Random Numbers

Ravula Karthik
AI21BTECH11024

1. Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat
Solution: The c code for 1.1 can be obtained from

```
wget https://github.com/karthik6281/AI1110-Assignments/blob/main/random_numbers/1/exrand.c
wget https://github.com/karthik6281/AI1110-Assignments/blob/main/random_numbers/1/coeffs.h
```

Execute the following lines

```
gcc exrand.c -lm
./a.out
```

1.2 Load the uni.dat file into Python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: The python code for 1.2 can be obtained from

```
wget https://github.com/karthik6281/AI1110-Assignments/blob/main/random_numbers/1/cdf_plot.py
```

Execute the following lines

```
python3 1_2.py
```

1.3 Find a theoretical expression for $F_U(x)$
Solution: The PDF of U is given by

$$p_U(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

The CDF of U is given by

$$F_U(x) = \Pr(U \le x) = \int_{-\infty}^{x} p_U(x)\, dx \tag{1.3}$$

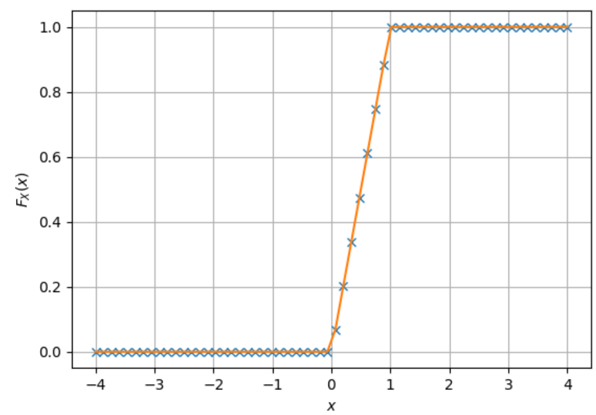


Fig. 1.2. The CDF of U

If $x < 0$,

$$\int_{-\infty}^{x} p_U(x)\, dx = \int_{-\infty}^{x} 0\, dx = 0 \tag{1.4}$$

If $x \in [0,1]$,

$$\int_{-\infty}^{x} p_U(x)\, dx = \int_{-\infty}^{0} 0\, dx + \int_{0}^{x} 1\, dx \tag{1.5}$$
$$= 0 + x \tag{1.6}$$
$$= x \tag{1.7}$$

If $x > 1$,

$$\int_{-\infty}^{x} p_U(x)\, dx$$
$$= \int_{-\infty}^{0} 0\, dx + \int_{0}^{1} 1\, dx + \int_{1}^{x} 0\, dx \tag{1.8}$$
$$\int_{-\infty}^{x} p_U(x)\, dx = 0 + 1 + 0 \tag{1.9}$$
$$= 1 \tag{1.10}$$

Therefore, we obtain the CDF of U as

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \tag{1.11}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.12}$$

and its variance as

$$\text{Var}[U] = E[U - E[U]]^2 \tag{1.13}$$

Write a C program to find the mean and variance of U

Solution: The c code for 1.4 can be obtained from

```
wget https://github.com/karthik6281/AI1110-
    Assignments/blob/main/random_numbers
    /1/1_4.c
wget https://github.com/karthik6281/AI1110-
    Assignments/blob/main/random_numbers
    /1/coeffs.h
```

Execute the following lines

```
gcc 1_4.c -lm
./a.out
```

$$\text{Mean} = 0.500137 \tag{1.14}$$
$$\text{Variance} = 0.083251 \tag{1.15}$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k \mathrm{d}F_U(x) \tag{1.16}$$

Solution:

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.17}$$

$$dF_U(x) = dx \tag{1.18}$$
$$\therefore E[U^k] = \int_{-\infty}^{\infty} x^k dx \tag{1.19}$$
$$E[U] = \int_0^1 x dx = \frac{1}{2} \tag{1.20}$$
$$E[U^2] = \int_0^1 x^2 dx = \frac{1}{3} \tag{1.21}$$
$$\because P_X(x) = 0, \forall x \in (1,\infty) \cap (-\infty, 0) \tag{1.22}$$
$$Var(X) = E[U^2] - (E[U])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \tag{1.23}$$

2. Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: The c code for 2.1 can be obtained from

```
wget https://github.com/karthik6281/
    AI1110-Assignments/blob/main/
    random_numbers/2/2_1.c
wget https://github.com/karthik6281/
    AI1110-Assignments/blob/main/
    random_numbers/2/coeffs.h
```

