Digital Signal Processing

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Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt-get update
sudo apt-get install libffi-dev libsndfile1 python3
    -scipy python3-numpy python3-matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://github.com/karthik6281/Signal-
    Processing/blob/main/sig-pro/codes/
    Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```python
import soundfile as sf
from scipy import signal

#read .wav file
input_signal,fs = sf.read('Sound_Noise.wav'
    )

#sampling frequency of Input signal
sampl_freq=fs

#order of the filter
order=4

#cutoff frquency 4kHz
cutoff_freq=4000.0

#digital frequency
Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and denominator
    polynomials respectively
b, a = signal.butter(order,Wn, 'low')

#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a,
    input_signal)
#output_signal = signal.lfilter(b, a,
    input_signal)

#write the output signal into .wav file
sf.write('Sound_With_ReducedNoise.wav',
    output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?

Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.
Solution: The following code yields Fig. 3.1.

```
wget https://github.com/karthik6281/Signal-Processing/blob/main/sig-pro/codes/3_1.py
```

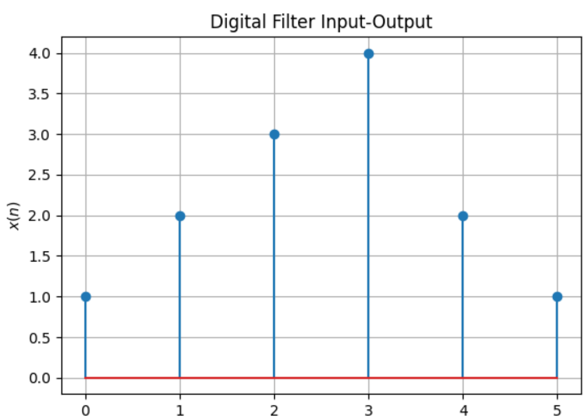

Fig. 3.1

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2), \quad y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following code yields Fig. 3.2.

```
wget https://github.com/karthik6281/Signal-Processing/blob/main/sig-pro/codes/3_2.py
```

3.3 Repeat the above exercise using a C code.
Solution: The c code can be obtained from

```
wget https://github.com/karthik6281/Signal-Processing/blob/main/sig-pro/codes/3_3.c
```

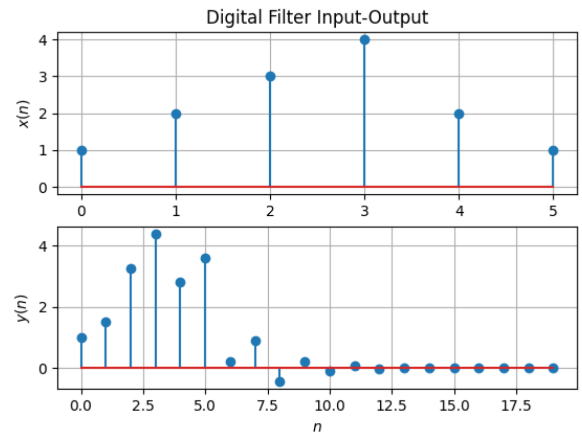

Fig. 3.2

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.6}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem **??**.
Solution: Z-transform of x(n),$X(z)$ is given by

$$\mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.7}$$

$$= \sum_{n=0}^{5} x(n)z^{-n} \tag{4.8}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \tag{4.9}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.10}$$

from (3.2) assuming that the Z-transform is a linear operation.

Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.11}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.12}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.13}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.14}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.15}$$

Solution: The Z-transform of $\delta(n)$ is defined as

$$\mathcal{Z}\{\delta(n)\} = \sum_{n=-\infty}^{\infty} \delta(n)z^{-n} \tag{4.16}$$

$$= \delta(0)z^{-0} \tag{4.17}$$

$$= 1 \tag{4.18}$$

Hence we can say that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.19}$$

and from (4.14),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.20}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.21}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.22}$$

Solution:

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} a^n u(n)z^{-n} \tag{4.23}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} \tag{4.24}$$

$$= \sum_{n=0}^{\infty} (z^{-1}a)^n \tag{4.25}$$

$$= \frac{1}{1 - az^{-1}}, \quad \left|z^{-1}a\right| < 1 \tag{4.26}$$

$$= \frac{1}{1 - az^{-1}}, \quad |z| > |a| \tag{4.27}$$

using the fomula for the sum of an infinite geometric progression.

