Digital Image Processing (DIP) - Processing of images which are digital in nature by using digital computers.

Digital System - It is a system which is driven or made with electro mechanical elements and controlled by electronic circuits.

Ex:- computers, Pendrives etc..,

Image consists of finite number of samples and these elements are called as pixels (or) pels

represented as $f(x,y) = \begin{bmatrix} f(0,0) & f(0,1) & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots \end{bmatrix}$ pixel.

Origin of DIP -

First application of digital images was in the newspaper industry, here the pictures are sent by submarine cable (which is a time taking process, more than a week)

lateron, Bartlane cable picture transmission system in early 1920's reduced the time required for the transport of pictures.

The visual quality is not good in the intial methods.

next, a technique based on photographic reproduction made from tapes perforated at the telegraph receiving terminal. In this, tonal quality is improved and resolution also.

In Bartlane systems, coding of images is done in five distinct gray levels. This capability was increased to 15 levels in 192c

A system for developing a film plate via light beams that were modulated by the coded picture tape improved the reproduction process.

of supporting technologies that include data storage, display and transmission.

Next John von Neumann introduced two key concepts (i) a memory to hold a stored program and data (ii) conditional branching. These two concepts gave the idea for foundation of CPU and other key advances led to computers powerful enough to be used for digital image processing.

The first computers powerful enough to carryout meaningful processing tasks appeared in 1960's. This is used to correct various types of distortions in the image. And used to enhance and restore the images of moon & in many Space applications.

Digital Image processing techniques which began in the late 1960's and early 1970's are used in medical engineering, remote earth resources observations and astronomy., Computer Tomograp -hy (CT) is one of the most important events in the application of image processing in medical diagnosis.

Computer procedures in DIP are used to enhance the contrast or code the intensity levels into colour for easier interpretation of X-rays and other images used in industry, medicine and the biological sciences.

This Digital image processing techniques are used in many fields such as automatic processing of finger prints, screening of X-rays and blood samples etc., The increase in the performance of computer and expansion of networking gave scope for the growth of digital Image Processing.

Biometrics - It refers to the way of identifying human-beings based on physiological and behavioral characteristics physiological characteristics implies finger prints, face, DNA and Iris etc.,

Image processing is used to analyse and recognize finger print, face, DNA and Iris.

Medical Imaging -

Image processing is very useful in interpreting medical images, from simple diagnosis to advanced telesurgical applications etc.,

This is used in X-rays, CT, MRI, PET and ultrasound and also for combining image modalities.

Factory Automation -

Automated visual inspection is a vast field where image processing is used by industries such as aerospace, food, textiles and plastic for automated surface testing.

Factory automation includes measurement of belt width, surface quality inspection, fiber analysis etc.,

Remote sensing - The role of ~~[illegible]~~ image processing in remote sensing applications is quite immense.

weather forecasting and prediction of atmospheric changes etc.,

Environmental monitoring applications have been developed to monitor deserts, forest etc.,

Defence / Military Applications -

Many applications such as military reconnaissance systems use image processing technology. Thermal images have the ability to acquire useful images at night and under atmospheric conditions such as fog & smoke.

photography -

Imaging processing helps in creating special effects such as warping, blending, animation and other visual effects.

Entertainment -

Photography is an excellent example of how image processing is helpful to common man. The applications are video conferencing, video phones, video editing, animation and image morphing etc., extracting image components that are useful in the representation and description of shape. Here the inputs are images and outputs are attributes extracted from those images.

Segmentation - Here partition of an image into its constituent parts or objects is done. A rugged segmentation procedure gives successful solution of imaging problems.

Representation and Description - The output of the segmentation stag is usually a raw pixel data, constituting either the boundary of a region or the points in the region itself. So, to convert the data to more suitable form for computer processing and to decide whether the data should be represented as a boundary or as a complete region, appropriate regional representation should be there.

Description is the feature selection, which deals with extracting attributes.

Recognition - Here we assigns a label or any symbol to an object based on its descriptors. We can recognise the image by using some of the coding techniques.

Components of an Image Processing System -

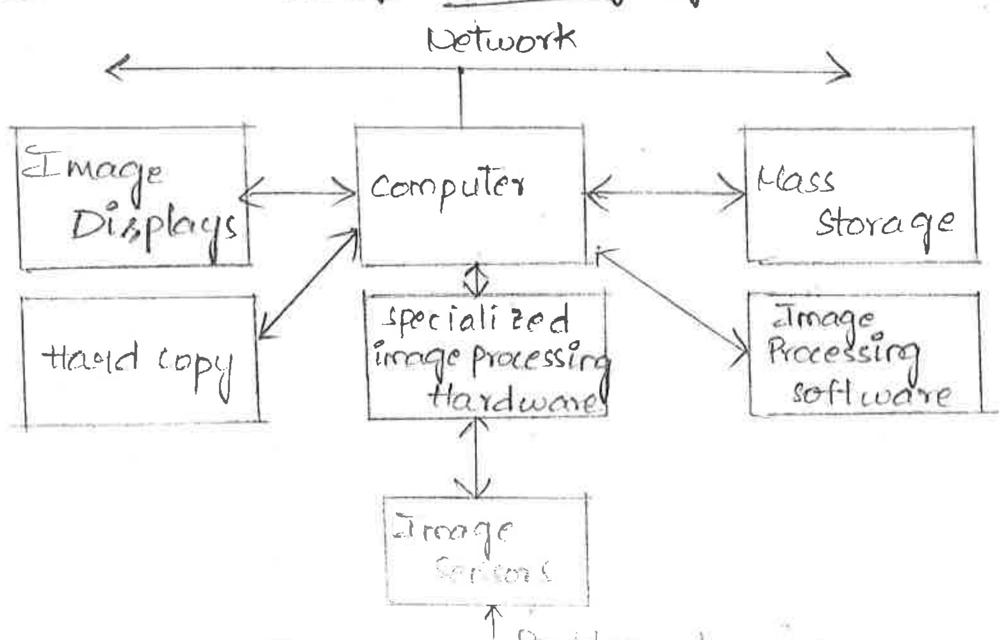


Image sensors- Generally two elements are needed [illegible]
digital images. One is physical device i.e., we can sense the image by with the energy radiated from the object. And other element is 'digitizer' which is used to convert the sensed image in to digital format.

Specialised Image Processing Hardware- It contains hardware which is used to perform arithematic and logical operations and some other operations on all images at the same time. It gives very fast outputs.

Computer- It is a general purpose computer which can range from a PC to a super computer. Depending on level of performance needed, we use different computers i.e., PC or super computer or custom computer etc. But now, we use well-equipped PC-type machine which is more suitable for off-line image programming tasks.

Image Processing Software- This contains specialized hardware modules to perform specific task. More sophisticated software packages allow the integration of those modules.

Mass Storage- If the image is not compressed, it requires lot of storage space, a single image may need 1 Megabyte of storag i.e., depending on the intensity level of each pixel in the image. So to provide adequate and efficient storage, we compress the image.

Image displays- Image displays are mainly colour T.V Monitors, which are driven by the outputs of images and graphic display cards, In some cases, we use stereo displays.

Hard Copy devices- These are used for recording images, such as laser printers, film cameras, heat-sensitive devices, optical and CD ROM disks etc.) Generally, Films are preferrable because these provide highest possible resolution.

according to our application, we use them.

Ex:- Thresholding, clipping etc..,

Image Restoration -

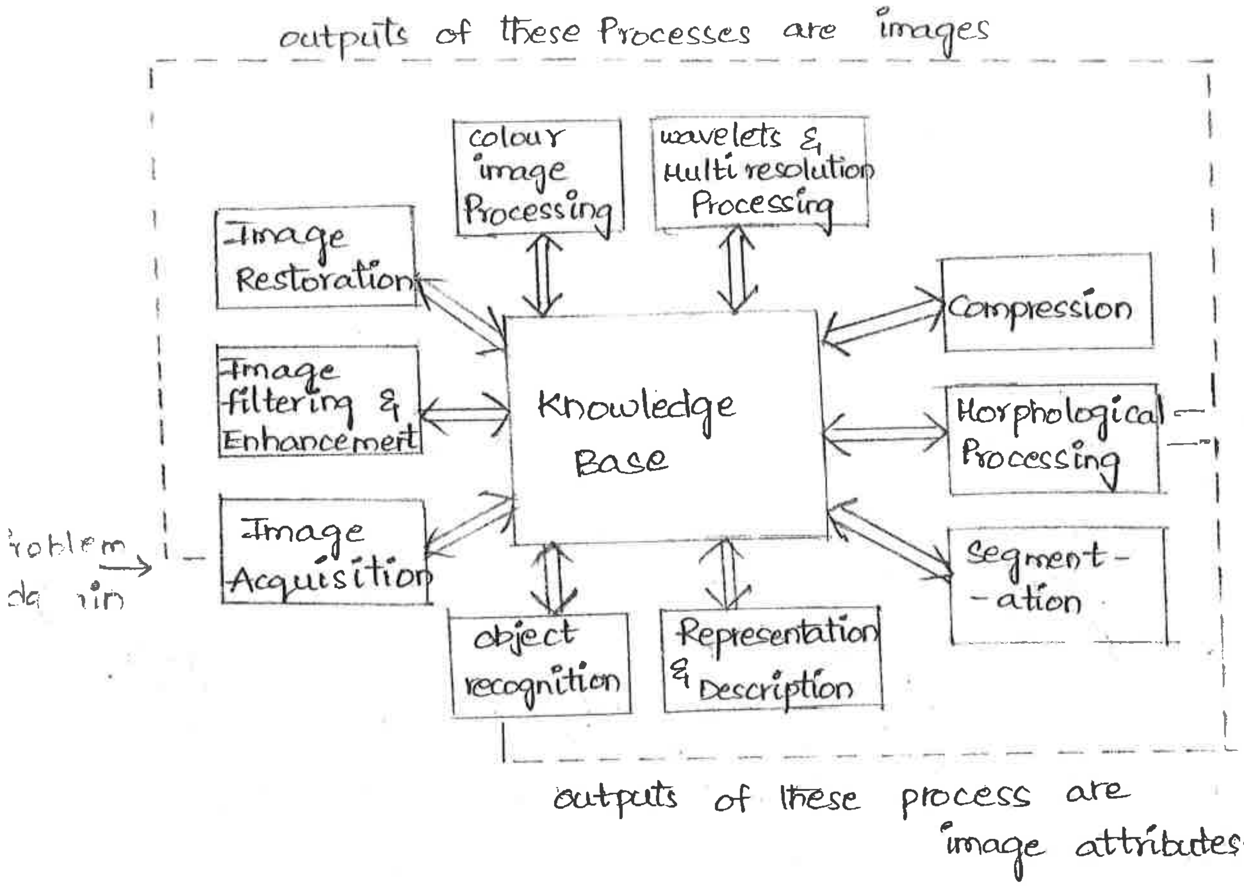
It is the process which also deals with improving the appearance of an image. But this is not similar to Enhancement technique, In Enhancement, we process the image based on human subjective preferences; where as in Restoration technique, we use mathematical or probabilistic models of image degradation ie., we use Transforms (FFT ....), Filters etc..)

Colour Image Processing - This is the area which has been gaining importance because of significant increase in the use of digital images over the internet. By using this colour, human can easily identify and analyse the image. There are different Colour Image Processing techniques.

Wavelets and Multiresolution Processing - Wavelets are the foundation for representing images in various degrees of resolution. Here different wavelet transform techniques are used to make the images compress, transmit and analyse easily. Multiresolution (or) is concerned with the representation and analysis of images at more than one resolution. We use this process for image data compression and for pyramidal representation, in which images are sub divided successively into smaller regions.

Compression - This reduces the storage required to save an image, or the bandwidth required to transmit it. We use this in the Internet, which are characterized by significant pictorial content. Image compression is familiar to most users of computers in the form of Image file extensions, such as jpg file extension used in JPEG (Joint Photographic Experts Group) image compression standard

Fundamental steps in DIP -



Some of the methods which are mentioned above have images as both input and output, some of the methods have images as their input whose outputs are attributes extracted from those images.

Let us have a brief overview of all the above mentioned process

Image Acquisation - This is the first process ie; we sense the image here. Generally images are generated by the combination of an "illumination" source and the reflection or absorption of energy from the source by the elements of the "scene" being imaged. Generally, the image acquisation stage involves Preprocessing sucha scaling.

