Robust Identifiability in Linear Structural Equation Models of Causal Inference

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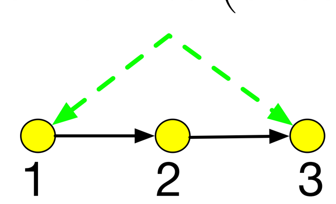
Overview

- Study the **numerical stability** of LSEM parameter recovery via **condition number** on **general path graphs**
- A sufficient condition when parameter recovery problem is stable
- Random models satisfy the condition with substantial probability
- Experimental results

Linear Structural Equations (LSEM)

A **mixed** graph on the n (observable) variables.



- $\mathbf{\Lambda} \in \mathbb{R}^{n \times n}$ - matrix of edge weights of the DAG (strength of causal effect).
- $\mathbf{X} \in \mathbb{R}^{n \times 1}$ - random variables corresponding to the observable variables in the system with covariance $\mathbf{\Sigma} \in \mathbb{R}^{n \times n}$.
- $\eta \in \mathbb{R}^{n \times 1}$ - zero-mean Gaussian noises whose covariance matrix is $\mathbf{\Omega} \in \mathbb{R}^{n \times n}$.

LSEM assumes the following relationship between the random variables in $\mathbf{X}$.

$$\mathbf{X} = \mathbf{\Lambda}^T \mathbf{X} + \eta.$$

Gaussian assumption on η implies $\mathbf{X}$ is a multi-variate Gaussian with covariance

$$\mathbf{\Sigma} = (\mathbf{I} - \mathbf{\Lambda})^{-T} \, \mathbf{\Omega} \, (\mathbf{I} - \mathbf{\Lambda})^{-1}.$$

Typical setting. Experimenter estimates co-variance matrix $\mathbf{\Sigma}$ from finite samples, has a causal hypothesis represented as a mixed graph. Uses a parameter recovery algorithm, such as [2], to obtain the matrices $\mathbf{\Lambda}$ and $\mathbf{\Omega}$.

Challenges. Finite samples, noisy data to estimate $\mathbf{\Sigma}$. Recovery can potentially be bad. **We answer when can it be good?**

Condition number

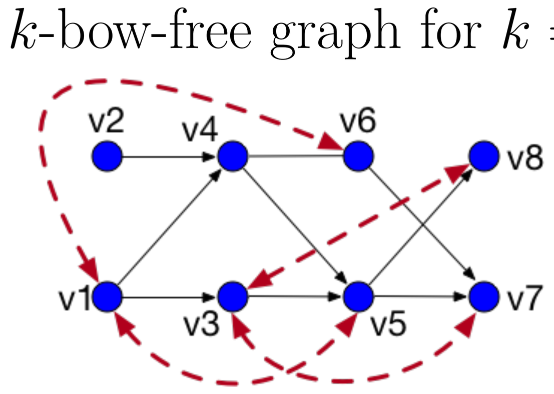
$$\kappa(\mathbf{\Lambda}, \mathbf{\Sigma}) := \lim_{\gamma \to 0^+} \sup_{\tilde{\mathbf{\Sigma}}_\gamma \in \mathcal{F}_\gamma} \frac{\mathrm{Rel}(\mathbf{\Lambda}, \tilde{\mathbf{\Lambda}}_\gamma)}{\mathrm{Rel}(\mathbf{\Sigma}, \tilde{\mathbf{\Sigma}}_\gamma)},$$

where

- $\mathrm{Rel}(\mathbf{A}, \mathbf{B}) := \max_{1 \le i \le n, 1 \le j \le m : |A_{i,j}| \neq 0} \frac{|A_{i,j} - B_{i,j}|}{|A_{i,j}|}$.
- $\mathcal{F}_\gamma$ - set of matrices $\tilde{\mathbf{\Sigma}}_\gamma$ such that $\mathrm{Rel}(\mathbf{\Sigma}, \tilde{\mathbf{\Sigma}}_\gamma) \le \gamma$.
- For any $\tilde{\mathbf{\Sigma}}_\gamma \in \mathcal{F}_\gamma$, $\tilde{\mathbf{\Lambda}}_\gamma$ is the corresponding recovered parameter