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.28}$$

Plot $|H\left(e^{\jmath\omega}\right)|$. Is it periodic? If so, find the period. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $h(n)$.

Solution: $H(e^{jw})$ is given by

$$H(e^{jw}) = \frac{1 + (e^{jw})^{-2}}{1 + \frac{1}{2}(e^{jw})^{-1}} \tag{4.29}$$

$$= 2\frac{1 + \cos(-2\omega) + j\sin(-2\omega)}{2 + \cos(-\omega) + j\sin(-\omega)} \tag{4.30}$$

$$= 2\frac{1 + \cos(2\omega) - j\sin(2\omega)}{2 + \cos(\omega) - j\sin(\omega)} \tag{4.31}$$

$$= 2\frac{2\cos^2(\omega) - 2j\sin(\omega)\cos(\omega)}{2 + \cos(\omega) - j\sin(\omega)} \tag{4.32}$$

$$= 4\cos(\omega)\frac{\cos(\omega) - j\sin(\omega)}{2 + \cos(\omega) - j\sin(\omega)} \tag{4.33}$$

$$= 4|\cos(\omega)|\frac{e^{jw}}{2 + e^{jw}} \tag{4.34}$$

So,

$$|H(e^{jw})| = 4|\cos(\omega)|\frac{|e^{jw}|}{|2 + e^{jw}|} \tag{4.35}$$

$$= \frac{|4\cos(\omega)|}{5 + 4\cos(\omega)} \tag{4.36}$$

$|H\left(e^{\jmath\omega}\right)|$ is periodic with period π.(The LCM of the period of $|\cos(\omega)|$ and $5 + 4\cos(\omega)$ is 2π) The graph of $|H\left(e^{\jmath\omega}\right)|$ is symmetric with respect to y-axis. It is continuous over ω. The following code plots Fig. 4.6.

```
wget https://github.com/
    karthik6281/Signal–
    Processing/blob/main/
    Assignment1/codes/4_6.
    py
```

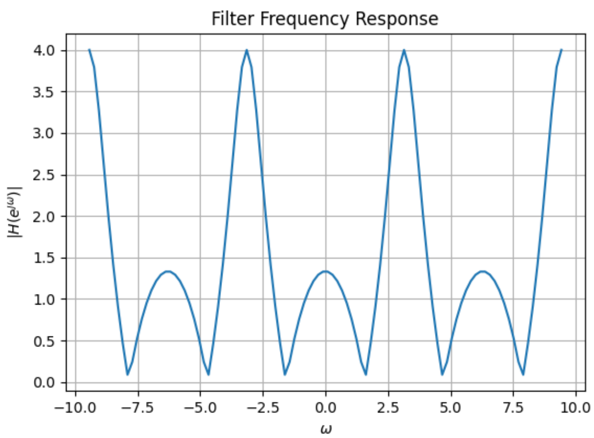

Fig. 4.6: $|H(e^{j\omega})|$

4.7 Express $h(n)$ in terms of $H(e^{j\omega})$.
Solution:

$$\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega k}d\omega = \sum_{-\infty}^{\infty} h(n) \int_{-\pi}^{\pi} e^{j\omega n}e^{-j\omega k}d\omega \tag{4.37}$$

$$= \sum_{-\infty}^{\infty} h(n) \int_{-\pi}^{\pi} (\cos(n-k) + i\sin(n-k))d\omega \tag{4.38}$$

$$\int_{-\pi}^{\pi} (\cos(n-k) + i\sin(n-k))d\omega = \begin{cases} 2\pi & n = k \\ 0 & n \neq k \end{cases} \tag{4.39}$$

$$\therefore h(n) = \frac{\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega}{2\pi} \tag{4.40}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.12).
Solution: $H(z)$ is given by

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} = \frac{2 + 2z^{-2}}{2 + z^{-1}} \tag{5.2}$$

$$\begin{array}{r} 2z^{-1} - 4 \\ z^{-1} + 2 \,\overline{)\, 2z^{-2} + 2} \\ 2z^{-2} + 4z^{-1} \\ \hline -4z^{-1} + 2 \\ -\,4z^{-1} - 8 \\ \hline 10 \end{array}$$

So,

$$H(z) = 2z^{-1} - 4 + \frac{10}{z^{-1} + 2} \tag{5.3}$$

$$= 2z^{-1} - 4 + \frac{5}{\frac{1}{2}z^{-1} + 1} \tag{5.4}$$

$$= 2z^{-1} - 4 + 5\sum_{n=0}^{\infty}\left(-\frac{z^{-1}}{2}\right)^n \tag{5.5}$$

$$= 1 - \frac{1}{2}z^{-1} + 5\sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^n z^{-n} \tag{5.6}$$

So,h(n) will be given by

$$h(n) = \begin{cases} 5 \times \left(-\frac{1}{2}\right)^n & n \geq 2 \\ \left(-\frac{1}{2}\right)^n & 2 > n \geq 0 \\ 0 & n < 0 \end{cases} \tag{5.7}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.8}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: From (4.12),

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.9}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.10}$$

using (4.22) and (4.6).