Image Enhancement - It is the process of manipulating an image so that the processed image is more suitable than the original image for a specific application. When an image is processed for visual interpretation, the viewer is the ultimate judge of how well

Light and Electromagnetic Spectrum –

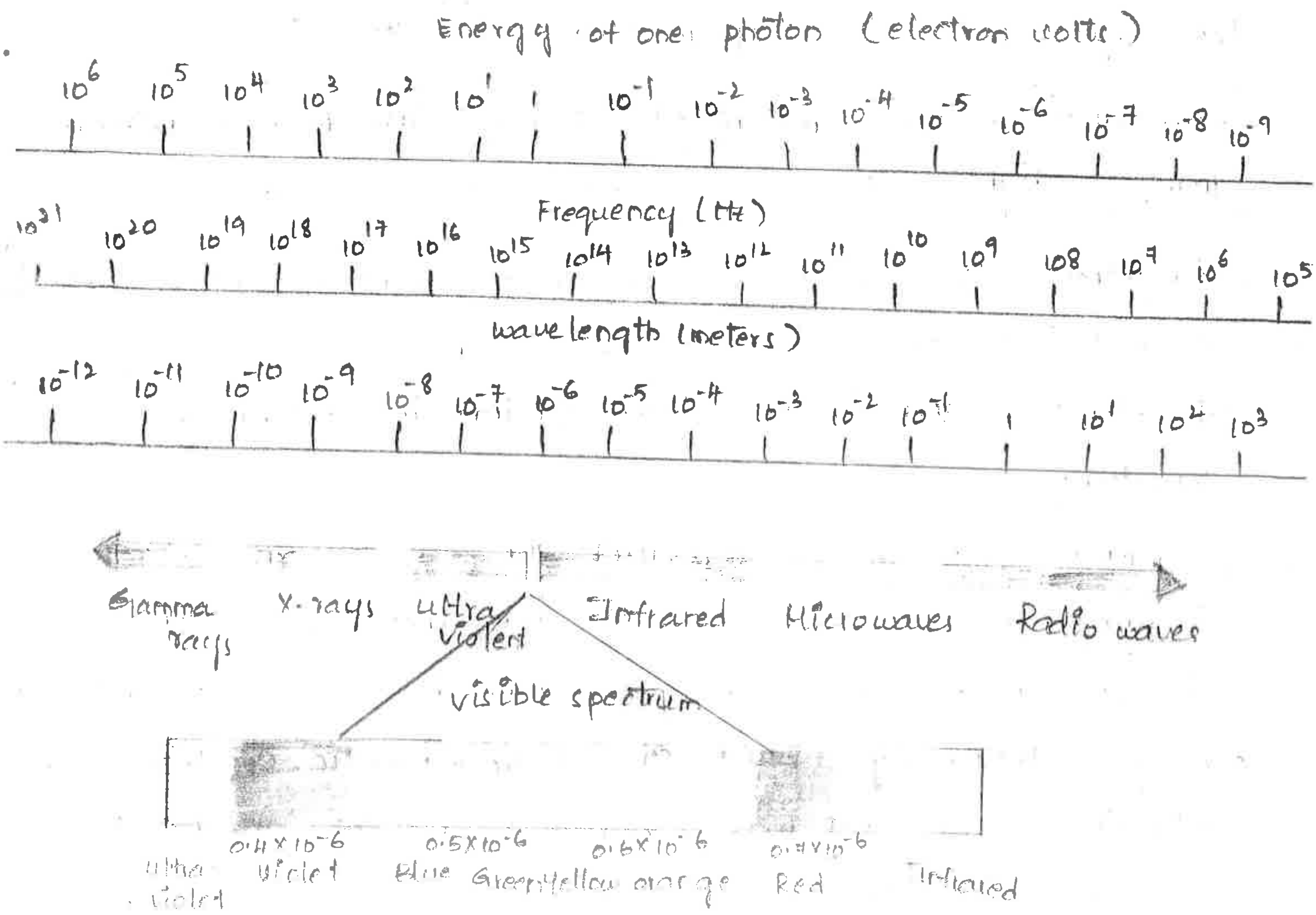
when Sunlight is passed through a glass prism, the emer-ging beam from the glass is a continous spectrum of colours ranging from violet at one end to red at the other end.

Electromagnetic spectrum can be expressed interms of wavelength, frequency or energy. wavelength and frequency are related by the expression

$$\boxed{\lambda = \frac{c}{v}} \qquad c = 3 \times 10^8 \text{ m/sec.}$$

The energy of the various components of the electromagnetic spectrum is given by

$$\boxed{E = h\nu}$$



Generally, Electromagnetic waves can be viewed as sinusoidal waves with wavelength 'λ' and these can be treated as a stream of massless particle with some bundle of energy. Each bundle of energy is called a "photon".

Light is a particular type of electromagnetic radiation that can be sensed by the human eye. The visible (colour) spectrum is divided in to six broad regions - violet, blue, green, yellow, orange and red.

Light that is void of colour is called Monochromatic light. The intensity of monochromatic light is perceived to vary from black to Grays and finally to white.

To describe the quality of a chromatic light source, three basic quantities are there - Radiance, luminance and brightness.

'Radiance' is the total amount of energy that flows from the light source, usually measured in watts.

'Luminance' is the amount of energy an observer perceives from a light source., usually measured in lumens.

'Brightness' is subjective descriptor of light perception that is practically impossible. It embodies the achromatic notion of intensity and is one of the factors in describing colour sensation.

Gamma radiation is important for medical and astronomical imaging. X-rays are used in industrial applications.

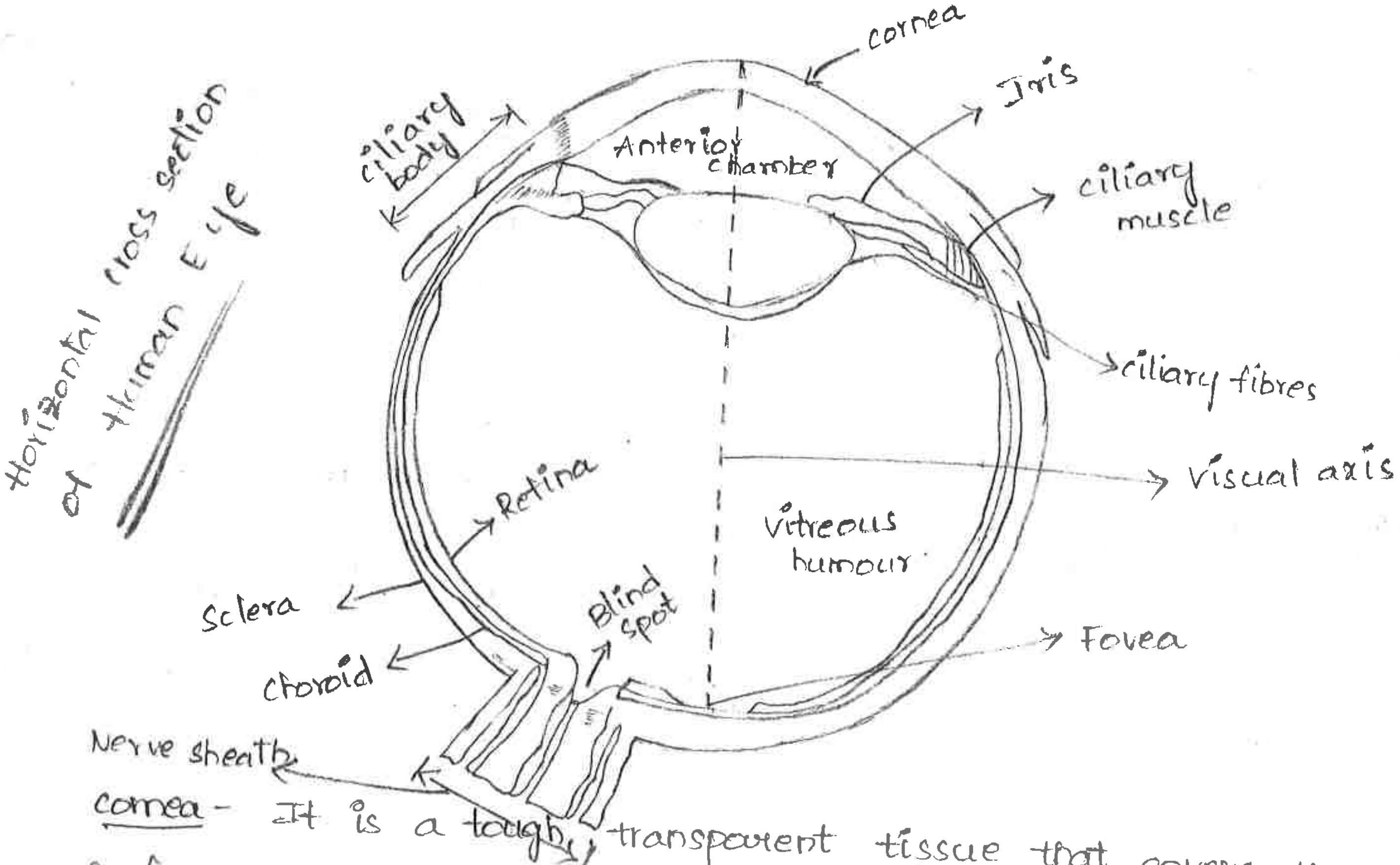
If a sensor can be developed that is capable of detecting energy radiated by a band of electromagnetic spectrum. The wavelength of an EM wave required to see an object must be of same size or smaller than the object.

Elements of Visual perception -

Generally human eye has the capability to sense the image and store it. It just acts as a camera. We should know how images are formed and perceived by humans. So, we should understand The human Visual perception.

structure of human eye -

The human eye is nearly in the form of sphere with diameter of approximately 20 mm. There are 3 membranes which enclose the eye, they are the 'cornea and sclera', the 'choroid', and 'Retina'



cornea - It is a tough, transparent tissue that covers the anterior surface of the eye.

Sclera - It is an opaque membrane that makes the eye to be in sphere shape. (optic globe)

choroid - It lies directly below the sclera. This membrane contains a network of blood vessels that serve as a major source of nutrition to the eye. If any injury to choroid, leads to restriction of flow of blood (blood circulation stops).

in to ciliarybody and Iris.

These help the eye to contract or expand inorder to control the amount of light.

<u>lens</u> - lens is made up of concentric layers of fibrous cells and is suspended by the ciliary fibres. It contain 60-70% water, 6%-fat and more protein than any other tissue in the eye.

Generally, these lens are coloured by slightly Yellow pigmentation and the colour increases with the age. In some cases, excessive clouding of lens caused by affliction is referred as 'Cataracts'.

<u>Retina</u> - The inner most membrane of the eye is Retina, which lines inside of wall's entire posterior portion. While viewing an object, the light from the object outside the eye is imaged on retina.

There are 2 receptors
- Cones
- Rods

<u>Cones</u> - These cones are between 6 to 7 million in number. These lies in the central portion of the retina, which are called '<u>Fovea</u>' These are highly sensitive to colour.

<u>Rods</u> - These are larger in number ie., 75 to 150 million are distributed over the retinal surface. These give the overall picture c field of view. These are not involved in colour vision and are sensitive to low levels of illumination.

The absence of receptors results in so-called '<u>Blind Spot</u>'.

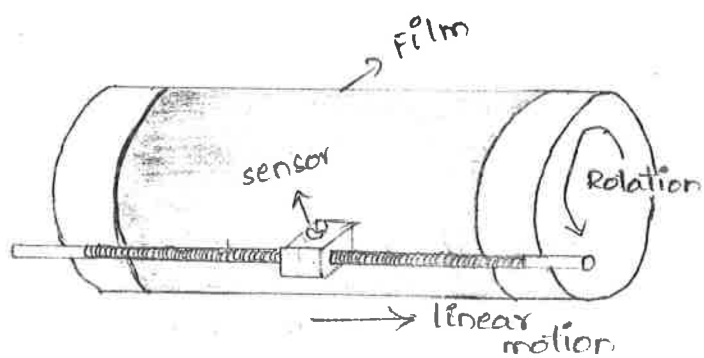
Image sensing and Acquisition -

Generally, images are generated by the combination of an "illumination" source and reflection or absorption of energy from that source by the elements of the "scene" being image. Now we can sence the image.

There are three principal sensor arrangements used to transform illumination energy in-to digital images. The basic idea is "the incoming energy from the source is transformed into the voltage by the combination of input electrical power and sensor material that is responsible to the particular type of energy being detected. The output voltage wave-form is the response of sensor and a digital quantity is obtained from each sensor by digitizing its response".

- Image Acquisation using single sensor -

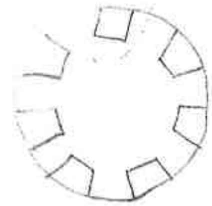
Here, we use single sensor, generally which is a photo--diode, which is constructed of silicon materials and whose output voltage waveform is proportional to light.



To generate a 2-D image, we use a single - sensor, a film. Here, a film negative is mounted on to a drum whose mechanical rotation provides displacement in one dimension. The single sensor is mounted on a lead screw that provides motion in perpendicular direction. This method is very expensive to obtain high-resolution images.

This consists of an in-line arrangement of sensors in the form of a sensor strip. The strip provides imaging elements in one direction. Motion perpendicular to the strip provides imaging in the other direction.

linear sensor strips

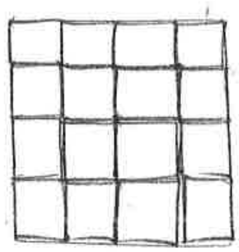


circular sensor strip.

These In-line sensors are used in airborne imaging applications. Sensor strips in ring configuration are used in medical and indus-trial imaging to obtain cross-sectional images of 3D-objects.

- Image Acquisation using Sensor Arrays -

In this, the individual sensors are arranged in the form of a 2D-array. This is also the predominant arrangement found in digital cameras. In cameras, we use CCD array which is used in other light sensing instruments.



sensor arrays.

The main principle in this is to collect incoming energy from the scene element and focus it onto an image plane. If the illumination is light, the front end of the imaging system is an optical lens that projects the viewed scene onto the lens focal plane. The sensor array, which is coincident with the focal plane, produces output proportional to the integral of the light received at each sensor.

To process an image, it should be in digital form. But the output of most sensors is a continous wave-form. To have a digital image, we have to convert the continous wave-form to digital form. This conversion involves two processes

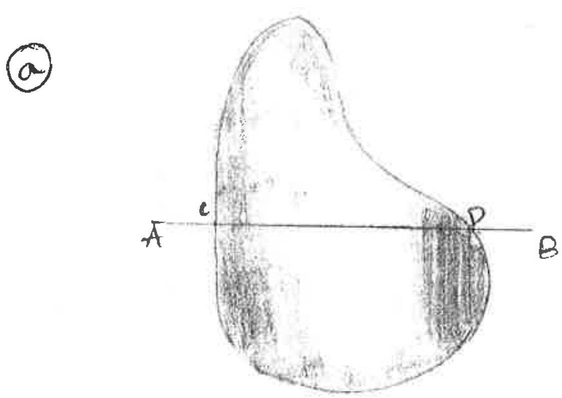
- Sampling
- Quantization

An image may be continous with respect to x and y coordinates and also in amplitude. To convert it in to digital form, we have to sample the function in both coordinates and in amplitude.

