Indian Institute of Information Technology
Design and Manufacturing, Kancheepuram
Chennai 600 127, India
An Autonomous Institute under MHRD, Govt of India
An Institute of National Importance
www.iiitdm.ac.in
COM205T Discrete Structures for Computing-Lecture Notes

Instructor
N.Sadagopan
Scribe: P.Renjith

Graph Theory

Motivation

1. Given a road network, find a minimum number of policemen so that every road is monitored. (policemen are placed at junctions, and in case of accidents at road r it will be addressed by the policeman standing at junction on any one end of r)
2. Design a router network so that it can handle all 2-node failures. (Fault tolerance level is 2)
3. Consider the interaction between processor and resources. Design an inter-process resource network so that there are no cyclic interactions (deadlock)

Graphs

- An abstract representation of a system under study (system: computer network, road network, router network)
- used as a model to understand the system better
- it is a binary relation
- graphs consist of vertices (nodes) and edges (links/arcs)

Basic Definitions and Simple Counting

$V(G) = \{v_1, v_2, \ldots, v_n\}$, the set of vertices.
$E(G)$, the set of edges
Ex: $V = \{1, 2, 3, 4\}$ $E = \{(1, 1), (1, 3), (3, 4)\}$

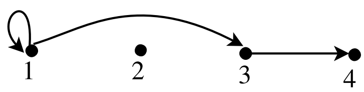


Fig. 1. A Graph

1. How many different graphs on n-vertices are possible? Note that graphs are precisely relations or represents a relation in graphical form.
   Ans. The number of n-vertex graphs = number of binary relations possible on a set of size $n = 2^{n^2}$
   Definition *Simple graphs* are graphs with no self loops and no multiple edges.
   Ex: $V = \{1, 2, 3, 4\}$ $E = \{(1, 3), (3, 4)\}$

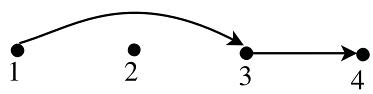


Fig. 2. A Simple Graph

2. How many directed simple graphs are there on n-vertices?
   Ans. The number of such graphs are equivalent to the number of irreflexive relations on a set of size $n = 2^{n^2-n}$
   Largely, we work with simple undirected graphs.
   Ex: $V = \{1,2,3,4\}$ $E = \{\{1,3\},\{2,4\}\}$

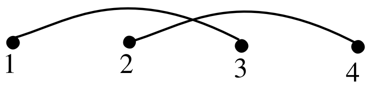


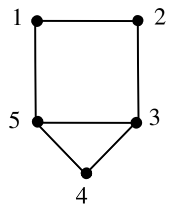
Fig. 3. An Undirected Graph

   Note: An undirected graph is a special case of directed graphs

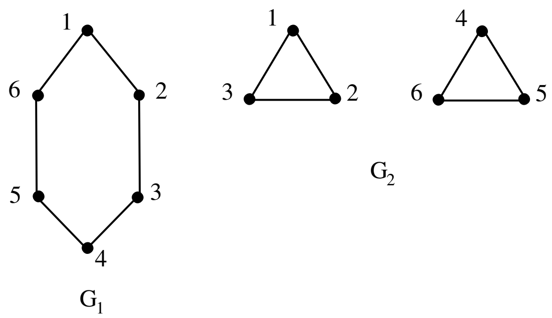
3. How many different undirected simple graphs are there on n-vertices?
   Ans. The number of such graphs are equivalent to the number of irreflexive and symmetric relations on a set of size $n = 2^{\binom{n}{2}}$

Undirected simple graphs and some more definitions

Neighborhood of a vertex v is $N_G(v) = \{u \mid \{u,v\} \in E(G)\}$.
Eg: $N_G(3) = \{2,4,5\}$, $N_G(4) = \{3,5\}$



Degree of a vertex $d_G(v)$ is the number of edges incident on v. $d_G(v) = |N_G(v)|$
The degree sequence of G is the sequence representing the degrees of $V(G)$.
Example: Consider the above graph G, with $d_G(1) = 2, d_G(2) = 2, d_G(3) = 3, d_G(4) = 2, d_G(5) = 3$. The degree sequence is (3 3 2 2 2).
For the degree sequence $(2,2,2,2,2,2)$, the two associated graphs are given below;