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.
Solution: The following code plots Fig. 5.3.

```
wget https://github.com/
    karthik6281/Signal–
    Processing/blob/main/
    Assignment1/codes/5_3.
    py
```

on simplfying we get h(n) as

$$\begin{cases} 5\times\left(-\frac{1}{2}\right)^n & n\geq 2 \\ \left(-\frac{1}{2}\right)^n & 2>n\geq 0 \\ 0 & n<0 \end{cases} \tag{5.11}$$

$$\because 5\times\left(-\frac{1}{2}\right)^n \to 0 \quad \text{for} \quad n\to\infty \tag{5.12}$$

So, we can conclude that h(n) is bounded.

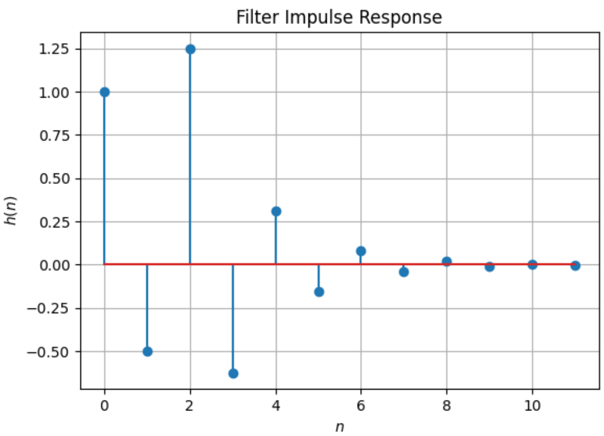

Fig. 5.3: $h(n)$ wrt n

5.4 Convergent? Justify using the ratio test.
Solution: A sequence $\{x_n\}$ is convergent if

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| < 1 \tag{5.13}$$

This is known as Ratio test.
In this case the limit will become,

$$\lim_{n\to\infty}\left|\frac{h\left(n+1\right)}{h\left(n\right)}\right| = \lim_{n\to\infty}\left|\frac{5\left(\frac{-1}{2}\right)^{n+1}}{5\left(\frac{-1}{2}\right)^{n}}\right| \tag{5.14}$$

$$= \frac{1}{2} < 1 \tag{5.15}$$

$\therefore h\left(n\right)$ is convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.16}$$

Is the system defined by (3.2) stable for the impulse response in (5.8)?
Solution: Taking $h(n)$ as defined in (5.7) Then

$$\sum_{n=-\infty}^{\infty} h(n) = +\sum_{n=-\infty}^{-1} 0 + \sum_{n=0}^{1}\left(-\frac{1}{2}\right)^n + \sum_{n=2}^{\infty} 5\times\left(-\frac{1}{2}\right)^n \tag{5.17}$$

$$= \frac{4}{3} \tag{5.18}$$

Since the sum is finite so the system is stable for impulsive response

5.6 Verify the above result using a python code.
Solution: The above result is verified using the below python code

```
wget https://github.com/karthik6281/Signal−
    Processing/tree/main/Assignment1/codes/5
    _6.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.19}$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. 5.7. Note that this is the same as Fig. 3.1.

```
wget https://github.com/karthik/Signal−
    Processing/tree/main/Assignment1/codes/5
    _7.py
```

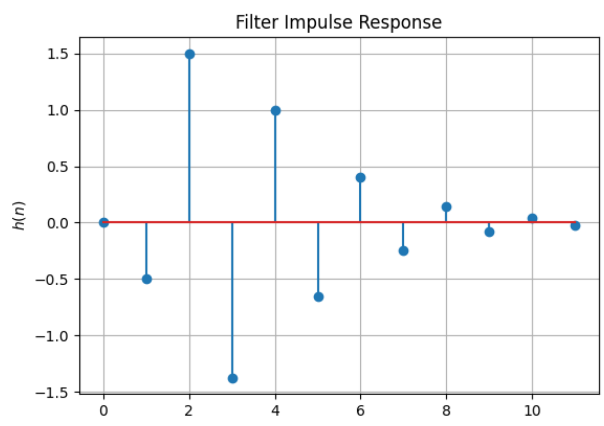

Fig. 5.7: $h(n)$ from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.20}$$

Comment. The operation in (5.20) is known as *convolution*.

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.1.

