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COM205T Discrete Structures for Computing-Lecture Notes

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Relations

Objective: In this module, we shall introduce sets, their properties, and relationships among their elements. We shall also look at in detail the properties of relations. Further we also count sets that satisfy specific properties.

Basic Definitions

A set is a *well defined* collection of distinct objects.
The cross product is defined as $A \times A = \{(a, b) \mid a, b \in A\}$.
Given two similar sets A and B, $A \times B = \{(x, y) \mid x \in A, y \in B\}$.
In general, given the sets $A_1, A_2, \ldots, A_n$, we can define $A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in A_i, 1 \le i \le n\}$.
Note that if $|A| = n$, then $|A \times A| = n^2$ and $|A_1 \times A_2 \times \ldots \times A_n| = m_1.m_2 \ldots m_n$ where $|A_i| = m_i$, $1 \le i \le n$.
Note: Empty set ϕ is not well defined. If $B = \phi$, then $A \times B$ is also not well defined. So, cross product is defined only for non-empty sets.

Binary Relation

A binary relation R from A to B is defined as $R \subseteq A \times B$. A binary relation R' defined on A is such that $R' \subseteq A \times A$.

Example 1. let $A = \{1, 2, 3\}$, R_1 to R_4 are binary relations defined on A.
$R_1 = \{(1, 1), (2, 2), (3, 3)\}$ $\quad$ $R_2 = \{(1, 1), (2, 1), (3, 2)\}$
$R_3 = \phi$ $\quad$ $R_4 = A \times A$

Example 2. $S = \{$DM,DSA,ALG,OOPS,C++,JAVA$\}$.
$R_5 = \{(x, y) \mid x, y \in S$ and x is a prerequisite for $y\}$.
$R_5 = \{$(DSA,ALG), (DM,ALG), (DM,OOPS), (OOPS,C++)$\}$.

Example 3. Let $I = \{i_1, i_2, \ldots, i_m\}$ be the items in a supermarket. A ternary relation R_3 on I is defined as $R_3 = \{(i_1, i_4, i_3), (i_2, i_1, i_7), (i_7, i_8, i_9)\}, R_3 \subseteq I \times I \times I$.

Claim. Consider a set A with $|A| = n$. The maximum number of binary relations on A is 2^{n^2}.

Proof. Let $A \times A = \{x_1, x_2, \ldots, x_{n^2}\}$. A relation is a subset of $A \times A$. The number of subsets of a set of size n^2 is 2^{n^2}. Thus, the claim follows. $\square$

A **Unary** relation R on A is $R \subseteq A$. The maximum number of unary relations is 2^n where $n = |A|$.

Remark: Familiar examples for relations are tables in databases or a spreadsheet. A row (or the entire spreadsheet) corresponds to a subset of the cross product of columns and each row is a relation by defintion.

Properties of Relations

Consider a binary relation $R \subseteq A \times A$, we define the following on R
(1) R is *reflexive* if $\forall a \in A, ((a,a) \in R)$
(2) R is *symmetric* if $\forall a, b \in A, ((a,b) \in R \rightarrow (b,a) \in R)$
(3) R is *transitive* if $\forall a, b, c \in A, ((a,b) \in R$ and $(b,c) \in R \rightarrow (a,c) \in R)$
(4) R is *Asymmetric* if $\forall a, b \in A, ((a,b) \in R \rightarrow (b,a) \notin R)$
(5) R is *Antisymmetric* if $\forall a, b \in A, [((a,b) \in R \wedge (b,a) \in R) \rightarrow a = b]$
(6) R is *Irreflexive* if $\forall a \in A, ((a,a) \notin R)$

Example 4. Let $A = \{1,2,3\}$. Identify which of the following relations defined on A are reflexive, symmetric and transitive.

$R_1 = \{(1,1),(2,2),(3,3)\}$ $\quad$ $R_2 = \{(1,1),(1,2),(1,3)\}$
$R_3 = \phi$ $\quad$ $R_4 = A \times A$
$R_5 = \{(2,2),(3,3),(1,2)\}$ $\quad$ $R_6 = \{(2,3),(1,2)\}$

Property	R_1	R_2	R_3	R_4	R_5	R_6
Reflexive	✓	X	X	✓	X	X
Symmetric	✓	X	✓	✓	X	X
Transitive	✓	✓	✓	✓	✓	X

Example 5. $R_7 = \{(a,b) \mid a, b \in \mathbb{I}$ and a divides $b\}$. **Claim:** R_7 is Reflexive and transitive.

Proof. Clearly, for all $a \in \mathbb{I}$, a divides a. Therefore, $(a,a) \in R_7$ $\forall a \in \mathbb{I}$. For all $a,b,c \in \mathbb{I}$, if $(a,b) \in R$ and $(b,c) \in R$, then we prove that $(a,c) \in R$. Let $\frac{b}{a} = c_1 \geq 1$ and $\frac{c}{b} = c_2 \geq 1$. This implies $b = c_1 \cdot a$ and $c = c_2 \cdot b$. Therefore, $c = c_2 \cdot c_1 \cdot a = c_3 \cdot a$ where $c_3 = c_2 \cdot c_1$. Thus, $\frac{c}{a} = c_3 \geq 1$ and a divides c, $(a,c) \in R_7$. Note: R_7 is not symmetric as $(1,3) \in R$ and $(3,1) \notin R$. $\square$

Example 6. $A = \{1,2,3,4\}$. Identify the properties of relations.

$R_1 = \{(1,1),(2,2),(3,3),(2,1),(4,3),(4,1),(3,2)\}$
$R_2 = A \times A, \quad R_3 = \phi, \quad R_4 = \{(1,1),(2,2),(3,3),(4,4)\}$
$R_5 = \{(1,1),(2,2),(3,3),(4,4),(1,2),(2,1),(4,3),(3,4)\}$

Relation	Reflexive	Symmetric	Asymmetric	Antisymmetric	Irreflexive	Transitive
R_1	×	×	×	✓	×	×
R_2	✓	✓	×	×	×	✓
R_3	×	✓	✓	✓	✓	✓
R_4	✓	✓	×	✓	×	✓
R_5	✓	✓	×	×	×	✓

Counting Special Relations

We shall now count relations satisfying specific properties such as reflexivity, symmetricity, etc. Let $A = \{a_1, a_2, \ldots, a_n\}$ and we represent $A \times A$ as a matrix such that i^{th} row j^{th} column represents (a_i, a_j), $1 \leq i, j \leq n$.

