatrix} = x_1 r_1^T + \cdots + x_m r_m^T$$

Vector-Matrix Multiplication

$$\begin{bmatrix} 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} + (-1) \begin{bmatrix} 2 & 4 & 6 \end{bmatrix}$$

Linear combination of the rows

$$1 \times m \qquad\qquad m \times n \qquad\qquad 1 \times n$$

$$\begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix} \begin{bmatrix} - & r_1^T & - \\ & \vdots & \\ - & r_m^T & - \end{bmatrix} = x_1 r_1^T + \cdots + x_m r_m^T$$

Vector-Matrix Multiplication

$$\begin{bmatrix} 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} + (-1)\begin{bmatrix} 2 & 4 & 6 \end{bmatrix}$$

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$$\begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix} \begin{bmatrix} — & r_1^T & — \\ & \vdots & \\ — & r_m^T & — \end{bmatrix} = x_1 r_1^T + \cdots + x_m r_m^T$$

$$\mathbb{R}^m \qquad M_{m\times n}(\mathbb{R}) \qquad \mathbb{R}^n$$

Vector-Vector Multiplication (Inner Product)

$$\begin{bmatrix} 1 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 3 \end{bmatrix} = -2$$

Dot product

Vector-Vector Multiplication (Inner Product)

$$\begin{bmatrix} 1 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 3 \end{bmatrix} = -2$$

Dot product

$$\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + \cdots + x_n y_n$$

Vector-Vector Multiplication (Inner Product)

$$\begin{bmatrix} 1 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 3 \end{bmatrix} = -2$$

Dot product

$$1 \times n \qquad n \times 1 \qquad 1 \times 1$$

$$\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + \cdots + x_n y_n$$

Vector-Vector Multiplication (Inner Product)

$$\begin{bmatrix} 1 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 3 \end{bmatrix} = -2$$

Dot product

$$1 \times n \qquad n \times 1 \qquad 1 \times 1$$

$$\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + \cdots + x_n y_n$$

$$\mathbb{R}^n \qquad \mathbb{R}^n \qquad \mathbb{R}$$

Vector-Vector Multiplication (Outer Product)

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 5 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 7 \\ 10 & 12 & 14 \\ 15 & 18 & 21 \end{bmatrix}$$

Outer Product

Vector-Vector Multiplication (Outer Product)

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 5 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 7 \\ 10 & 12 & 14 \\ 15 & 18 & 21 \end{bmatrix}$$

Outer Product

$$\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1 y_1 & \cdots & x_1 y_n \\ \vdots & & \vdots \\ x_m y_1 & \cdots & x_m y_n \end{bmatrix}$$

Vector-Vector Multiplication (Outer Product)

$$\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}5 & 6 & 7\end{bmatrix} = \begin{bmatrix}5 & 6 & 7\\10 & 12 & 14\\15 & 18 & 21\end{bmatrix}$$

Outer Product

$m \times 1$ $\quad 1 \times n$ $\quad m \times n$

$$\begin{bmatrix}x_1\\\vdots\\x_m\end{bmatrix}\begin{bmatrix}y_1 & \cdots & y_n\end{bmatrix} = \begin{bmatrix}x_1y_1 & \cdots & x_1y_n\\\vdots & & \vdots\\x_my_1 & \cdots & x_my_n\end{bmatrix}$$

Vector-Vector Multiplication (Outer Product)

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 5 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 7 \\ 10 & 12 & 14 \\ 15 & 18 & 21 \end{bmatrix}$$