Note: Given a degree sequence, one can construct the associated graph in more than one way. The above two graphs G_1 and G_2 are one such example. We shall see the properties of these two graphs in detail. We next introduce *connectedness*.
Definition: *Connectedness* A graph G is connected if for every $u, v \in V(G)$ there exist a path

between u and v
In the above figure, G_1 is connected whereas G_2 is disconnected with two components.
Connected component is a maximal connected subgraph of a graph. Note that maximal is with respect to a property, and here it is connectedness.
We see a natural extension of the previous question as follows. Given a degree sequence $(d_1, d_2, \ldots, d_n)$ can you construct the associated connected graph uniquely?
Ans. No. Consider the degree sequence $(3, 2, 2, 2, 1)$, there are two associated graphs as shown below;

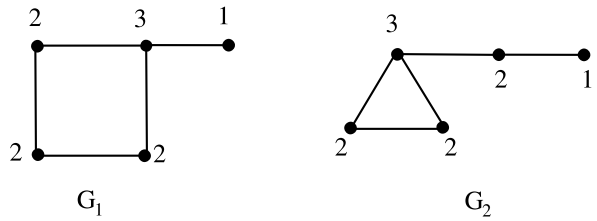


Fig. 4. Two non isomorphic representation of $(3, 2, 2, 2, 1)$

Question: Given two graphs G_1 and G_2, how can you determine that they are different?
Isomorphism
Two graphs G and H are isomorphic if and only if there exist a bijection from $V(G)$ to $V(H)$. $f : V(G) \to V(H)$ such that $\{u, v\} \in E(G)$ if and only if $\{f(u), f(v)\} \in E(H)$
In Figure 4, there does not exist such a bijection from $V(G) \to V(H)$.

Questions

1. Given $(d_1, d_2, \ldots, d_n)$, how will you construct G.
2. Given G and H, how do you check whether they are isomorphic or not? Also, produce the associated bijection, if it exists.
3. In a group of n people, how many handshakes are possible?- Ans: $\binom{n}{2}$
4. Are there graphs with the degree sequence
   (i) (3,3,3,3,3) (ii) (3,3,3,4,4,2) (iii) (1,2,2,2)

5. Does there exist a graphic degree sequence $(5, 4, 3, 2, 1)$- No.

6. Verify: In any group of n-people, there exist at least two with equal number of friends.

Some Structural observations on Graphs

Claim 1: $\sum_{i=1}^{n} d_i = \text{Even}$
Claim 2: $\sum_{i=1}^{n} d_i = 2|E|$
Induction on $m = |E(G)|$

Proof. Base case: $m = 1$, $\sum_{i=1}^{n} d_i = 2$ is even
Induction Hypothesis: Assume that the claim is true for graphs with less than m edges, $m \geq 2$.

Induction Step: Consider the graph with m-edges; $m \geq 2$ and let $\{u, v\} \in E(G)$.
Consider the graph $G - \{u, v\}$. $V(G - \{u, v\}) = V(G)$ and $E(G - \{u, v\}) = E(G) \backslash \{u, v\}$.
Since $|E(G - \{u, v\})| = m - 1$, we can bring in the induction hypothesis.
By the Induction hypothesis, in $G - \{u, v\}$, $\sum_{i=1}^{n} d_i = 2m' = 2(m - 1)$.
Add $\{u, v\}$ to $G - \{u, v\}$. Consider the degree sequence $d_1 + d_2 + \ldots + d_u + d_v + \ldots + d_n$.
$d_u = d_{u'} + 1$, $d_v = d_{v'} + 1$, u' and v' are the vertices corresponding to u and v in $G - \{u, v\}$. By introducing the edge $\{u, v\}$, the degree of u' (v') increases by one.
$d_1 + d_2 + \ldots + d_{u'} + 1 + d_{v'} + 1 + \ldots + d_n$.
$= d_1 + d_2 + \ldots + d_{u'} + d_{v'} + \ldots + d_n + 2 = 2(m - 1) + 2 = 2m$. □