Claim: The number of reflexive binary relations possible in A is $2^{n(n-1)}$.

Proof: Observe that the diagonal elements of the matrix are mandatory in any reflexive binary relation. Therefore, the diagonal elements along with any subset from the remaining $n^2 - n$ elements is a reflexive relation. So, the number of such sets is $2^{n(n-1)}$.

Claim: The number of symmetric binary relations possible in A is $2^{(n(n+1))/2}$.

Proof: Consider the elements other than the diagonal elements, we divide them into lower triangle elements $(i > j)$ and upper triangle elements $(i < j)$. Notice that in any symmetric relation, if there exist an element, say (a_i, a_j) from the lower triangle, then the element (a_j, a_i) from the upper triangle is also in the relation (this element is forced into the relation). There are $n^2 - n$ such pairs of elements. Also, it is to be noted that any subset of the diagonal elements together with a subset from lower triangle (upper triangle) is a symmetric relation. Therefore, the number of symmetric relations is $2^n \cdot 2^{(n^2-n)/2} = 2^{(n^2+n)/2}$.

Claim: The number of antisymmetric binary relations possible in A is $2^n \cdot 3^{(n^2-n)/2}$.

Proof: Consider an antisymmetric binary relation and note that, if there exist an element, say (a_i, a_j) from the lower triangle, then the element (a_j, a_i) from the upper triangle should not be present in the relation and vice versa. Therefore there exist 3 possibilities for an (i, j) pair. i.e., either (a_i, a_j) is in the relation, or (a_j, a_i) is in the relation, or none of $(a_i, a_j), (a_j, a_i)$ are present in the relation. There are $(n^2 - n)/2$ pairs of (a_i, a_j) such that $i \neq j$. Therefore, there exist $3^{(n^2-n)/2}$ relations. Also, observe that any subset of the diagonal elements are possible in an antisymmetric relation. Therefore, the number of antisymmetric binary relations is $2^n \cdot 3^{(n^2-n)/2}$.

Claim: The number of binary relations in A which are both symmetric and antisymmetric is 2^n.

Proof: Suppose $(a_i, a_j), i \neq j$ is in the relation, then due to symmetry, (a_j, a_i) is also in the relation. However, this violates antisymmetric property. Therefore, for all $i \neq j$, (a_i, a_j) is not in the relation. Thus, the only possible elements for consideration are the diagonal elements. Observe that any subset of the diagonal elements are symmetric and antisymmetric. Therefore, the number of binary relations which are both symmetric and antisymmetric is 2^n.

Claim: The number of binary relations in A which are both symmetric and asymmetric is one.

Proof: Let R be a symmetric and asymmetric binary relation on any A. For all $a \in A$, none of the (a, a) elements are in the relation as R is asymmetric. Clearly, non-diagonal elements (a, b), where $a \neq b$ are not present in R. Therefore, there does not exist any of the diagonal and non-diagonal elements in R, and it follows that there is only one relation $R = \phi$, which is symmetric and asymmetric.

Claim: The number of binary relations which are both reflexive and antisymmetric in the set A is $3^{(n^2-n)/2}$.

Proof: Since all diagonal elements are part of the reflexive relation and there are 3 possibilities for the remaining $(n^2-n)/2$ elements, there exist $3^{(n^2-n)/2}$ binary relations which are reflexive and antisymmetric.

Claim: The number of asymmetric binary relations possible in the set A is $3^{(n^2-n)/2}$.

Proof: Similar to the argument for antisymmetric relations, note that there exist $3^{(n^2-n)/2}$ asymmetric binary relations, as none of the diagonal elements are part of any asymmetric binary relations.

Note: If $A = \phi$ and $R \subseteq A \times A$, then reflexive and irreflexive relations are same. This is the only set and the relation having this property.

Definitions:

R is an *Equivalence relation*, if R satisfies **R**eflexivity, **S**ymmetricity and **T**ransitivity. (RST properties)
R is a *Partial Order*, if R is **R**eflexive, **A**ntisymmetric and **T**ransitive. (RAT properties)

Operations on relations:

$R_1 - R_2 = \{(a,b) \mid (a,b) \in R_1 \text{ and } (a,b) \notin R_2\}$
$R_2 - R_1 = \{(a,b) \mid (a,b) \in R_2 \text{ and } (a,b) \notin R_1\}$
$R_1 \cup R_2 = \{(a,b) \mid (a,b) \in R_1 \text{ or } (a,b) \in R_2\} = \{(a,b) \mid a < b \text{ or } a > b\}$
$R_1 \cap R_2 = \{(a,b) \mid (a,b) \in R_1 \text{ and } (a,b) \in R_2\} = \phi$

Example 7. Let A be a set on integers and $R_1, R_2 \subseteq A \times A$ where $R_1 = \{(a,b) \mid a < b\}$ and $R_2 = \{(a,b) \mid a > b\}$. Check whether the following relations satisfy the properties of relations.

Relation	Reflexive	Symmetric	Asymmetric	Antisymmetric	Transitive
R_1	$\times$	$\times$	✓	✓	✓
R_2	$\times$	$\times$	✓	✓	✓
$R_1 \cup R_2$	$\times$	✓	$\times$	$\times$	✓
$R_1 \cap R_2$	$\times$	✓	✓	✓	✓
$R_1 - R_2$	$\times$	$\times$	✓	✓	✓
$R_2 - R_1$	$\times$	$\times$	✓	✓	✓

For the following theorems, we work with a set A and $R_1, R_2 \subseteq A \times A$.

Theorem 1. *If R_1 and R_2 are reflexive, and symmetric, then $R_1 \cup R_2$ is reflexive, and symmetric.*

Proof. Clearly, for all $a \in A$, there exist $(a,a) \in R_1$ and thus, $(a,a) \in R_1 \cup R_2$. Therefore, $R_1 \cup R_2$ is reflexive. We can claim that if $(a,b) \in R_1 \cup R_2$, then $(b,a) \in R_1 \cup R_2$. *Case 1:* if $(a,b) \in R_1$, then $(b,a) \in R_1$ as R_1 is symmetric and this implies that $(b,a) \in R_1 \cup R_2$. *Case*

2: if $(a,b) \in R_2$, then $(b,a) \in R_2$ as R_2 is symmetric and this implies that $(b,a) \in R_1 \cup R_2$. Therefore, we can conclude that $R_1 \cup R_2$ is symmetric and thus the theorem follows. $\square$

Claim: If R_1 is transitive and R_2 is transitive, then $R_1 \cup R_2$ need not be transitive.

