Fourier Series

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Contents

***Abstract*—This manual provides a simple introduction to Fourier Series**

1 Periodic Function

Let

$$x(t) = A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.1}$$

1.1 Plot $x(t)$.
Solution:

```
wget https://raw.githubusercontent.com/
    kumarsuraj151/EE3900/master/charger/
    codes/1.1.py
```

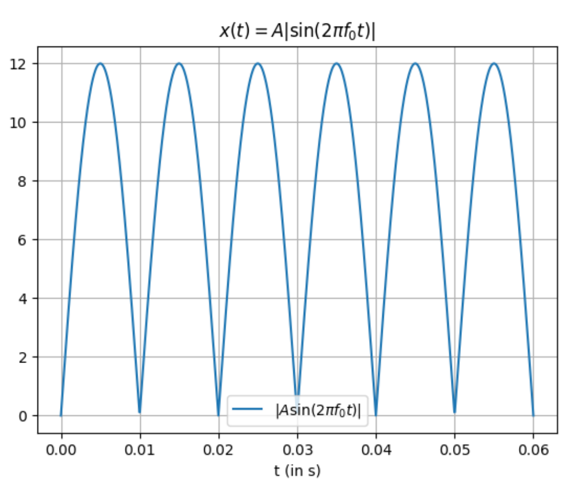


Fig. 1.1

1.2 Show that $x(t)$ is periodic and find its period.
Solution: If a signal $x(t)$ is periodic then

$$x(t+T) = x(t) \tag{1.2}$$

where T is known as fundamental period. Since $|sin\theta|$ function is periodic, x(t) is also periodic.

$$\text{Fundamental Period} = T = \frac{1}{2}\left(\frac{2\pi}{2\pi f_0}\right) \tag{1.3}$$
$$= \frac{1}{2f_0} \tag{1.4}$$

2 Fourier Series

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations.

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.1}$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi k f_0 t}\, dt \tag{2.2}$$

Solution: From (2.1),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.3}$$

Mulitply $e^{-\jmath 2\pi l f_0 t}$ on both sides

$$x(t)e^{-\jmath 2\pi l f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} e^{-\jmath 2\pi l f_0 t} \tag{2.4}$$

Integrate on both sides with respect to 't' between $-T$ to T where T is fundamental time period of x(t).
Using (1.4),

$$T = \frac{1}{2f_0} \tag{2.5}$$

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-\jmath 2\pi k f_0 t}\,dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi (k-l) f_0 t}\,dt \tag{2.6}$$

$$= \sum_{k=-\infty}^{\infty} c_k \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{\jmath 2\pi (k-l) f_0 t}\,dt \tag{2.7}$$

The above integral:

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{\jmath 2\pi (k-l) f_0 t}\,dt = \begin{cases} 0 & k \neq l \\ \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} 1\,dt & k = l \end{cases} \tag{2.8}$$

$$\therefore \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-\jmath 2\pi k f_0 t}\,dt = \left(\frac{1}{f_0}\right) c_k \tag{2.9}$$

$$\therefore c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-\jmath 2\pi k f_0 t}\,dt \tag{2.10}$$

2.2 Find c_k for (1.1)
Solution: c_k can be calculated even simpler by using

$$c_k = 2f_0 \int_0^{\frac{1}{2f_0}} x(t)e^{-\jmath 2\pi k f_0 t}\,dt \tag{2.11}$$

$x(t) = A_0 \sin(2\pi f_0 t)$ in 0 to $\frac{1}{2f_0}$ region.
Also,

$$\sin\theta = \frac{e^{\jmath\theta} - e^{-\jmath\theta}}{2\jmath} \tag{2.12}$$

Using (2.12),

$$c_k = 2f_0 \int_0^{\frac{1}{2f_0}} A_0 \left(\frac{e^{\jmath 2\pi f_0 t} - e^{-\jmath 2\pi f_0 t}}{2\jmath}\right) e^{-\jmath 2\pi k f_0 t}\,dt \tag{2.13}$$

$$= A_0 f_0 \int_0^{\frac{1}{2f_0}} \left(\frac{e^{\jmath 2\pi (1-k) f_0 t} - e^{\jmath 2\pi (-1-k) f_0 t}}{\jmath}\right) dt \tag{2.14}$$

