# Pingala Series

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### Contents

*Abstract*—**This manual provides a simple introduction to Transforms**



Fig. 1.1



Fig. 1.2

## 1 JEE 2019

Let

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{1.1}$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{1.2}$$

Verify the following using a python code.

1.1

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{1.3}$$

**Solution:** : Run the below python code

```
https://github.com/Shivanshu6281/
Signal−Processing/tree/main/
PINGALA/codes/1.py
```

Use the following command in the terminal to run the code

```
python3 1.py
```

1.2

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{1.4}$$

**Solution:** : Run the below python code

```
https://github.com/Shivanshu6281/
Signal−Processing/tree/main/
PINGALA/codes/1.py
```

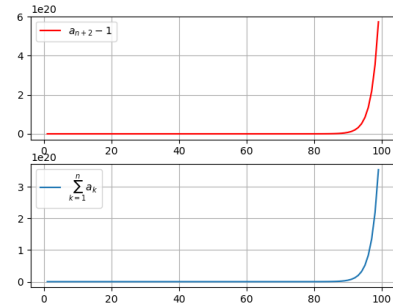Use the following command in the terminal to run the code

```
python3 1.py
```

1.3

$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{1.5}$$

**Solution:** : Run the below python code

```
https://github.com/Shivanshu6281/
Signal−Processing/tree/main/
PINGALA/codes/1.py
```

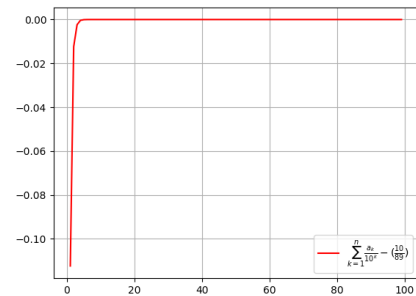Use the following command in the terminal to run the code
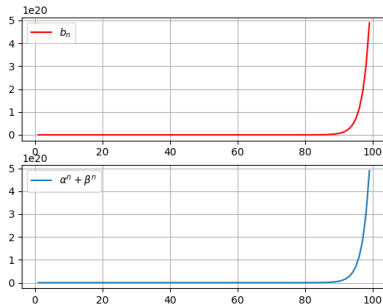
```
python3 1.py
```

Fig. 1.3

### 1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \qquad (1.6)$$

**Solution:** : Run the below python code

https://github.com/Shivanshu6281/
Signal−Processing/tree/main/
PINGALA/codes/1.py

Use the following command in the terminal to run the code
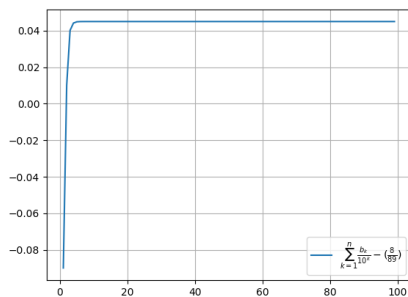
python3 1.py



Fig. 1.4

Clearly from the figure this isnt true, it actually converges to $\frac{12}{89}$

## 2 PINGALA SERIES

### 2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n}, \quad z \in \mathbb{C} \qquad (2.1)$$

### 2.2 The *Pingala* series is generated using the difference equation

$$x(n + 2) = x(n + 1) + x(n)$$
$$x(0) = x(1) = 1, n \geq 0$$

Generate a stem plot for $x(n)$.
**Solution:** Run the below python code

https://github.com/Shivanshu6281/
Signal−Processing/tree/main/
PINGALA/codes/2.py

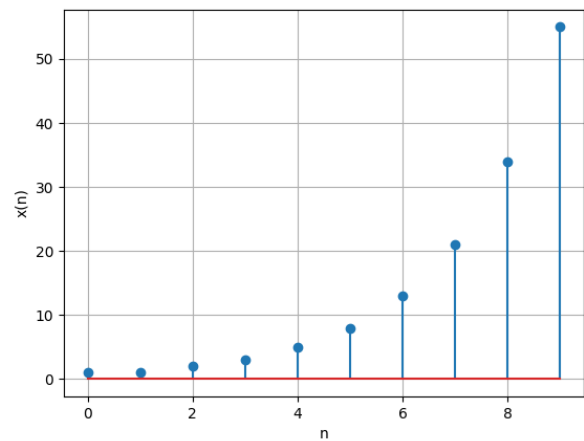Use the following command in the terminal to run the code

python3 2.py



Fig. 2.2

### 2.3 Find $X^+(z)$.
**Solution:** :

$$x(n + 2) = x(n + 1) + x(n) \qquad (2.2)$$

applying positive Z-transform on both sides,also wkt it is a linear operator

$$\sum_{k=0}^{\infty} x(k + 2)z^{-k} = \sum_{k=0}^{\infty} x(k + 1)z^{-k} + \sum_{k=0}^{\infty} x(k)z^{-k} \qquad (2.3)$$

$$z^2(X^+(z) - x(0) - x(1)z^{-1}) = X^+(z) + z(X^+(z) - x(0)) \qquad (2.4)$$

$$\implies X^+(z) = \frac{z^2}{z^2 - z - 1} \qquad (2.5)$$

$$\implies X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} \qquad (2.6)$$