We shall next present another inductive proof of the above claim; induction on $|V(G)|$

Proof. Base case: $n = 1$, $\sum_{i=1}^{n} d_i = 0$ is even
Induction hypothesis: Assume the claim is true for graphs with $(n - 1)$-vertices, $n \geq 2$.
Induction Step: Let G be a graph on n-vertices $n \geq 2$
$V(G) = \{u_1, u_2, u_3, \ldots, u_n\}$
Let u_i be a vertex with minimum degree ($\delta(G)$)
Consider the graph $G - u_i$. $|V(G - \{u_i\})| = n - 1$ and $|E(G - \{u_i\})| = |E(G)| - d_{u_i}$
By the induction hypothesis, the claim is true in $G - u_i$
i.e., $d_{u_1} + d_{u_2} + \ldots + d_{u_{i-1}} + d_{u_{i+1}} + \ldots + d_{u_n} = 2m'$
$d_{u_1} + d_{u_2} + \ldots + d_{u_{i-1}} + d_{u_{i+1}} + \ldots + d_{u_n} = 2(m - \delta(G))$, due to the removal $\delta(G)$ edges incident on the minimum degree vertex.
By introducing u_i in $G - u_i$, we can see that every vertex $v \in N_G(u_i)$, $d_G(v)$ is increased by one.
Now, $d_{u_1} + d_{u_2} + \ldots + d_{v_1} + d_{v_2} + \ldots + d_{v_{\delta(G)}} + \ldots + d_{u_n} + d_{u_i}$ where $\{v_1, v_2, \ldots, v_{\delta(G)}\} = N_G(u_i)$.
Also, note that d_{u_i} is added to the sum as u_i is added.
$d_{u_1} + d_{u_2} + \ldots + d_{v'_1} + 1 + d_{v'_2} + 1 + \ldots + d_{v'_{\delta(G)}} + 1 + \ldots + d_{u_n} + d_{u_i}$
$= d_{u_1} + d_{u_2} + \ldots + d_{v'_1} + d_{v'_2} + \ldots + d_{\delta'(G)} + \ldots + d_{u_n} + 1 + 1 \ldots + 1 + d_{u_i}$ [no.of 1's$=\delta(G)$ and $d_{u_i} = \delta(G)$]
By I.H. $\implies 2(m - \delta(G)) + \delta(G) + \delta(G) = 2m$. This completes the proof. □

Based on the above claim, here is an interesting corollary; Let $V_{\text{ODD}} = \{u \mid d_G(u) : 2k+1, k \geq 0\}$ and $V_{\text{EVEN}} = \{u \mid d_G(u) : 2k, k \geq 0\}$
$\sum_{u \in V_{\text{ODD}}} + \sum_{u \in V_{\text{EVEN}}} = 2m$
implies $\sum_{u \in V_{\text{ODD}}}$ is even

Claim 3: The number of odd degree vertices in any graph is always even.
Corollary of claim 2.

Some Special Graphs

Path graph P_n on n vertices.
$|V(P_n)| = n, |E(P_n)| = n - 1$
Cycle graph C_n on n vertices
$|V(C_n)| = n, |E(C_n)| = n$
Regular graph: G is k-regular if for every $v \in V(G), d_G(v) = k$
Ex: C_n is 2-regular.

The number of edges in a k regular graph on n vertices $= \frac{nk}{2}$
Are there 3-regular graph on 7 vertices - No
Complete graph K_n
The number of edges in a complete graph on n vertices $= \frac{n(n-1)}{2}$
Tree: is a connected acyclic graph.
Bipartite graph:
G is a bipartite graph if there exist a partition V_1, V_2 of $V(G)$ such that $V(G) = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. For every edge $e = \{u, v\} \in E(G)$, $u \in V_1$ and $v \in V_2$

Example bipartite graphs include P_n, C_{2n}, and all trees. K_n and C_{2n+1} are not bipartite.

Question: Does there exist a characterization for a graph to be bipartite ?

Claim 4: G is bipartite if and only if G is odd-cycle free.