Minimal counter example: Let $A = \{1,2\}$ such that $R_1 = \{(1,2)\}$ and $R_2 = \{(2,1)\}$. $R_1 \cup R_2 = \{(1,2),(2,1)\}$ and $(1,1) \notin R_1 \cup R_2$ implies that $R_1 \cup R_2$ is not transitive.

Note: From the above claim and theorem, it follows that if R_1 and R_2 are equivalence relations, then $R_1 \cup R_2$ need not be an equivalence relation. Is $R_1 \cap R_2$ an equivalence relation ?

Theorem 2. *If R_1 and R_2 are equivalence relations, then $R_1 \cap R_2$ is an equivalence relation.*

Proof. Clearly, for all $a \in A$, there exist $(a,a) \in R_1$ and $(a,a) \in R_1$. Therefore, for all $a \in A$, $(a,a) \in R_1 \cap R_2$. Therefore, $R_1 \cap R_2$ is reflexive. We now claim that if $(a,b) \in R_1 \cap R_2$, then $(b,a) \in R_1 \cap R_2$. Note that $(a,b) \in R_1$ and $(a,b) \in R_2$. Since R_1 and R_2 are symmetric $(b,a) \in R_1$ and $(b,a) \in R_2$. Therefore $(b,a) \in R_1 \cap R_2$. If $(a,b),(b,c) \in R_1 \cap R_2$, then $(a,b),(b,c) \in R_1$ and $(a,b),(b,c) \in R_2$. Since R_1 is transitive, $(a,c) \in R_1$. Similarly, since R_2 is transitive, $(a,c) \in R_2$. This implies that $(a,c) \in R_1 \cap R_2$. Therefore, we can conclude that $R_1 \cup R_2$ is transitive and thus, the theorem follows. $\square$

Remark:
If R_1 and R_2 are equivalence relations on A,

1. $R_1 - R_2$ is not an equivalence relation.
2. $R_1 - R_2$ is not a partial order since $R_1 - R_2$ is not reflexive.
3. $R_1 \oplus R_2 = R_1 \cup R_2 - (R_1 \cap R_2)$

Claim: If R_1 is antisymmetric and R_2 is antisymmetric, then $R_1 \cup R_2$ need not be antisymmetric.
Minimal counter example: Let $A = \{1,2\}$ such that $R_1 = \{(1,2)\}$ and $R_2 = \{(2,1)\}$. $(1,2),(2,1) \in R_1 \cup R_2$ and $1 \neq 2$ implies that $R_1 \cup R_2$ is not antisymmetric.

Note: From the above claim it follows that if R_1, R_2 are partial order, then $R_1 \cup R_2$ need not be a partial order. Is $R_1 \cap R_2$ a partial order ?

Theorem 3. *If R_1 and R_2 are partial order, then $R_1 \cap R_2$ is a partial order.*

Proof. From the proof of Theorem 2, if R_1 and R_2 are reflexive, transitive, then $R_1 \cap R_2$ is reflexive, transitive. Now we shall show that if R_1 and R_2 are antisymmetric, then $R_1 \cap R_2$ is antisymmetric. On the contrary, assume that $R_1 \cap R_2$ is not antisymmetric. I.e., there exist $(a,b),(b,a) \in R_1 \cap R_2$ such that $a \neq b$. Note that $(a,b),(b,a) \in R_1$ and $(a,b),(b,a) \in R_2$ and it follows that R_1 and R_2 are not antisymmetric, which is a contradiction. Therefore, our assumption is wrong and $R_1 \cap R_2$ is antisymmetric. This implies that $R_1 \cap R_2$ is a partial order. This completes the proof of the theorem. $\square$

Questions:

1. Count the number of transitive relations in the set A.
2. Count the number of equivalence relations and partial ordered sets in the set A.
3. Prove using mathematical induction that $2^n \cdot 3^{(n^2-n)/2} \leq 2^{n^2}$.
4. Prove or disprove: If $A \neq \phi$ and $R \subseteq A \times A$, then R cannot be both reflexive and irreflexive. Give an example for which R is neither reflexive nor irreflexive.
5. Count the number of relations which are neither reflexive nor irreflexive.

MYSTERIOUS 22 [1]

Select any three-digit number with all digits different from one another. Write all possible two-digit numbers that can be formed from the three-digits selected earlier. Then divide their sum by the sum of the digits in the original three-digit number. See the result!!!

Composition of Relations

Let $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$, Composition of R_2 on R_1, denoted as $R_1 \circ R_2$ or simply R_1R_2 is defined as $R_1 \circ R_2 = \{(a, c) \mid a \in A, c \in C \wedge \exists b \in B \text{ such that } ((a, b) \in R_1, (b, c) \in R_2)\}$.
Note: If $R_1 \subseteq A \times B$ and $R_2 \subseteq C \times D$, then $R_1 \circ R_2$ is undefined.
Let $R_1 \subseteq A \times B$, $R_2, R_3 \subseteq B \times C$, $R_4 \subseteq C \times D$.

Theorem 4. $R_1(R_2 \cup R_3) = R_1R_2 \cup R_1R_3$

Proof. Consider $(a, c) \in R_1(R_2 \cup R_3) \iff$ by definition, $\exists b \in B$ such that $(a, b) \in R_1 \wedge (b, c) \in R_2 \cup R_3$.
$\iff \exists b[(a, b) \in R_1 \wedge ((b, c) \in R_2 \vee (b, c) \in R_3)]$
$\iff \exists b[((a, b) \in R_1 \wedge (b, c) \in R_2) \vee ((a, b) \in R_1 \wedge (b, c) \in R_3)]$ (distribution law)
$\iff \exists b[((a, b) \in R_1 \wedge (b, c) \in R_2)] \vee \exists b[((a, b) \in R_1 \wedge (b, c) \in R_3)]$
$\iff (a, c) \in R_1R_2 \vee (a, c) \in R_1R_3 \iff (a, c) \in R_1R_2 \cup R_1R_3$ □

