

Random Numbers

Ravula Karthik
AI21BTECH11024

1. UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

- 1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat

Solution: The c code for 1.1 can be obtained from

```
wget https://github.com/karthik6281/AI1110-
Assignments/blob/main/random_numbers
/1/exrand.c
wget https://github.com/karthik6281/AI1110-
Assignments/blob/main/random_numbers
/1/coeffs.h
```

Execute the following lines

```
gcc exrand.c -lm
./a.out
```

- 1.2 Load the uni.dat file into Python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution: The python code for 1.2 can be obtained from

```
wget https://github.com/karthik6281/AI1110-
Assignments/blob/main/random_numbers
/1/cdf_plot.py
```

Execute the following lines

```
python3 1_2.py
```

- 1.3 Find a theoretical expression for $F_U(x)$

Solution: The PDF of U is given by

$$p_U(x) = \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \quad (1.2)$$

The CDF of U is given by

$$F_U(x) = \Pr(U \leq x) = \int_{-\infty}^x p_U(x) dx \quad (1.3)$$

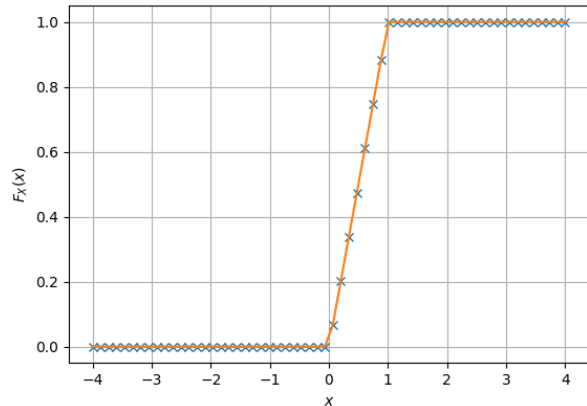


Fig. 1.2. The CDF of U

If $x < 0$,

$$\int_{-\infty}^x p_U(x) dx = \int_{-\infty}^x 0 dx = 0 \quad (1.4)$$

If $x \in [0, 1]$,

$$\int_{-\infty}^x p_U(x) dx = \int_{-\infty}^0 0 dx + \int_0^x 1 dx \quad (1.5)$$

$$= 0 + x \quad (1.6)$$

$$= x \quad (1.7)$$

If $x > 1$,

$$\begin{aligned} \int_{-\infty}^x p_U(x) dx \\ = \int_{-\infty}^0 0 dx + \int_0^1 1 dx + \int_1^x 0 dx \end{aligned} \quad (1.8)$$

$$\int_{-\infty}^x p_U(x) dx = 0 + 1 + 0 \quad (1.9)$$

$$= 1 \quad (1.10)$$

Therefore, we obtain the CDF of U as

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \quad (1.11)$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.12)$$

and its variance as

$$\text{Var}[U] = E[U - E[U]]^2 \quad (1.13)$$

Write a C program to find the mean and variance of U

Solution: The c code for 1.4 can be obtained from

```
wget https://github.com/karthik6281/AI1110-
Assignments/blob/main/random_numbers
/1/1_4.c
wget https://github.com/karthik6281/AI1110-
Assignments/blob/main/random_numbers
/1/coeffs.h
```

Execute the following lines

```
gcc 1_4.c -lm
./a.out
```

$$\text{Mean} = 0.500137 \quad (1.14)$$

$$\text{Variance} = 0.083251 \quad (1.15)$$

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.16)$$

Solution:

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.17)$$

$$dF_U(x) = dx \quad (1.18)$$

$$\therefore E[U^k] = \int_{-\infty}^{\infty} x^k dx \quad (1.19)$$

$$E[U] = \int_0^1 x dx = \frac{1}{2} \quad (1.20)$$

$$E[U^2] = \int_0^1 x^2 dx = \frac{1}{3} \quad (1.21)$$

$$\therefore P_X(x) = 0, \forall x \in (1, \infty) \cap (-\infty, 0) \quad (1.22)$$

$$\text{Var}(X) = E[U^2] - (E[U])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \quad (1.23)$$

2. CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: The c code for 2.1 can be obtained from

```
wget https://github.com/karthik6281/
AI1110-Assignments/blob/main/
random_numbers/2/2_1.c
wget https://github.com/karthik6281/
AI1110-Assignments/blob/main/
random_numbers/2/coeffs.h
```

Execute the following lines

```
gcc 2_1.c -lm
./a.out
```

2.2 Load gau.dat in Python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The python code for 2.2 can be obtained from

```
wget https://github.com/karthik6281/
AI1110-Assignments/blob/main/
random_numbers/2/2_2.py
```

Execute the following lines

```
python3 2_2.py
```

- $\Phi(x) = P(Z \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left\{-\frac{u^2}{2}\right\} du$
- $\lim_{x \rightarrow \infty} \Phi(x) = 1, \lim_{x \rightarrow -\infty} \Phi(x) = 0$
- $\Phi(0) = \frac{1}{2}$
- $\Phi(-x) = 1 - \Phi(x)$

2.3 Load gau.dat in Python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

Solution: The python code for 2.3 can be obtained from

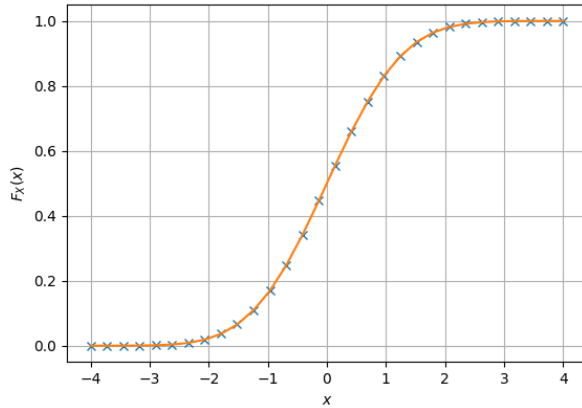


Fig. 2.5. The CDF of X

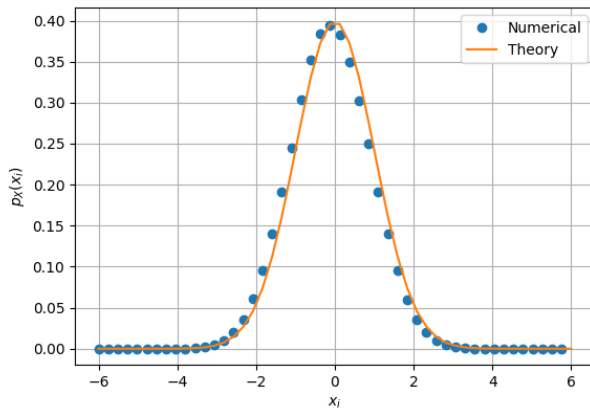


Fig. 2.5. The PDF of X

```
wget https://github.com/karthik6281/
AI1110-Assignments/blob/main/
random_numbers/2/2_3.py
```

Execute the following lines

```
python3 2.3.py
```

Every PDF is bounded between 0 and 1 and

$$\int_{-\infty}^{\infty} p_X(x) dx = 1 \quad (2.3)$$

In this case, the PDF is symmetric about $x = 0$ and graph is bell shaped

2.4 Find the mean and variance of X by writing a C program

Solution: the c code for 2.4 can be obtained from

```
wget https://github.com/karthik6281/
AI1110-Assignments/blob/main/
random_numbers/2/2_4.c
wget https://github.com/karthik6281/
AI1110-Assignments/blob/main/
random_numbers/2/coeffs.h
```

execute the following lines

```
gcc 2_4.c -lm
./a.out
```

$$\text{Mean} = 0.000294 \quad (2.4)$$

$$\text{Variance} = 0.999561 \quad (2.5)$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.6)$$

repeat the above exercise theoretically

Solution: The mean of X is given by

$$E[X] = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.7)$$

$\Rightarrow xe^{-\frac{x^2}{2}}$ is a odd function

$$\therefore E[X] = \int_{-\infty}^{\infty} g(x) dx = 0 \quad (2.8)$$

Now,

$$E[X^2] = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.9)$$

$$= 2 \int_0^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.10)$$

$\Rightarrow \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$ is an even function