## 2.4 Find $x(n)$.

**Solution:** :

$$X^+(z) = \frac{1}{(1 - \alpha z^{-1})(1 - \beta z^{-1})} \tag{2.7}$$

where $\alpha, \beta$ are the roots of equation

$$z^2 - z - 1 = 0 \tag{2.8}$$

coefficient of $z^{-k}$ in the above expression is $x(k)$ by comparing LHS and RHS

$$X^+(z) = \frac{1}{\alpha - \beta}\left(\frac{\alpha}{1 - \alpha z^{-1}} - \frac{\beta}{1 - \beta z^{-1}}\right) \tag{2.9}$$

$\therefore$ using binomial theorem we get

$$x(k) = \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} \tag{2.10}$$

## 2.5 Sketch

$$y(n) = x(n-1) + x(n+1), \quad n \geq 0 \tag{2.11}$$

**Solution:** Run the below python code

> https://github.com/Shivanshu6281/
> Signal−Processing/tree/main/
> PINGALA/codes/2.py

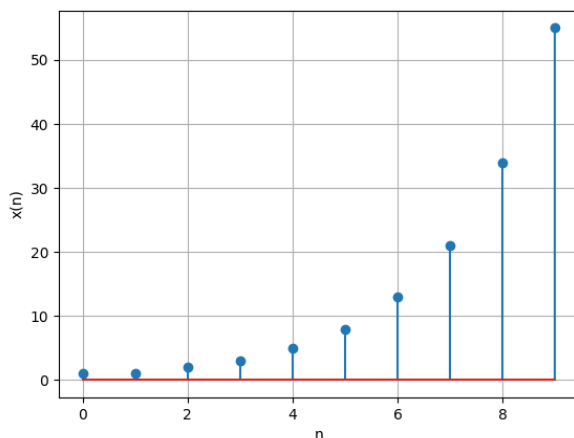Use the following command in the terminal to run the code

> python3 2.py



Fig. 2.5

## 2.6 Find $Y^+(z)$.

**Solution:** : Taking +ve Z-transform on both

sides of equation

$$\sum_{k=0}^{\infty} y(k)z^{-k} = \sum_{k=0}^{\infty} x(k+1)z^{-k} + \sum_{k=0}^{\infty} x(k-1)z^{-k} \tag{2.12}$$

$$Y^+(z) = z(X^+(z) - x(0)) + z^{-1}X^+(z) \tag{2.13}$$

$\because x(-1) = 0$

$$Y^+(z) = \frac{z + z^{-1}}{1 - z^{-1} - z^{-2}} - z \tag{2.14}$$

$$\therefore Y^+(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{2.15}$$

## 2.7 Find $y(n)$.

**Solution:** : Coefficient of $z^{-n}$ in $Y^+(z)$ will be $y(n)$

$$Y^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} + \frac{2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{2.16}$$

$$y(k) = \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} + 2\frac{\alpha^k - \beta^k}{\alpha - \beta} \tag{2.17}$$

$$= \frac{\alpha^{k+2} + \alpha^k - \beta^k - \beta^{k+2}}{\alpha - \beta} \tag{2.18}$$

$$= \frac{\alpha^{k+2} - \beta^{k+2} + \alpha\beta^{k+1} - \beta\alpha^{k+1}}{\alpha - \beta}[\because \alpha\beta = -1] \tag{2.19}$$

$$\therefore y(k) = \alpha^{k+1} + \beta^{k+1} \tag{2.20}$$

# 3 POWER OF THE Z TRANSFORM

## 3.1 Show that

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) = x(n) * u(n-1) \tag{3.1}$$

