Random Numbers

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1. Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat
Solution: The c code for 1.1 can be obtained from

```
wget https://github.com/karthik6281/AI1110-Assignments/blob/main/random_numbers/1/exrand.c
wget https://github.com/karthik6281/AI1110-Assignments/blob/main/random_numbers/1/coeffs.h
```

Execute the following lines

```
gcc exrand.c -lm
./a.out
```

1.2 Load the uni.dat file into Python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: The python code for 1.2 can be obtained from

```
wget https://github.com/karthik6281/AI1110-Assignments/blob/main/random_numbers/1/cdf_plot.py
```

Execute the following lines

```
python3 1_2.py
```

1.3 Find a theoretical expression for $F_U(x)$
Solution: The PDF of U is given by

$$p_U(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

The CDF of U is given by

$$F_U(x) = \Pr(U \le x) = \int_{-\infty}^{x} p_U(x)\, dx \tag{1.3}$$

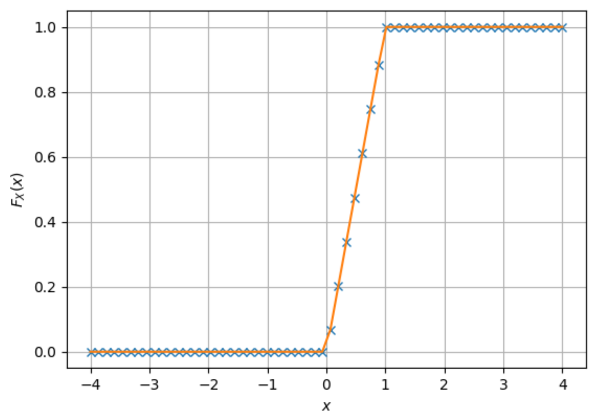

Fig. 1.2. The CDF of U

If $x < 0$,

$$\int_{-\infty}^{x} p_U(x)\, dx = \int_{-\infty}^{x} 0\, dx = 0 \tag{1.4}$$

If $x \in [0,1]$,

$$\int_{-\infty}^{x} p_U(x)\, dx = \int_{-\infty}^{0} 0\, dx + \int_{0}^{x} 1\, dx \tag{1.5}$$

$$= 0 + x \tag{1.6}$$

$$= x \tag{1.7}$$

If $x > 1$,

$$\int_{-\infty}^{x} p_U(x)\, dx$$

$$= \int_{-\infty}^{0} 0\, dx + \int_{0}^{1} 1\, dx + \int_{1}^{x} 0\, dx \tag{1.8}$$

$$\int_{-\infty}^{x} p_U(x)\, dx = 0 + 1 + 0 \tag{1.9}$$

$$= 1 \tag{1.10}$$

Therefore, we obtain the CDF of U as

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \tag{1.11}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.12}$$

and its variance as

$$\text{Var}[U] = E[U - E[U]]^2 \tag{1.13}$$

Write a C program to find the mean and variance of U
Solution: The c code for 1.4 can be obtained from

```
wget https://github.com/karthik6281/AI1110-Assignments/blob/main/random_numbers/1/1_4.c
wget https://github.com/karthik6281/AI1110-Assignments/blob/main/random_numbers/1/coeffs.h
```

Execute the following lines

```
gcc 1_4.c -lm
./a.out
```

$$\text{Mean} = 0.500137 \tag{1.14}$$
$$\text{Variance} = 0.083251 \tag{1.15}$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k \mathrm{d}F_U(x) \tag{1.16}$$

Solution:

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.17}$$

$$dF_U(x) = dx \tag{1.18}$$
$$\therefore E[U^k] = \int_{-\infty}^{\infty} x^k dx \tag{1.19}$$
$$E[U] = \int_0^1 x dx = \frac{1}{2} \tag{1.20}$$
$$E[U^2] = \int_0^1 x^2 dx = \frac{1}{3} \tag{1.21}$$
$$\because P_X(x) = 0, \forall x \in (1,\infty)\cap(-\infty,0) \tag{1.22}$$
$$Var(X) = E[U^2] - (E[U])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \tag{1.23}$$

2. Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: The c code for 2.1 can be obtained from

```
wget https://github.com/karthik6281/AI1110-Assignments/blob/main/random_numbers/2/2_1.c
wget https://github.com/karthik6281/AI1110-Assignments/blob/main/random_numbers/2/coeffs.h
```