Digitizing the x- and y- coordinate values (for 2D) is called "Sampling"

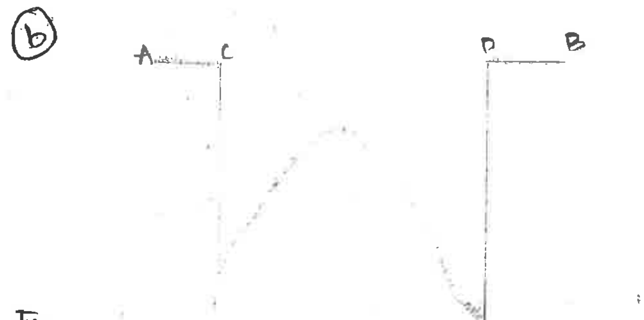
Digitizing along the amplitude values is called "Quantization"

Consider a 2D image,



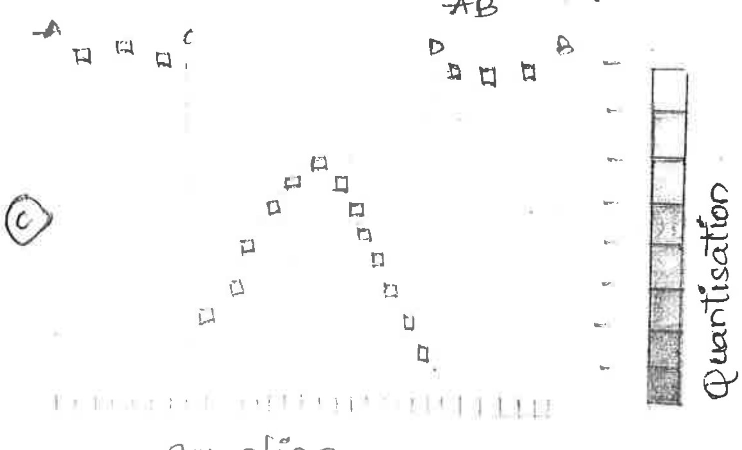
continous image ↑

Here consider along line segment AB.



From A to C, it is white ↑ and then slowly the colour variation occurs due to noise.

equally spaced samples along AB



Digital image along AB

Sampling

Fig (a) shows the continous image. Fig (b) is the plot of amplitude values of continous image along the line segment 'AB'. There are some random variations which are due to noise.

Fig(c) shows the equally spaced samples along bothe the axis and the intensity scale divided in-to eight discrete intervals ranging from black to white.

Fig (d) shows the digital samples resulting from both sampling and Quantisation.

Representation of Digital images -

Consider f(s,t) is a continous image and it is sampled in to a 2D-array f(x,y), where x & y are coordinates and it contains 'M' rows & 'N' columns.

Generally, we use integer values for discrete coordinates ie,

$x = 0, 1, 2 \ldots M-1$ and $y = 0, 1, 2 \ldots N-1$.

Hence the image is represented as

$$f(x,y) = \begin{bmatrix} f(0,0) & f(0,1) & \cdots & f(0,N-1) \\ f(1,0) & f(1,1) & \cdots & f(1,N-1) \\ \vdots & & & \\ f(M-1,0) & f(M-1,1) & \cdots & f(M-1,N-1) \end{bmatrix}$$

Each element of this matrix is called an "image element", picture element, pixel or pel".

The number of intensity levels typically is an integer power of 2.

$\boxed{L = 2^k}$ where 'k' is quantised number.

and the interval, of intensity values is $[0 \quad L-1]$ for $M \times N$ 2-d image

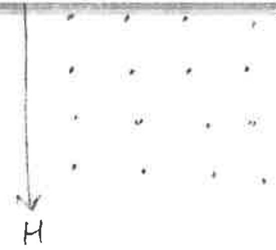
Number of bits required to store a digitized image is

$\boxed{b = M \times N \times k}$.

L = 256 then k = ?

$256 = 2^k$

$\therefore \boxed{k = 8}$ Hence the image is 8-bit image.



2D - representation of image.

Some Basic Relationships between pixels -

neighbours -

consider a two dimensional image $f(x,y)$ and let $P(x,y)$ is the centre pixel of that image.

This centre pixel have four neighbours ie, two horizontal and two vertical.

$$\begin{array}{ccc} & P(x-1,y) & \\ P(x,y-1) & \boxed{P(x,y)} & P(x,y+1) \\ & P(x+1,y) & \end{array}$$

The set of these pixels is called 4-neighbours of $P(x,y)$. and is denoted as "$N_4(P)$"

This center pixel have diagnol pixels ie, it have 4 diagnol pixels and these are denoted as "$N_D(P)$"

$$\begin{array}{ccc} P(x-1,y-1) & & P(x-1,y+1) \\ & \boxed{P(x,y)} & \\ P(x+1,y-1) & & P(x+1,y+1) \end{array}$$

combination of both $N_4(P)$ and $N_D(P)$ ie, two horizontal, two vertical and four diagnol neighbours gives 8-neighbourhood of P. denoted as $N_8(P)$

$$\boxed{N_8(P) = N_4(P) + N_D(P)}$$

$$\begin{array}{ccc} P(x-1,y-1) & P(x-1,y) & P(x-1,y+1) \\ P(x,y-1) & \boxed{P(x,y)} & P(x,y+1) \\ P(x+1,y-1) & P(x+1,y) & P(x+1,y+1) \end{array}$$

- **Adjacency** - Consider 'V' be a set of some intensity values

(a) **4-adjacency** - Let P and q are two pixels from 'V' and are said to be 4-adjacent if q is in set $N_4(P)$ $(q \in N_4(P))$

and are said to be 8-adjacent if q is in the set of $N_8(P)$
$(q \in N_8(P))$

ⓒ m-adjacency - let P and q are two pixel values from 'V' and are said to be m-adjacent if

- q is in $N_4(P)$ or
- q is in $N_D(P)$ and the set ~~$N_4(P) \cap N_D(P$~~ $N_4(P) \cap N_4(q)$ has no pixels whose values are from V.

- Connectivity - This connectivity is an important factor in Image processing.

we can connect two pixels when (conditions for connectivity)

* The pixels are adjacency
* And those pixels must have same intensity value ie; same Gray level.

4-connectivity -

consider's' some set of pixels, Here we can connect the pixels, if they are adjacent, must have same intensity value and $S \in N_4(P)$

centre pixel

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

8-connectivity -

Here, we connect the pixels which are adjacent and diagnol having the same intensity values. $(S \in N_D(P))$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

M-connectivity -

Intially, we check the adjacent case, if there are adjacent pixels with same intensity value, we connect them. If there are no adjacent pixels, then we check the diagnol pixels and we connect

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & \boxed{1} & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$S \in (N_4(P) \cap N_D(P))$

of intensity values of the image and the pixels must be connected.

Two regions are said to be ~~connected~~ Adjacent, if their union forms a connected set.

- **Boundary** - This should be a set of intensity values in an image and some pixels are not connected.

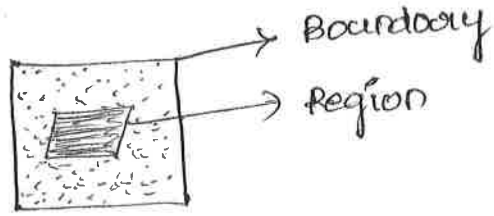


Image.

- **Distance Measurement** →
  - Euclidean distance
  - D_4 distance (or) city block distance
  - D_8 distance (or) chess board distance.

Consider two pixels $p(x,y)$ and $q(s,t)$

Euclidean distance - The Euclidean distance between P and q is

$$\boxed{D_e(P,q) = \sqrt{(x-s)^2 + (y-t)^2}}$$

D_4 distance (or) City Block distance - The D_4 distance between P and q is

$$\boxed{D_4(P,q) = |x-s| + |y-t|}$$

The pixels having a D_4 distance from (x,y) form a diamond centered at (x,y) (less than or equal to some value)

let,

D_4 distance ≤ 2 forms the diamond shape.

$$\begin{array}{ccccc} & & 2 & & \\ & 2 & 1 & 2 & \\ 2 & 1 & 0 & 1 & 2 \\ & 2 & 1 & 2 & \\ & & 2 & & \end{array}$$

D_8 distance (or) chess board distance -

The D_8 distance between P and q is given as

$$\boxed{D_8(P,q) = \max(|x-s|, |y-t|)}$$

the pixels with D_8 distance from (x,y) less than or equal to some value 'r' form a square centered at (x,y)

Ex: consider D_8 distance ≤ 2 $\Rightarrow$

This structure looks like chess board.

2	2	2	2	2
2	1	1	1	2
2	1	0	1	2
2	1	1	1	2
2	2	2	2	2

Introduction to the mathematical tools used in DIP -

We can perform different mathematical operations on the images. Some of such operations are

Array versus Matrix Operations -

Generally, array operation is carried out on a pixel-by-pixel basis.

Images can be viewed in the form of Matrices (ie., all set of intensity values) & operations can be performed on Matrix also.

But there is quite difference between Array operations and Matrix operation.

Ex:- Consider two images of size 2x2

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Matrix product

$$\text{array Product} = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} \\ a_{21}b_{21} & a_{22}b_{22} \end{bmatrix} \qquad = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix}$$

To raise the intensity values of the image, we use this operation.

Arithematic operations between images are array operations and these are carried out between corresponding pixel pairs.

Addition - when the two images are added ie, the corresponding pixels will add up.

$$s(x,y) = f(x,y) + g(x,y)$$

($f(x,y)$ and $g(x,y)$ are two images.)

$s(x,y)$ ↙ resultant image

Here, the resultant image have more intensity values.

Subtraction - when the images are subtracted, the resultant image have low intensity values.

$$d(x,y) = f(x,y) - g(x,y)$$

Multiplication - when images are multiplied, the corresponding pixels are multiplied. The resultant image have more and more intensity values.

$$m(x,y) = f(x,y) * g(x,y)$$

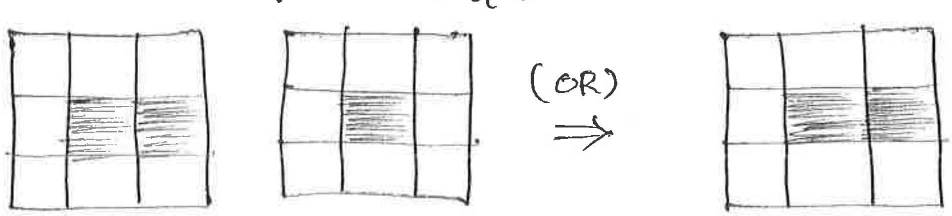
Division - Division operation is performed on the corresponding pixels of two images. Now, the resultant image have very low intensity values.

$$r(x,y) = f(x,y) / g(x,y)$$

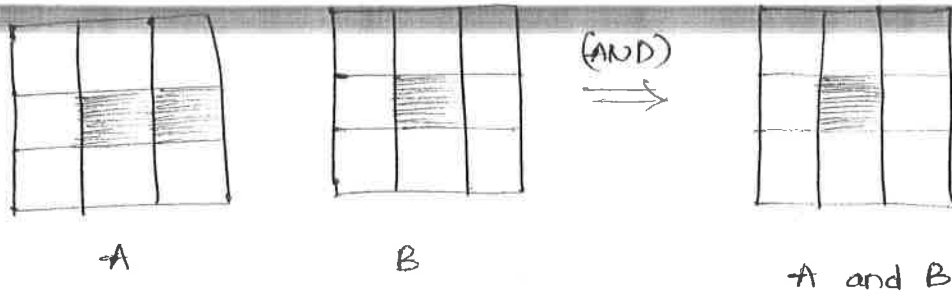
logical operations -

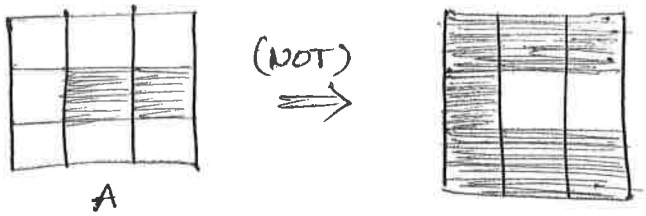
logical operations includes AND, OR, NOT, XOR, NAND etc,

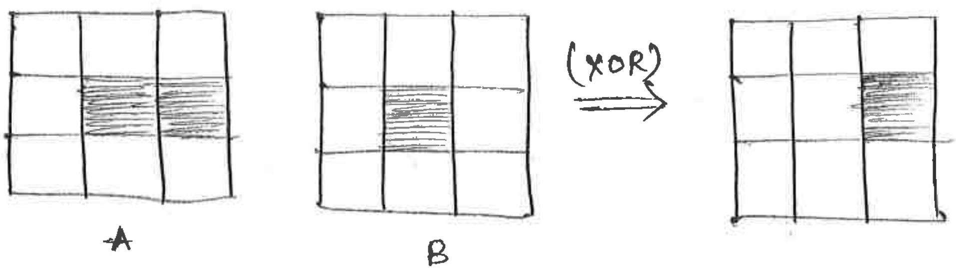
consider two images A & B



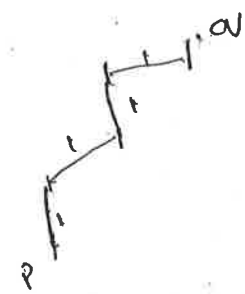
A B A (or) B







Here ▨ represent '1' and □ represent '0'.



Transformations applied on an image -

Consider an image ie, $f(x,y)$

If we multiply the image with identity matrix, then there is no change in the resultant image.

$$\underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{\text{Resultant image}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{Identity matrix}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{Initial image}}$$

$x' = x$; $y' = y$. (Processed one is same as original one)

To have image scaling, we multiply the image with some value

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} c_x & 0 & 0 \\ 0 & c_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ 1 \end{bmatrix}$$

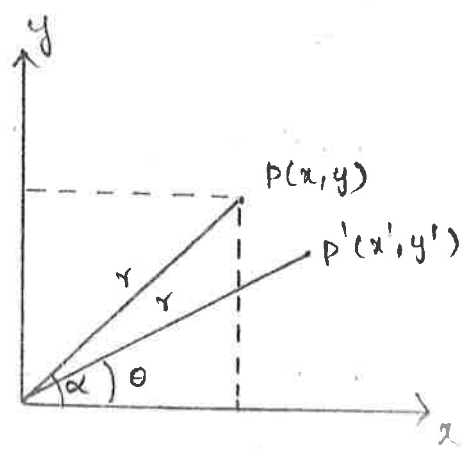
like wise image scaling is done.

$x' = c_x x_0$; $y' = c_y y_0$; $1 = 1$

Rotation of an image -

Consider an image in 2-dimensional which is at a distance 'r' and it makes an angle 'α' with x-axis.

later, the image is moved to some other point and it makes an angle 'θ' with x-axis.