```
wget https://github.com/karthik/Signal−Processing/tree/main/Assignment1/codes/5_8.py
```

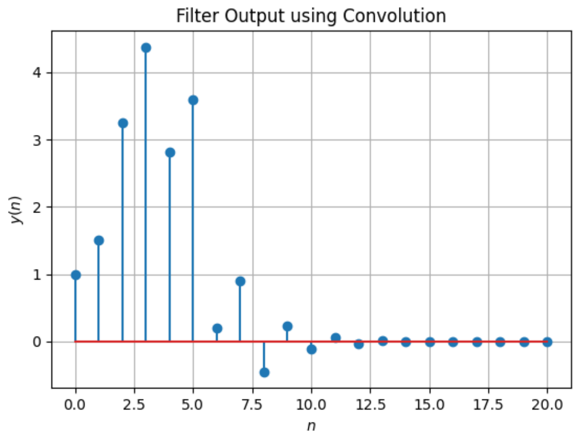

Fig. 5.8: $y(n)$ from the definition of convolution

5.9 Express the above convolution using a Toeplitz matrix.
Solution:

```
wget https://github.com/karthik6281/Signal−Processing/tree/main/Assignment1/codes/5_9.py
```

From (5.20),we express $y(n)$ as

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \tag{5.21}$$

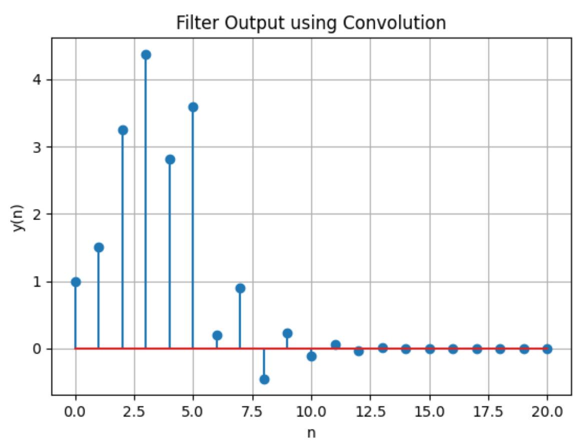

Fig. 5.9: Convolution of $x(n)$ and $h(n)$ using toeplitz matrix

To understand how we can use a Toeplitz matrix, we will see what we are doing in (5.20)

$$y(0) = x(0)\, h(0) \tag{5.22}$$
$$y(1) = x(0)\, h(1) + x(1)\, h(0) \tag{5.23}$$
$$y(2) = x(0)\, h(2) + x(1)\, h(1) + x(2)\, h(0) \tag{5.24}$$

.

.

The same thing can be written as,

$$y(0) = \begin{pmatrix} h(0) & 0 & 0 & . & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.25}$$

$$y(1) = \begin{pmatrix} h(1) & h(0) & 0 & 0 & . & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.26}$$

$$y(2) = \begin{pmatrix} h(2) & h(1) & h(0) & 0 & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.27}$$

.

.

Using Toeplitz matrix of $h(n)$ we can simplify it as,

$$y(n) = \begin{pmatrix} h(0) & 0 & 0 & . & . & .0 \\ h(1) & h(0) & 0 & . & . & .0 \\ h(2) & h(1) & h(0) & . & . & .0 \\ & & .. & & & \\ & & .. & & & \\ 0 & 0 & 0 & . & . & .h(m-1) \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.28}$$

Now from (3.1) we will take n

$$x(n) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{5.29}$$

And from (5.7) we will take some values of n,

$$h(n) = \begin{pmatrix} 1 \\ -0.5 \\ 1.25 \\ . \\ . \end{pmatrix} \tag{5.30}$$

Now using (5.28),

$$y(n) = x(n) * h(n) \tag{5.31}$$

$$= \begin{pmatrix} 1 & 0 & 0 & . & . & .0 \\ -0.5 & 1 & 0 & . & . & .0 \\ 1.25 & -0.5 & 1 & . & . & .0 \\ & & .. & & & \\ & & .. & & & \\ 0 & 0 & 0 & . & . & . \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.32}$$

$$= \begin{pmatrix} 1 \\ 1.5 \\ 3.25 \\ . \\ . \\ . \end{pmatrix} \tag{5.33}$$

5.10 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.34}$$

Solution: From (5.20)

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.35}$$

Replacing n-k with a,we get

$$y(n) = \sum_{n-a=-\infty}^{\infty} x(n-a)h(a) \tag{5.36}$$

$$= \sum_{-a=-\infty}^{\infty} x(n-a)h(a) \tag{5.37}$$

$$= \sum_{a=-\infty}^{\infty} x(n-a)h(a) \tag{5.38}$$

6 DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N} \quad k = 0, 1, \dots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$

Solution: The python code can be obtained from