Execute the following lines

```
gcc 2_1.c -lm
./a.out
```

2.2 Load gau.dat in Python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The python code for 2.2 can be obtained from

```
wget https://github.com/karthik6281/AI1110-Assignments/blob/main/random_numbers/2/2_2.py
```

Execute the following lines

```
python3 2_2.py
```

- $\Phi(x) = P(Z \le x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left\{-\frac{u^2}{2}\right\} du$
- $\lim_{x\to\infty} \Phi(x) = 1, \quad \lim_{x\to-\infty} \Phi(x) = 0$
- $\Phi(0) = \frac{1}{2}$
- $\Phi(-x) = 1 - \Phi(x)$

2.3 Load gau.dat in Python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The python code for 2.3 can be obtained from

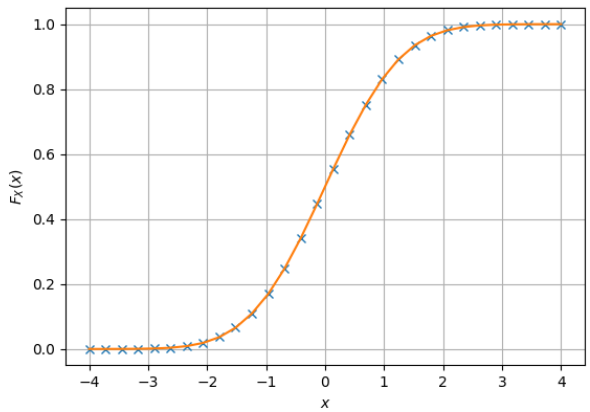

Fig. 2.5. The CDF of X

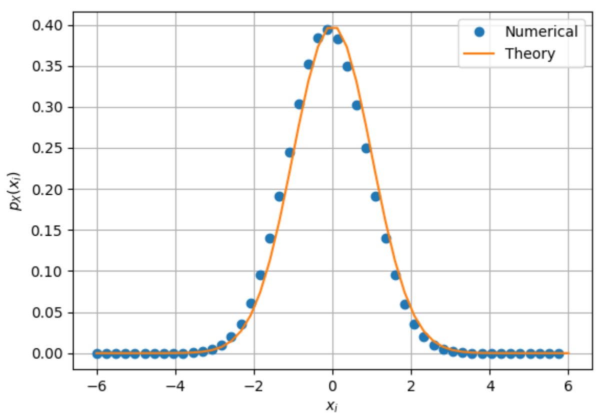

Fig. 2.5. The PDF of X

```
wget https://github.com/karthik6281/
    AI1110−Assignments/blob/main/
    random_numbers/2/2_3.py
```

Execute the following lines

```
python3 2.3.py
```

Every PDF is bounded between 0 and 1 and

$$\int_{-\infty}^{\infty} p_X(x)\, \mathrm{d}x = 1 \tag{2.3}$$

In this case, the PDF is symmetric about $x = 0$ and graph is bell shaped

2.4 Find the mean and variance of X by writing a C program
Solution: the c code for 2.4 can be obtained from

```
wget https://github.com/karthik6281/
    AI1110−Assignments/blob/main/
    random_numbers/2/2_4.c
wget https://github.com/karthik6281/
    AI1110−Assignments/blob/main/
    random_numbers/2/coeffs.h
```

execute the following lines

```
gcc 2_4.c −lm
./a.out
```

$$\text{Mean} = 0.000294 \tag{2.4}$$
$$\text{Variance} = 0.999561 \tag{2.5}$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.6}$$

repeat the above exercise theoretically
Solution: The mean of X is given by

$$E[X] = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.7}$$

$\implies xe^{-\frac{-x^2}{2}}$ is a odd function

$$\therefore E[X] = \int_{-\infty}^{\infty} g(x) \mathrm{d}x = 0 \tag{2.8}$$

Now,

$$E\left[X^2\right] = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.9}$$
$$= 2\int_{0}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.10}$$

$\implies \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$ is an even function
By integration by parts,

$$E\left[X^2\right] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} x \cdot x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.11}$$

$$= \sqrt{\frac{2}{\pi}} \left(x \int x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \right)\Bigg|_0^{\infty} - \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} 1 \cdot \int x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.12}$$

