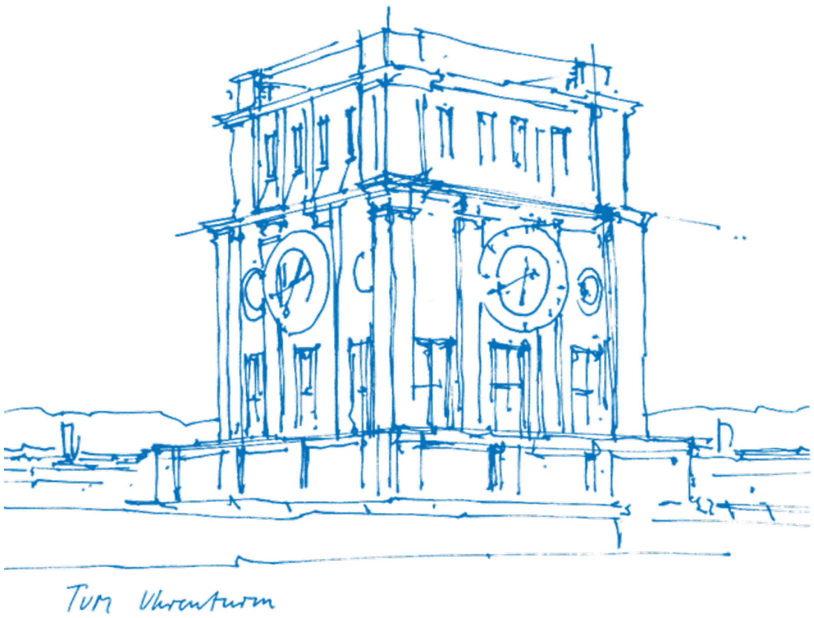
Dynamical systems and bifurcation theory

Felix Dietrich



Today

Dynamical systems and bifurcation theory

We will study qualitative changes of dynamical systems over changes of their parameters. These changes in the qualitative behavior of the system are called *bifurcations*, “to divide into two, like a fork”.

Parameters in crowd dynamics

- Desired speed for individuals
- Distance to obstacles
- Width of doors
- Number of pedestrians in a room
- Target distribution or sequence
- ...

All of them influence the crowd, and by changing them the behavior changes.
When is a change “relevant”?

Today

Outline

1. Dynamical systems: introduction, phase portraits, orbits, topological equivalence
2. Bifurcations: theory, examples in one, two, and three dimensions

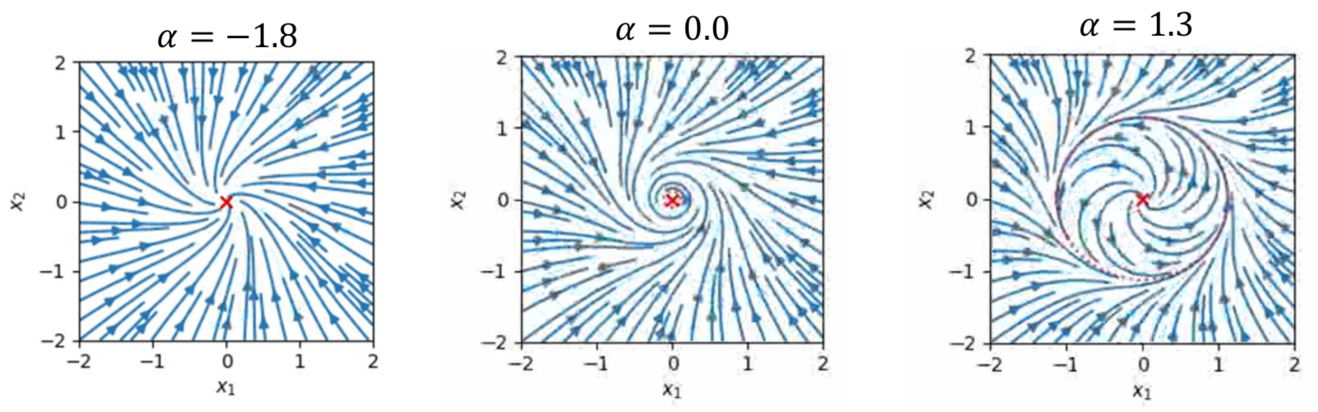


Figure: Bifurcation at parameter $a = 0$, visualized through phase portraits. For positive values of a, a limit cycle exists.