Execute the following lines

```
gcc 2_1.c -lm
./a.out
```

2.2 Load gau.dat in Python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The python code for 2.2 can be obtained from

```
wget https://github.com/karthik6281/
    AI1110-Assignments/blob/main/
    random_numbers/2/2_2.py
```

Execute the following lines

```
python3 2_2.py
```

- $\Phi(x) = P(Z \le x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left\{-\frac{u^2}{2}\right\} du$
- $\lim_{x\to\infty} \Phi(x) = 1, \quad \lim_{x\to-\infty} \Phi(x) = 0$
- $\Phi(0) = \frac{1}{2}$
- $\Phi(-x) = 1 - \Phi(x)$

2.3 Load gau.dat in Python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x) \tag{2.2}$$

What properties does the PDF have?

Solution: The python code for 2.3 can be obtained from

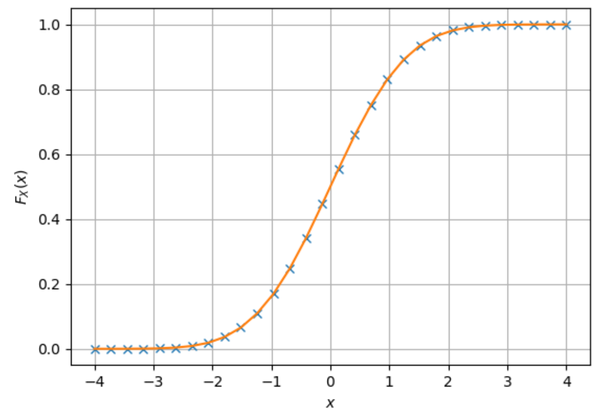


Fig. 2.5. The CDF of X

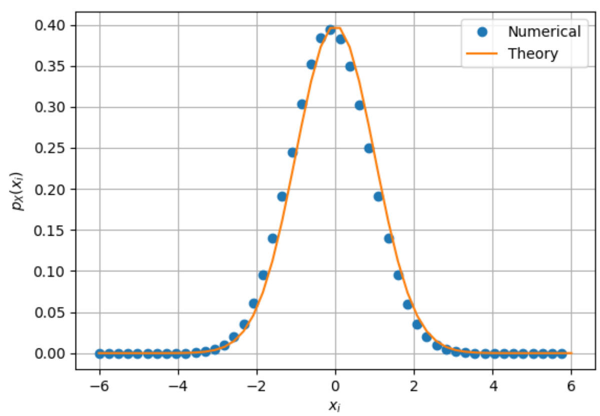


Fig. 2.5. The PDF of X

```
wget https://github.com/karthik6281/
    AI1110−Assignments/blob/main/
    random_numbers/2/2_3.py
```

Execute the following lines

```
python3 2.3.py
```

Every PDF is bounded between 0 and 1 and

$$\int_{-\infty}^{\infty} p_X(x)\ \mathrm{d}x = 1 \tag{2.3}$$

In this case, the PDF is symmetric about $x = 0$ and graph is bell shaped

2.4 Find the mean and variance of X by writing a C program
Solution: the c code for 2.4 can be obtained from

```
wget https://github.com/karthik6281/
    AI1110−Assignments/blob/main/
    random_numbers/2/2_4.c
wget https://github.com/karthik6281/
    AI1110−Assignments/blob/main/
    random_numbers/2/coeffs.h
```

execute the following lines

```
gcc 2_4.c −lm
./a.out
```

$$\text{Mean} = 0.000294 \tag{2.4}$$
$$\text{Variance} = 0.999561 \tag{2.5}$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.6}$$

repeat the above exercise theoretically
Solution: The mean of X is given by

$$E\left[X\right] = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.7}$$

$\implies xe^{-\frac{-x^2}{2}}$ is a odd function

$$\therefore E\left[X\right] = \int_{-\infty}^{\infty} g(x) \mathrm{d}x = 0 \tag{2.8}$$

Now,

$$E\left[X^2\right] = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.9}$$
$$= 2\int_{0}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.10}$$

$\implies \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$ is an even function
By integration by parts,

$$E\left[X^2\right] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} x \cdot x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.11}$$

$$= \sqrt{\frac{2}{\pi}} \left(x \int x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x\right)\Bigg|_{0}^{\infty} - \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} 1 \cdot \int x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.12}$$