Proof. Necessity: Since G is bipartite, there exist V_1, V_2 such that $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, and for every edge $e = \{u, v\} \in E(G), u \in V_1$ and $v \in V_2$
Consider $u \in V_1$ and a cycle C starting and ending at u.
Since any cycle C that starts and ends at u visits vertices of V_1 and V_2 alternately, the length of C is clearly even.
Note: for any $\{x, y\} \subseteq V_1$, distance between x and y is $2k+1, k \geq 1$. Therefore, the length of cycle C is $2k+1+1$, which is even.
Sufficiency: G is odd cycle free. To show that G is 2-partite, we need to exhibit a bipartition.

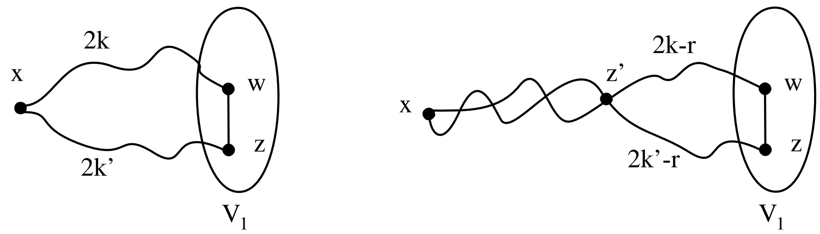


Fig. 5. An illustration for the proof of Claim 3

Let x be a vertex in G.
Consider $V_1 = \{u \mid distance(x, u) = even\}$
$V_2 = \{u \mid distance(x, u) = odd\}$
Claim 1: $V = V_1 \cup V_2$
Claim 2: $V_1 \cap V_2 = \emptyset$
Claim 3: For each $w, z \in V_1, \{w, z\} \notin E(G)$.
Proof: Suppose $\{w, z\} \in E(G)$, then $|P_{xw}| = 2k$ and $|P_{zx}| = 2k'$ for some $k', k \geq 1$ as shown in figure. $(P_{xw}, \{w, z\}, P_{zx})$ is a cycle of length $2k + 2k' - 1 = 2l + 1$ for some $l \geq 1$. Therefore, G contains an odd cycle and this is a contradiction to the premise. Our assumption that there exist $\{w, z\} \in E(G)$ is wrong and $\{w, z\} \notin E(G)$. Therefore, V_1 is an independent set. Similar arguments hold true if V_2 is an independent set.
Suppose, $V(P_{xw}) \cap V(P_{zx}) \neq \emptyset$, then identify the last vertex z' such that $z' \in P_{xw}$ and $z' \in P_{zx}$. Let length of $|P_{xz'}| = r$. Note that $P_{xz'} \subseteq P_{xw}$ and $P_{xz'} \subseteq P_{xz}$ are of length r. Suppose

$|P_{xz'}| < r$, then it contradicts the fact that P_{xz} is a shortest path. It follows that $|P_{zz'}| = 2k' - r$ and $|P_{z'w}| = 2k - r$. Length of cycle $(P_{z'w}, \{w, z\}, P_{zz'})$ is $2k - r + 2k' - r - 1 = 2l + 1$ for some $l \geq 1$. Therefore, there exist an odd cycle which is a contradiction. Therefore, the assumption is wrong, and the claim follows. $\square$

Questions:
A graph is 3-partite if and only if $---$*?*
What about a necessary and sufficient condition for a graph to be k-partite ?

Graph Coloring

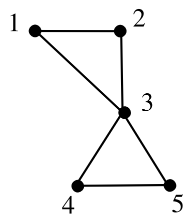
- An assignment of colors to vertices of a graph
- *Proper coloring* - adjacent vertices receive different colors.
- G is k-colorable if and only if there exist $c : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that for all $e = \{u, v\}, c(u) \neq c(v)$
- *Chromatic Number* $\chi(G)$ is the minimum number of colors required to properly color a graph.

$\chi(K_n) = n$
$\chi(P_n) = 2$
$\chi(Tree) = 2$
$\chi(C_{2n}) = 2$
$\chi(C_{2n+1}) = 3$
$\chi(\textit{bipartite graph}) = 2$
G is bipartite if and only if G is 2-colorable.
The following statements are equivalent.