Dynamical systems

What is a dynamical system?

“The notion of a dynamical system is the mathematical formalization of the general scientific concept of a deterministic process.” [Kuznetsov, 2004]

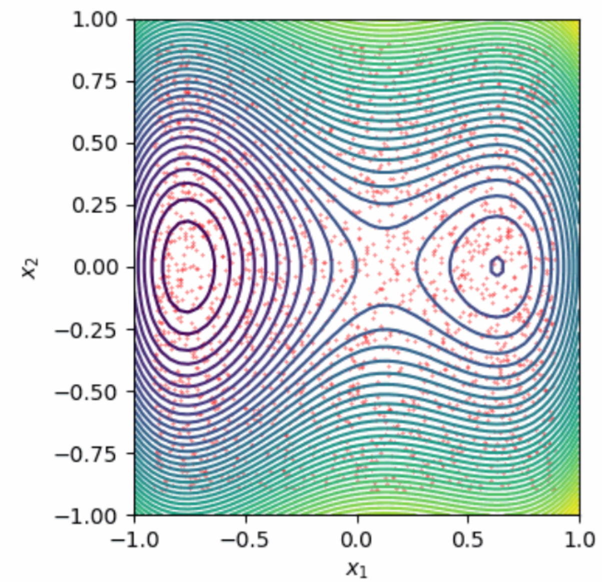


Figure: Each point represents a state of the system, and is moving towards one of the three steady states over time.

Starlings:
`https://www.youtube.com/watch?v=eakKfY5aHmY`
Vortex street:
`https://www.youtube.com/watch?v=IDeGDFZSYo8`

Dynamical systems

What is a dynamical system?

1. Introduction
2. Phase portraits
3. Orbits
4. Topological equivalence

Dynamical systems

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A dynamical system is a triple (T, X, ϕ), with

- the state space X (Euclidean space $\mathbb{R}^n$, manifold $\mathscr{M}$, metric space,...),
- the time T (continuous $\mathbb{R}$, $\mathbb{R}_0^+$, discrete $\mathbb{Z}$, $\mathbb{N}$, ...), and
- the evolution operator $\phi : T \times X \to X$, with the following properties for all $x \in X$:

P1 $\phi(0, x) = \mathrm{Id}(x) = x$,

P2 $\phi(t+s, x) = \phi(t, \phi(s, x)) = (\phi_t \circ \phi_s)(x)$ for all $t, s \in T$.

A good book covering dynamical systems and numerical analysis is written by [Stuart and Humphries, 1996].

Dynamical systems

Notation

The evolution operator can be specified explicitly (as a map), or implicitly, as a

1. recurrence relation (here, $\phi(n, x_0)$ may be difficult to state explicitly):

$$x_{n+1} = \phi(1, x_n)$$

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1. recurrence relation (here, $\phi(n, x_0)$ may be difficult to state explicitly):

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2. differential equation (here, $\phi(t, x)$ may be difficult to state explicitly):

$$\left.\frac{d\phi}{dt}\right|_{t=0}(x) = v(x),$$

   where v is called *vector field*.

The time derivative of the evolution operator at $t = 0$ has several notations,

$$\frac{d}{dt}\phi^t(x),\ \frac{d}{dt}x,\ \text{or } \dot{x}. \tag{1}$$

Dynamical systems

Orbits

Given a system (I, X, ϕ) and a state $x \in X$, the orbit containing x is the set

$$\mathscr{O}(x) := \{\phi(t, x) \in X | t \in I\}. \tag{2}$$

Orbits are also called trajectories (usually if the time information is kept).

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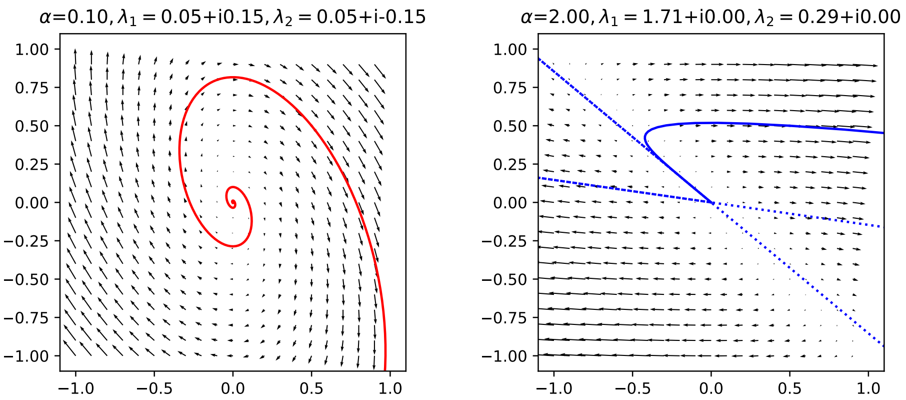

unstable focus (a) unstable node (b)