Theorem 5. $R_1(R_2 \cap R_3) \subset R_1R_2 \cap R_1R_3$

Proof. Let $(a, c) \in R_1(R_2 \cap R_3)$ by definition, $\exists b((a, b) \in R_1 \wedge (b, c) \in R_2 \cap R_3)$
$\iff \exists b[(a, b) \in R_1 \wedge (b, c) \in R_2 \wedge (b, c) \in R_3]$
$\iff \exists b[(a, b) \in R_1 \wedge (b, c) \in R_2 \wedge (a, b) \in R_1 \wedge (b, c) \in R_3]$
$\implies \exists b[(a, b) \in R_1 \wedge (b, c) \in R_2] \wedge \exists b[(a, b) \in R_1 \wedge (b, c) \in R_3]$
Note that, the biconditional operator is changed to implication as existential quantifier respects implication with respect to 'and' operator.
$\implies (a, c) \in R_1R_2 \wedge (a, c) \in R_1R_3 \implies (a, c) \in R_1R_2 \cap R_1R_3$ □

Theorem 6. $R_1 \subseteq A \times B$, $R_2 \subseteq B \times C$, $R_3 \subseteq C \times D$. $(R_1R_2)R_3 = R_1(R_2R_3)$

Proof. Let $(a, d) \in (R_1R_2)R_3$ by definition, $\exists c((a, c) \in R_1R_2 \wedge (c, d) \in R_3)$
$\iff \exists c\ [\exists b[(a, b) \in R_1 \wedge (b, c) \in R_2] \wedge (c, d) \in R_3]$
$\iff \exists c\ \exists b\ [(a, b) \in R_1 \wedge (b, c) \in R_2 \wedge (c, d) \in R_3]$
$\iff \exists b\ \exists c\ [(a, b) \in R_1 \wedge (b, c) \in R_2 \wedge (c, d) \in R_3]$
$\iff \exists b\ [(a, b) \in R_1 \wedge \exists c\ [(b, c) \in R_2 \wedge (c, d) \in R_3]]$
$\iff \exists b\ [(a, b) \in R_1 \wedge (b, d) \in R_2R_3]$
$\iff (a, d) \in R_1(R_2R_3)$ □

Claim: If R_1 and R_2 are both reflexive, then $R_1 \circ R_2$ is reflexive.
Note that for all a, $(a, a) \in R_1$, and $(a, a) \in R_2$.
This implies that for all a, $(a, a) \in R_1 \circ R_2$.

$\square$

Remark:

Prove the following using counter examples.

- If R_1 and R_2 are both symmetric, then $R_1 \circ R_2$ need not be symmetric.
  $R_1 = \{(1, 2), (2, 1)\}$, $R_2 = \{(2, 3), (3, 2)\} \implies R_1 \circ R_2 = \{(1, 3)\}$

- If R_1 and R_2 are both antisymmetric, then $R_1 \circ R_2$ need not be antisymmetric.
  $R_1 = \{(1, 2), (3, 4)\}$, $R_2 = \{(2, 3), (4, 1)\} \implies R_1 \circ R_2 = \{(1, 3), (3, 1)\}$

- If R_1 and R_2 are both transitive, then $R_1 \circ R_2$ need not be transitive.
  $R_1 = \{(1, 2), (3, 4)\}$, $R_2 = \{(2, 3), (4, 1)\} \implies R_1 \circ R_2 = \{(1, 3), (3, 1)\}$

- If R_1 and R_2 are both irreflexive, then $R_1 \circ R_2$ need not be irreflexive.
  $R_1 = \{(1, 2)\}$, $R_2 = \{(2, 1)\} \implies R_1 \circ R_2 = \{(1, 1)\}$

Closure of a Relation

Let $R \subseteq A \times A$ be a non reflexive relation. To make R a reflexive relation, one can add the elements from $R' = (A \times A) \backslash R$ to R so that $R \cup R' (\subseteq A \times A)$ is reflexive. An interesting question is what will be the minimum cardinality of the set R' such that $R \cup R'$ is a relexive relation. Closure operation deals with such minimum cardinality sets.

Let A be a finite set and $R \subseteq A \times A$.

Reflexive closure of R, denoted as $r(R)$ is a relation $R' \subseteq A \times A$ such that
(i) $R' \supseteq R$
(ii) R' is reflexive.
(iii) For any reflexive relation R'' $(\neq R')$ such that $R'' \supset R$, then $R'' \supset R'$.

Similarly, we can define symmetric closure $s(R)$ of R and transitive closure $t(R)$ of R.

Note:
$r(R) = R \cup E$ where $E = \{(x, x) \mid x \in A\}$.
$s(R) = R \cup R^c$ where $R^c = \{(a, b) \mid (b, a) \in R\}$.

Example:

$A = \{1, 2, 3\}$	r(R)	s(R)	t(R)
$R_1 = \{(1, 1), (2, 2), (3, 3)\}$	R_1	R_1	R_1
$R_2 = \{(1, 1), (2, 1)\}$	$\{(1, 1), (2, 1), (2, 2), (3, 3)\}$	$\{(1, 1), (2, 1), (1, 2)\}$	R_2
$R_3 = \phi$	$\{(1, 1), (2, 2), (3, 3)\}$	R_3	R_3
$R_4 = A \times A$	R_4	R_4	R_4
$R_5 = \{(1, 1), (2, 1), (2, 3)\}$	$\{(1, 1), (2, 1), (2, 3), (2, 2), (3, 3)\}$	$\{(1, 1), (2, 1), (2, 3), (1, 2), (3, 2)\}$	R_5

Some more examples: $A = \{1, 2, 3, 4\}$ $R_6 = \{(1, 2), (2, 1), (2, 3), (3, 4)\}$
$t(R_6) = \{(1, 2), (2, 1), (2, 3), (3, 4), (1, 1), (2, 4), (1, 4), (1, 3), (2, 2)\}$

$A = \{1, 2, 3, 4, 5\}$ $R_7 = \{(1, 2), (3, 4), (4, 5), (5, 3), (2, 1)\}$
$t(R_7) = \{(1, 2), (3, 4), (4, 5), (5, 3), (2, 1), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (4, 3), (3, 5), (5, 4)\}$

Relation as a graph

Note that each binary relation can be expressed as a directed graph. An example is illustrated below.
$A = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (2, 1), (2, 3), (3, 4)\}$

We define $R^2 = R \circ R$ and for all $i > 2$, R^i is composition of R on R^{i-1}. i.e., $R^i = R^{i-1} \circ R$.