$$= A_0 f_0 \left[\frac{e^{\jmath 2\pi (1-k) f_0 t}}{-2\pi(1-k) f_0}\Bigg|_0^{\frac{1}{2f_0}} - \frac{e^{\jmath 2\pi (-1-k) f_0 t}}{-2\pi(-1-k) f_0}\Bigg|_0^{\frac{1}{2f_0}} \right] \tag{2.15}$$

$$= A_0 \left[\frac{e^{j\pi(1-k)} - 1}{2\pi(k-1)} - \frac{e^{-j\pi(1+k)} - 1}{2\pi(k+1)} \right] \tag{2.16}$$

$$= \begin{cases} \frac{2A_0}{\pi(1-k^2)} & k = even \\ 0 & k = odd \end{cases} \tag{2.17}$$

2.3 Verify (1.1) using python.
Solution:

```
wget https://raw.githubusercontent.com/
    kumarsuraj151/EE3900/master/charger/
    codes/2.3.py
```

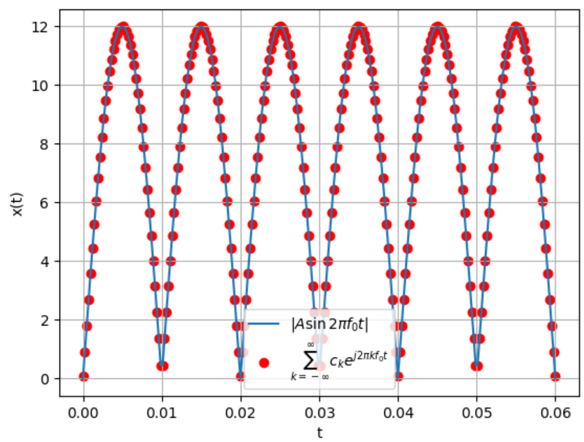


Fig. 2.3

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} (a_k \cos \jmath 2\pi k f_0 t + b_k \sin \jmath 2\pi k f_0 t) \tag{2.18}$$

and obtain the formulae for a_k and b_k. **Solution:** Using (2.1),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.19}$$

As,

$$e^{\jmath 2\pi k f_0 t} = \cos(2\pi k f_0 t) + j\sin(2\pi k f_0 t) \tag{2.20}$$

Substituting leads to

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \left[\cos(2\pi k f_0 t) + j\sin(2\pi k f_0 t)\right] \tag{2.21}$$

$$= \sum_{k=-\infty}^{\infty} c_k \cos(2\pi k f_0 t) + jc_k \sin(2\pi k f_0 t) \tag{2.22}$$

$$= \sum_{k=-\infty}^{-1} \left[c_k \cos(2\pi k f_0 t) + jc_k \sin(2\pi k f_0 t)\right] + c_0 + \sum_{k=1}^{\infty} \left[c_k \cos(2\pi k f_0 t) + jc_k \sin(2\pi k f_0 t)\right] \tag{2.23}$$

$$= \sum_{k=1}^{\infty} \left[c_{-k} \cos(2\pi k f_0 t) - jc_{-k} \sin(2\pi k f_0 t)\right] + c_0 + \sum_{k=1}^{\infty} \left[c_k \cos(2\pi k f_0 t) + jc_k \sin(2\pi k f_0 t)\right] \tag{2.24}$$

$$= c_0 + \sum_{k=1}^{\infty} \left[(c_k + c_{-k}) \cos(2\pi k f_0 t) + j(c_k - c_{-k}) \sin(2\pi k f_0 t)\right] \tag{2.25}$$

Replacing $(c_k + c_{-k}) \to a_k$ and $j(c_k - c_{-k}) \to b_k$,

$$= c_0 + \sum_{k=1}^{\infty} (a_k \cos 2\pi k f_0 t + b_k \sin 2\pi k f_0 t) \tag{2.26}$$

$$= \sum_{k=0}^{\infty} (a_k \cos 2\pi k f_0 t + b_k \sin 2\pi k f_0 t) \tag{2.27}$$

$$\therefore a_k = \begin{cases} c_k + c_{-k} & k \neq 0 \\ c_0 & k = 0 \end{cases} \tag{2.28}$$

$$b_k = j(c_k - c_{-k}) \tag{2.29}$$

Using (2.2),

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi k f_0 t}\, dt \tag{2.30}$$

$$c_{-k} = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{\jmath 2\pi k f_0 t}\, dt \tag{2.31}$$

$$a_k = c_k + c_{-k} = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) \left[e^{-\jmath 2\pi k f_0 t} + e^{\jmath 2\pi k f_0 t}\right] dt \tag{2.32}$$

$$= 2f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) \cos(2\pi k f_0 t)\, dt \tag{2.33}$$