By integration by parts,

$$E[X^2] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} x \cdot x \exp\left(-\frac{x^2}{2}\right) dx \quad (2.11)$$

$$= \sqrt{\frac{2}{\pi}} \left(x \int x \exp\left(-\frac{x^2}{2}\right) dx \right) \Big|_0^{\infty} - \sqrt{\frac{2}{\pi}} \int_0^{\infty} 1 \cdot \int x \exp\left(-\frac{x^2}{2}\right) dx \quad (2.12)$$

Substitute $t = \frac{x^2}{2} \Rightarrow dt = x dx$

$$\int x \exp\left(-\frac{x^2}{2}\right) dx = \int \exp(-t) dt \quad (2.13)$$

$$= -\exp(-t) \quad (2.14)$$

$$= -\exp\left(-\frac{x^2}{2}\right) \quad (2.15)$$

$$\int_0^\infty -\exp\left(-\frac{x^2}{2}\right) dx \quad (2.16)$$

$$\xleftrightarrow{x=t\sqrt{2}} \int_0^\infty -\exp(-t^2) dt \sqrt{2} \quad (2.17)$$

$$= -\sqrt{2} \int_0^\infty \exp(-t^2) dt \quad (2.18)$$

$$= -\sqrt{\frac{\pi}{2}} \quad (2.19)$$

Therefore,

$$E[X^2] = 0 - \sqrt{\frac{2}{\pi}} \left(-\sqrt{\frac{\pi}{2}}\right) \quad (2.20)$$

$$= 1 \quad (2.21)$$

$$\therefore \text{Var}[X] = E[X^2] - (E[X])^2 \quad (2.22)$$

$$= 1 - 0 \quad (2.23)$$

$$= 1 \quad (2.24)$$

3. FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF

Solution: The c code for 3.1 can be obtained from

```
wget https://github.com/karthik6281/AI1110-
Assignments/blob/main/random_numbers
/3/3_1.c
```

Execute the following lines

```
gcc 3.1.c -lm
./a.out
```

The python code for cdf can be obtained from

```
wget https://github.com/karthik6281/AI1110-
Assignments/blob/main/random_numbers
/3/3_1.py
```

execute the following lines

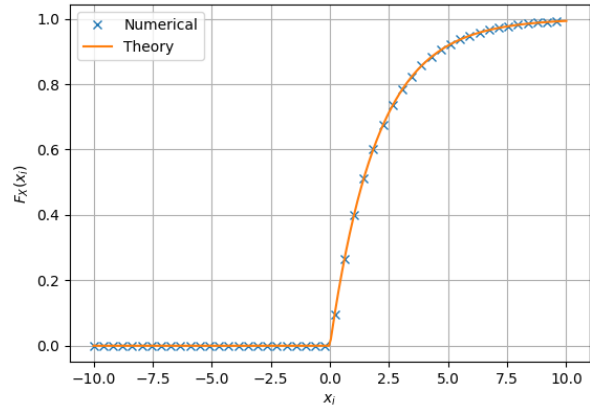


Fig. 3.5. The CDF of V

```
python3 3_1.py
```

3.6 Find a theoretical expression for $F_V(x)$

Solution:

$$F_V(x) = P(V \leq x) \quad (3.2)$$

$$= P(-2 \ln(1 - U) \leq x) \quad (3.3)$$

$$= P(1 - e^{-\frac{x}{2}} \geq U) \quad (3.4)$$

$$P(U < x) = \int_0^x dx = x \quad (3.5)$$

$$\therefore P(1 - e^{-\frac{x}{2}} \geq U) = 1 - e^{-\frac{x}{2}}, \forall x \geq 0 \quad (3.6)$$

4. TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: The c code for 4.1 can be obtained from

```
wget https://github.com/karthik6281/
AI1110-Assignments/blob/main/
random_numbers/4/4_1.c
```

execute the following lines

```
gcc 4_1.c -lm
./a.out
```

4.2 Find the CDF of T

Solution:

$$p_T(x) = p_{U_1+U_2}(x) = p_{U_1}(x) * p_{U_2}(x) \quad (4.2)$$

$$p_T(x) = \int_{-\infty}^{\infty} p_{U_1}(\tau) p_{U_2}(x - \tau) \quad (4.3)$$

$$p_T(x) = \int_0^1 p_{U_2}(x - \tau) \quad (4.4)$$

$$p_T(x) = \begin{cases} 0 & x \leq 0 \\ \int_0^x 1 d\tau & 0 < x < 1 \\ \int_{x-1}^1 1 d\tau & 1 \leq x < 2 \\ 0 & x > 2 \end{cases} \quad (4.5)$$

$$p_T(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 2 - x & 1 \leq x < 2 \\ 0 & x > 2 \end{cases} \quad (4.6)$$

Expression for CDF can be obtained by integrating $p_T(x)$ w.r.t. X

$$F_T(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x^2}{2} & 0 < x < 1 \\ -\frac{x^2}{2} + 2x - 1 & 1 \leq x < 2 \\ 1 & x > 2 \end{cases} \quad (4.7)$$

4.3 Find the PDF of T

Solution: The PDF of T is given by

$$p_T(t) = \frac{d(F_T(t))}{dt} \quad (4.8)$$

$$\therefore p_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2 - t & 1 < t < 2 \\ 0 & t \geq 2 \end{cases} \quad (4.9)$$

4.4 Find the theoretical expressions for the PDF and CDF of T

Solution:

$$P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2 - t & 0 < t \leq 2 \\ 0 & t > 2 \end{cases} \quad (4.10)$$

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 2t - \frac{t^2}{2} - 1 & 1 < t \leq 2 \\ 1 & t > 2 \end{cases} \quad (4.11)$$

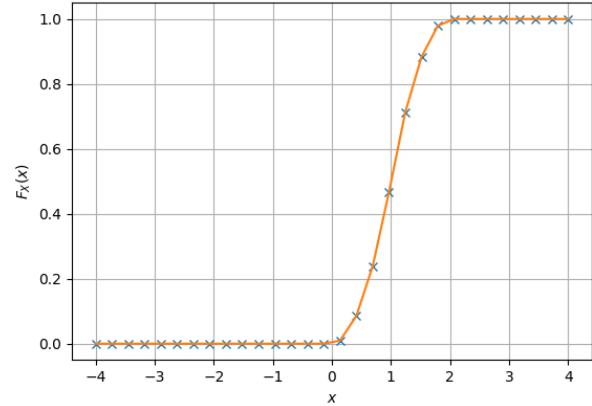


Fig. 4.6. The CDF of T

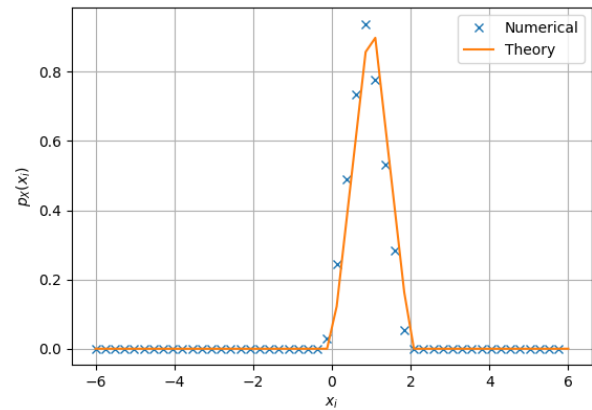


Fig. 4.6. The PDF of T

4.5 Verify your results through a plot

Solution: The codes that plots cdf and pdf can be obtained from

```
wget https://github.com/karthik6281/
  AI1110-Assignments/blob/main/
  random_numbers/4/4_cdf.py
wget https://github.com/karthik6281/
  AI1110-Assignments/blob/main/
  random_numbers/4/4_pdf.py
```

Execute the following lines

```
python3 4_cdf.py
python3 4_pdf.py
```