From figure,

$$\cos\alpha = \frac{x}{r} \quad ; \quad \sin\alpha = \frac{y}{r}$$

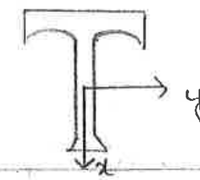
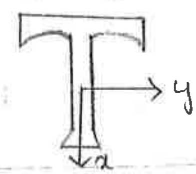
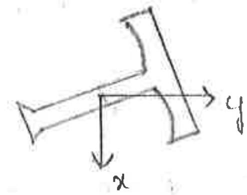
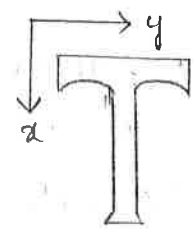
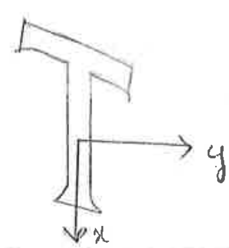
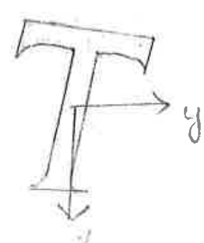
$$y' = r\sin(\alpha-\theta) = \underbrace{r\sin\alpha}_{y}\cos\theta - \underbrace{r\cos\alpha}_{x}\sin\theta = y\cos\theta - x\sin\theta.$$

$$x' = r\cos(\alpha-\theta) = \underbrace{r\cos\alpha}_{x}\cos\theta + \underbrace{r\sin\alpha}_{y}\sin\theta = x\cos\theta + y\sin\theta.$$

$$\begin{bmatrix} y' \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ 1 \end{bmatrix}$$ — Rotation Matrix

The image can be translated by multiplying the image with

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$ — translation matrix.

Transformation name	Transformation matrix	coordinate eqns	Example.
Identity	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x' = x$ $y' = y$	
Scaling	$\begin{bmatrix} c_x & 0 & 0 \\ 0 & c_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x' = x_0 c_x$ $y' = y_0 c_y$	
Rotation	$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x' = x\cos\theta + y\sin\theta$ $y' = y\cos\theta - x\sin\theta$	
Translation	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix}$	$x' = x + t_x$ $y' = y + t_y$.	
Shear (vertical)	$\begin{bmatrix} 1 & 0 & 0 \\ s_v & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x' = x + s_v y$ $y' = y$	
Shear (horizontal)	$\begin{bmatrix} 1 & s_h & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x' = x$ $y' = s_h x + y$.	

The underlined values are Eigen values, as already determined.

The co-variance matrix c_y consists of significant diagonal values with all other values almost zero, indicating that the element of the transformed vector is made independent.

Inverse Hotelling transform

$$X = A^T y + m_x$$

$$X_1 = \begin{bmatrix} 0.8165 & 0.5774 & 0.7071 \\ 0.4082 & -0.5774 & -0.7071 \\ 0.4082 & -0.5774 & 0 \end{bmatrix} \begin{bmatrix} -0.8165 \\ -0.1444 \\ -0.3535 \end{bmatrix} + \begin{bmatrix} 3/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} -0.25 \\ 0 \\ 0 \end{bmatrix}$$

$$X_2 = A^T \begin{bmatrix} 0 \\ 0.4331 \\ 0.3535 \end{bmatrix} \begin{bmatrix} 3/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ -0.25 \\ 0 \end{bmatrix}$$

lll'y x_3, x_4 are computed as.

$$X_3 = A^T \begin{bmatrix} 0.4082 \\ -0.144 \\ -0.3535 \end{bmatrix} + \begin{bmatrix} 3/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0.4 + 3/4 \\ 0.49 + 0.25 \\ 0.3 + 0.25 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.75 \\ 0.55 \end{bmatrix}$$

$$X_4 = A^T \begin{bmatrix} 0.4082 \\ -0.1444 \\ -0.3535 \end{bmatrix} + \begin{bmatrix} 3/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0.49 + 3/4 \\ 0 + 1/4 \\ 0.25 + 1/4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 0.25 \\ 0.5 \end{bmatrix}$$

Applications of KL Transform

1) Binary image alignment.
2) Image Compression.

$$+\begin{bmatrix}0\\0.4331\\0.3535\end{bmatrix}\begin{bmatrix}0 & 0.4331 & 0.3535\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\0 & 0.188 & 0.1531\\0 & 0.153 & 0.125\end{bmatrix}$$

$$+\begin{bmatrix}0.4082\\-0.1444\\-0.3535\end{bmatrix}\begin{bmatrix}0.4082 & -0.144 & -0.3535\end{bmatrix}+\begin{bmatrix}0.167 & -0.06 & -0.144\\-0.06 & 0.02 & 0.05\\-0.14 & 0.05 & 0.125\end{bmatrix}$$

$$+\begin{bmatrix}0.167 & -0.06 & 0.144\\-0.06 & 0.02 & -0.05\\0.14 & -0.05 & 0.125\end{bmatrix}+\begin{bmatrix}0.4082\\-0.1444\\0.3535\end{bmatrix}\begin{bmatrix}0.4082 & -0.1444 & 0.3535\end{bmatrix}$$

$$C_y = \frac{1}{4}\begin{bmatrix}0.994 & 0 & 0\\0 & 0.248 & 0.2\\0 & 0.2 & 0.5\end{bmatrix}$$

Covarience matrix of y can also be computed using AC_xA^T
'C_x' is the covarience matrix of x.

$$C_x = \begin{bmatrix}3/16 & 1/16 & 1/16\\1/16 & 3/16 & -1/16\\1/16 & 1/16 & 1/16\end{bmatrix}$$

A is the transformation matrix

$$A = \begin{bmatrix}0.8165 & 0.4082 & 0.4082\\0.5774 & -0.5774 & -0.5774\\0.7071 & -0.7071 & 0\end{bmatrix}$$

$$C_y = AC_xA^T = \begin{bmatrix}\underline{0.24998} & 0 & 0.03\\0 & \underline{0.0625} & 0.10\\0.07 & 0.05 & \underline{0.125}\end{bmatrix}$$

Image transforms: Image transform is basically a representation of an image. There are two for transforming an image from one representation to another. First the transformation may isolate critical components of the image pattern so that they are directly accessible for analysis. second, the transformation may place the image data in a more compact form so that they can be stored and transmitted efficiently.

Types of Image transforms

1) Fourier transform
2) Walsh transform
3) Hadamard transform
4) Slant transform
5) Discrete cosine transform
6) KL transform
7) Haar transform
8) Discrete sine transform

Classification of Image transforms

Image transforms can be classified based on the nature of the basis function as

i) transforms with orthogonal basis functions
ii) transforms with non sinusoidal orthogonal basis functions
iii) transforms whose basis function depend on the statics of input data.
iv) Directional transforms.

Image transforms

- orthogonal sinusoidal basis function
- Non sinusoidal orthogonal basis function.
- Basis function depending on statistics of i/p signal
- Directional transformation

orthogonal sinusoidal basis function:

* Fourier transform
* Discrete cosine transform
* Discrete sine transform

Non sinusoidal orthogonal basis function

* Haar transform
* walsh transform
* Hadamard transform
* slant transform

Basis function depending on statistics of input signal

* KL transform
* singular value decomposition.

Directional transformation

* Hough transform
* Random transform
* Ridgelet transform
* Contourlet transform.

Introduction:

Image transforms are extensively used in image processing and image analysis. Transform is basically a mathematical tool, which allows us to move from one domain to another domain (time ~~freqe~~ domain to frequency domain). The reason to migrate from one domain to another domain is to perform the task in an easier manner

Need for transform:

Transform is basically a mathematical tool to represent a signal. The need for transform is given as follows

* Mathematical Convenience
* To extract more information.

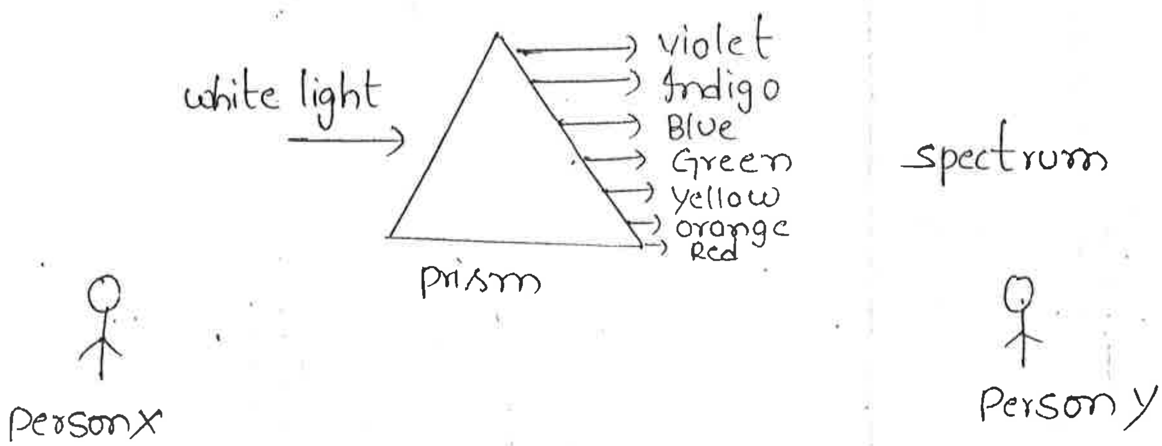
1) Mathematical convenience: Every action in time domain will have an impact in the frequency domain.

Convolution in time domain <—> Multiplication in frequency domain

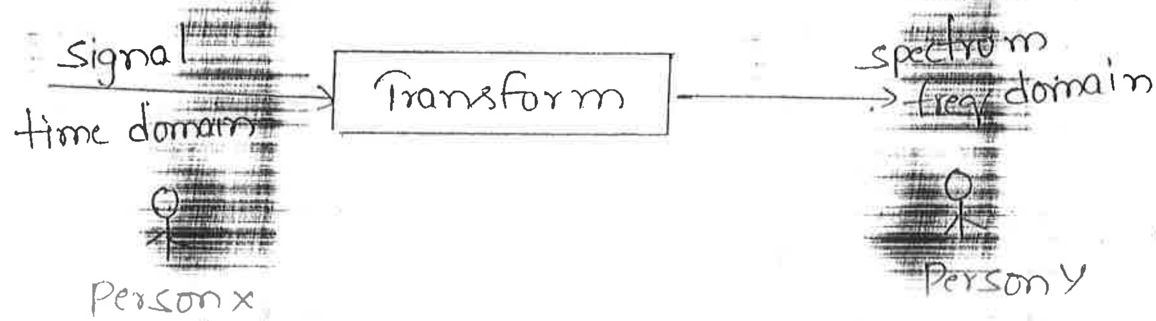
2) To extract more information: Transforms allow us to extract more information. Consider the following Ex

person x is on the left-hand side of the prism. where as the person y is on the right-hand side of the prism as illustrated in fig.

person x sees the light as white light where as the person y sees the white light as a combination of seven colors (VIBGYOR) obviously the person y is getting more information than the person x by using the prism. similarly a transform is a tool that allows one to extract more information from a signal. Here the person x is in the time domain and the person y is in the frequency domain. The tool which allows us to move from time domain to frequency domain.



spectrum of white light.



Concept of transformation.

-) The transform which is widely used in the field of image Compression is discrete cosine transform.

-) The Haar transform is the simplest example of a wavelet transform.

-) one of the important advantages of wavelet transform is that signals can be represented in diff resolutions.

-) The KL transform is considered to be the best among all linear transforms with respect to energy compaction.

Fourier transform for 1D

Fourier transform is widely used in the field of image processing. An image is a spatially varying function. one way to analyse spatial variations is to decompose an image into a set of orthogonal functions. A fourier transform is used to transform an intensity image into domain of spatial frequency.

Let us assume Continous function $f(x)$. The variable x represents distance. The fourier transform of continous function is denoted as $F(U)$, where U represents the spatial frequency.

$$F(U) = \int_{-\alpha}^{\alpha} f(x) \left[\cos(2\pi Ux) - \sin(2\pi Ux)\right] dx$$

This can be expressed in conscise manner in exponential form

$$F(U) = \int_{-\alpha}^{\alpha} f(x)\, e^{-2\pi Ux}\, dx$$

Inverse fourier transform for 1D

$$F(u) = \int_{-\alpha}^{\alpha} f(x) \left[\cos(2\pi ux) + j\sin(2\pi ux)\right] dx.$$

In exponential form, it is expressed as

$$f(x) = \int_{-\alpha}^{\alpha} F(u)\, e^{+j2\pi ux}\, du$$

The fourier transform can be extended to 2D functions also.

Fourier transform for 2-D

$$F(u,v) = \int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} f(x,y)\, e^{-j2\pi(ux+vy)}\, dx\, dy$$

Inverse fourier transform for 2-D

$$f(x,y) = \int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} F(u,v)\, e^{+j2\pi(u_x+vy)}\, du\, dv$$

Discrete fourier transform

since the images are digitized it is necessary to have a discrete formulation of the fourier transform. This is achieved by the Discrete fourier transform (DFT), which takes regularly spaced data values, and return the value of the fourier transform by replacing the integral by a summation.

Properties of 2D-DFT :-

1) separable property
2) spatial shift property
3) periodicity property
4) Convolution property
5) Correlation property
6) scaling property
7) Conjugate symmetry property
8) Rotation property.

Separable property : This property allows a 2D transform to be computed in two steps by successive 1D operations on rows and columns of an image.

Mathematically it is represented as

$$F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{\frac{-j2\pi ux}{M}} e^{\frac{-j2\pi vy}{N}}$$

$$= \frac{1}{M} \sum_{x=0}^{M-1} e^{\frac{-j2\pi ux}{M}} \frac{1}{N} \sum_{y=0}^{N-1} f(x,y) e^{\frac{-j2\pi vy}{N}}$$

$$= \frac{1}{M} \sum_{x=0}^{M-1} e^{\frac{-j2\pi ux}{M}} f(x,v)$$

$$= F(u,v)$$

Note: The location of the factor $\frac{1}{MN}$ does not matter as some authors use it as part of inverse transform instead of the forward transform.