Parallely,

$$b_k = -2jf_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) \sin(2\pi k f_0 t)\, dt \tag{2.34}$$

2.5 Find a_k and b_k for (1.1)
Solution: Using (2.28) and (2.29) with (2.17),

$$a_k = c_k + c_{-k} = \begin{cases} \frac{4A_0}{\pi(1-k^2)} & k = even \\ \frac{2A_0}{\pi} & k = 0 \\ 0 & k = odd \end{cases} \tag{2.35}$$

$$b_k = j(c_k - c_{-k}) = 0 \tag{2.36}$$

2.6 Verify (2.18) using python.
Solution:

```
wget https://raw.githubusercontent.com/
    kumarsuraj151/EE3900/master/charger/
    codes/2.6.py
```

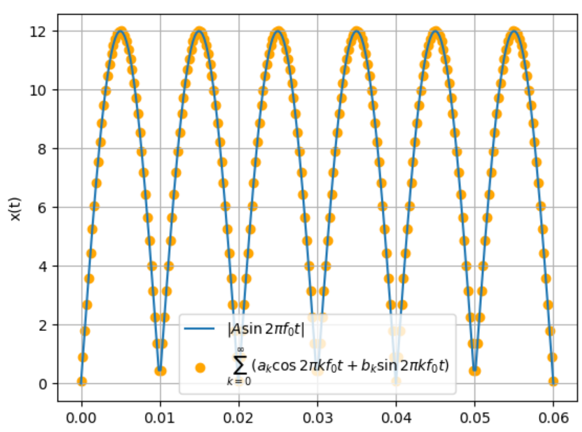


Fig. 2.6

3 Fourier Transform

3.1

$$\delta(t) = 0, \quad t \neq 0 \tag{3.1}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{3.2}$$

3.2 The Fourier Transform of $g(t)$ is

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}\,dt \tag{3.3}$$

3.3 Show that

$$g(t-t_0) \xleftrightarrow{\mathcal{F}} G(f)e^{-j2\pi ft_0} \tag{3.4}$$
$$\tag{3.5}$$

Solution:

$$\mathcal{F}\{g(t-t_0)\} = \int_{-\infty}^{\infty} g(t-t_0)e^{-j2\pi ft}\,dt \tag{3.6}$$
$$= e^{-j2\pi ft_0}\int_{-\infty}^{\infty} g(t-t_0)e^{-j2\pi f(t-t_0)}\,dt \tag{3.7}$$
$$= G(f)e^{-j2\pi ft_0} \tag{3.8}$$

3.4 Show that

$$G(t) \xleftrightarrow{\mathcal{F}} g(-f) \tag{3.9}$$

Solution: From the definition of Inverse Fourier Transform

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}\,df \tag{3.10}$$

Replace t → f,

$$g(f) = \int_{-\infty}^{\infty} G(t)e^{j2\pi ft}\,dt \tag{3.11}$$

Replace f → -f,

$$g(-f) = \int_{-\infty}^{\infty} G(t)e^{-j2\pi ft}\,dt \tag{3.12}$$
$$= \mathcal{F}\{G(t)\} \tag{3.13}$$

$$\therefore G(t) \xleftrightarrow{\mathcal{F}} g(-f) \tag{3.14}$$

3.5 $\delta(t) \xleftrightarrow{\mathcal{F}} ?$
Solution:

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft}\,dt \tag{3.15}$$
$$= \int_{-\infty}^{\infty} \delta(t)\,dt \tag{3.16}$$
$$\tag{3.17}$$

Since $e^{-j2\pi ft} = 1$ for t=0 and remaining integrand is zero for t ≠ 0.