```
wget https://github.com/
    karthik6281/Signal-
    Processing/tree/main/
    Assignment1/codes/6_1.
    py
```

Execute the following commands

```
python3 6_1.py
```

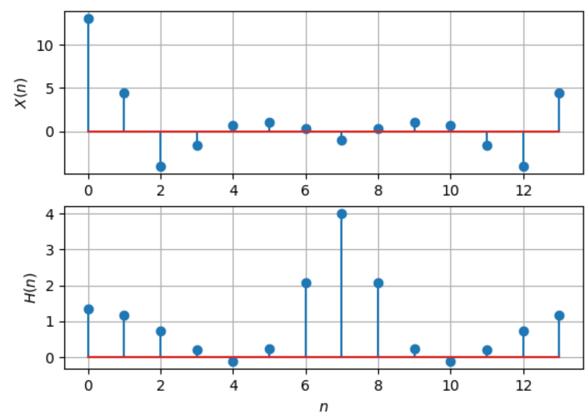

Fig. 6.1: Plots of the real parts of the discrete Fourier transforms of $x(n)$ and $h(n)$

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution: The python code can be obtained from

```
wget https://github.com/
    karthik6281/Signal-
    Processing/tree/main/
    Assignment1/codes/6_2.
    py
```

Execute the following commands

```
python3 6_2.py
```

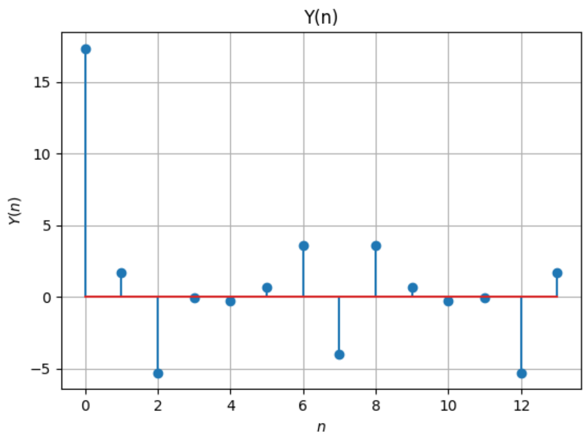

Fig. 6.2: Plot of $Y(k)$

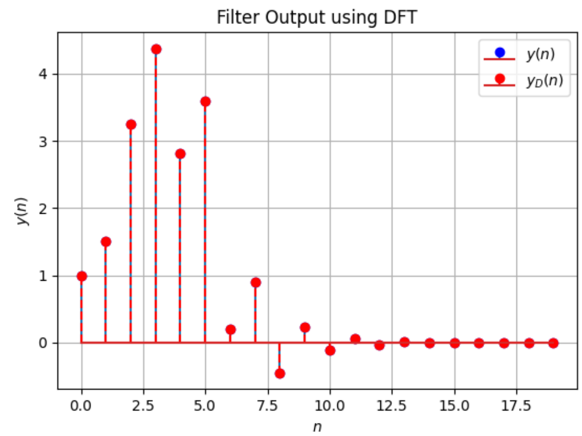

Fig. 6.3: Plot of the inverse discrete Fourier transform of $Y(k)$

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\, e^{j2\pi kn/N} \quad n = 0, 1, \ldots, N-1 \tag{6.3}$$

Solution: The python code can be obtained from

```
wget https://github.com/karthik6281/Signal-Processing/tree/main/Assignment1/codes/6_3.py
```

Execute the following commands