Figure: Vector fields and orbits of a parametrized dynamical system with state $x \in \mathbb{R}^2$. The vector field is $v_\alpha(x) = A_\alpha x = (\alpha x_1 + \alpha x_2, -0.25x_1)$, with $\alpha = 0.1$ (left) and $\alpha = 2.0$ (right). Orbits are shown in color, the eigenvalues of the matrix A_α are shown in the title.

Dynamical systems

Phase portraits

A phase portrait visualizes qualitative features of a dynamical system by showing representative orbits and vectors.

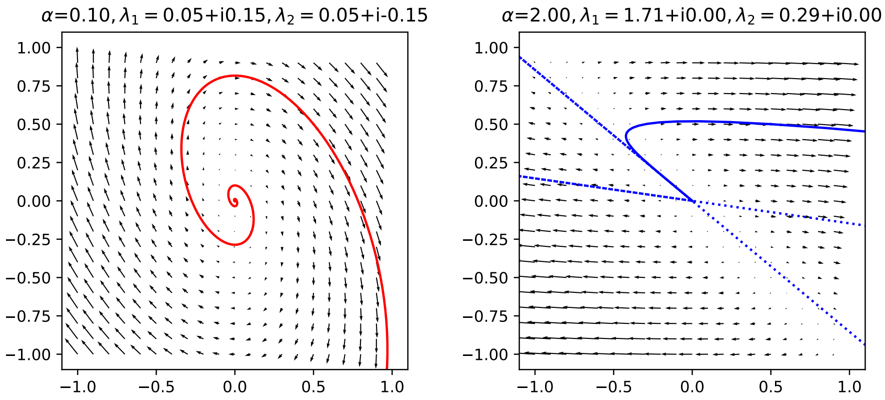

Figure: Phase portraits of a dynamical system with state $x \in \mathbb{R}^2$ and parametrized vector field $v_\alpha(x) = A_\alpha x = (\alpha x_1 + \alpha x_2, -0.25 x_1)$, with $\alpha = 0.1$ (left) and $\alpha = 2.0$ (right). Orbits are shown in color, the eigenvalues of the matrix A_α are shown in the title.

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The notion of qualitative difference is formalized through the notion of topological equivalence, i.e. a system is *qualitatively the same* as another system, if it is *topologically equivalent*:

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A dynamical system (I, X, ϕ) is topologically equivalent to another dynamical system (I, Y, ψ) if there is a homeomorphism $h : X \to Y$ (continuous, and with continuous inverse), such that h is mapping orbits of the first system onto orbits of the second system, preserving the direction of time.

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Examples in crowd dynamics

1. If two models produce exactly the same trajectories, they are topologically equivalent systems.
2. If the trajectories are the same, but the speed along them is different, the systems are still topologically equivalent.
3. If one model moves one pedestrian in a circle, but the other one moves them in a straight line, the models are not topologically equivalent (one is recurrent, the other is not).

Bifurcation theory

What is a bifurcation?