**Solution:** :

$$x(k) = a(k + 1) \tag{3.2}$$

$$\implies \sum_{k=0}^{n-1} x(k) = \sum_{k=0}^{n-1} a(k + 1) \tag{3.3}$$

$$\implies \sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) \tag{3.4}$$

$$x(n) * u(n - 1) = \sum_{k=-\infty}^{\infty} x(k)u(n - 1 - k) \tag{3.5}$$

$$u(n - 1 - k) = \begin{cases} 0 & k > n - 1 \\ 1 & k >= n - 1 \end{cases} \tag{3.6}$$

$$x(k) = 0 \forall k < 0 \tag{3.7}$$

$$\therefore x(n) * u(n - 1) = \sum_{k=0}^{n-1} x(k) \tag{3.8}$$

**3.2 Show that**

$$a_{n+2} - 1, \quad n \geq 1 \tag{3.9}$$

can be expressed as

$$[x(n + 1) - 1] u(n) \tag{3.10}$$

**Solution:** :

$$x(k) = a(k + 1) \tag{3.11}$$

$$\implies x(k + 1) = a(k + 2) \tag{3.12}$$

$$a(k + 2) - 1 = x(k + 1) - 1 \tag{3.13}$$

$$\therefore = [x(k + 1) - 1]u(k)[\because \forall n >= 1] \tag{3.14}$$

**3.3 Show that**

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \tag{3.15}$$

**Solution:** :

$$X^+(z) = \sum_{k=0}^{\infty} x(k)z^{-k} = z \sum_{k=1}^{\infty} a(k)z^{-k} \tag{3.16}$$

$$z = 10 \tag{3.17}$$

$$\implies 10 \sum_{k=1}^{\infty} \frac{a_k}{10^k} = \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = X^+(10) \tag{3.18}$$

$$\implies \sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \tag{3.19}$$

**3.4 Show that**

$$\alpha^n + \beta^n, \quad n \geq 1 \tag{3.20}$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \tag{3.21}$$

and find $W(z)$.

**Solution:** : applying Z-transform on both sides,

$$W(z) = \sum_{n=-\infty}^{\infty} \left(\alpha^{n+1} + \beta^{n+1}\right) u(n)z^{-n} \tag{3.22}$$

$$= \sum_{n=0}^{\infty} \left(\alpha^{n+1} + \beta^{n+1}\right) z^{-n} \tag{3.23}$$

$$= \alpha \sum_{n=0}^{\infty} (\alpha z^{-1})^n + \beta \sum_{n=0}^{\infty} (\beta z^{-1})^n \tag{3.24}$$

$$\text{ROC:}|z| > \max(\alpha, \beta) \tag{3.25}$$

$$= \frac{\alpha}{1 - \alpha z^{-1}} + \frac{\beta}{1 - \beta z^{-1}} \tag{3.26}$$

$$= \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{3.27}$$

**3.5 Show that**

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \tag{3.28}$$

**Solution:** :

$$y(k) = b(k + 1) \tag{3.29}$$

$$\sum_{k=0}^{\infty} y(k)z^{-k} = \sum_{k=0}^{\infty} b(k + 1)z^{-k} = Y^+(z) \tag{3.30}$$

$$\sum_{k=0}^{\infty} y(k)z^{-k} = z \sum_{k=1}^{\infty} b(k)z^{-k} = Y^+(z) \tag{3.31}$$

$$z = 10 \tag{3.32}$$

$$\implies \sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \tag{3.33}$$

**3.6 Solve the JEE 2019 problem.**

**Solution:** :

$$X^+(z) = z \sum_{k=1}^{\infty} a(k)z^{-k} = \frac{1}{1 - z^{-1} - z^{-2}} \qquad (3.34)$$

$$z = 10 \qquad (3.35)$$

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10\left(1 - \frac{1}{10} - \frac{1}{100}\right)} \qquad (3.36)$$

$$= \frac{10}{89} \qquad (3.37)$$

$$y(k) = \alpha^{k+1} + \beta^{k+1} \qquad (3.38)$$

$$y(k) = b(k+1) \qquad (3.39)$$

$$\implies b(k) = \alpha^k + \beta^k \qquad (3.40)$$

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} Y^+(10) \qquad (3.41)$$

$$= \frac{1}{10}\left[\frac{1 + \frac{2}{10}}{1 - \frac{1}{10} - \frac{1}{100}}\right] \qquad (3.42)$$

$$\because Y^+(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \qquad (3.43)$$

$$= \frac{12}{89} \qquad (3.44)$$

Run the following code to get the expressions of $x(n)$ and $y(n)$

| https://github.com/Shivanshu6281/ Signal−Processing/tree/main/ PINGALA/codes/Xk.py |
|---|

Use the following command in the terminal to run the code

| python3 Xk.py |
|---|