Substitute $t = \frac{x^2}{2} \implies \mathrm{d}t = x\mathrm{d}x$

$$\int x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x = \int \exp(-t)\mathrm{d}t \quad (2.13)$$
$$= -\exp(-t) \quad (2.14)$$
$$= -\exp\left(-\frac{x^2}{2}\right) \quad (2.15)$$

$$\int_0^\infty -\exp\left(-\frac{x^2}{2}\right)\mathrm{d}x \quad (2.16)$$
$$\xleftrightarrow{x=t\sqrt{2}} \int_0^\infty -\exp(-t^2)\mathrm{d}t\,\sqrt{2} \quad (2.17)$$
$$= -\sqrt{2}\int_0^\infty \exp(-t^2)\mathrm{d}t \quad (2.18)$$
$$= -\sqrt{\frac{\pi}{2}} \quad (2.19)$$

Therefore,

$$E\left[X^2\right] = 0 - \sqrt{\frac{2}{\pi}}\left(-\sqrt{\frac{\pi}{2}}\right) \quad (2.20)$$
$$= 1 \quad (2.21)$$
$$\therefore \mathrm{Var}\left[X\right] = E\left[X^2\right] - \left(E\left[X\right]\right)^2 \quad (2.22)$$
$$= 1 - 0 \quad (2.23)$$
$$= 1 \quad (2.24)$$

3. From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \quad (3.1)$$

and plot its CDF

Solution: The c code for 3.1 can be obtained from

```
wget https://github.com/karthik6281/AI1110−
    Assignments/blob/main/random_numbers
    /3/3_1.c
```

Execute the following lines

```
gcc 3.1.c −lm
./a.out
```

The python code for cdf can be obtained from

```
wget https://github.com/karthik6281/AI1110−
    Assignments/blob/main/random_numbers
    /3/3_1.py
```

execute the following lines

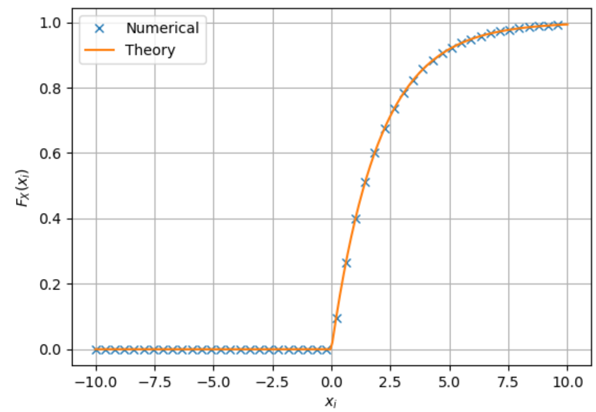


Fig. 3.5. The CDF of V

```
python3 3_1.py
```

3.6 Find a theoretical expression for $F_V(x)$

Solution:

$$F_V(x) = P(V \leq x) \quad (3.2)$$
$$= P(-2ln(1-U) \leq x) \quad (3.3)$$
$$= P(1 - e^{\frac{-x}{2}} \geq U) \quad (3.4)$$
$$P(U < x) = \int_0^x dx = x \qquad 0 < x < 1 \quad (3.5)$$
$$\therefore P(1 - e^{\frac{-x}{2}} \geq U) = 1 - e^{\frac{-x}{2}}, \forall x \geq 0 \quad (3.6)$$

4. Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: The c code for 4.1 can be obtained from

```
wget https://github.com/karthik6281/
    AI1110−Assignments/blob/main/
    random_numbers/4/4_1.c
```

execute the following lines

```
gcc 4_1.c −lm
./a.out
```

4.2 Find the CDF of T

Solution:

$$p_T(x) = p_{U_1+U_2}(x) = p_{U_1}(x) * p_{U_2}(x) \quad (4.2)$$

$$p_T(x) = \int_{-\infty}^{\infty} p_{U_1}(\tau)p_{U_2}(x-\tau) \quad (4.3)$$

$$p_T(x) = \int_0^1 p_{U_2}(x-\tau) \quad (4.4)$$

$$p_T(x) = \begin{cases} 0 & x \leq 0 \\ \int_0^x 1d\tau & 0 < x < 1 \\ \int_{x-1}^1 1d\tau & 1 \leq x < 2 \\ 0 & x > 2 \end{cases} \quad (4.5)$$

$$p_T(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 2-x & 1 \leq x < 2 \\ 0 & x > 2 \end{cases} \quad (4.6)$$