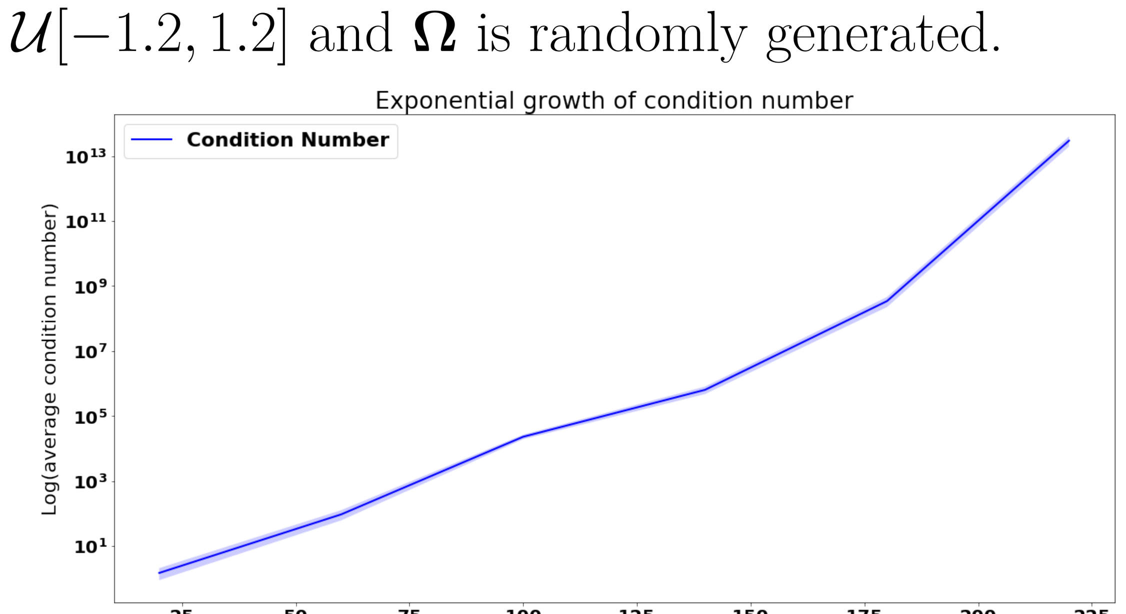
k-bow-free layered graph

The underlying DAG forms a directed layered graph and the mixed graph is bow-free [1]. Here we depict a k-bow-free graph for $k = 2$.



Exponential condition number

For every directed edge $i \to j$ we have $\lambda_{i,j} \sim \mathcal{U}[-1.2, 1.2]$ and $\mathbf{\Omega}$ is randomly generated.



Perturbation model

Let $\epsilon \in \mathbb{R}^{n \times n}$ denote the matrix of perturbations such that every entry (i, j) we have $\varepsilon_{i,j} \le \frac{\gamma}{\sqrt{k}} \Sigma_{i,j}$. Fix a small $0 < \gamma < \frac{1}{n^4}$. The perturbed matrix is $\tilde{\mathbf{\Sigma}} := \mathbf{\Sigma} + \epsilon$.

Sufficient condition for stability

$\mathbf{\Sigma} \succeq 0$ is an $n \times n$ symmetric matrix satisfying the following conditions for some $0 \le \alpha \le 1$. $\|\mathbf{\Sigma}_{[\mathrm{pa}(v),\, v]}\| \le \alpha \|\mathbf{\Sigma}_{[\mathrm{pa}(v),\, \mathrm{pa}(v)]}\|$, $\|\mathbf{\Sigma}_{[\mathrm{spa}(v),\, \mathrm{pa}(v)]}\| \le \alpha \|\mathbf{\Sigma}_{[\mathrm{pa}(v),\, \mathrm{pa}(v)]}\|$ and $\|\mathbf{\Sigma}_{[\mathrm{spa}(v),\, v]}\| \le \alpha \|\mathbf{\Sigma}_{[\mathrm{pa}(v),\, \mathrm{pa}(v)]}\|$ and $\kappa(\mathbf{\Sigma}_{[\mathrm{pa}(v),\, \mathrm{pa}(v)]}) \le \kappa_0 \le \frac{1}{2\gamma}$. The *true* parameter $\mathbf{\Lambda}$, corresponding to the $\mathbf{\Sigma}$, satisfies the following. For every i, j,

$$|\Lambda_{i,j}| \neq 0 \to \frac{1}{n^2} \le \frac{1}{\lambda} \le |\Lambda_{i,j}| \le \beta < 1.$$

If the above conditions hold, we have the following bound on the condition number for bow-free paths with n vertices.

$$\kappa(\mathbf{\Lambda}, \mathbf{\Sigma}) \le O(n^2).$$

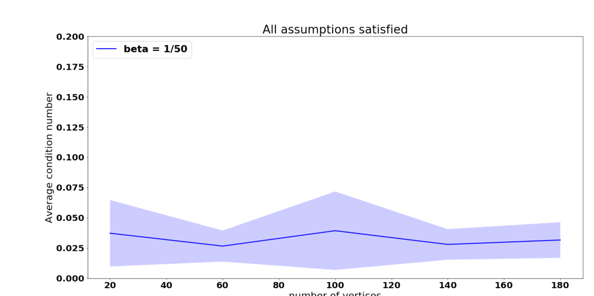
Condition satisified on $1 - \frac{1}{\mathtt{poly}(n)}$ measure of parameters

Generative Model.