```
python3 6_3.py
```

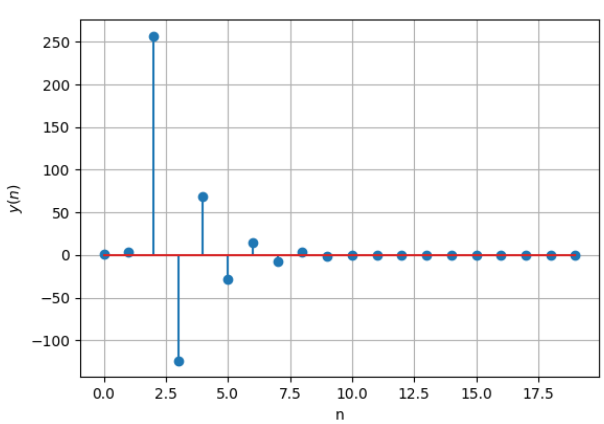

Fig. 6.3: Plot using difference equation of $Y(k)$

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.

Solution: The python code can be obtained from

```
wget https://github.com/karthik6281/Signal-Processing/tree/main/Assignment1/codes/6_4.py
```

Execute the following commands

```
python3 6_4.py
```

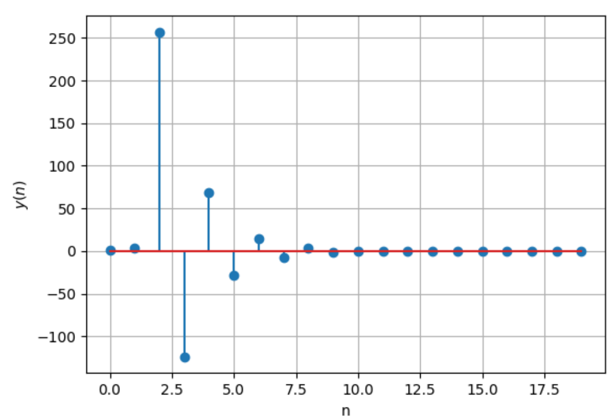

Fig. 6.4: Plot using difference equation of $Y(k)$

7 FFT

1. The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{7.1}$$

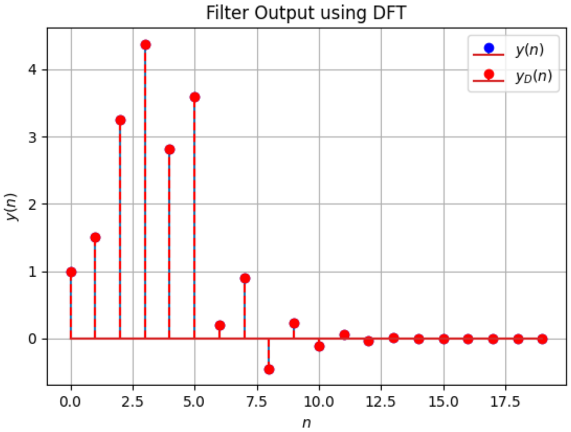

Fig. 6.4: Plot of the inverse discrete Fourier transform of $Y(k)$

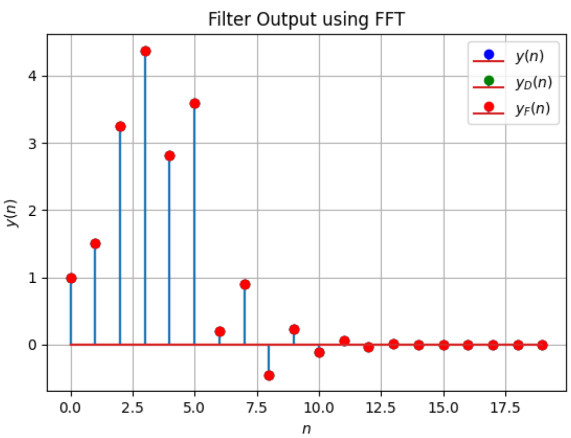

Fig. 6.4: Plot of $y(n)$ by fast Fourier transform

2. Let
$$W_N = e^{-j2\pi/N} \tag{7.2}$$
Then the N-point $DFT matrix$ is defined as
$$\mathbf{F}_N = \left[W_N^{mn}\right], \quad 0 \le m, n \le N-1 \tag{7.3}$$
where W_N^{mn} are the elements of $\mathbf{F}_N$.
3. Let
$$\mathbf{I}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^2 & \mathbf{e}_4^3 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.4}$$
be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as
$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.5}$$
4. The 4 point $DFT diagonalmatrix$ is defined as
$$\mathbf{D}_4 = diag\begin{pmatrix} W_8^0 & W_8^1 & W_8^2 & W_8^3 \end{pmatrix} \tag{7.6}$$
5. Show that
$$W_N^2 = W_{N/2} \tag{7.7}$$
Solution: From (7.2),
$$W_N = e^{-j2\pi/N} \tag{7.8}$$
Consider,
$$W_N^2 = \left(e^{-j2\pi/N}\right)^2 \tag{7.9}$$
$$= e^{-j2\pi/(N/2)} \tag{7.10}$$
$$= W_{N/2} \tag{7.11}$$
Hence proved.
6. Show that
$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.12}$$
Solution: From the eq (7.5),
$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.13}$$
Clearly $\mathbf{P}_4$ is an elementary matrix of $\mathbf{I}_4$, so on multiplication with a matrix it will interchange the rows/columns of matrix depending on positions of unit vectors.
Generalising the condition ,
$$\mathbf{P}_N^2 = \mathbf{I}_N \tag{7.14}$$
So it is similar to prove that,
$$\mathbf{F}_4\mathbf{P}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \tag{7.15}$$
Now from (7.3),
$$\mathbf{F}_2 = \begin{bmatrix} W_2^{0.0} & W_2^{0.1} \\ W_2^{1.0} & W_2^{1.1} \end{bmatrix} \tag{7.16}$$
$$= \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} \tag{7.17}$$
Using the result (7.11), we can write
$$\mathbf{F}_2 = \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{bmatrix} \tag{7.18}$$
And $\mathbf{D}_2$ is a diagonal matrix,
$$\mathbf{D}_2 = diag\left(W_4^0, W_4^1\right) \tag{7.19}$$
$$= diag\left(1, W_4\right) \tag{7.20}$$

Then,

$$\mathbf{D}_2\mathbf{F}_2 = \begin{bmatrix}1 & 0\\ 0 & W_4^1\end{bmatrix}\begin{bmatrix}W_4^0 & W_4^0\\ W_4^0 & W_4^2\end{bmatrix} \tag{7.21}$$