Expression for CDF can be obtained by integrating $p_T(x)$ w.r.t. X

$$F_T(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x^2}{2} & 0 < x < 1 \\ -\frac{x^2}{2} + 2x - 1 & 1 \leq x < 2 \\ 1 & x > 2 \end{cases} \quad (4.7)$$

4.3 Find the PDF of T

Solution: The PDF of T is given by

$$p_T(t) = \frac{d(F_T(t))}{dt} \quad (4.8)$$

$$\therefore p_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2-t & 1 < t < 2 \\ 0 & t \geq 2 \end{cases} \quad (4.9)$$

4.4 Find the theoretical expressions for the PDF and CDF of T

Solution:

$$P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2-t & 0 < t \leq 2 \\ 0 & t > 2 \end{cases} \quad (4.10)$$

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 2t - \frac{t^2}{2} - 1 & 1 < t \leq 2 \\ 1 & t > 2 \end{cases} \quad (4.11)$$

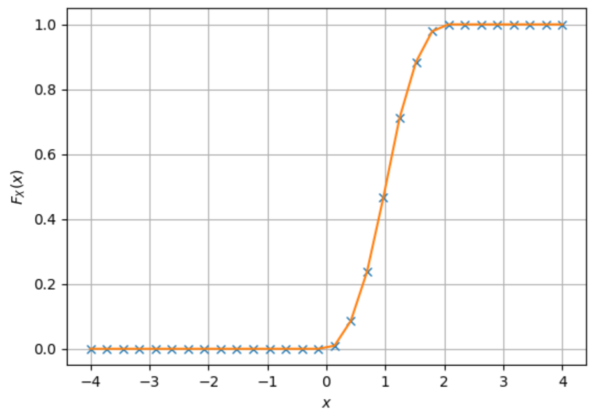


Fig. 6. The CDF of T

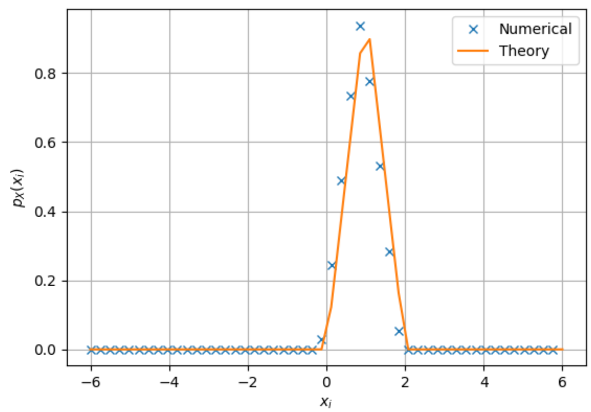


Fig. 6. The PDF of T

4.5 Verify your results through a plot

Solution: The codes that plots cdf and pdf can be obtained from

```
wget https://github.com/karthik6281/
    AI1110−Assignments/blob/main/
    random_numbers/4/4_cdf.py
wget https://github.com/karthik6281/
    AI1110−Assignments/blob/main/
    random_numbers/4/4_pdf.py
```

Execute the following lines

```
python3 4_cdf.py
python3 4_pdf.py
```

5. Maximum Likelihood

5.1 Generate equiprobable $X \in \{-1, 1\}$

Solution: The c code for 5.1 can obtained from

```
wget https://github.com/karthik6281/AI1110-
    Assignments/blob/main/random_numbers
    /5/5_1.c
wget https://github.com/karthik6281/AI1110-
    Assignments/blob/main/random_numbers
    /5/coeffs.h
```

execute the following commands

```
gcc 5_1.c -lm
    ./a.out
```

5.2 Generate

$$Y = AX + N \tag{5.1}$$

where $A = 5$ dB, $X \in \{-1, 1\}$ is Bernoulli and $N \sim \mathcal{N}(0, 1)$
Solution: The c code for 5.2 can obtained from

```
wget https://github.com/karthik6281/AI1110-
    Assignments/blob/main/random_numbers
    /5/5_2.c
wget https://github.com/karthik6281/AI1110-
    Assignments/blob/main/random_numbers
    /5/coeffs.h
```

execute the following commands

```
gcc 5_2.c -lm
    ./a.out
```

5.3 Plot Y
Solution: Download the following Python code that plots Fig. 5.3

```
https://github.com/karthik6281/AI1110-
    Assignments/blob/main/random_numbers
    /5/5_3_plot.py
```

Run the code by executing

```
python3 5_3_plot.py
```

5.4 Guess how to estimate X from Y
Solution:

$$X = \begin{cases} 1 & Y > 0 \\ -1 & Y < 0 \end{cases} \tag{5.2}$$

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.3}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.4}$$