DFT for one-dimensional

$$F(U) = \sum_{x=0}^{M-1} f(x)\left[\cos\left(\frac{2\pi Ux}{M}\right) - j\sin\left(\frac{2\pi Ux}{M}\right)\right]$$

In exponential form

$$F(U) = \cancel{\frac{1}{M}} \sum_{x=0}^{M-1} f(x)\, e^{\frac{-j2\pi Ux}{M}} \times \cancel{\frac{1}{M}}$$

Inverse DFT for one-dimensional

$$F(x) = \sum_{x=0}^{m-1} f(u)\left[\cos\left(\frac{2\pi Ux}{m}\right) + i\sin\left(\frac{2\pi Ux}{m}\right)\right]$$

$$F(x) = \sum_{x=0}^{m-1} F(U)\, e^{\frac{j2\pi Ux}{M}}$$

DFT for Two-dimensional

$$F(U V) = \cancel{\frac{1}{MN}} \sum_{x=0}^{m-1} \sum_{y=0}^{N-1} f(x, y)\, e^{-j2\pi\left(\frac{Ux}{M} + \frac{Vy}{N}\right)}$$

for $U = 0 \cdots M-1$, & $V = 0 \cdots N-1$.

Inverse DFT for two-dimesional

$$f(x, y) = \sum_{U=0}^{m-1} \sum_{V=0}^{N-1} F(U, V)\, e^{j2\pi\left(\frac{Ux}{m} + \frac{Vy}{N}\right)}$$

If Images are sampled in square array for $M = N$

$$F(U, V) = \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x, y)\, e^{-j2\pi(Ux + Vy)/N}$$

$$f(x, y) = \sum_{U=0}^{N-1} \sum_{V=0}^{N-1} F(U, V)\, e^{j2\pi(Ux + Vy)/N}$$

spatial shift property

The 2D DFT of a shifted version of the image $f(x,y)$ i.e, $f(x-x_0, y)$ is given by

where x_0 represents the number of times that the function $f(x,y)$ is shifted.

proof: Adding and subtracting x_0 to $e^{\frac{-j2\pi ux}{N}}$ in equation.

$$DFT\left[f(x-x_0,y)\right] = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x-x_0,y)\, e^{\frac{-j2\pi u(x-x_0+x_0)}{M}}\, e^{\frac{-j2\pi vy}{N}}$$

→ splitting $e^{\frac{-j2\pi u(x-x_0+x_0)}{N}}$ into $e^{\frac{-j2\pi(x-x_0)u}{M}}\, e^{\frac{-j2\pi ux_0}{M}}$

$$= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x-x_0,y)\, e^{\frac{-j2\pi(x-x_0)u}{M}}\, e^{\frac{-j2\pi ux_0}{M}}\, e^{\frac{-j2\pi vy}{N}}$$

$$= \left[\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x-x_0,y)\, e^{\frac{-j2\pi(x-x_0)u}{M}}\, e^{\frac{-j2\pi vy}{N}}\right] e^{\frac{-j2\pi ux_0}{M}} \quad \text{--- (1)}$$

from the definition of 2D- DFT we can write

$$\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x-x_0,y)\, e^{\frac{-j2\pi(x-x_0)u}{M}}\, e^{\frac{-j2\pi vy}{N}} = F(u,v) \quad \text{--- (2)}$$

sub (2) in (1) we get

$$DFT\left[f(x-x_0),y\right] = e^{\frac{-j2\pi ux_0}{M}}\, F(u,v)$$

This proves that the DFT of a shifted function is unaltered except for a linearly varying phase factor.

Periodicity property

The 2D-DFT of a function $f(x,y)$ is said to be periodic with a period N if

$$F(u,v) \longrightarrow F(u+pM,\ v+qN) \quad — (1)$$

Proof:

$$F(u+pM,\ v+qN) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, e^{\frac{-j2\pi x}{M}(u+pM)}\, e^{\frac{-j2\pi y}{N}(v+qN)} \quad \rightarrow (2)$$

$$F(u+pM,\ v+qN) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, e^{\frac{-j2\pi ux}{M}}\, e^{\frac{-j2\pi xpM}{M}}\, e^{\frac{-j2\pi yv}{N}}\, e^{\frac{-j2\pi yqN}{N}} \quad \rightarrow (3)$$

By taking $e^{\frac{-j2\pi ux}{M}}$, $e^{\frac{-j2\pi vy}{N}}$

$$F(u+pM,\ v+qN) = \left[\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, e^{\frac{-j2\pi ux}{M}}\, e^{\frac{-j2\pi vy}{N}}\right] e^{-j2\pi xp}\, e^{-j2\pi yq} \quad \rightarrow (4)$$

we know

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, e^{\frac{-j2\pi ux}{M}}\, e^{\frac{-j2\pi vy}{N}} \quad \rightarrow (5)$$

(5) in (4) we get.

$$F(u+pM,\ v+qN) = F(u,v)\, e^{-j2\pi xp}\, e^{-j2\pi yq}$$

The $e^{-j2\pi xp}$, $e^{-j2\pi yq}$ values are always '1' for any integer value of x, p, q and y.

$$\therefore\ F(u+pM,\ v+qN) = F(u,v) \times 1$$

$$F(u+pM,\ v+qN) = F(u,v)$$

Convolution property

Convolution is one of the most powerful operations in digital image processing. Convolution in spatial domain is equal to multiplication in freq domain.

Convolution of 2 sequences $x(n)$ and $h(n)$ is

$$x(n) * h(n) = \sum_{k=-\alpha}^{\alpha} x(k)\, h(n-k)$$

Two-dimensional convolution of two arrays (or) matrices $f(x,y)$ and $g(x,y)$ is given as.

$$f(x,y) * g(x,y) = \sum_{a=0}^{M-1} \sum_{b=0}^{N-1} f(a,b)\, g(x-a, y-b) \quad \text{---} (1)$$

Proof: DFT of convolution of 2 sequences $f(x,y)$ and $g(x,y)$ is given by

$$DFT\{f(x,y) * g(x,y)\} = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left\{ \sum_{a=0}^{M-1} \sum_{b=0}^{N-1} f(a,b)\, g(x-a, y-b) \right\} e^{\frac{-j2\pi ux}{M}} e^{\frac{-j2\pi vy}{N}} \quad \text{---} (2)$$

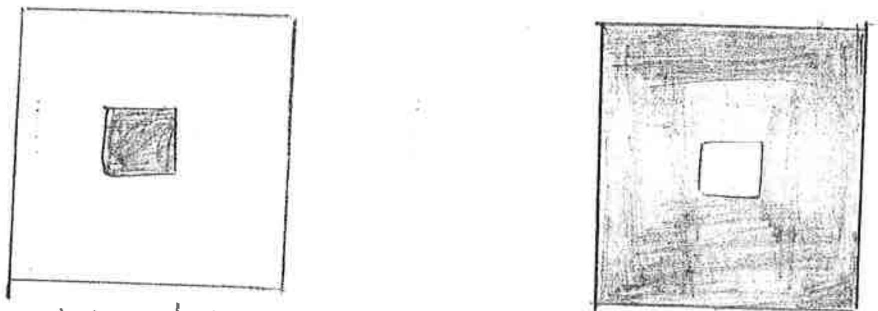
$$= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \sum_{a=0}^{M-1} \sum_{b=0}^{N-1} f(a,b)\, g(x-a, y-b)\, e^{\frac{-j2\pi}{M}(x-a+a)u}\, e^{\frac{-j2\pi}{N}(y-b+b)v} \quad \text{---} (3)$$

$$= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \sum_{a=0}^{M-1} \sum_{b=0}^{N-1} f(a,b)\, g(x-a, y-b)\, e^{\frac{-j2\pi u(x-a)}{M}}\, e^{\frac{-j2\pi au}{M}}\, e^{\frac{-j2\pi v(y-b)}{N}}\, e^{\frac{-j2\pi vb}{N}}$$

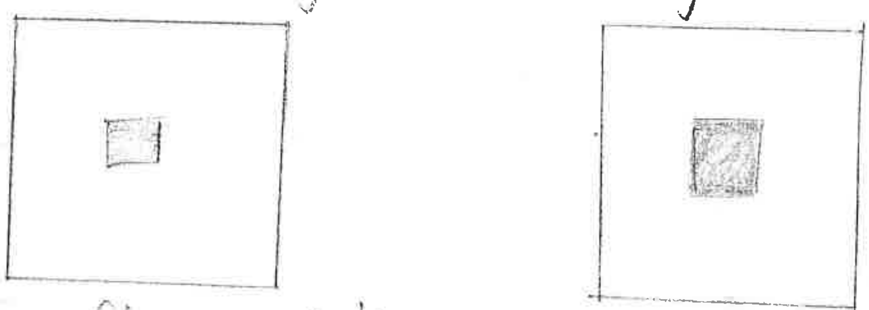
$$= \underbrace{\sum_{a=0}^{M-1} \sum_{b=0}^{N-1} f(a,b)\, e^{\frac{-j2\pi au}{M}}\, e^{\frac{-j2\pi bv}{N}}}_{F(u,v)} \; \underbrace{\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} g(x-a, y-b)\, e^{\frac{-j2\pi u(x-a)}{M}}\, e^{\frac{-j2\pi v(y-b)}{N}}}_{G(u,v)}$$

$$DFT\left\{f(x,y) * g(x,y)\right\} = F(u,v) \times G(u,v)$$

The convolution theorem tells us that the convolution of two functions in the spatial domain correspons to multiplication in the freq domain and vice-versa.



(a) original image A b) original image B



(c) image after convolution operation (d) Image after spectral multiplication.

Correlation property: Correlation is basically used to find the relative similarity between two signals. The process of finding similarity of a signal to itself is auto correlation, where as the process of finding of the similarity between two different signals is cross correlation

Proof: The DFT of correlation of two sequences $x(n)$ and $y(n)$ is defined as

$$DFT\{R_{x,h}\} = \sum_{m=0}^{N-1} \left\{ \sum_{n=0}^{N-1} x(n)(h(n+m)) \right\} e^{\frac{-j2\pi}{N} mk} \quad \text{--- (1)}$$

Here $R_{x,h}$ denotes the correlation b/w signals $x(n)$ & $h(n)$.

By adding & subtracting to the power of the exponential term $e^{\frac{-j2\pi mk}{N}}$ in eq (1) we get

$$DFT\{R_{x,h}\} = \sum_{m=0}^{N-1} \left\{ \sum_{n=0}^{N-1} x(n) h(n+m) \right\} e^{\frac{-j2\pi}{N}(m+n-n)k} \quad \text{--- (2)}$$

$e^{\frac{-j2\pi}{N}(m+n-n)k}$ in to $e^{\frac{-j2\pi(m+n)k}{N}}$ & $e^{+\frac{j2\pi}{N} nk}$

$$= \sum_{m=0}^{N-1} \left\{ \sum_{n=0}^{N-1} x(n) h(n+m) \right\} e^{\frac{-j2\pi(m+n)k}{N}} e^{\frac{j2\pi}{N} nk}$$

from the definition of DFT, we can write.

$$DFT\{R_{x,h}\} = H(k) \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi}{N} n(-k)}$$

which is reduced to

$$DFT\{R_{x,h}\} = H(k) X(-k)$$

The correlation property tell us that the correlation of two sequences in time domain is equal to the multiplication of DFT of one sequence and time reversal of the DFT of another sequence in the frequency domain.

Scaling property:-

Scaling is basically used to increase ~~aux~~ or decrease the size of an image. According to this property, the expansion of a signal in one domain is equal to compression of the signal in another domain.

The 2D DFT of a function $f(m,n)$ is defined as

$$f(m,n) \xrightarrow{DFT} F(k,l)$$

If DFT of $f(m,n)$ is $F(k,l)$ then $DFT\left[f(am,bn)\right]$

$$= \frac{1}{|ab|} F\left(k/a, l/b\right)$$

Proof:- DFT of func^n $f(am, bn)$ is given by

$$DFT\left\{f(am,bn)\right\} = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(am,bn)\, e^{\frac{-j2\pi}{N}mk}\, e^{\frac{-j2\pi}{N}nl} \quad \text{--- (1)}$$

Mul & div the power of exponential term

$e^{\frac{-j2\pi}{N}mk}$ with 'a'

$e^{\frac{-j2\pi}{N}nl}$ with 'b'

$$DFT\left\{f(am,bn)\right\} = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(am,bn)\, e^{\frac{-j2\pi}{N}mk\left(a/a\right)}\, e^{\frac{-j2\pi}{N}nl\left(b/b\right)} \quad \text{--- (2)}$$

$$DFT\left\{f(am,bn)\right\} = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(am,bn)\, e^{\frac{-j2\pi}{N}ma\left(k/a\right)}\, e^{\frac{-j2\pi}{N}nb\left(l/b\right)} \quad \text{--- (3)}$$

By sub (3) in

$$F(k,l) = \sum_{m=0}^{m-1} \sum_{n=0}^{N-1} f(m,n)\, e^{\frac{-j2\pi}{m}nk}\, e^{\frac{-j2\pi}{N}nl}$$

we get

$$DFT\{f(am, bn)\} = \frac{1}{ab} F(k/a, l/b)$$

The scaling theorem tell us that compression in one domain produces a corresponding expansion in the fourier domain and vice versa.

Conjugate symmetry

If the DFT of $f(m,n)$ is $F(k,l)$ then the

$$DFT[f^*(m,n)] = F^*(-k,-l) \quad — (1)$$

Proof: The DFT of function $f(x,y)$ is defined as

$$F(k,l) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(m,n) e^{-j\frac{2\pi}{N}mk} e^{-j\frac{2\pi}{N}nl} \quad — (2)$$

By applying complex conjugate to $F(k,l)$ we get

$$F^*(k,l) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(m,n) e^{j\frac{2\pi}{N}mk} e^{j\frac{2\pi}{N}nl}$$

By applying reversal to $F^*(k,l)$ in eq (2) we get

$$F^*(-k,-l) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(m,n) e^{j\frac{2\pi}{N}mk} e^{j\frac{2\pi}{N}nl}.$$

By applying reversal to $F^*(k,l)$ in eq (2) we get

$$F^*(-k,-l) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(m,n) e^{j\frac{2\pi}{N}m(-k)} e^{j\frac{2\pi}{N}n(-l)}$$

Orthogonality property:

The orthogonality property of a 2D DFT is given as

$$\frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} a_{k,l}(m,n)\, a^{*}_{k',l'}(m,n) = \delta(k-k',\ l-l') \quad \text{--- (1)}$$

where $\delta(k-k', l-l')$ is the kronecker delta. This orthogonality condition can be used to derive the formula for the IDFT from the definition of the DFT.