Outer Product

$$m \times 1 \qquad 1 \times n \qquad m \times n$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1 y_1 & \cdots & x_1 y_n \\ \vdots & & \vdots \\ x_m y_1 & \cdots & x_m y_n \end{bmatrix}$$

$$\mathbb{R}^m \qquad \mathbb{R}^n \qquad M_{m \times n}(\mathbb{R})$$

Matrix-Matrix Multiplication

$$\boxed{AB = C}$$

$m \times n$ $\quad$ $n \times p$ $\quad$ $m \times p$

Matrix-Matrix Multiplication

$$\boxed{AB = C}$$

$$m \times n \qquad\qquad n \times p \qquad\qquad m \times p$$

- Only matrices of compatible dimensions can be multiplied
  - # columns of A = # rows of B
- Matrix multiplication is not commutative
  - In general $AB \neq BA$
  - If $AB = BA$, we say that A and B commute

Matrix-Matrix Multiplication

$$AB = C$$

$m \times n$ $\quad$ $n \times p$ $\quad$ $m \times p$

Matrix-Matrix Multiplication

$$AB = C$$

$$m \times n \qquad n \times p \qquad m \times p$$

Matrix-Vector

$$A\begin{bmatrix} | & & | \\ b_1 & \cdots & b_p \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ Ab_1 & \cdots & Ab_p \\ | & & | \end{bmatrix}$$

Matrix-Matrix Multiplication

$$AB = C$$

$$m \times n \qquad n \times p \qquad m \times p$$

Vector-Matrix

$$\begin{bmatrix} — & a_1^T & — \\ & \vdots & \\ — & a_m^T & — \end{bmatrix} B = \begin{bmatrix} — & a_1^T B & — \\ & \vdots & \\ — & a_m^T B & — \end{bmatrix}$$

Matrix-Matrix Multiplication

$$AB = C$$

$$m \times n \qquad n \times p \qquad m \times p$$

Vector-Vector (Inner Product)

$$\begin{bmatrix} — & a_1^T & — \\ & \vdots & \\ — & a_m^T & — \end{bmatrix} \begin{bmatrix} | & & | \\ b_1 & \cdots & b_p \\ | & & | \end{bmatrix} = \begin{bmatrix} & \vdots & \\ \cdots & a_i^T b_j & \cdots \\ & \vdots & \end{bmatrix}$$

Matrix-Matrix Multiplication

$$AB = C$$

$$m \times n \qquad n \times p \qquad m \times p$$

Vector-Vector (Outer Product)

$$\begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} \begin{bmatrix} — & b_1^T & — \\ & \vdots & \\ — & b_n^T & — \end{bmatrix} = a_1 b_1^T + \cdots + a_n b_n^T$$

Matrix-Matrix Multiplication

$$AB = C$$

$$m \times n \qquad n \times p \qquad m \times p$$

Matrix-Vector

$$A\begin{bmatrix} | & & | \\ b_1 & \cdots & b_p \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ Ab_1 & \cdots & Ab_p \\ | & & | \end{bmatrix}$$

Vector-Vector (Inner Product)

$$\begin{bmatrix} — & a_1^T & — \\ & \vdots & \\ — & a_m^T & — \end{bmatrix}\begin{bmatrix} | & & | \\ b_1 & \cdots & b_p \\ | & & | \end{bmatrix} = \begin{bmatrix} & \vdots & \\ \cdots & a_i^T b_j & \cdots \\ & \vdots & \end{bmatrix}$$

Vector-Matrix

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Vector-Vector (Outer Product)

$$\begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}\begin{bmatrix} — & b_1^T & — \\ & \vdots & \\ — & b_n^T & — \end{bmatrix} = a_1 b_1^T + \cdots + a_n b_n^T$$

Special Matrices

square matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$A \rightarrow n \times n$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Diagonal

$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & d_n \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Scalar

$$S = \begin{bmatrix} c & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & c \end{bmatrix}$$

$$S = cI$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Identity

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Special Matrices

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Upper Triangular

$$U = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ \mathbf{0} & & a_{nn} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ -1 & 2 & 3 \end{bmatrix}$$

Lower Triangular

$$L = \begin{bmatrix} a_{11} & & \mathbf{0} \\ \vdots & \ddots & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