$$= \int_{-\infty}^{\infty} \delta(t)\,dt \tag{3.18}$$
$$= 1 \tag{3.19}$$

3.6 $e^{-j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} ?$
Solution:

$$\mathcal{F}\left\{e^{-j2\pi f_0 t}\right\} = \int_{-\infty}^{\infty} 1 \cdot e^{-j2\pi(f+f_0)t}\,dt \tag{3.20}$$
$$= \int_{-\infty}^{\infty} \mathcal{F}\{\delta(t)\}\, e^{-j2\pi(f+f_0)t}\,dt \tag{3.21}$$

Using (3.9),

$$= \delta(-(f+f_0)) = \delta(f+f_0) \tag{3.22}$$

3.7 $\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} ?$
Solution:

$$\mathcal{F}\{\cos(2\pi f_0 t)\} = \mathcal{F}\left\{\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\right\} \tag{3.23}$$

Using (3.22),

$$= \frac{\delta(f+f_0) + \delta(f-f_0)}{2} \tag{3.24}$$

3.8 Find the Fourier Transform of $x(t)$ and plot it. Verify using python.
Solution:

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\sum_{k=-\infty}^{\infty} c_k e^{\mathrm{j}2\pi k f_0 t}\right\} \tag{3.25}$$
$$X(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - kf_0) \tag{3.26}$$

```
wget https://raw.githubusercontent.com/
    kumarsuraj151/EE3900/master/charger/
    codes/3.8.py
```

3.9 Show that

$$\mathrm{rect}\,(t) \xleftrightarrow{\mathcal{F}} \mathrm{sinc}\,(t) \tag{3.27}$$

Verify using python.
Solution:

$$\mathcal{F}\{\mathrm{rect}\,(t)\} = \int_{-\infty}^{\infty} \mathrm{rect}\,(t)e^{-j2\pi ft}\,dt \tag{3.28}$$
$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} 1 \cdot e^{-j2\pi ft}\,dt \tag{3.29}$$

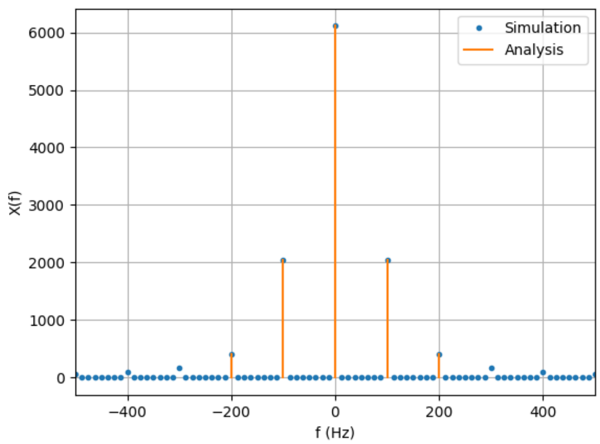


Fig. 3.8

$$= \frac{e^{-j2\pi f t}}{-j2\pi f}\Big|_{-\frac{1}{2}}^{\frac{1}{2}} \tag{3.30}$$
$$= \text{sinc}\,(t) \tag{3.31}$$

```
wget https://raw.githubusercontent.com/
    kumarsuraj151/EE3900/master/
    charger/codes/3.9.py
```

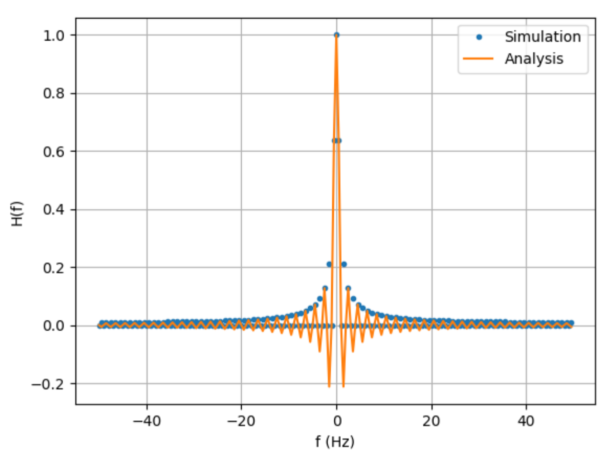


Fig. 3.9

3.10 $\text{sinc}\,(t) \xleftrightarrow{\mathcal{F}} ?$. Verify using python.
Solution: Using (3.31), (3.14) and even property of rect function,

$$\text{sinc}\,(t) \xleftrightarrow{\mathcal{F}} \text{rect}\,(t) \tag{3.32}$$

```
wget https://raw.githubusercontent.com/
    kumarsuraj151/EE3900/master/charger/
    codes/3.10.py
```