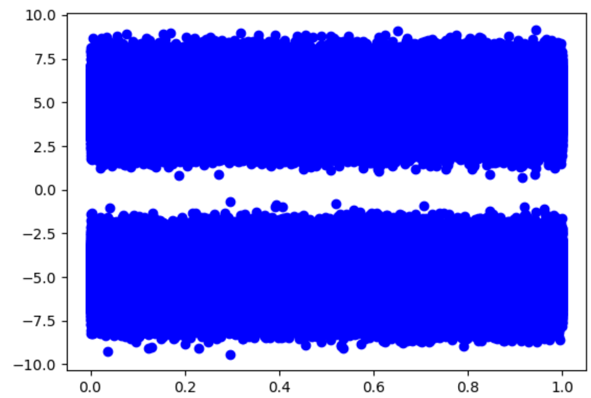


Fig. 5.3. Plot of Y

Solution: $\hat{X} = -1 \implies Y < 0$
Given $X = +1$, $Y < 0 \implies A + N < 0 \implies N < -A$

$$\Pr\left(\hat{X} = -1 | X = 1\right) = \Pr\left(N < -A\right) \tag{5.5}$$
$$= 1 - \Pr\left(N > -A\right) \tag{5.6}$$
$$= 1 - Q(-A) \tag{5.7}$$
$$= Q(A) \tag{5.8}$$

Similarly $\hat{X} = 1 \implies Y > 0$
Given $X = -1$, $Y > 0 \implies -A + N > 0 \implies N > A$

$$\Pr\left(\hat{X} = 1 | X = -1\right) = \Pr\left(N > A\right) \tag{5.9}$$
$$= Q(A) \tag{5.10}$$

5.6 Find P_e assuming that X has equiprobable symbols
Solution:

$$P_e = \Pr\left(X = -1\right) P_{e|0} + \Pr\left(X = 1\right) P_{e|1} \tag{5.11}$$

From question, $\Pr\left(X = -1\right) = \Pr\left(X = 1\right) = \frac{1}{2}$

$$P_e = 2 \cdot (\frac{1}{2} Q(A)) \tag{5.12}$$
$$= Q(A) \tag{5.13}$$

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB
Solution: $\hat{X} = 1 \implies Y > 0$
then, $X = -1$, $Y > 0 \implies -A + N > 0 \implies N > A$

$$P_{e|0} = \Pr\left(N > A\right) = Q(A) \tag{5.14}$$

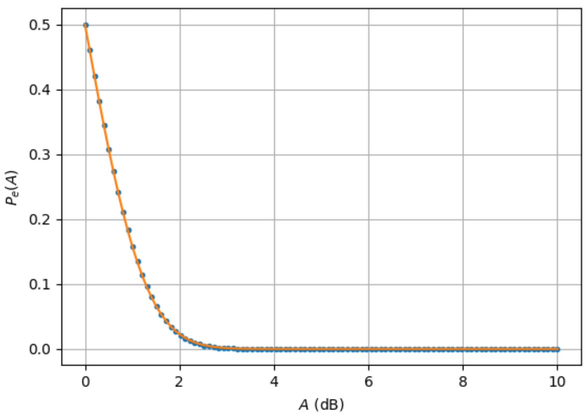


Fig. 5.7. Plot of P_e

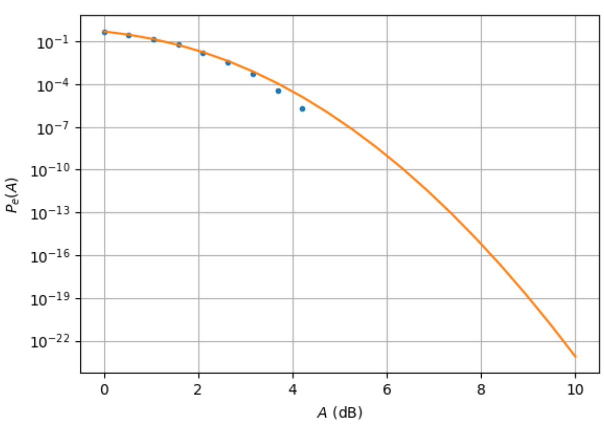


Fig. 5.7. Plot of P_e

similarly ;

$$\Pr(N < -A) = Q(A) \tag{5.15}$$

The python code for 5.7 can be obtained from

```
wget https://github.com/karthik6281/AI1110-
    Assignments/blob/main/random_numbers
    /5/5_7.py
```