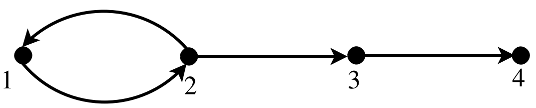


Fig. 1. Directed graph corresponding to R

Using the above definition iteratively, we get

$R^2 = \{(1, 1), (2, 2), (1, 3), (1, 4)\}$
$R^3 = \{(1, 2), (2, 1), (2, 3), (1, 4)\}$
$R^4 = \{(1, 1), (2, 2), (1, 3), (1, 4)\}$
$R^5 = \{(1, 2), (2, 1), (2, 3), (1, 4)\}$
$t(R)$ can be formulated as $t(R) =$ minimum i such that $R \cup \bigcup_{j=2}^{i} R^j$ is transitive.
Here $R_2 = R_4$, $R_3 = R_5$ and $t(R) = R \cup R^2 \cup R^3$.

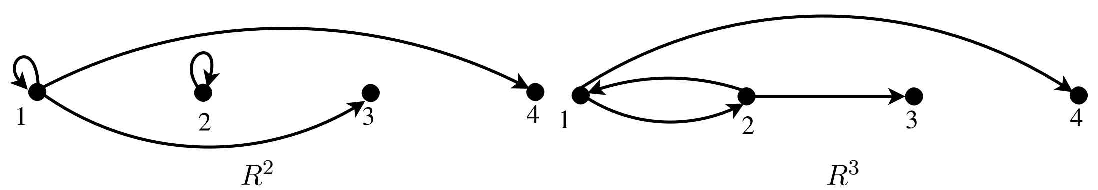


Fig. 2. Directed graph corresponding to R^2 and R^3

Note that for a finite set of size n there are at most 2^{n^2} distinct relations, therefore $t(R)$ in the worst case contain all 2^{n^2} relations which is $A \times A$. Otherwise, $t(R) \subset A \times A$.

Questions:

1. *Prove:* $(R_2 \cup R_3)R_4 = R_2R_4 \cup R_3R_4$
2. *Prove:* $(R_2 \cap R_3)R_4 \subset R_2R_4 \cap R_3R_4$

Amazing 1089 [1]

Select a non-palindrome 3 digit number xyz. Find their difference, say $abc = xyz - zyx$*. See the value of* $abc + cba$*!!!*

More on Transitive Closure

Given a relation R, a transitive closure of R is a minimal superset of R that is transitive. One approach to find $t(R)$ is to check whether $R^1 \cup R^2$ is transitive. If not, check $R^1 \cup R^2 \cup R^3$ is transitive and so on. It is certain that this approach terminates after some time. In fact, if the underlying directed graph is connected, then the longest path between any two nodes can not exceed n and due to which in the worst case $R^1 \cup R^2 \cup \ldots \cup R^n$ is transitive.
For the example given in Figure 1, $t(R) = R^1 \cup R^2 \cup R^3$. However, if the underlying directed graph is disconnected, then we observe the following. Consider the illustration given in Figure 3, R^1 has two components. Transitive closure is $t(R) = R^1 \cup R^2 \cup R^3$, which is **max(m,n)** where m and n are the number of nodes in the two components.

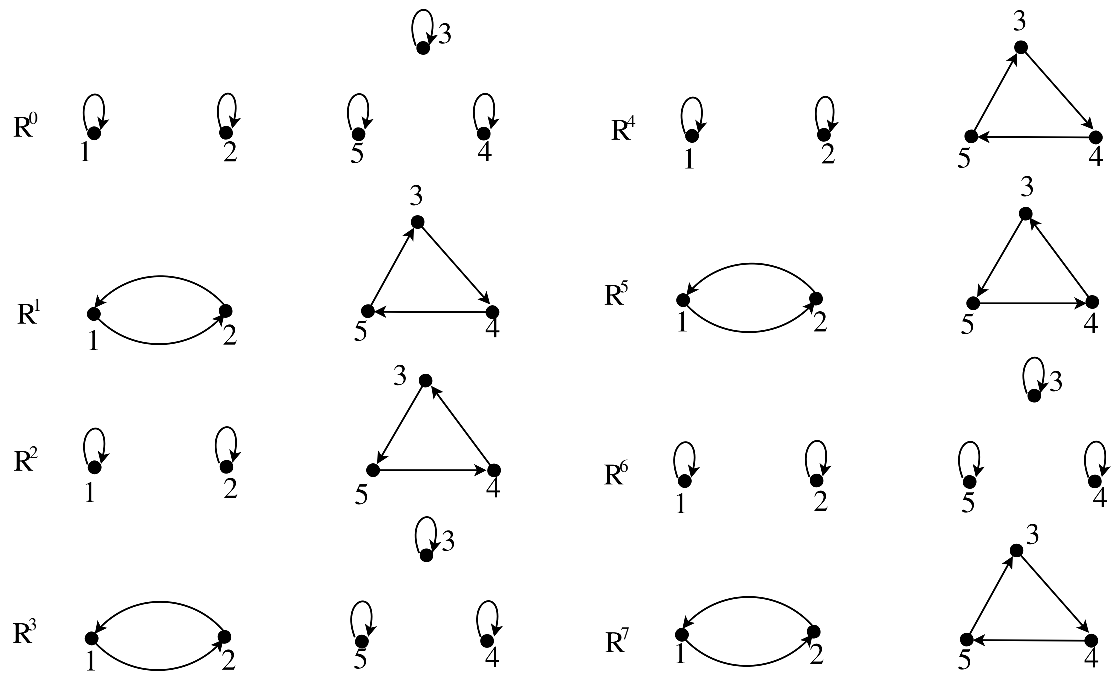


Fig. 3. R^1 to R^7 of $R = R^1$

Also, note that R^0 refers to equality relation or pure reflexive relation. Further, the smallest integers x, y such that $R^x = R^y$ in the graph is $x = 0$, and $y = 6$. In general, it is **LCM(m,n)**.

Remark: Let us consider the relation $R = \{(a, b) \mid b = a + 1, a, b \in \mathbb{R}\}$. Note that the relation is infinite and $t(R) = \bigcup_{i=1}^{\infty} R^i$. Transitive closure of an infinite set is infinite.