$$= \begin{bmatrix}W_4^0 & W_4^0\\ W_4^1 & W_4^3\end{bmatrix} \tag{7.22}$$

And for $k \in \mathcal{N}$ and N be a even integer we know that,

$$W_N^{Nk} = 1 \tag{7.23}$$

$$W_N^{Nk+N/2} = -1 \tag{7.24}$$

Using that we can write,

$$-\mathbf{D}_2\mathbf{F}_2 = \begin{bmatrix}W_4^2 & W_4^6\\ W_4^3 & W_4^9\end{bmatrix} \tag{7.25}$$

And from (7.3),

$$\mathbf{F}_4 = \begin{bmatrix}W_4^0 & W_4^0 & W_4^0 & W_4^0\\ W_4^0 & W_4^1 & W_4^2 & W_4^3\\ W_4^0 & W_4^2 & W_4^4 & W_4^6\\ W_4^0 & W_4^3 & W_4^6 & W_4^9\end{bmatrix} \tag{7.26}$$

And

$$\mathbf{F}_4\mathbf{P}_4 = \begin{bmatrix}W_4^0 & W_4^0 & W_4^0 & W_4^0\\ W_4^0 & W_4^2 & W_4^1 & W_4^3\\ W_4^0 & W_4^4 & W_4^2 & W_4^6\\ W_4^0 & W_4^6 & W_4^3 & W_4^9\end{bmatrix} \tag{7.27}$$

This is same as,

$$\begin{bmatrix}\mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2\\ \mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2\end{bmatrix} \tag{7.28}$$

$$\implies \begin{bmatrix}\mathbf{I}_2 & \mathbf{D}_2\\ \mathbf{I}_2 & -\mathbf{D}_2\end{bmatrix}\begin{bmatrix}\mathbf{F}_2 & 0\\ 0 & \mathbf{F}_2\end{bmatrix} \tag{7.29}$$

Hence proved.

7. Show that

$$\mathbf{F}_N = \begin{bmatrix}\mathbf{I}_{N/2} & \mathbf{D}_{N/2}\\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2}\end{bmatrix}\begin{bmatrix}\mathbf{F}_{N/2} & 0\\ 0 & \mathbf{F}_{N/2}\end{bmatrix}\mathbf{P}_N \tag{7.30}$$

Solution: For N even ;
We already know ;

$$\mathbf{F}_N = \left[W_N^{mn}\right], \quad 0 \le m, n \le N-1 \tag{7.31}$$

$$\mathbf{D}_N\mathbf{F}_N = \left[W_N^{m.(2k+1)}\right], \quad 0 \le m, k \le \frac{N}{2} - 1 \tag{7.32}$$

$$\mathbf{F}_N\mathbf{P}_N = \begin{bmatrix}W_N^{2mk} & W_N^{m.(2k+1)}\\ W_N^{2mk+Nk} & W_N^{m.(2k+1)+\frac{N}{2}.(2k+1)}\end{bmatrix}$$

$$0 \le m, k \le \frac{N}{2} - 1$$

From (7.23) and (7.24) ;

$$\mathbf{F}_N\mathbf{P}_N = \begin{bmatrix}W_N^{2mk} & W_N^{m.(2k+1)}\\ W_N^{2mk} & -W_N^{m.(2k+1)}\end{bmatrix} \tag{7.33}$$

from (7.7) ;

$$\mathbf{F}_N\mathbf{P}_N = \begin{bmatrix}W_{N/2}^{mk} & W_{N/2}^{m.(k+1/2)}\\ W_{N/2}^{mk} & -W_N^{m.(k+1/2)}\end{bmatrix} \tag{7.34}$$

$$\mathbf{F}_N\mathbf{P}_N = \begin{bmatrix}\mathbf{F}_{N/2} & \mathbf{D}_{N/2}\mathbf{F}_{N/2}\\ \mathbf{F}_{N/2} & -\mathbf{D}_{N/2}\mathbf{F}_{N/2}\end{bmatrix} \tag{7.35}$$

Following (7.14) ;

$$\mathbf{F}_N = \begin{bmatrix}\mathbf{F}_{N/2} & \mathbf{D}_{N/2}\mathbf{F}_{N/2}\\ \mathbf{F}_{N/2} & -\mathbf{D}_{N/2}\mathbf{F}_{N/2}\end{bmatrix}\mathbf{P}_N \tag{7.36}$$

From above it follows ;

$$\mathbf{F}_N = \begin{bmatrix}\mathbf{I}_{N/2} & \mathbf{D}_{N/2}\\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2}\end{bmatrix}\begin{bmatrix}\mathbf{F}_{N/2} & 0\\ 0 & \mathbf{F}_{N/2}\end{bmatrix}\mathbf{P}_N \tag{7.37}$$

8. Find

$$\mathbf{P}_4\mathbf{x} \tag{7.38}$$

Solution: From (7.5),

$$\mathbf{P}_4 = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{7.39}$$