Execute the following commands

```
python3 5_7.py
```

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that minimizes the theoretical P_e
Solution: To estimate X from Y, we now consider the following:

$$X = \begin{cases} 1, & Y > \delta \\ -1, & Y < \delta \end{cases} \tag{5.16}$$

Therefore, $\hat{X} = -1 \implies Y < \delta$
then, $X = 1$, $Y > 0 \implies A + N < \delta \implies N < \delta - A$

$$P_{e|0} = \Pr(N < \delta - A) \tag{5.17}$$
$$= \int_{-\infty}^{\delta - A} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \tag{5.18}$$
$$= \int_{A-\delta}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \tag{5.19}$$
$$= Q_N(A - \delta) \tag{5.20}$$

Similarly

$$P_{e|1} = Q_N(A + \delta) \tag{5.21}$$

$$P_e = \frac{1}{2}(Q_N(A - \delta) + Q_N(A + \delta)) \tag{5.22}$$

To minimise P_e, we differentiate w.r.t δ:

$$0 = \frac{d}{d\delta}\left(\frac{1}{2}(Q_N(A - \delta) + Q_N(A + \delta))\right) \tag{5.23}$$
$$= \frac{1}{2}\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{(\delta - A)^2}{2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(A+\delta)^2}{2}}\right) \tag{5.24}$$

From which we obtain

$$\implies \delta = A = -A \implies \delta = 0 \tag{5.25}$$

5.9 Repeat the above exercise when

$$p_X(-1) = p \tag{5.26}$$

Solution:

$$P_e = p_X(1)P_{e|0} + p_X(-1)P_{e|1} \tag{5.27}$$
$$= (1 - p)Q(A - \delta) + pQ(A + \delta) \tag{5.28}$$

To minimise P_e, we differentiate w.r.t δ:

$$0 = p\frac{1}{\sqrt{2\pi}} e^{-\frac{(\delta-A)^2}{2}} - (1 - p)\frac{1}{\sqrt{2\pi}} e^{-\frac{(A+\delta)^2}{2}} \tag{5.29}$$

Taking ln on both sides we have:

$$\ln p - \frac{(\delta - A)^2}{2} = \ln 1 - p + \frac{(\delta + A)^2}{2} \tag{5.30}$$
$$\implies \delta = \frac{1}{2A} \ln \frac{1 - p}{p} \tag{5.31}$$

5.10 Repeat the above exercise using the MAP criterion

Solution: Taking $\Pr(X = -1) = p$, and $\Pr(X = 1) = (1 - p)$.

$$p_Y(y) = p \times p_{(-A+N)}(y) + (1-p) \times p_{(A+N)}(y) \tag{5.32}$$

where $p_Y(y)$ is the pdf of Y. Now, $p_{(-A+N)}$ is just the pdf of a shifted normal distribution, and therefore:

$$p_Y(y) = p\frac{e^{-\frac{(y+A)^2}{2}}}{\sqrt{2\pi}} + (1-p)\frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}} \tag{5.33}$$

finding $p_{X|Y}(x|y)$. We already know that

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x) \times p_X(x)}{p_Y(y)} \tag{5.34}$$

$X = 1$:

$$p_{X|Y}(1|y) = \frac{p_{Y|X}(y|1) \times p_X(1)}{p_Y(y)} \tag{5.35}$$

$$= \frac{(1-p)\frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}}}{p\frac{e^{-\frac{(y+A)^2}{2}}}{\sqrt{2\pi}} + (1-p)\frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}}} \tag{5.36}$$

$$= \frac{(1-p)e^{2yA}}{p + (1-p)e^{2yA}} \tag{5.37}$$

Similarly,$X = -1$:

$$p_{X|Y}(-1|y) = \frac{p}{p + (1-p)e^{2yA}} \tag{5.38}$$

Therefore, when $p_{X|Y}(1|y) > p_{X|Y}(-1|y)$, we have:

$$\frac{(1-p)e^{2yA}}{p + (1-p)e^{2yA}} > \frac{p}{p + (1-p)e^{2yA}} \tag{5.39}$$

$$e^{2yA} > \frac{p}{(1-p)} \tag{5.40}$$

$$y > \frac{1}{2A}\ln\frac{p}{(1-p)} \tag{5.41}$$

Therefore, when Eq. (5.41), we can assert that $X = 1$, and $X = -1$ otherwise. Now, consider when $p = \frac{1}{2}$. We have:

$$y > \frac{1}{2A}\ln\frac{p}{(1-p)} \tag{5.42}$$

$$= \frac{1}{2A}\ln 1 \tag{5.43}$$

$$= 0 \tag{5.44}$$

Therefore, when $y > 0$, we choose $X = 1$, and we choose $X = -1$ otherwise.