Theorem 7. *Let R be an infinite relation.* $t(R) = \bigcup_{i=1}^{\infty} R^i$.

Proof. By definition $t(R)$ is transitive. Observe that if $(a, b) \in R^m$ and $(b, c) \in R^n$, then $(a, c) \in R^{m+n}$. To show that $\bigcup_{i=1}^{\infty} R^i$ is transitive, we use the above observation. Since $t(R)$ is minimal, transitive, and it contains R, $t(R)$ will be a subset of any transitive superset. I.e., R is such that

$t(R) \supset R$ and $\bigcup_{i=1}^{\infty} R^i \supset R$, clearly $t(R) \subset \bigcup_{i=1}^{\infty} R^i$.

We prove $\bigcup_{i=1}^{\infty} R^i \subset t(R)$ by induction. *Base case:* $t(R) \supset R^1$. *Induction hypothesis:* Assume that for all $k >= 1$, $t(R) \supset \bigcup_{i=1}^{k} R^i$. *Anchor step:* Let $(a,b) \in R^{k+1} \implies$ there exists c such that $(a,c) \in R^k$ and $(c,b) \in R$. Notice that $(a,c) \in t(R)$ from the induction hypothesis and $(c,b) \in t(R)$ from the base case and by transitivity, it follows that $(a,b) \in t(R)$. Therefore, $t(R) \supset \bigcup_{i=1}^{k} R^i$ for all $k \geq 1$. It can be concluded that $t(R) = \bigcup_{i=1}^{\infty} R^i$ $\square$

Equivalence Class

We shall revisit equivalence relation in this section and introduce equivalence classes. We also explore paritition of a set and its connection to equivalence classes.

Consider the following equivalence relations defined on $A = \{1,2,3,4,5\}$.

$R^1 = \{(1,1),(2,2),(3,3),(4,4),(5,5)\}$
$R^2 = \{(1,1),(2,2),(3,3),(4,4),(5,5),(2,1),(1,2)\}$
$R^3 = \{(1,1),(2,2),(3,3),(4,4),(5,5),(1,3),(3,1),(2,4),(4,2)\}$
$R^4 = A \times A$

Definition: *Equivalence class* of $a \in A$ is defined as $[a]_R = \{x \mid (x,a) \in R\}$. If R is clear from the context, we drop the subscript R and shall denote the equivalence class of a as $[a]$

$[1]_{R^1} = \{1\}$ $[1]_{R^2} = \{1,2\}$ $[1]_{R^3} = \{1,3\}$
$[2]_{R^1} = \{2\}$ $[2]_{R^2} = \{1,2\}$ $[2]_{R^3} = \{2,4\}$
$[3]_{R^1} = \{3\}$ $[3]_{R^2} = \{3\}$ $[3]_{R^3} = \{1,3\}$
$[4]_{R^1} = \{4\}$ $[4]_{R^2} = \{4\}$ $[4]_{R^3} = \{2,4\}$
$[5]_{R^1} = \{5\}$ $[5]_{R^2} = \{5\}$ $[5]_{R^3} = \{5\}$

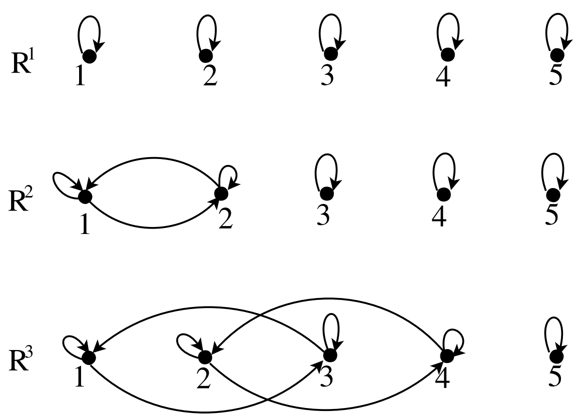


Fig. 4.

Properties: We consider equivalence relation R on set A.

1. $\bigcup_{\forall a \in A} [a] = A$
2. For every $a, b \in A$ such that $a \in [b]$, $a \neq b$, it follows that $[a] = [b]$.
3. $\sum_{\forall x \in A} |[x]| = |R|$.

Note: We can also interpret item 2 as for any two equivalence class $[a]$ and $[b]$, either $[a] = [b]$ or $[a] \cap [b] = \phi$.

Theorem 8. *If* $[a] \cap [b] \neq \phi$, *then* $[a] = [b]$

Proof. Given $[a] \cap [b] \neq \phi$. There exist $c \in [a] \cap [b]$. i.e., $c \in [a]$ and $c \in [b]$ and by definition $a \in [a]$, $b \in [b]$. We observe the following.

(1)	$(c, a), (c, b) \in R$	definition
(2)	$(a, c), (b, c) \in R$	(1) and symmetricity
(3)	$(a, b), (b, a) \in R$	(1), (2) and transitivity

Consider an arbitrary element $x \in [a]$. By definition $(x, a) \in R$. Since $(x, a) \in R$ and from (3) $(a, b) \in R$, by transitivity it follows that $(x, b) \in R$. This implies $x \in [b]$ and thus $[a] \subset [b]$. Similarly, consider an arbitrary element $y \in [b]$. By definition $(y, b) \in R$. Since $(y, b) \in R$ and from (3) $(b, a) \in R$, by transitivity it follows that $(y, a) \in R$. This implies $y \in [a]$ and thus $[b] \subset [a]$. Therefore, we conclude $[a] = [b]$. $\square$

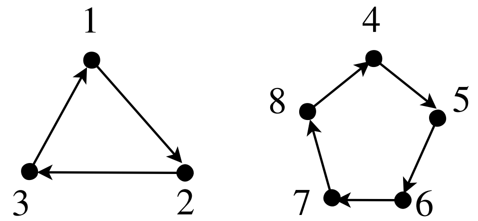
Theorem 9. $\bigcup_{x \in A} [x] = A$

Proof. Let $y \in \bigcup_{x \in A} [x]$. Clearly, $y \in [y]$ and since $[y] \in A$, $y \in A$. This implies $\bigcup_{x \in A} [x] \subset A$. Consider $y \in A$. $y \in [c]$ for some $c \in A$. Since $[c] \subset \bigcup_{x \in A} [x], y \in \bigcup_{x \in A} [x]$ and it follows that $A \subset \bigcup_{x \in A} [x]$. Therefore we conclude $\bigcup_{x \in A} [x] = A$ $\square$

Definition: *Clique* is a completely connected subgraph of a graph. Given an equivalence relation and its directed graph representation, we observe that the vertices corresponding to equivalence classes induces a clique. See Figure 4.