Transpose

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Transpose

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

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$$A \to m \times n$$

$$A^T \to n \times m$$

Transpose

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

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$$A \to m \times n$$

$$A^T \to n \times m$$

$$\left(A^T\right)_{ij} = A_{ji}$$

Transpose

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

$$A \to m \times n$$

$$\left(A^T\right)_{ij} = A_{ji}$$

$$A^T \to n \times m$$

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$\left(A^T\right)^T = A$$

$$(AB)^T = B^T A^T$$

Symmetric and Skew-symmetric

Symmetric

$$A^T = A$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 1 \end{bmatrix}$$

Symmetric and Skew-symmetric

Symmetric

$$A^T = A$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 1 \end{bmatrix}$$

Skew-Symmetric

$$A^T = -A$$

$$A = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix}$$

Symmetric and Skew-symmetric

Symmetric

$$A^T = A$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 1 \end{bmatrix}$$

Skew-Symmetric

$$A^T = -A$$

$$A = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix}$$

For any square matrix A

$$A = \underbrace{\frac{A + A^T}{2}}_{\text{symmetric}} + \underbrace{\frac{A - A^T}{2}}_{\text{skew-symmetric}}$$

Inverse

$$A \rightarrow n \times n \qquad\qquad B \rightarrow n \times n$$

$$AB = BA = I \implies B = A^{-1} \text{ and } A = B^{-1}$$

Inverse

$$A \to n \times n \qquad\qquad B \to n \times n$$

$$AB = BA = I \implies B = A^{-1} \text{ and } A = B^{-1}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ if } ad - bc \neq 0$$

Inverse

$$A \rightarrow n \times n \qquad\qquad B \rightarrow n \times n$$

$$AB = BA = I \implies B = A^{-1} \text{ and } A = B^{-1}$$

$$\left(A^{-1}\right)^{-1} = A$$

$$(cA)^{-1} = \frac{1}{c} \cdot A^{-1} \quad (c \neq 0)$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$\left(A^{T}\right)^{-1} = \left(A^{-1}\right)^{T}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ if } ad - bc \neq 0$$

Determinants

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Determinants

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

Determinants

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32} \qquad M_{12} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31}$$

Determinants

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

$$M_{12} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31}$$

$$M_{13} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{21}a_{32} - a_{22}a_{31}$$

Determinants

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = (-1)^{(1+1)} a_{11} M_{11} + (-1)^{(1+2)} a_{12} M_{12} + (-1)^{(1+3)} a_{13} M_{13} = a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13}$$

$$M_{11} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{22} a_{33} - a_{23} a_{32} \qquad M_{12} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{21} a_{33} - a_{23} a_{31}$$

$$M_{13} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{21} a_{32} - a_{22} a_{31}$$

Determinants

$M_{ij} \rightarrow$ minor

$$|A| =$$

M_{ij} = determinant of the matrix formed by deleting row i, column j

Determinants

M_{ij}→minor

$$|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

Expanding along row i

M_{ij} = determinant of the matrix formed by deleting row i, column j

Determinants

$M_{ij} \rightarrow$ minor

$$|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

Expanding along row i Expanding along column j

$M_{ij} =$ determinant of the matrix formed by deleting row i, column j

Determinants

$C \rightarrow$ co-factor matrix

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$$A \text{ is invertible if and only if } \det(A) \neq 0 \longrightarrow A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

Determinants and Row Operations

- Swapping two rows changes the sign of the determinant.
- Scaling a row by a constant scales the determinant by the same constant.
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Consequences:

- If a matrix has a zero row, its determinant is zero
- If two rows of a matrix are the same, its determinant is zero.
- If a row of a matrix is a linear combination of other rows, its determinant is zero.

Determinants

- $|AB| = |A| \cdot |B|$
- $|A^T| = |A|$
- $|A^{-1}| = \dfrac{1}{|A|}$, if $|A| \neq 0$
- $|cA| = c^n |A|$

- If A is upper triangular or lower triangular its determinant is the product of the diagonal entries.
- Specifically, if A is diagonal, its determinant is the product of its diagonal entries.