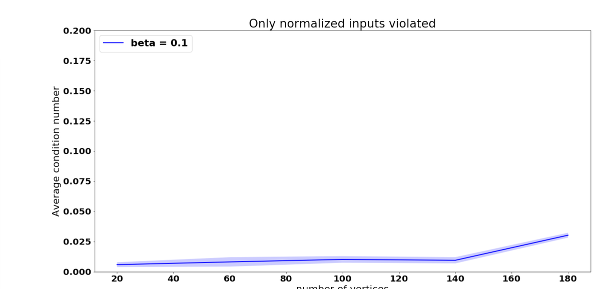
- $\mathbf{\Lambda} \sim \mathcal{U}\left[-\frac{1}{2k\mu}, \frac{1}{2k\mu}\right] \setminus \left[-\frac{1}{n^2}, \frac{1}{n^2}\right]$ when non-zero.
- Sample n-dimensional vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^d$ independently such that each vector $\mathbf{v}_i$ is a uniform sample from the sub-space perpendicular to $\mathtt{SPAN}(\{\mathbf{v}_j\}_{j \in V_{I-1}})$ and $\Omega_{i,j} = \langle \mathbf{v}_i, \mathbf{v}_j \rangle$. $V_I :=$ vertices in layer I.

When $\mu \ge 10(k+1)$ the generative model satisfies the sufficient condition with high-probability.

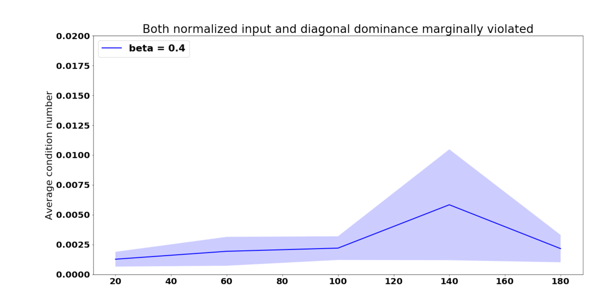
Experiments



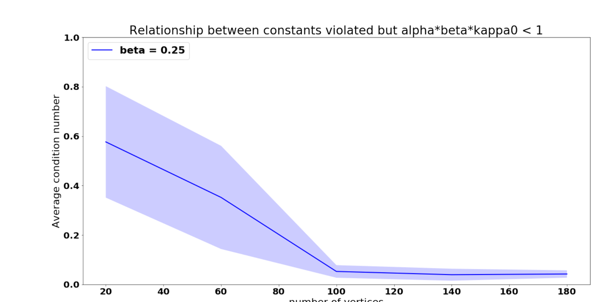
All assumptions satisfied



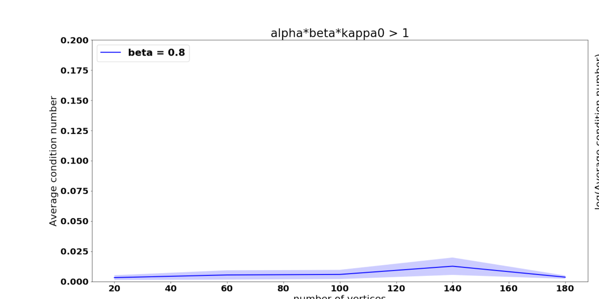
Only normalized input violated.



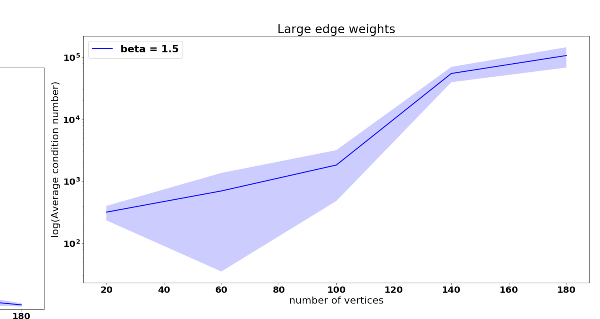
Normalized input and diagonal dominance marginally violated.



Relationship between the constants violated.



$\alpha * \beta * \kappa_0 > 1$.



$\beta > 1$ (y-axis is in log-scale).

Thus, only the requirement that $|\lambda_{i,j}| < 1$ is truly required assumption in practice. Rest are required to prove pessimistic theoretical bounds.

References

[1] Brito, C., and Pearl, J. A new identification condition for recursive models with correlated errors. *Structural Equation Modeling 9*, 4 (2002), 459–474.

[2] Foygel, R., Draisma, J., and Drton, M. Half-trek criterion for generic identifiability of linear structural equation models. *Annals of Statistics 40*, 3 (2012), 1682–1713.