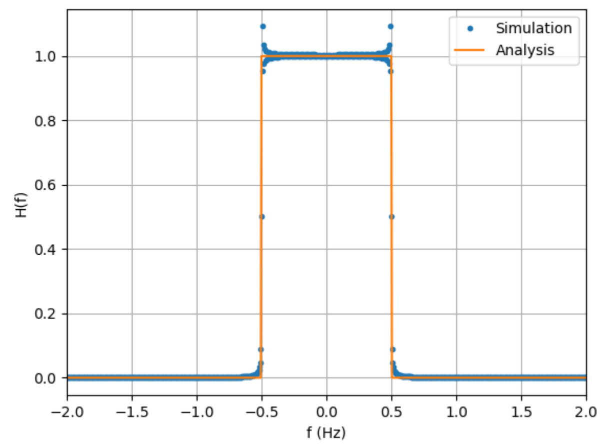
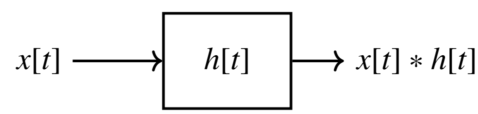
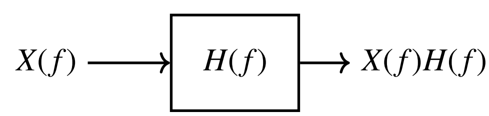


Fig. 3.10

4 Filter

4.1 Find $H(f)$ which transforms $x(t)$ to DC 5V. **Solution:**





$$X(f)H(f) = V_0\delta(f) \tag{4.1}$$

Above equation indicates that H(f) will pass X(f) for f=0.
$\therefore$ H(f) should be a low pass filter.

$$|H(f)| = \frac{V_0}{\left(\frac{2A_0}{\pi}\right)} = \frac{V_0\pi}{2A_0} \tag{4.2}$$

$$H(f) = \frac{V_0\pi}{2A_0} \quad in \quad -2f_0 \leq f \leq 2f_0 \tag{4.3}$$

$$H(f) = \frac{V_0\pi}{2A_0}\text{rect}\left(\frac{f}{4f_0}\right) \tag{4.4}$$

4.2 Find $h(t)$. **Solution:** Using (4.4) and (3.32),

$$h(t) = \frac{2V_0\pi f_0}{A_0}\text{sinc}\,(4f_0t) \tag{4.5}$$

4.3 Verify your result using through convolution. **Solution:**

```
wget https://raw.githubusercontent.com/
    kumarsuraj151/EE3900/master/charger/
    codes/4.3.py
```

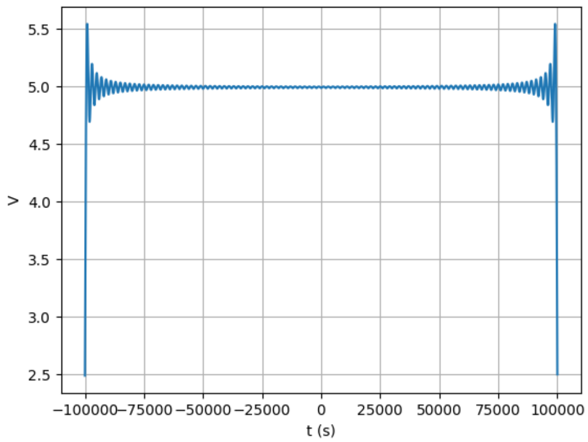


Fig. 4.3

5 Filter Design

5.1 Design a Butterworth filter for $H(f)$. **Solution:**

$$|H(f)| = \frac{1}{\sqrt{1+\left(\frac{f}{f_c}\right)^{2n}}} \tag{5.1}$$

n = Order of the filter

$$A = 10log_{10}|H(f)|^2 \tag{5.2}$$

$$A_1 = -10log_{10}\left[1+\left(\frac{f_1}{f_c}\right)^{2n}\right] \tag{5.3}$$

$$A_2 = -10log_{10}\left[1+\left(\frac{f_2}{f_c}\right)^{2n}\right] \tag{5.4}$$

Solving for n,

$$n = \frac{log\left(\frac{10^{-A_1/10}-1}{10^{-A_2/10}-1}\right)}{2log\left(\frac{f_1}{f_2}\right)} \tag{5.5}$$