1. Introduction
2. Normal forms
3. Examples: 1D, 2D, one and more parameters

Bifurcation theory

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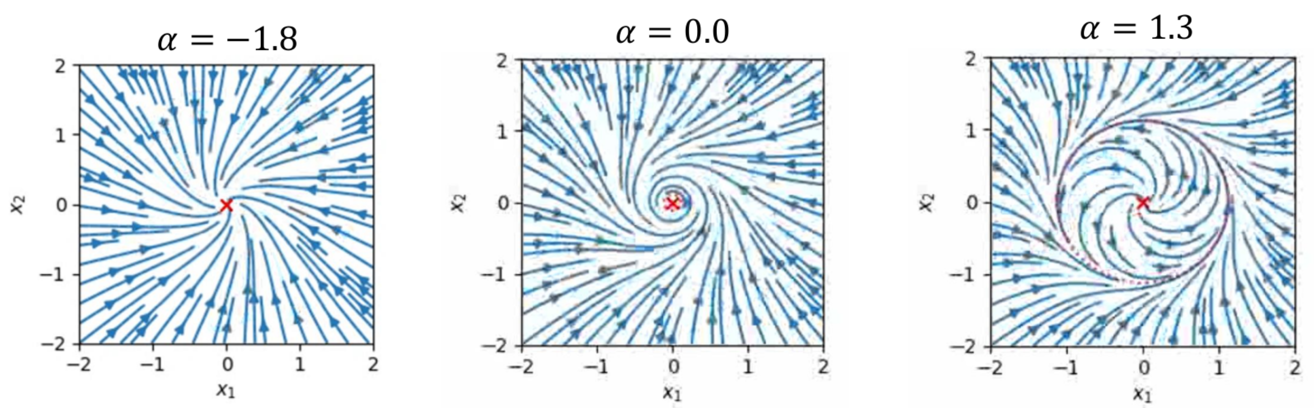

Figure: Bifurcation at parameter $a = 0$, visualized through phase portraits. For positive values of a, a limit cycle exists.

Bifurcation theory

Normal forms (informal definition)

The normal form of a dynamical system around a fixed point is a polynomial vector field, with

- the smallest degree,
- the smallest number of coordinates, and
- the smallest number of parameters,
- such that the two systems are topologically equivalent locally around the fixed point (in state space), and locally around the bifurcation point (in parameter space).

For formal definitions of normal forms and topological equivalence including the parameter space, see [Kuznetsov, 2004, p.63ff].

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Most important statement of the lecture today:

The normal form is locally topologically equivalent to **all dynamical systems with that normal form**. That means once we know that a particular system has a normal form, we do not need to study the specific system anymore - we understand it! At least around the given steady state...