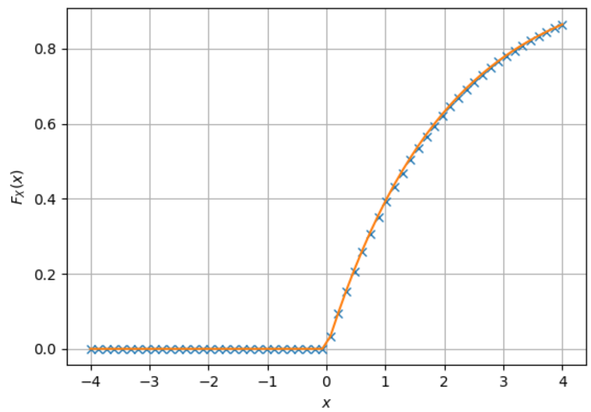


Fig. 6.1. CDF of V

6. Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

Solution: The c code to generate data can be obtained from

```
wget https://github.com/karthik6281/AI1110−
    Assignments/blob/main/random_numbers
    /6/6_1.c
wget https://github.com/karthik6281/AI1110−
    Assignments/blob/main/random_numbers
    /6/coeffs.h
```

Execute the following commands

```
gcc 6_1.c −lm
./a.out
```

The codes that plot cdf and pdf can be obtained from

```
wget https://github.com/karthik6281/AI1110−
    Assignments/blob/main/random_numbers
    /6/6_1_cdf.c
wget https://github.com/karthik6281/AI1110−
    Assignments/blob/main/random_numbers
    /6/6_1_pdf.c
```

Run the code by executing

```
python3 6_1.cdf.py
python3 6_1.pdf.py
```

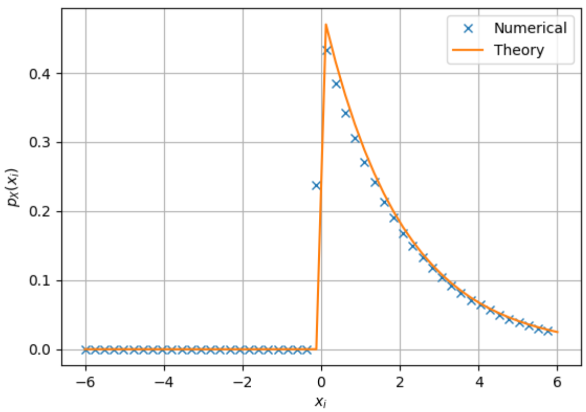


Fig. 6.1. PDF of V

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.2}$$

find α

Solution: Let $R \geq 0, \Theta \in [0, 2\pi]$

$$X_1 = R\cos\Theta \tag{6.3}$$

$$X_2 = R\sin\Theta \tag{6.4}$$

such that $V = X_1^2 + X_2^2 = R^2$

The Jacobian matrix transforming R, Θ to X_1, X_2 is defined as

$$\vec{J} = \begin{pmatrix} \frac{\partial X_1}{\partial R} & \frac{\partial X_1}{\partial \Theta} \\ \frac{\partial X_2}{\partial R} & \frac{\partial X_2}{\partial \Theta} \end{pmatrix} \tag{6.5}$$

$$= \begin{pmatrix} \cos\Theta & -R\sin\Theta \\ \sin\Theta & R\cos\Theta \end{pmatrix} \tag{6.6}$$