$$\mathbf{x} = \begin{pmatrix}1\\ 2\\ 3\\ 4\\ 2\\ 1\end{pmatrix} \tag{7.40}$$

After proper zero padding of $\mathbf{P}_4$,

$$\mathbf{P}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.41}$$

$$\mathbf{P}_4\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.42}$$

$$= \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 0 \\ 0 \end{pmatrix} \tag{7.43}$$

9. Show that

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{7.44}$$

where $\mathbf{x}, \mathbf{X}$ are the vector representations of $x(n), X(k)$ respectively.

Solution: Given $\mathbf{x}, \mathbf{X}$ are the vector representations of $x(n), X(k)$ respectively.

$$\mathbf{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \tag{7.45}$$

$$\mathbf{X} = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} \tag{7.46}$$

$$\mathbf{F}_N = \begin{bmatrix} 1 & 1 & ! & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \tag{7.47}$$

As

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \tag{7.48}$$

Upon linear transformation over k,

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{N-1} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \tag{7.49}$$

$\therefore \mathbf{X} = \mathbf{F}_N\mathbf{x}$

10. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.50}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.51}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.52}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.53}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.54}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.55}$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{7.56}$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{7.57}$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{7.58}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.59}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.60}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.61}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.62}$$

11. For

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.63}$$

compute the DFT using (7.45)

12. Repeat the above exercise using the FFT after zero padding **x**.
13. Write a C program to compute the 8-point FFT.

8 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

8.1 The command

```
output_signal = signal.
    lfilter(b, a, input_signal
    )
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \tag{8.1}$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.

8.2 Repeat all the exercises in the previous sections for the above a and b.

8.3 What is the sampling frequency of the input signal?
Solution: Sampling frequency(fs)=44.1kHZ.

8.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.

8.5 Modifying the code with different input parameters and to get the best possible output.