Bifurcation theory

Examples - 1D space, 1 parameter

The **Pitchfork bifurcation**

$$\dot{x} = x\alpha - x^3 \tag{3}$$

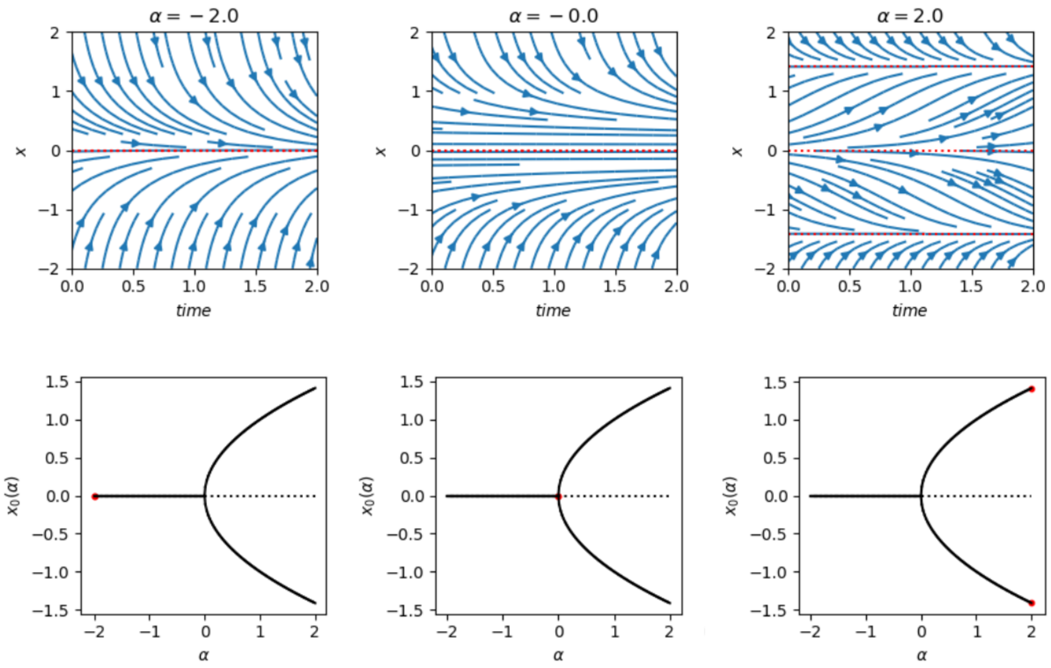


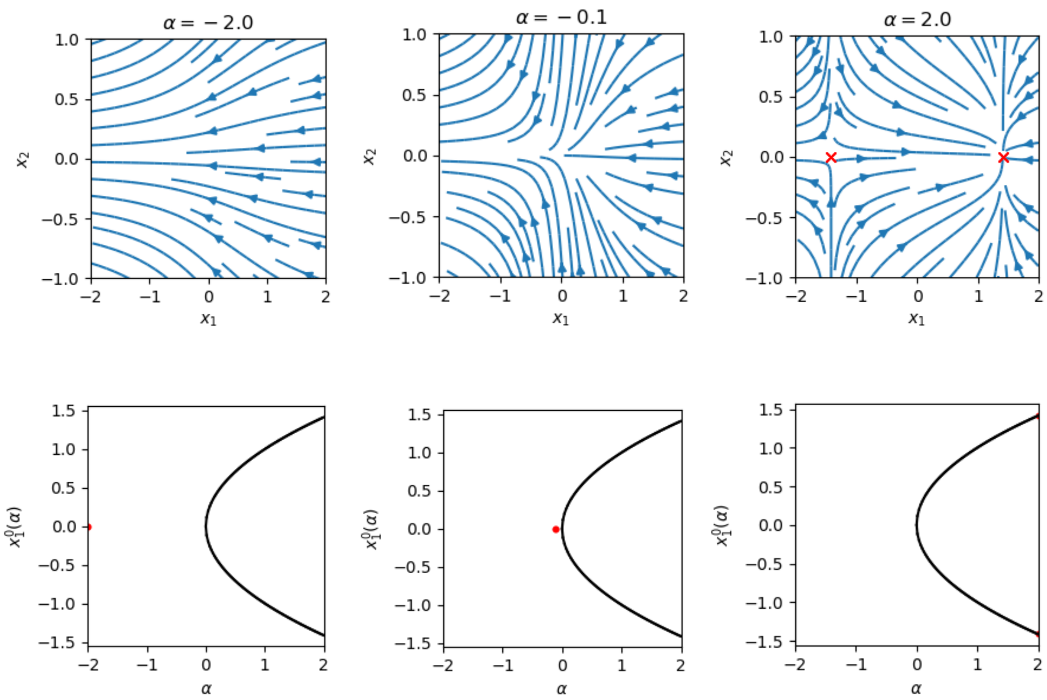
Figure: Phase portraits and bifurcation diagram for the pitchfork bifurcation.

Bifurcation theory

Examples - 1D space, 1 parameter

The **Saddle-node bifurcation**

$$\dot{x} = \alpha - x^2 \tag{4}$$



Phase portraits and bifurcation diagram for the saddle-node bifurcation. Here, the x_2 direction does not contribute to the bifurcation (its dynamic is $\dot{x}_2 = -x_2$).

Bifurcation theory

Examples - 2D space, 1 parameter

The **Andronov-Hopf bifurcation**

$$\begin{aligned} \dot{x}_1 &= \alpha x_1 - x_2 - x_1(x_1^2 + x_2^2), \\ \dot{x}_2 &= x_1 + \alpha x_2 - x_2(x_1^2 + x_2^2). \end{aligned} \tag{5}$$