Assuming values,

$$A_1 = -1dB \tag{5.6}$$
$$A_2 = -10dB \tag{5.7}$$
$$f_1 = 50Hz \tag{5.8}$$
$$f_2 = 100Hz \tag{5.9}$$

We get,

$$n = 2.5596 \approx 3 \tag{5.10}$$

Putting n = 3 in above equations, We get

$$f_1^{'} = 62.628Hz \tag{5.11}$$
$$f_2^{'} = 69.336Hz \tag{5.12}$$
$$f_c = \sqrt{f_1^{'}f_2^{'}} = 65.897Hz \tag{5.13}$$

5.2 Design a Chebyschev filter for $H(f)$. **Solution:**

$$|H_n(f)| = \frac{1}{\sqrt{1+\epsilon^2 c_n^2\left(\frac{f}{f_0}\right)}} \tag{5.14}$$

where
c_n=nth order Chebyshev polynomial,
ϵ=ripple factor which is related to passband ripple in δ as $\sqrt{10^{\delta/10}-1}$

$$A_1 = -10\log_{10}\left[1+\epsilon^2 c_n^2\left(\frac{f}{f_0}\right)\right] \tag{5.15}$$

$$\implies c_n\left(\frac{f}{f_0}\right) = \frac{\sqrt{10^{-\frac{A_1}{10}}-1}}{\epsilon} \tag{5.16}$$

$$\implies n = \frac{\cosh^{-1}\left(\frac{\sqrt{10^{-\frac{A_1}{10}}-1}}{\epsilon}\right)}{\cosh^{-1}\left(\frac{f}{f_0}\right)} \tag{5.17}$$

Considering

$$f_0 = 65Hz \tag{5.18}$$
$$f = 120Hz \tag{5.19}$$
$$A_1 = 10dB \tag{5.20}$$
$$\delta = 0.2dB \tag{5.21}$$
$$\epsilon = 0.217 \tag{5.22}$$

On solving we get n=2.71302, $\implies$ n=3.

5.3 Design a circuit for your Butterworth filter. **Solution:** Using Cauer Topology,

$$C_k = 2sin\left[\frac{(2k-1)}{2n}\pi\right] \tag{5.23}$$

$$L_k = 2sin\left[\frac{(2k-1)}{2n}\pi\right] \tag{5.24}$$

5.4 Design a circuit for your Chebyschev filter. **Solution:** the normalised values obtained are $C_1 = 1.2276F, L_2 = 1.1525H, C_3 =$

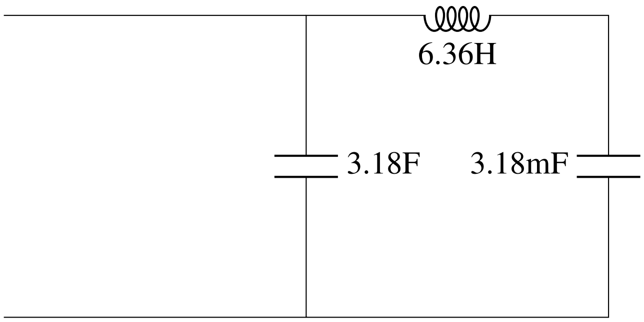


Fig. 5.3

$1.2276F$. De-normalizing the values we get:

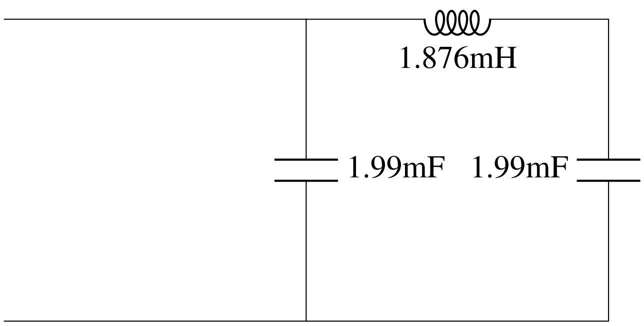


Fig. 5.4