Multiplication by Exponential :-

If the DFT of $f(m,n)$ is $F(k,l)$ then

$$DFT\left[e^{j\frac{2\pi}{N}mk_0}\, e^{j\frac{2\pi}{N}nl_0} f(m,n)\right]$$

$$= \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}mk_0}\, e^{j\frac{2\pi}{N}nl_0} f(m,n)\, e^{-j\frac{2\pi}{N}mk}\, e^{-j\frac{2\pi}{N}nl} \rightarrow (1)$$

Proof:- from the definition of a 2D-DFT, we can write

$$DFT\left(e^{j\frac{2\pi}{N}mk_0}\, e^{j\frac{2\pi}{N}nl_0} f(m,n)\right)$$

$$= \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}mk_0}\, e^{j\frac{2\pi}{N}nl_0} f(m,n)\, e^{-j\frac{2\pi}{N}mk}\, e^{-j\frac{2\pi}{N}nl} \quad \text{--- (2)}$$

By combining $e^{j\frac{2\pi}{N}mk_0}$, $e^{-j\frac{2\pi}{N}mk}$ and $e^{-j\frac{2\pi}{N}nl}$, $e^{j\frac{2\pi}{N}nl_0}$ into a single exponential function in eq-(2) we get

$$DFT\left[e^{\frac{j2\pi}{N}mk_0} e^{\frac{j2\pi}{N}ml_0} f(m,n)\right]$$

$$= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(m,n)\, e^{\frac{j2\pi}{N}m(k-k_0)} e^{\frac{j2\pi}{N}n(l-l_0)} \quad — (3)$$

By sub 3 in

$$F(k,l) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n)\, e^{\frac{-j2\pi}{N}mk} e^{\frac{-j2\pi}{N}nl}$$

This theorem proves that multiplication of a function $f(m,n)$ with an exponential in the spatial domain leads to a freq shift we get

$$DFT\left[e^{\frac{j2\pi}{N}mk_0} e^{\frac{j2\pi}{N}nl_0} f(m,n)\right] = F(k-k_0, l-l_0)$$

Rotation property:

The rotation property states that if a function is rotated by the angle, its fourier transform also rotate by an amount.

$$f(m,n) \rightarrow f(r\cos\theta, r\sin\theta)$$

$$DFT\left[f(r\cos\theta, r\sin\theta)\right] \rightarrow F\left[R\cos\phi, R\sin\phi\right]$$

$$DFT\left[f(r\cos(\theta+\theta_0), r\sin(\theta+\theta_0)\right]$$

$$\rightarrow F\left[R\cos(\phi+\phi_0), R\sin(\phi+\phi_0)\right]$$

Property	Sequence	Transform
spatial shift Property	$f(x-x_0, y)$	$e^{\frac{-j2\pi u x_0}{m}} F(u,v)$
Periodicity	$F(k+pN, l+qN)$	$= F(k,l)$
Convolution	$f(m,n) * g(m,n)$	$F(k,l) \times G(k,l)$
scaling	$f(am, bn)$	$\frac{1}{\lvert ab \rvert} F(k/a, l/b)$
Conjugate symmetry	$F(k,l)$	$= F^*(-k,-l)$
multiplication by exponential	$e^{j\frac{2\pi}{N}mk_0} e^{j\frac{2\pi}{N}ml_0} f(m,n)$	$F(k-k_0, l-l_0)$
Rotation Property	$f(r\cos(\theta+\theta_0), r\sin(\theta+\theta_0))$	$F[R\cos(\phi+\phi_0), R\sin(\phi+\phi_0)]$

Example :

Compute 2D DFT of the 4x4 gray scale image given below

$$\underset{\text{Image}}{f[m,n]} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Sol:- The 2D-DFT of the image $f(m,n)$ is rep as $F[k,l]$

$$F[k,l] = \text{kernel} \times f[m,n] \times (\text{kernel})^T \quad \text{--- (1)}$$

The kernel or basis of the fourier transform for N=4 is given by

The DFT basis for N=4 is given by $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$ --- (2)

sub (2) in (1) we get

$$F(k,l) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$F(k,l) = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Walsh transform :-

Fourier analysis is basically the representation of a signal by a set of orthogonal sinusoidal waveforms. The coefficients of this representation are called frequency components and the waveforms are ordered by frequency. It is a complete set of orthonormal square wave functions to represent these functions. The computational simplicity of the walsh function is due to the fact that walsh functions are real and they take only two values which are either +1 or 1

1 - Dimensional walsh kernal

$$g(x,u) = \frac{1}{N} \prod_{i=0}^{N-1} (-1)^{b_i(x)\, b_{n-1-i}(u)}$$

1 - Dimensional transform of walsh

$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x)\, g(x,u)$$

$$= \frac{1}{N} \sum_{x=0}^{N-1} f(x) \prod_{i=0}^{N-1} (-1)^{b_i(x)\, b_{n-1-i}(u)}$$

2 - Dimensional walsh kernal

$$g(x,y,u,v) = \frac{1}{N} \prod_{i=0}^{N-1} (-1)^{b_i(x)\, b_{n-1-i}(u) + b_i(y)\, b_{n-1-i}(v)}$$

2 - D walsh transform

$$F(u,v) = \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y)\, g(x,y,u,v)$$

$$F(u,v) = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y) \prod_{i=0}^{n-1} (-1)^{b_i(x) b_{n-1-i}(u) + b_i(y) b_{n-1-i}(v)}$$

1-d inverse walsh kernal

$$h(x,u) = \frac{1}{N} \prod_{i=0}^{n-1} (-1)^{b_i(x) b_{n-1-i}(u)}$$

1-d inverse walsh transform

$$f(x) = \frac{1}{N} \sum_{u=0}^{N-1} f(u) \prod_{i=0}^{n-1} (-1)^{b_i(x) b_{n-1-i}(u)}$$

2-d Inverse walsh kernal

$$h(x,y,u,v) = \frac{1}{N} \prod_{i=0}^{n-1} (-1)^{b_i(x) b_{n-1-i}(u) + b_i(v) b_{n-1-i}(v)}$$

2-d inverse walsh transform

$$f(x,y) = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} f(u,v)\ h(x,y,u,v)$$

$$f(x,y) = \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} f(u,v) \prod_{i=0}^{n-1} (-1)^{b_i(x) b_{n-1-i}(u) + b_i(y) b_{n-1-i}(v)}$$

Ex: Walsh transform for N=4

$N = 4$

$N = 2^n$

$n = 2$

Decimal value

n	$b_1(n)$	$b_0(n)$
0	$b_1(0)=0$	$b_0(0)=0$
1	$b_1(1)=0$	$b_0(1)=1$
2	$b_1(2)=1$	$b_0(2)=0$
3	$b_1(3)=1$	$b_0(3)=1$

$$g(x,u)=\frac{1}{N}\prod_{i=0}^{n-1}(-1)^{b_i(x)\,b_{n-1-i}(u)}$$

$$g(0,0)=\frac{1}{4}\prod_{i=0}^{1}(-1)^{b_i(0)\,b_{1-i}(0)}$$

$$=\frac{1}{4}\left\{(-1)^{b_0(0)\,b_1(0)}\times(-1)^{b_1(0)\,b_0(0)}\right\}$$

$$=\frac{1}{4}\left\{(-1)^0\times(-1)^0\right\}=1/4$$

Similarly

$$g(0,1)=\frac{1}{4}\left\{\prod_{i=0}^{1}(-1)^{b_0(0)\,b_1(1)}\times(-1)^{b_1(0)\,b_0(1)}\right\}=1/4$$

$$g(1,0)=\frac{1}{4}\left\{\prod_{i=0}^{1}(-1)^{b_0(1)\,b_1(0)}\times(-1)^{b_1(1)\,b_0(0)}\right\}=1/4$$

$$g(1,1)=\frac{1}{4}\left\{\prod_{i=0}^{1}(-1)^{b_0(1)\,b_0(1)}\times(-1)^{b_1(1)\,b_0(1)}\right\}=1/4$$

$$g(0,2)=\frac{1}{4}\left\{\prod_{i=0}^{1}(-1)^{b_i(0)\,b_{1-i}(2)}\right\}$$

$$=\frac{1}{4}\left\{(-1)^{b_0(0)\,b_1(2)}\times(-1)^{b_1(0)\,b_0(2)}\right\}=1/4$$

$$g(2,1) = \frac{1}{4}\left\{(-1)^{b_0(2)b_1(1)} \times (-1)^{b_1(2)b_0(1)}\right\}$$

$$= \frac{1}{4}\left\{(1)\times(-1)\right\} = -1/4$$

By calculating all the values in similar manner we get

$u \backslash x$	0	1	2	3	Sequency
0	$1/4$	$1/4$	$1/4$	$1/4$	zero sign change
1	$1/4$	$1/4$	$-1/4$	$-1/4$	one sign change
2	$1/4$	$-1/4$	$1/4$	$-1/4$	Three sign changes
3	$1/4$	$-1/4$	$-1/4$	$1/4$	Two sign changes.

$$g(u,x) = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & -1/4 & -1/4 \\ 1/4 & -1/4 & 1/4 & -1/4 \\ 1/4 & -1/4 & -1/4 & 1/4 \end{bmatrix}$$

-) Walsh transform for N=8. It holds the same magnitude But it is lengthy process. So We go for short cut method

Algorithm for short cut method of Walsh transform

1) find the binary representation of x.
2) Find the binary values of u and consider those values in reverse binary form.
3) check for the number of overlaps of 1 between u and x.
4) If the number of overlaps of between n and K
   i) Zero overlaps then the sign is positive.
   ii) Even number of overlaps then the sign is positive.
   iii) odd then the sign is negative

For Example we can go for x = 4 and u = 3

Step 1:- write the binary representation of x = 4.
and its binary representation is 100

step 2:- write the binary representation of u = 3(011) in reverse order and it is of 110 (by reversing).

step 3:- check for the number of overlaps between n and K

$$\begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{matrix}$$

only one overlap;

It means odd num of overlaps so the sign is 'negative'

for $N = 8$

$$g(x,u) = \begin{bmatrix} +\frac{1}{8} & +\frac{1}{8} & \frac{1}{8} & +\frac{1}{8} & +\frac{1}{8} & +\frac{1}{8} & +\frac{1}{8} & +\frac{1}{8} \\ +\frac{1}{8} & +\frac{1}{8} & \frac{1}{8} & +\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} \\ +\frac{1}{8} & +\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & +\frac{1}{8} & +\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} \\ +\frac{1}{8} & +\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & +\frac{1}{8} & +\frac{1}{8} & +\frac{1}{8} \\ +\frac{1}{8} & -\frac{1}{8} & +\frac{1}{8} & -\frac{1}{8} & +\frac{1}{8} & -\frac{1}{8} & +\frac{1}{8} & -\frac{1}{8} \\ +\frac{1}{8} & -\frac{1}{8} & +\frac{1}{8} & +\frac{1}{8} & -\frac{1}{8} & +\frac{1}{8} & -\frac{1}{8} & +\frac{1}{8} \\ +\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & +\frac{1}{8} & +\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & +\frac{1}{8} \\ +\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & +\frac{1}{8} & -\frac{1}{8} & +\frac{1}{8} & +\frac{1}{8} & -\frac{1}{8} \end{bmatrix}$$

Advantage of walsh transform

The advantage of walsh transform is fourier transform is based on the trignometric terms, where as walsh transform consists of a series expansion of basis functions whose values are only +1 & -1. These functions can be implemented more efficiently in a digital environment than the exponential basis functions of the fourier transform.

Hadamard transform :-

The Hadamard transform is basically the same the walsh transform except the rows of the transform matrix are re-ordered. The Elements of the mutually orthogonal basis vectors of a Hadamard transform are either +1 & -1, which results in very low Computational Complexity in the calculation of the transform coefficients. Hadamard matrices are easily constructed for $N = 2^n$.

one-dimensional Hadamard kernal

$$g(x,u) = \frac{1}{N} (-1)^{\sum_{i=0}^{n-1} b_i(x)\, b_i(u)}$$

1-D hadamard transform

$$F(u) = \sum_{x=0}^{N-1} f(x)\, g(x,u)$$

$$= \frac{1}{N} \sum_{x=0}^{N-1} f(x)\, (-1)^{\sum_{i=0}^{n-1} b_i(x)\, b_i(u)}$$

1-D Hadamard inverse kernal

$$h(x,u) = \frac{1}{N} (-1)^{\sum_{i=0}^{n-1} b_i(x)\, b_i(u)}$$

$$f(x) = \sum_{u=0}^{n-1} F(u)\, (-1)^{\sum_{i=0}^{n-1} b_i(x)\, b_i(u)}$$

2-D Hadamard kernal

$$g(x,y,u,v) = \frac{1}{N} (-1)^{\sum_{i=0}^{n-1} b_i(x)\, b_i(u) + b_i(y)\, b_i(v)}$$

2-D. Hadamard transform

$$f(u,v) = \sum_{i=0}^{n-1} f(x,y)\ g(x,y,u,v)$$

$$= \frac{1}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y)\ (-1)^{\sum_{i=0}^{n-1} b_i(x) b_i(u) + b_i(y) b_i(v)}$$

2-D Hadamard Inverse kernal

$$h(x,y,u,v) = \frac{1}{N} (-1)^{\sum_{i=0}^{n-1} b_i(x)\ b_i(u) + b_i(y) b_i(v)}$$

$$f(x,y) = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} f(u,v)\ h(x,y,u,v)$$

$$= \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} (-1)^{\sum_{i=0}^{n-1} b_i(x)\ b_i(u) + b_i(y) b_i(v)}$$

Ex:- for $N=2$ Hadamard kernal

$N = 2$
$n = \log_2^N = 1$

$$g(x,u) = \frac{1}{N} (-1)^{\sum_{i=0}^{n-1} b_i(x)\ b_i(u)}$$

$$= \frac{1}{2} (-1)^{\sum_{i=0}^{0} b_0(x) b_0(u)}$$

$$g(0,0) = \frac{1}{2} (-1)^{b_0(0) b_0(0)} = 1/2$$

$$g(1,0) = \frac{1}{2} (-1)^{b_0(1) b_0(0)} = 1/2$$

$$g(0,1) = \frac{1}{2} (-1)^{b_0(0) b_0(1)} = 1/2$$

$$g(1,1) = \frac{1}{2} (-1) = -1/2$$

$$H_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

for Hadamard transform for $N=4$

$$H_4 = \frac{1}{4}\begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix}$$

$$H_4 = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & +1 \end{bmatrix}$$

for $N=8$

$$H_8 = \frac{1}{8}\begin{bmatrix} H_4 & H_4 \\ H_4 & -H_4 \end{bmatrix}$$

$$H_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & +1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & +1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

The main difference between walsh and Hadamard is only in the order of the basis function.