Questions:

1. *Prove using counting technique and PHP (Pigeon hole principle) or use mathematical induction:* Let R be a finite relation and G be the directed graph corresponding to R. $t(R) = \bigcup_{i=1}^{n} R^i$, where $R^i = R^{i-1}R$, $1 \leq i \leq n$ and n is the length of longest path in G.
2. Find R^1 to R^{16} for the figure shown below.



Counting Equivalence Relations

MATH WONDERS [1]

Find the digit corresponding to each letter

$$\begin{array}{rcccccc} & & S & E & N & D \\ + & & M & O & R & E \\ = & M & O & N & E & Y \end{array}$$

In this section, we shall ask; How many equivalence relations are possible on a given set A?
Note: Number of equivalence relations on A = Number of partitions of A, as partition of A corresponds to equivalence classes of the equivalence relation. This observation is due to the structural theorem presented in the previous section. We now count the number of partitions on a set of size n.

Let B_n be the number of partitions (equivalence relations) possible in a set A of n elements. We now present a recurrence relation which will count the number B_n. Consider a partition of n elements, say for example, $P = \{\{1,2\},\{3,4,5\},\{6,\ldots,n\}\}$. In general $P = \{A_1, A_2, \ldots, A_p\}$ and $A_i \subset \{1,\ldots,n\}$ and each A_i is distinct. In P, consider A_i containing the element n and assume that $k = |A_i \setminus \{n\}|$. These k elements of A_i can be any subset in $\{1,\ldots,n-1\}$. Therefore, the number of such possible sets for A_i is $\binom{n-1}{k}, k \in \{0,\ldots,n-1\}$. Now to establish a recursive relation we focus on the remaining $n-k-1$ elements which are distributed among $A_1,\ldots,A_{i-1},A_{i+1},\ldots,A_n$. Interestingly, there are B_{n-k-1} partitions among $n-k-1$ elements. Thus, we get $B_n = \sum\limits_{k=0}^{n-1} \binom{n-1}{k} B_{n-k-1}$.

Note: $B_1 = 1$, $B_2 = 2$, $B_3 = 5$.

The number B_n is known as Bell's number in the literature.

Rank of an equivalence relation is the number of distinct equivalence classes.

For example, consider the relations R^1 to R^4 given below, their ranks are;
$R^1 = \{(1,1),(2,2),(3,3),(4,4),(5,5)\}$
$R^2 = \{(1,1),(2,2),(3,3),(4,4),(5,5),(2,1),(1,2)\}$
$R^3 = \{(1,1),(2,2),(3,3),(4,4),(5,5),(1,3),(3,1),(2,4),(4,2)\}$
$R^4 = A \times A$

Relation	Rank
R^1	5
R^2	4
R^3	3
R^4	1

Revisit: Partial Order

In this section, we shall revisit partial order and discuss some more properties in detail. Recall, a relation R is a partial order if it satisfies Reflexivity, Antisymmetricity, and Transitivity.
Further, R is a *Quasi order* if R is transitive and irreflexive.
Trichotomy Property: For all elements $a, b \in A$, exactly one of the following holds:
(i). $(a,b) \in R$. (ii). $(b,a) \in R$. (iii). $(a = b)$.
Remark: A relation is a *Total order* if it is a partial order and satisfies the trichotomy property.

Question 1 Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 3), (1, 3)\}$. Find a minimum augmentation relation R' such that $R \cup R'$ is a total order.
Ans: $R' = \{(3, 4), (2, 4)\}$
Question 2 Let $A = \{\{1\}, \phi\}$, $P(A) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$ and $R = \{(A, B) \mid A \subseteq B, A, B \in P(A)\}$. Find $R' \subseteq R$ which is a total order.
Ans: Such a relation does not exist as $(\{1\}, \{2\}), (\{2\}, \{1\}) \notin R$. Also, not all partial orders can be converted into a total order by augmenting minimum pairs.

Example 1: Consider the relation R: a divides b on the set $A = \{1, 2, \ldots, 9\}$.
R is a partial order and not a total order as the elements $(3, 5), (6, 9) \notin R$.
Example 2: $(Z, \geq)$ is a partial order and a total order.
Example 3: $(Z, >)$ is a not a partial order as reflexivity fails.
Example 4: $(Z^+, |)$ ($|$ denotes 'divides') is a partial order but not a total order.
Example 5: Let $A = \{1, 2, \ldots, 6\}$ and $R = \{(a, b) \mid a \text{ divides } b\}$ is a partial order but not a total order as $(3, 8), (8, 3) \notin R$
Example 6: $R = \{(A, B) \mid A \subseteq B\}$ on the power set of $\{a, b, c\}$ is a partial order but not a total order as $(\{a, c\}, \{b, c\}), (\{b, c\}, \{a, c\}) \notin R$

Hasse Diagram

Hasse diagram is a graphical representation of partial order relations. In this diagram, self loops (due to reflexivity) and transitivity arcs (edges) will not be explicitly mentioned. Arcs due to antisymmetric pairs will alone be mentioned in Hasse diagram.

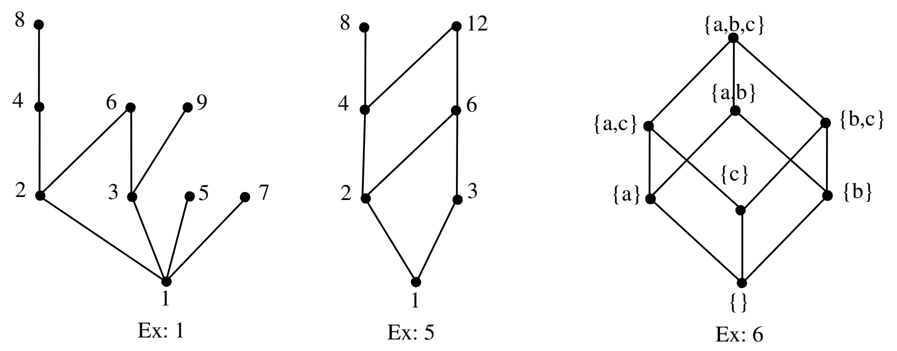


Fig. 5. Hasse diagram of Examples 1,5 and 6

Definition:
Let $(A, \preceq)$ be a poset and $B \subseteq A$. ($\preceq$ means some relation)
1. An element $b \in B$ is the *greatest element* of B if for every $b' \in B, b' \preceq b$.
2. An element $b \in B$ is the *least element* of B if for every $b' \in B, b \preceq b'$.
For Example 6,