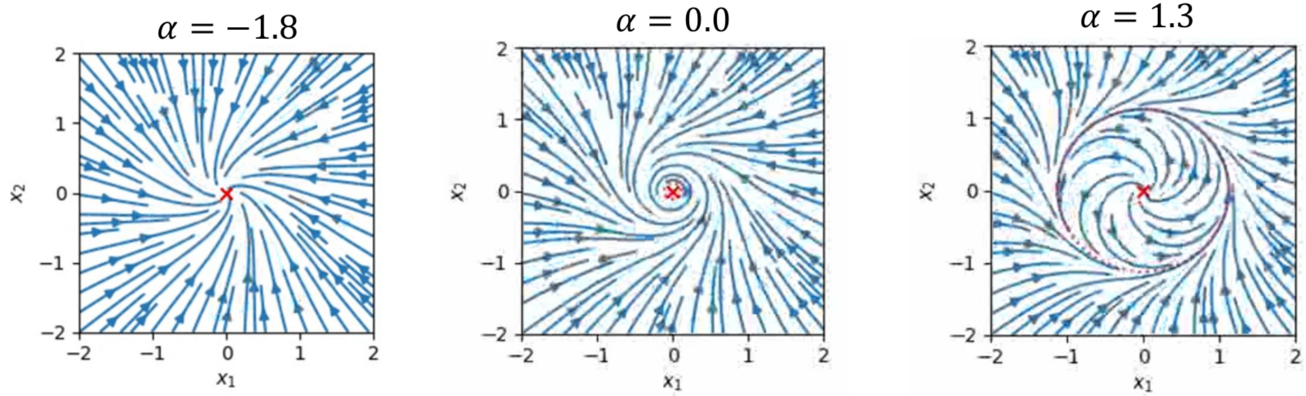


Figure: Hopf bifurcation at parameter $a = 0$, visualized through phase portraits. For positive values of a, a limit cycle exists.

Bifurcation theory

Examples - 3D space, 2 parameters

The **Blue sky catastrophe.** A stable limit cycle disappears, its length and period tend to infinity, while it remains bounded, and located at a finite distance from all equilibrium points. The normal form is

$$\begin{aligned} \dot{x} &= x[2+\alpha-10(x^2+y^2)]+z^2+y^2+2y, \\ \dot{y} &= -z^3-(y+1)(z^2+y^2+2y)-4x+\alpha y, \\ \dot{z} &= y^2(y+1)+x^2-\beta. \end{aligned} \tag{6}$$

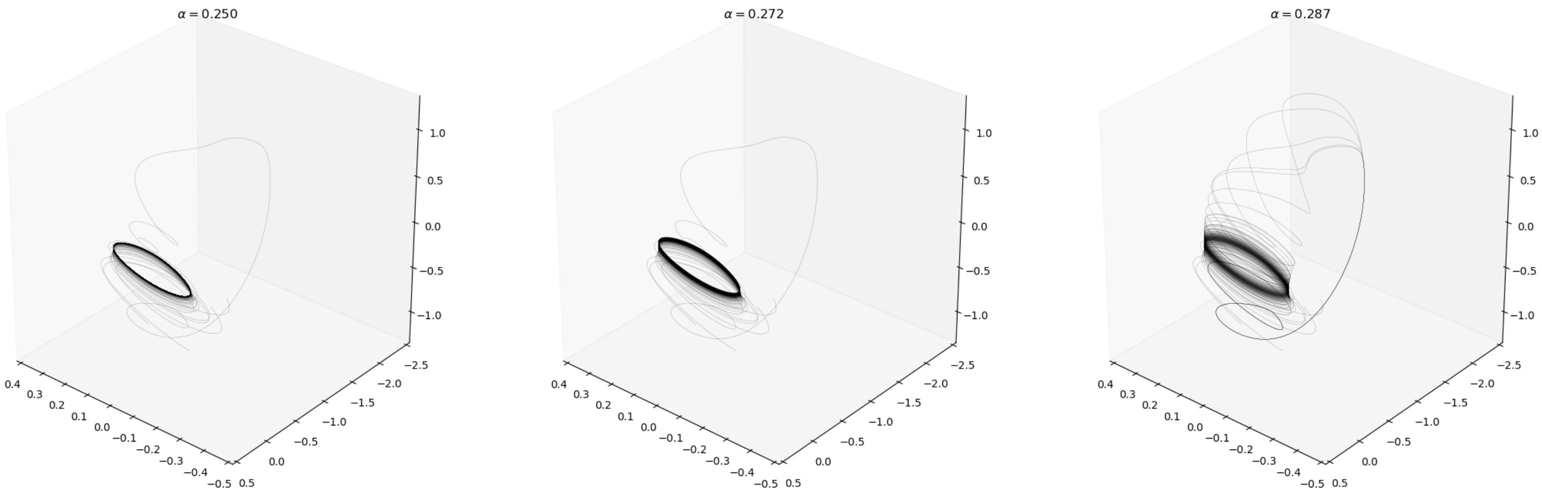


Figure: Stable limit cycle disappears into bounded orbit of infite length and period.

Literature I

Kuznetsov, Y. A. (2004).
Elements of Applied Bifurcation Theory.
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