Substitute $t = \frac{x^2}{2} \implies \mathrm{d}t = x\mathrm{d}x$

$$\int x \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x = \int \exp(-t)\mathrm{d}t \quad (2.13)$$
$$= -\exp(-t) \quad (2.14)$$
$$= -\exp\left(-\frac{x^2}{2}\right) \quad (2.15)$$

$$\int_0^\infty -\exp\left(-\frac{x^2}{2}\right)\mathrm{d}x \quad (2.16)$$
$$\xleftrightarrow{x=t\sqrt{2}} \int_0^\infty -\exp(-t^2)\mathrm{d}t\,\sqrt{2} \quad (2.17)$$
$$= -\sqrt{2}\int_0^\infty \exp(-t^2)\mathrm{d}t \quad (2.18)$$
$$= -\sqrt{\frac{\pi}{2}} \quad (2.19)$$

Therefore,

$$E\left[X^2\right] = 0 - \sqrt{\frac{2}{\pi}}\left(-\sqrt{\frac{\pi}{2}}\right) \quad (2.20)$$
$$= 1 \quad (2.21)$$
$$\therefore \mathrm{Var}[X] = E\left[X^2\right] - (E[X])^2 \quad (2.22)$$
$$= 1 - 0 \quad (2.23)$$
$$= 1 \quad (2.24)$$

3. From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1 - U) \quad (3.1)$$

and plot its CDF

Solution: The c code for 3.1 can be obtained from

```
wget https://github.com/karthik6281/AI1110-Assignments/blob/main/random_numbers/3/3_1.c
```

Execute the following lines

```
gcc 3.1.c -lm
./a.out
```

The python code for cdf can be obtained from

```
wget https://github.com/karthik6281/AI1110-Assignments/blob/main/random_numbers/3/3_1.py
```

execute the following lines

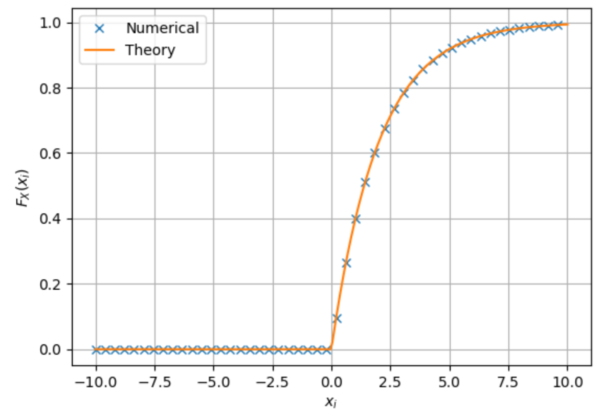

Fig. 3.5. The CDF of V

```
python3 3_1.py
```

3.6 Find a theoretical expression for $F_V(x)$

Solution:

$$F_V(x) = P(V \le x) \quad (3.2)$$
$$= P(-2ln(1 - U) \le x) \quad (3.3)$$
$$= P(1 - e^{\frac{-x}{2}} \ge U) \quad (3.4)$$
$$P(U < x) = \int_0^x dx = x \quad (3.5)$$
$$\therefore P(1 - e^{\frac{-x}{2}} \ge U) = 1 - e^{\frac{-x}{2}}, \forall x \ge 0 \quad (3.6)$$

4. Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: The c code for 4.1 can be obtained from

```
wget https://github.com/karthik6281/AI1110-Assignments/blob/main/random_numbers/4/4_1.c
```

execute the following lines

```
gcc 4_1.c -lm
./a.out
```

4.2 Find the CDF of T

Solution: The CDF of T is given by

$$F_T(t) = \Pr(T \le t) = \Pr(U_1 + U_2 \le t) \quad (4.2)$$

Since $U_1, U_2 \in [0,1] \implies U_1 + U_2 \in [0,2]$ Therefore, if $t \geq 2$, then $U_1+U_2 \leq t$ is always true and if $t < 0$, then $U_1 + U_2 \leq t$ is always false.
Now, fix the value of U_1 to be some x

$$x + U_2 \leq t \implies U_2 \leq t - x \tag{4.3}$$

If $0 \leq t \leq 1$, then x can take all values in $[0, t]$

$$F_T(t) = \int_0^t \Pr\left(U_2 \leq t - x\right) p_{U_1}(x) \mathrm{d}x \tag{4.4}$$
$$= \int_0^t F_{U_2}(t-x) p_{U_1}(x) \mathrm{d}x \tag{4.5}$$

$$0 \leq x \leq t \implies 0 \leq t - x \leq t \leq 1 \tag{4.6}$$
$$\implies F_{U_2}(t-x) = t - x \tag{4.7}$$

$$F_T(t) = \int_0^t (t-x) \cdot 1 \cdot \mathrm{d}x \tag{4.8}$$
$$= tx - \frac{x^2}{2}\Big|_0^t \tag{4.9}$$
$$= \frac{t^2}{2} \tag{4.10}$$