Haar transform :-

The Haar transform is based on a class of orthogonal matrices whose elements are either 1, -1, (or) 0 multiplied by powers of $\sqrt{2}$. The Haar transform is a computationally efficient transform as the transform of a N-point vector requires only $(2(N-1))$ additions and N multiplications

Algorithm for Haar transform

step1: Determine the order of N of the Haar basis.

2) Determine n where $n = \log_2 N$

3) Determine p and q

i) $0 \leq p < n-1$

ii) if $p=0$ then $q=0$ or 1

iii) If $p \neq 0$, $1 \leq q \leq 2^p$

4) Determine $k \Rightarrow k = 2^p + q - 1$

5) Determine z $\quad z \rightarrow [0,1] \Rightarrow \left[\frac{0}{N}, \frac{1}{N}, \frac{2}{N} \cdots \frac{N-1}{N}\right]$

6) If $k=0$ then $H(z) = \frac{1}{\sqrt{N}}$

$$H_k(z) = H_{pq}(z) = \frac{1}{\sqrt{N}} \begin{cases} +2^{p/2} & \frac{(q-1)}{2^p} \leq z < \frac{q-1/2}{2^p} \\ -2^{p/2} & \frac{q-1/2}{2^p} \leq z < \frac{q}{2^p} \\ 0 & \text{otherwise.} \end{cases}$$

Generate Haar basis for $N=2$

1) $N=2$
2) $n = \log_2 2 = 1$
3) i) since $n=1$, the only value of p is 0
   ii) So q takes the value of 0 (or) 1
4) Determine k $\quad k = 2^p + q - 1$

p	q	k
0	0	0
0	1	1

5) step 5: Determine z value $z \to [0,1] \Rightarrow \left[\frac{0}{2}, \frac{1}{2}\right]$

6) $\boxed{p=0}$

$q=0 \qquad q=1$

Case (i) if $k=0$ then $H(z) = \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{2}}$;

z \ k	0	1
0	$1/\sqrt{2}$	$1/\sqrt{2}$
1	—	—
2	0	1

← since the value for k is '0' for all 'z' $H(z)$ is $\frac{1}{\sqrt{2}}$

Case ii For $k=1$; $p=0$; $q=1$

Condition (i) $0 \leq z \leq 1/2$

Condition (ii) $1/2 \leq z < 1$

Condition (iii) otherwise.

$$H_k(z) = H_{pq}(z) = \frac{1}{\sqrt{2}} \begin{cases} +2^{p/2} & \left(\frac{q-1}{2^p}\right) \leq z < \frac{q-1/2}{2^p} \\ -2^{p/2} & \frac{q-1/2}{2^p} \leq z < \frac{q}{2^p} \\ 0 & \text{otherwise.} \end{cases}$$

For $z=0$ the boundary Condition (i) satisfied

$$H(z) = \frac{1}{\sqrt{2}} 2^{0/2} = 1/\sqrt{2}$$

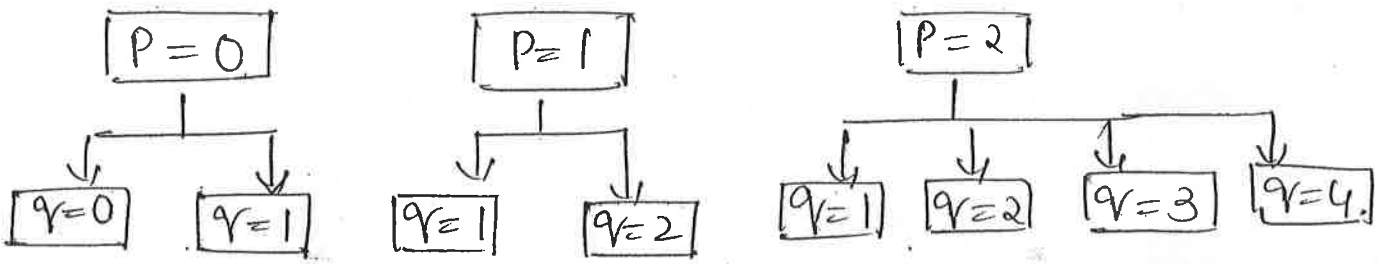
for $z=1/2$ Condition (ii) satisfied $H(z) = \frac{-1}{\sqrt{2}} 2^{0/2} = \frac{-1}{\sqrt{2}}$

The Haar basis for $N=2$ is given below

$k \backslash n$	0	1
0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
1	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$

Haar basis for $N=8$

1) Determine the order of $N=8$
2) Determine n where $n = \log_2 N = 3$
3) Determine p and q
   i) $0 \le p \le 2$
   ii) If $p=0$ then $q=0$ & $q=1$
   iii) If $p \neq 0$, $1 \le q \le 2^p$



k values for different combinations of p and q

Combination	p	q	$k = 2^p + q - 1$
0	0	0	0
1	0	1	1
2	1	1	2
3	1	2	3
4	2	1	4
5	2	2	5
6	2	3	6
7	2	4	7

5) Determine z

$z \to [0,1] \Rightarrow [0/8,\ 1/8,\ 2/8,\ 3/8,\ 4/8,\ 5/8,\ 6/8,\ 7/8]$.

6) If $k=0$ then $H(z) = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}$

otherwise

$$H_k(z) = H_{pq}(z) = \frac{1}{\sqrt{N}} \begin{cases} +2^{p/2} & \frac{(q-1)}{2^p} \le z < \frac{(q-1/2)}{2^p} \\ -2^{p/2} & \frac{(q-1/2)}{2^p} \le z < q/2^p \\ 0 & \text{otherwise,} \end{cases}$$

when $k=1$;

① $\underline{p=0;\ q=1}$ $\quad (2^p + q - 1)$

Condition i) $0 \le z < \frac{1}{2} \Rightarrow H_1(z) = \frac{1}{\sqrt{N}} 2^{p/2}$

ii) $\frac{1}{2} \le z < 1 \Rightarrow H_1(z) = \frac{-1}{\sqrt{N}} 2^{p/2}$

iii) otherwise $\Rightarrow H_1(z) = 0$.

a) for $z=0$ 1 condiⁿ satisfied so

$0 \le z < 1/2 \Rightarrow H_1(z) = \frac{1}{\sqrt{N}} 2^{p/2}$ $\quad 1/2\sqrt{2}$

(b) $z = 1/8$ 1 Condition satisfied.

$0 \le z < 1/2 \Rightarrow H_1(z) = \frac{1}{\sqrt{N}} 2^{p/2} = \frac{1}{2\sqrt{2}}$

c) for $z = 1/4$, the first condition satisfied.

$0 \le z < 1/2 \Rightarrow H_1(z) = \frac{1}{\sqrt{N}} 2^{p/2} = \frac{1}{2\sqrt{2}}$

d) $z = 3/8$, Condition 1 $\Rightarrow$ $H_1(z) = \frac{1}{\sqrt{N}} 2^{p/2} = \frac{1}{2\sqrt{2}}$

e) $z = 1/2$, Condition 2 $\Rightarrow$ $H_1(z) = \frac{-1}{\sqrt{N}} 2^{p/2} = -1/2\sqrt{2}$

f) $z = 5/8$, Condi^n 2 $\Rightarrow$ $H_1(z) = \frac{-1}{\sqrt{N}} 2^{p/2} = -1/2\sqrt{2}$.

g) $z = 3/4$, 2 Condi^n $\Rightarrow$ $H_1(z) = \frac{-1}{\sqrt{N}} 2^{p/2} = -1/2\sqrt{2}$

h) $z = 7/8$, 2 condi^n $\Rightarrow$ $H_1(z) = \frac{-1}{\sqrt{N}} 2^{p/2} = \frac{-1}{2\sqrt{2}}$

② When $k=2$ $p=1$; $q=1$

Conditions
i) $0 \le z < 1/4 \Rightarrow H_2(z) = \frac{1}{\sqrt{N}} 2^{p/2}$

ii) $1/4 \le z < 1/2 \Rightarrow H_2(z) = \frac{-1}{\sqrt{N}} 2^{p/2}$

iii) otherwise $\Rightarrow H_1(z) = 0$

a) For $z=0$, condi^n 1 $\Rightarrow$ $H_2(z) = \frac{1}{\sqrt{2}} \times 2^{p/2} = 1/2$

b) For $z = 1/8$, Condi^n 1 $\Rightarrow$ $H_2(z) = 1/2$

c) For $z = 1/4$, Condi^n 2 $\Rightarrow$ $H_2(z) = -1/2$

d) For $z = 3/8$, Condi^n 2 $\Rightarrow$ $H_2(z) = -1/2$

e) For $z = 1/2$, Condi^n 3 $\Rightarrow$ $H_2(z) = 0$

similarly

for $H_2(5/8) = H_2(3/4) = H_2(7/8) = 0$.

when $k=3$ $p=1$; $q=3$;

Conditions i) $\frac{1}{2} \le z < \frac{3}{4} \Rightarrow H_3(z) = \frac{1}{\sqrt{N}} 2^{p/2} = \frac{1}{2\sqrt{2}} \times \sqrt{2} = \frac{1}{2}$

ii) $\frac{3}{4} \le z < 1 \Rightarrow H_3(z) = \frac{-1}{\sqrt{N}} 2^{p/2} = -\frac{1}{2}$

iii) otherwise $\Rightarrow H_3(z) = 0$

a) for $z = 0, \frac{1}{4}, \frac{1}{8}, \frac{3}{8}$ satisfies 3 condition. $H_3(z) = 0$

b) for $z = \frac{1}{2}, \frac{5}{8}$ 1st condition satisfied

$$H_3(z) = \frac{1}{\sqrt{N}} 2^{p/2} = \frac{1}{2}$$

c) For $z = \frac{3}{4}, \frac{7}{8}$ the 2nd condin satisfied $= -\frac{1}{2}$

③ When $k=4$ $p=2$ $q=1$

Conditions i) $0 \le z < \frac{1}{8} \Rightarrow H_4(z) = \frac{1}{\sqrt{N}} 2^{p/2} = \frac{1}{\sqrt{2}}$

ii) $\frac{1}{8} \le z < \frac{1}{4} \Rightarrow H_4(z) = \frac{-1}{\sqrt{N}} 2^{p/2} = \frac{-1}{2\sqrt{2}} \times 2 = \frac{-1}{\sqrt{2}}$

otherwise $H_4(z) = 0$

a) for $z = 0$, 1st condition $\Rightarrow$ $H_4(z) = \frac{1}{\sqrt{2}}$

b) for. $z = \frac{1}{8}$, 2nd condition $\Rightarrow$ $H_4(z) = \frac{-1}{\sqrt{2}}$

c) for $z = \frac{1}{4}$, ~~3/8~~, ~~1/2~~, ~~5/8~~, $\frac{3}{8}$, $\frac{7}{8}$

3rd condition satisfied.

$$H_4(z) = 0$$

④ when $k=5,\ p=2;\ q=2$.

Condition i) $1/4 \le z < 3/8 \Rightarrow H_5(z) = \frac{1}{\sqrt{N}} 2^{p/2} = \frac{1}{2\sqrt{2}} \times 2 = \frac{1}{\sqrt{2}}$

ii) $3/8 \le z < 1/2 \Rightarrow H_5(z) = \frac{-1}{\sqrt{N}} 2^{p/2} = -1/\sqrt{2}$

iii) otherwise $\Rightarrow H_5(z) = 0$

a) for $z=0$, 3^{rd} condition $\rightarrow H_5(z) = 0$.

b) for $z = 1/4$, 1^{st} condition $\rightarrow H_5(z) = 1/\sqrt{2}$

c) $z = 3/8$, 2^{nd} condition $\rightarrow H_5(z) = \frac{-1}{\sqrt{2}}$

d) $z = 1/2, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}$, 3^{rd} condition $\rightarrow H_5(z) = 0$

⑤ when $k=6,\ p=2,\ q=3$

Condition i) $\frac{1}{2} \le z \le \frac{5}{8} \Rightarrow H_6(z) = \frac{1}{\sqrt{N}} 2^{p/2} = \frac{1}{2\sqrt{2}} \times 2 = \frac{1}{\sqrt{2}}$

ii) $\frac{5}{8} \le z < \frac{3}{4} \Rightarrow H_6(z) = \frac{-1}{\sqrt{N}} 2^{p/2} = \frac{-1}{\sqrt{2}}$

iii) otherwise $H_6(z) = 0$

a) $z = 0, 1/8, 1/4, 3/8, 3/4, 7/8 \rightarrow 3^{rd}$ condition is satisfied $\rightarrow H_6(z) = 0$.