Set B	greatest element	least element
$\{\{a\}, \{a, c\}\}$	$\{a, c\}$	$\{a\}$
$P(\{a, b, c\})$	$\{a, b, c\}$	$\{\}$
$\{\{a\}, \{b\}, \{c\}, \{\}\}$	NIL	$\{\}$
$\{\{a\}, \{b\}, \{c\}, \{a, b\}\}$	NIL	NIL
$\{\{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$	$\{a, b, c\}$	NIL

Theorem 10. *The greatest and least elements in a poset are unique.*

Proof. Let $R = A \times A$ be a poset. On the contrary, assume that there exist two greatest elements $g_1, g_2 \in A$ such that $g_1 \neq g_2$. Since g_1 is a greatest element, $g_1 \preceq g, \forall g \in A$ and it follows that $g_1 \preceq g_2$. Similarly, since g_2 is a greatest element, $g_2 \preceq g_1$. It follows that $(g_1, g_2) \in R$ and $(g_2, g_1) \in R$. This contradicts the fact that R is antisymmetric. Similar argument can be made for the fact that there exist a unique least element.

□

Definition: $(A, \preceq)$ is a *well order* if $(A, \preceq)$ is a total order and for all $A' \subseteq A, A' \neq \phi, A'$ has a least element.

Note: Every finite totally ordered set is well ordered.

Example:

Relation	Total order	Well-order
$(\mathbb{R}, \leq)$	✓	×
$(\mathbb{N}, \leq)$	✓	✓
$(\mathbb{I}, \leq)$	✓	×

Remark: If we consider a subset $A' \subseteq A$ such that $A' = \mathbb{I}$ or $A' = \mathbb{I}^-$, then $(A', \leq)$ does not have a least element. Also, there does not exist a least element for $(\mathbb{R}^+, \leq)$.

Note: Mathematical induction can be applied only to well ordered sets as there is a least element for every non-empty subset which implicitly gives an ordering among elements. This shows that, proving claims on well-ordered sets using mathematical induction proof technique is appropriate.

Remark: Consider a set A such that $|A| = n$. The number of total orders possible on $(A, \leq)$ is the number of paths possible in the hasse diagram. Note that this is equal to the number of permutations which is $n!$. i.e., # total orders = # paths = # permutations = $n!$

Lexicographic and Standard orderings

In this section, we introduce an ordering among elements of a set.
We first introduce lexicographic ordering:

Given $\sum$: finite alphabet, for example $\sum = \{a, b\}$

If $x, y \in \sum^*$, then $x \leq y$ in the lexicographic ordering
(x precedes y) of $\sum^*$ if
(i) x is a prefix of y (or)
(ii) $x = zu$ and $y = zv$ where $z \in \sum^*$ is the longest prefix common to x and y and u precedes v in the lexicographic ordering.

$\sum = \{a\}$ $\sum^* = \{a, aa, aaa, \ldots\}$ is a partial, total and well ordered set.
$\sum = \{a, b\}$ $\sum^* = \{a, aa, aaa, \ldots, aa\ldots ab, aa\ldots ba, \ldots, b, ba, baa, \ldots\}$ is a partial, and total ordered set but not a well ordered. For example, there is no least element for the set $B = \{b, ab, aab, aaaab, aa\ldots ab\}$.

Standard ordering
Notation: $\sum$: alphabet and $\sum^*$ is the set of all strings over $\sum$. $||x||$ is the length of string $x \in \sum^*$
Definition:
$x \leq y$ if
(i) $||x|| < ||y||$ or
(ii) $||x|| = ||y||$ and x precedes y in the lexicographic ordering of $\sum^*$

Note that standard ordering is a poset, total and well ordered set as we can order $\sum^*$ as $(a, b,\ aa, ab, ba, bb,\ aaa, aab, aba, abb, baa, bab, bba, bbb, \ldots)$

Special Elements in a poset

Definition: Let $(A, \preceq)$ be a poset, $B \subseteq A$

1. An element $b \in B$ is a maximal element of B if $b \in B$ and there does not exist $b' \in B$ such that $b \neq b'$ and $b \preceq b'$. Similarly, minimal elements of B can be defined.
2. An element $b \in A$ is upper bound for B if for every element $b' \in B$, $b' \preceq b$. Similarly, lower bound of B can be defined.
3. An element $b \in A$ is a least upper bound (lub) for B if b is an upper bound and for every upper bound b' of B, $b \preceq b'$.
4. An element $b \in A$ is a greatest lower bound (glb) for B if b is a lower bound and for every lower bound b' of B, $b' \preceq b$.

For Figure 5 associated with Example 5

Set B	minimal elements	maximal elements	Lower bound	Upper bound
A	$\{1\}$	$\{8, 12\}$	$\{1\}$	NIL
$\{2, 3, 4\}$	$\{2, 3\}$	$\{3, 4\}$	$\{1\}$	$\{12\}$

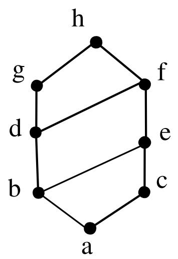


Fig. 6. Hasse diagram 2

For Figure 6

Set B	minimal elements	maximal elements	Lower bound	Upper bound	LUB	GLB
A	$\{a\}$	$\{h\}$	$\{a\}$	$\{h\}$	$\{h\}$	$\{a\}$
$\{b, c, d, e\}$	$\{b, c\}$	$\{d, e\}$	$\{a\}$	$\{f, h\}$	$\{f\}$	$\{a\}$
$\{a, b, c\}$	$\{a\}$	$\{b, c\}$	$\{a\}$	$\{e, f, h\}$	$\{e\}$	$\{a\}$
$\{d, e\}$	$\{d, e\}$	$\{d, e\}$	$\{b, a\}$	$\{f, h\}$	$\{f\}$	$\{b\}$

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Reading

1. Bell numbers.
2. H. Hasse (1898-1979) and Hasse diagrams

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