$$\implies \left|\vec{J}\right| = R\cos^2\Theta + R\sin^2\Theta = R \tag{6.7}$$

Then

$$p_{R,\Theta}(r,\theta) = p_{X_1,X_2}(x_1,x_2)\left|\vec{J}\right| \tag{6.8}$$

$$= \frac{R}{2\pi}\exp\left(-\frac{X_1^2 + X_2^2}{2}\right) \tag{6.9}$$

$$= \frac{R}{2\pi}\exp\left(-\frac{R^2}{2}\right) \tag{6.10}$$

we can find,

$$p_R(r) = \int_0^{2\pi} p_{R,\Theta}(r,\theta)d\theta \tag{6.11}$$

$$= R\exp\left(-\frac{R^2}{2}\right) \tag{6.12}$$

We can find cdf by

$$F_R(r) = \Pr(R \leq r) \tag{6.13}$$

$$= \int_0^r p_R(r)\mathrm{d}r \tag{6.14}$$

$$= \int_0^r r\exp\left(-\frac{r^2}{2}\right)\mathrm{d}r \tag{6.15}$$

$$= -\exp\left(-\frac{r^2}{2}\right)\Big|_0^r \tag{6.16}$$

$$= 1 - \exp\left(-\frac{r^2}{2}\right) \quad \text{for } r \geq 0 \tag{6.17}$$

But we need to find $F_V(x)$ which can be wriiten as

$$F_V(x) = F_R(\sqrt{x}) \tag{6.18}$$

$$= 1 - \exp\left(-\frac{x}{2}\right) \quad \text{for } x \geq 0 \tag{6.19}$$

And the PDF of V is given by

$$p_V(x) = \frac{\mathrm{d}}{\mathrm{d}x}F_V(x) \tag{6.20}$$

$$= \frac{1}{2}\exp\left(-\frac{x}{2}\right) \tag{6.21}$$

Therefore,

$$F_V(x) = \begin{cases} 1 - \exp\left(-\frac{x}{2}\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{6.22}$$

$$p_V(x) = \begin{cases} \frac{1}{2}\exp\left(-\frac{x}{2}\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{6.23}$$

$$\therefore \alpha = \frac{1}{2} \tag{6.24}$$

6.3 Plot the CDF and PDF of

$$A = \sqrt{V} \tag{6.25}$$

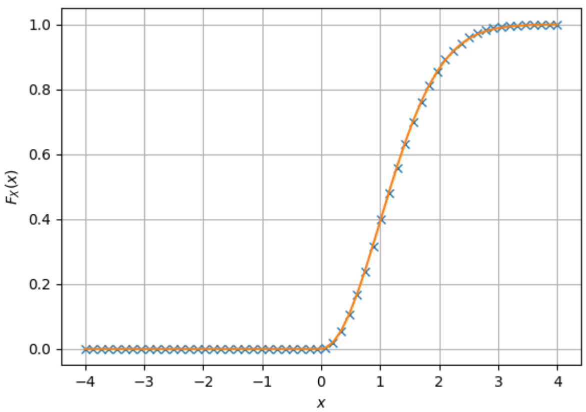
Solution: The c code to generate

```
wget https://github.com/karthik6281/AI1110−
    Assignments/blob/main/random_numbers
    /6/6_3.c
```

executing the following

```
gcc 6_3.c −lm
./a.out
```

The codes for cdf and pdf can be obtained from



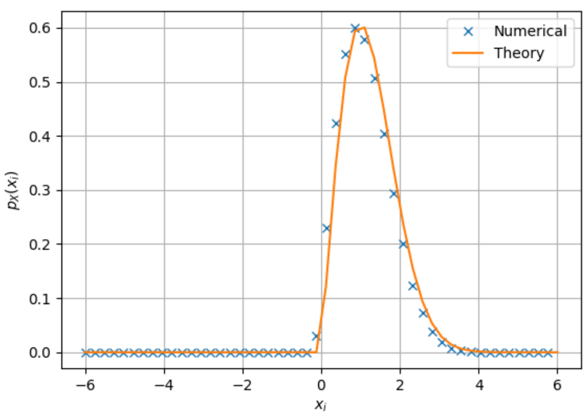


Fig. 6.3. PDF of A

```
wget https://github.com/karthik6281/AI1110−Assignments/blob/main/random_numbers/6/6_3_cdf.py
wget https://github.com/karthik6281/AI1110−Assignments/blob/main/random_numbers/6/6_3_cdf.py
```

execute the following

```
python3 6_3_cdf.py
python3 6_3_pdf.py
```

The CDF of A for $x > 0$:

$$F_A(x) = F_V(x^2) \tag{6.26}$$

$$= 1 - \exp\left(-\frac{x^2}{2}\right) \tag{6.27}$$

Pdf of A for $x > 0$:

$$p_A(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_A(x) \tag{6.28}$$

$$= x \exp\left(-\frac{x^2}{2}\right) \tag{6.29}$$

Therefore,

$$F_A(x) = \begin{cases} 1 - \exp\left(-\frac{x^2}{2}\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{6.30}$$

$$p_A(x) = \begin{cases} x \exp\left(-\frac{x^2}{2}\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{6.31}$$