b) $z = 1/2$, 1^{st} condition $\rightarrow H_6(z) = 1/\sqrt{2}$

c) $z = 5/8$, 2^{nd} condition $\rightarrow H_6(z) = \frac{-1}{\sqrt{2}}$

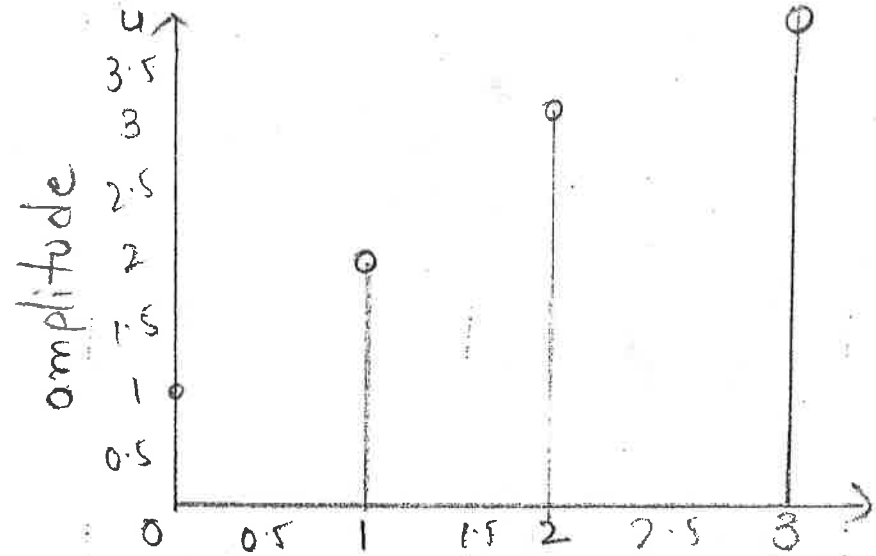
$$
= \frac{1}{2\sqrt{2}}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\frac{7}{\sqrt{21}} & \frac{5}{\sqrt{21}} & \frac{3}{\sqrt{21}} & \frac{1}{\sqrt{21}} & \frac{-1}{\sqrt{21}} & \frac{-3}{\sqrt{21}} & \frac{-5}{\sqrt{21}} & \frac{-7}{\sqrt{21}} \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
\frac{1}{\sqrt{5}} & \frac{-3}{\sqrt{5}} & \frac{3}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{-3}{\sqrt{5}} & \frac{3}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\
\frac{3}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & \frac{-3}{\sqrt{5}} & \frac{-3}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{3}{\sqrt{5}} \\
\frac{7}{\sqrt{105}} & \frac{-1}{\sqrt{105}} & \frac{-9}{\sqrt{105}} & \frac{-17}{\sqrt{105}} & \frac{17}{\sqrt{105}} & \frac{9}{\sqrt{105}} & \frac{1}{\sqrt{105}} & \frac{7}{\sqrt{105}} \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
\frac{1}{\sqrt{5}} & \frac{-3}{\sqrt{5}} & \frac{3}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & \frac{3}{\sqrt{5}} & \frac{-3}{\sqrt{5}} & \frac{1}{\sqrt{5}}
\end{bmatrix}
$$

Discrete Cosine transform :

A discrete cosine transform consists of a set of basis vectors that are sampled cosine functions. DCT is a technique for converting a signal into elementary frequency components and it is widely used in JPEG Lossy image compression. DCT only use real numbers and extended as periodically and symmetrically.

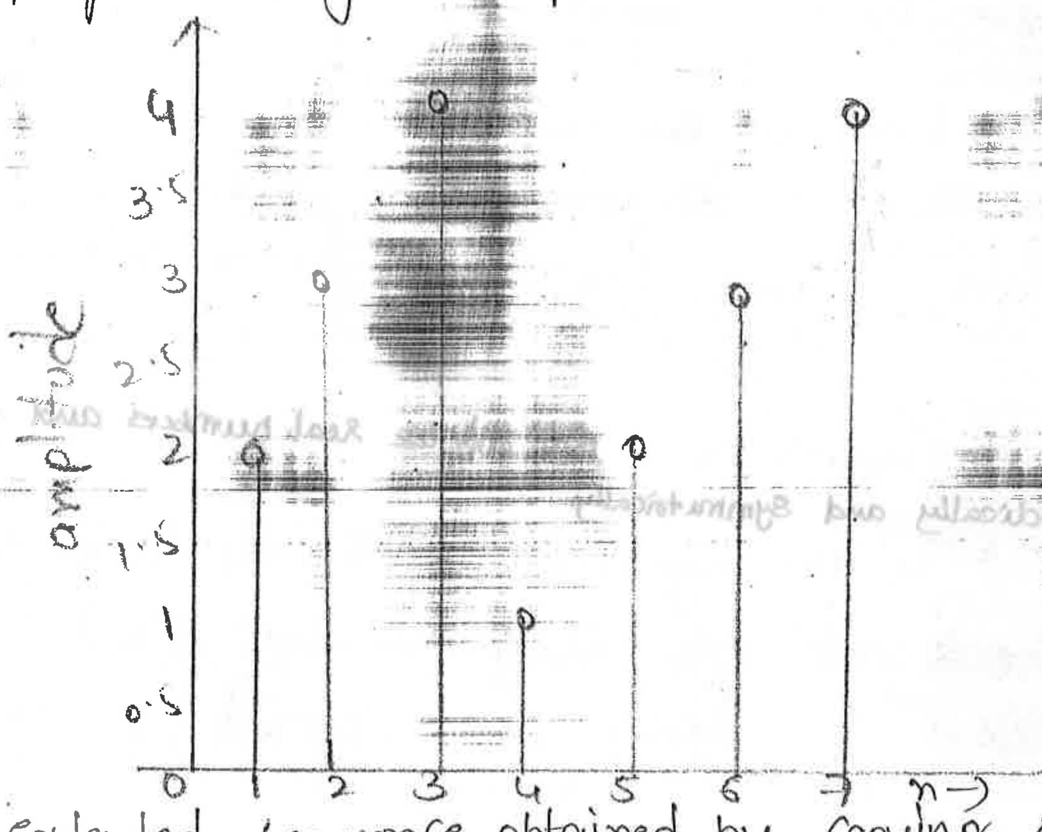
If $x(n)$ is the signal of length N, the fourier transform of the signal $x[n]$ is given by $x(k)$

$$x[k] = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$



original sequence of x[n].

The main drawback of this method is the variation in the value of the sample at n=3 and at n=4. since the variation is drastic the phenomenon of ringing is inevitable. To overcome this, a second method of obtaining the extended sequence is by copying the original sequence in a folded manner



Extended sequence obtained by copying original

KL Transform: (KARHUNEN - LOEVE Transform)

KL transform is known as Hotelling transform & eigen vector transform. It is based on stastical properties of an image.

KL Transform is used for compression of an image by decorrelating the neighbouring pixels of an image.

Procedure (or) algorithm to KL Transform:

(i) Find the mean vector and covariance of the matrix

(ii) Find the eigen values and eigen vectors of the covariance matrix

(iii) Create Transformation matrix T, such that rows of T are eigen values.

(iv) Find KL Transform.

For example:

$x_1 = (000)^T \quad x_2 = (100)^T \quad x_3 = (110)^T \quad x_4 = (101)^T$

Covariance matrix of vector population

$$C_x = \frac{1}{4} \sum_{k=1}^{4} x_k x_k^T - m_k m_k^T$$

where m_k = mean of matrix, i.e calculated as

$$m_k = \frac{1}{4}[x_1+x_2+x_3+x_4] = \frac{1}{4}\left[\begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}1\\0\\1\end{bmatrix}\right]$$

$$= \frac{1}{4}\begin{bmatrix}3\\1\\1\end{bmatrix}$$

$$m_k . m_k^T = \frac{1}{4}\begin{bmatrix}3\\1\\1\end{bmatrix} . \frac{1}{4}[3\ 1\ 1] = \frac{1}{16}\begin{bmatrix}9 & 3 & 3\\3 & 1 & 1\\3 & 1 & 1\end{bmatrix}$$

$$\frac{1}{4}\sum_{K=1}^{4} x_K x_K^T = \frac{1}{4}\left[\begin{bmatrix}0\\0\\0\end{bmatrix}[0\,0\,0]+\begin{bmatrix}1\\0\\0\end{bmatrix}[1\,0\,0]+\begin{bmatrix}1\\1\\0\end{bmatrix}[1\,1\,0]+\begin{bmatrix}1\\0\\1\end{bmatrix}[1\,0\,1]\right]$$

$$= \frac{1}{4}\begin{bmatrix}3 & 1 & 1\\ 1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}$$

The covariance matrix $= C_x = \begin{bmatrix}3/4 & 1/4 & 1/4\\ 1/4 & 1/4 & 0\\ 1/4 & 0 & 1/4\end{bmatrix} - \begin{bmatrix}9/16 & 3/16 & 3/16\\ 3/16 & 1/16 & 1/16\\ 3/16 & 1/16 & 1/16\end{bmatrix}$

$$C_x = \begin{bmatrix}3/16 & 1/16 & 1/16\\ 1/16 & 3/16 & 1/16\\ 1/16 & 1/16 & 1/16\end{bmatrix}$$

A is the transformation matrix obtained by

$$|C_x - \lambda I| = 0$$

$$\Rightarrow \left|\frac{1}{16}\begin{bmatrix}3 & 1 & 1\\ 1 & 3 & -1\\ 1 & 1 & 1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right| = 0$$

$$\Rightarrow \begin{vmatrix}\frac{3}{16}-\lambda & \frac{1}{16} & \frac{1}{16}\\ \frac{1}{16} & \frac{3}{16}-\lambda & -\frac{1}{16}\\ \frac{1}{16} & \frac{1}{16} & \frac{1}{16}-\lambda\end{vmatrix} = 0$$

$$\Rightarrow \lambda_1 = 0.25 \quad \lambda_2 = 0.065 \quad \lambda_3 = 0.125$$

Eigen vectors corresponding to eigen values $\lambda_1, \lambda_2, \lambda_3$ are

$$\begin{array}{c} \lambda_1 \rightarrow \\ \lambda_2 \rightarrow \\ \lambda_3 \rightarrow \end{array}\begin{bmatrix}0.8165 & 0.4082 & 0.4082\\ 0.5774 & 0.5774 & -0.5774\\ 0.7071 & -0.7071 & 0\end{bmatrix} = A$$

Transformed vector groups are obtained as

$y_1 = A(x_1 - m_x)$ $y_2 = A(x_2 - m_x)$

$y_3 = A(x_3 - m_x)$ $y_4 = A(x_4 - m_x)$

$$y_1 = \begin{bmatrix} 0.8165 & 0.4082 & 0.4082 \\ 0.5774 & -0.5774 & -0.5774 \\ 0.7071 & 0.7071 & 0 \end{bmatrix} \begin{bmatrix} 0 - 3/4 \\ 0 - 1/4 \\ 0 - 1/4 \end{bmatrix} = \begin{bmatrix} -0.8165 \\ -0.1444 \\ -0.3535 \end{bmatrix}$$

Similarly $y_2 = \begin{bmatrix} 0 \\ 0.4331 \\ 0.3535 \end{bmatrix}$ $y_3 = \begin{bmatrix} 0.4082 \\ -0.1444 \\ -0.35 \end{bmatrix}$ $y_4 = \begin{bmatrix} 0.4082 \\ -0.144 \\ 0.3535 \end{bmatrix}$

Covariance of transposed vectors

$$C_y = \frac{1}{4} \sum_{k=1}^{4} y_k y_k^T - m_y m_y^T$$

$$m_y = \frac{1}{4}\left[y_1 + y_2 + y_3 + y_4\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_y = \frac{1}{4}\left[y_1 y_1^T + y_2 y_2^T + y_3 y_3^T + y_4 y_4^T\right] - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_y = \frac{1}{4}\begin{bmatrix} 0.994 & 0 & 0 \\ 0 & 0.248 & 0 \\ 0 & 0.2 & 0.5 \end{bmatrix}$$

Inverse Transform $x = A^T y + m_x$

Applications of KL Transform:

→ Dimentional reduction

→ Removes Random Noise without blurring stationary & moving edges

→ Reduce noise in real time images

→ Extraction of signal corresponding to small breathing displacements of human chest.

SVD Transform: (Singular value Decomposition Transform):

The SVD transform is another popular image transform that has huge no. of applications in image Restoration, compression and object recognition. The SVD transform of an image 'f' is

$$g = SVD(A)$$

The SVD transform transforms the given matrix A into the product $U \times S \times V^{-1}$ i.e

$$A = U \times S \times V^{-1}$$

The matrix U is an orthogonal matrix. The column vectors from an orthogonal set. i.e

$$u_i^T \times u_j = \delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

The matrix V is an $n \times n$ orthogonal matrix and its columns form an orthonormal set. 'S' is the matrix of order $n \times n$ with singular values are

$$S = \begin{bmatrix} \sigma_1 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_3 & \cdots & 0 \\ & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & \sigma_n \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$\sigma_1, \sigma_2, \cdots \sigma_n$ are called singular values which are square roots of the eigen values and form diagonal of S.

The property of SVD is that the singular values are not unique. i.e $U = AA^{-1}$ $V = \bar{A}^{t} \cdot A$.

Therefore the image is expressed as

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T$$

$$\Rightarrow A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T$$

here r is called Rank of matrix

Here the Rank is nothing but no. of non zero diagonal elements.

The SVD transform is used for image compression. If the sum is truncated after r terms, the result is called r. approximation of original matrix.

The ~~distance~~ difference b/w original and approximation is called error.

<u>Ex:</u> Find SVD of image F?

$$F = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

<u>Sol:</u> <u>Step 1:</u> Here F is square matrix. Calculate eigen value & eigen vector & characteristic equation

$$|F - \lambda I| = 0$$

$$\Rightarrow (\lambda - 1)^2(\lambda + 2) = 0$$

$\lambda = 1, 1, -2$ and eigen vectors are

$x_1 = (1\,0\,1)$ $\quad x_2 = (1\,2\,-1)$ $\quad x_3 = (-1\,1\,1)$

Normalized the vector $\Rightarrow$ the normalized matrix & modal matrix S

$$S = \begin{bmatrix} \frac{1}{\sqrt{1^2+1^2}} & \frac{1}{\sqrt{1^2+2^2+4^2}} & \frac{1}{\sqrt{-1^2+2^2+1^2}} \\ \frac{1}{\sqrt{1^2+1^2}} & \frac{1}{\sqrt{1^2+2^2+4-1^2}} & \frac{1}{\sqrt{-1^2+2^2+1^2}} \\ \frac{1}{\sqrt{1^2+1^2}} & \frac{1}{\sqrt{1^2+2^2+8-1^2}} & \frac{1}{\sqrt{-1^2+2^2+1^2}} \end{bmatrix}$$

$$S = \begin{bmatrix} 1/\sqrt{2} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\Rightarrow \quad D = S \times F \times S$$

$$\Rightarrow \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$