If $1 < t < 2$, x can only take values in $[0, 1]$ as $U_1 \leq 1$

$$F_T(t) = \int_0^1 F_{U_2}(t-x) \cdot 1 \cdot \mathrm{d}x \tag{4.11}$$

$$0 \leq x \leq t - 1 \implies 1 \leq t - x \leq t \tag{4.12}$$
$$t - 1 \leq x \leq 1 \implies 0 < t - 1 \leq t - x \leq 1 \tag{4.13}$$

$$F_T(t) = \int_0^{t-1} 1 \mathrm{d}x + \int_{t-1}^{1} (t-x) \mathrm{d}x \tag{4.14}$$
$$= t - 1 + t(1 - (t-1)) - \frac{1}{2} + \frac{(t-1)^2}{2} \tag{4.15}$$
$$= t - 1 + 2t - t^2 - \frac{1}{2} + \frac{t^2}{2} + \frac{1}{2} - t \tag{4.16}$$
$$= -\frac{t^2}{2} + 2t - 1 \tag{4.17}$$

Therefore,

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 2t - \frac{t^2}{2} - 1 & 1 < t < 2 \\ 1 & t \geq 2 \end{cases} \tag{4.18}$$

4.3 Find the PDF of T
Solution: The PDF of T is given by

$$p_T(t) = \frac{d(F_T(t))}{dt} \tag{4.19}$$
$$\therefore p_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2 - t & 1 < t < 2 \\ 0 & t \geq 2 \end{cases} \tag{4.20}$$

4.4 Find the theoretical expressions for the PDF and CDF of T
Solution:

$$P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2 - t & 0 < t \leq 2 \\ 0 & t > 2 \end{cases} \tag{4.21}$$

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 2t - \frac{t^2}{2} - 1 & 1 < t \leq 2 \\ 1 & t > 2 \end{cases} \tag{4.22}$$

4.5 Verify your results through a plot
Solution: The codes that plots cdf and pdf can be obtained from

```
wget https://github.com/karthik6281/
    AI1110−Assignments/blob/main/
    random_numbers/4/4_cdf.py
wget https://github.com/Ankit−Saha−2003/
    AI1110/raw/main/Random−Numbers/
    codes/4.3.py
```

Execute the following lines

```
python3 4_cdf.py
python3 4_pdf.py
```

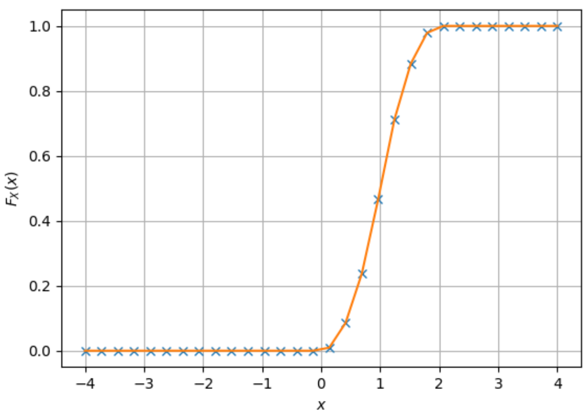

Fig. 4.6. The CDF of T

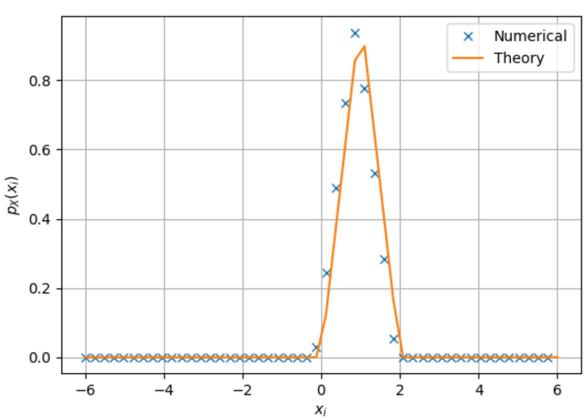

Fig. 4.6. The PDF of T