- G is bipartite
- G is 2-colorable
- G is odd-cycle free

Problem Session

1. Consider the Adjacency matrix representation of a simple graph. What does entries A^2 denote? What about A^3, A^k?



G

$$A = \begin{bmatrix} 0&1&1&0&0 \\ 1&0&1&0&0 \\ 1&1&0&1&1 \\ 0&0&1&0&1 \\ 0&0&1&1&0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 2&1&1&1&1 \\ 1&2&1&1&1 \\ 1&1&4&1&1 \\ 1&1&1&2&1 \\ 1&1&1&1&2 \end{bmatrix} \quad A^3 = \begin{bmatrix} 2&3&5&2&2 \\ 3&2&5&2&2 \\ 5&5&4&5&5 \\ 2&2&5&2&3 \\ 2&2&5&3&2 \end{bmatrix}$$

Consider the adjacency matrix representation of the above graph.
Note that $A^k[i, j]$ represents the number of paths from i to j of length k.
For example, there exist three paths of length 3 from 1 to 2.
I.e., $A^3[1, 2] = 3$ and the paths are $(1-2-1-2), (1-3-1-2), (1-2-3-2)$.

2. If G is a simple graph with 15 edges and $\overline{G}$ has 13 edges, how many vertices does G have?
   Total number of edges possible $= \binom{n}{2} = 15 + 13 = 28. \implies n = 8$
3. The maximum number of edges in a simple graph with 10 vertices and 4 components is
   Ans: 21. Three components with K_1's and one component with K_7. Number of edges $= \binom{7}{2} = 21$
4. For which value of k an acyclic graph G with 17 vertices, 8 edges and k components exist?
   Let $e_i, n_i, 1 \leq i \leq k$ represents the number of edges, and number of vertices in component i, respectively.
   Given $\sum_{i=1}^{k} e_i = 8$
   Since the graph is acyclic it follows that $e_i = n_i - 1, 1 \leq i \leq k$.
   Therefore, $\sum_{i=1}^{k} (n_i - 1) = 8 \implies \sum_{i=1}^{k} n_i - k = 8$
   Since $\sum_{i=1}^{k} n_i = 17, k = 17 - 8 = 9$
5. Find the minimum number of vertices in a simple graph with 13 edges and having 5 vertices of degree 4 and the rest having degree less than 3.
   Ans: 8. $\sum_{i=1}^{n} d_i = 13 \times 2 = 26$. Number of 4 degree vertices $= 5$. This contributes 20 to the degree. For the rest, we can use vertices of degree less than 3. i.e., there should exist three 2-degree vertices. Therefore, the total number of vertices is at least $5 + 3 = 8$.
6. Does there exist a simple graph with degree sequence $(7, 7, 6, 6, 5, 5, 4, 4, 3, 3, 2, 2, 1, 1)$?
   Ans: Yes.

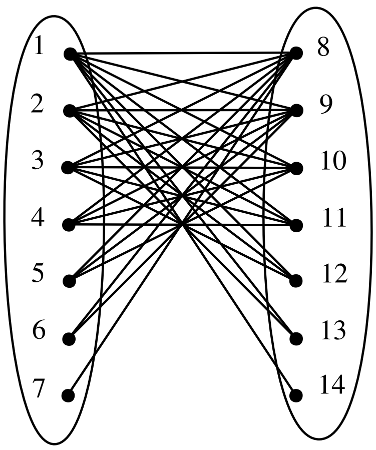


Fig. 6. Bipartite graph with the given degree sequence

7. A simple graph G has degree sequence $(3, 3, 2, 1, 1)$. What is the degree sequence of G^c ?
   Ans: $(1, 1, 2, 3, 3)$
8. If the degree sequence of a simple graph G is $(4, 3, 3, 2, 2)$, what is the degree sequence of G^c ?
   Ans: $(0, 1, 1, 2, 2)$
9. The maximum number of edges in a bipartite graph with n vertices is ...
   Ans: If n is even, then complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$ has maximum edges, which equals $\frac{n^2}{4}$.
   If n is odd, then $K_{\frac{n-1}{2},\frac{n+1}{2}}$ has maximum edges which is $\frac{n^2-1}{4}$.
10. What is the chromatic number of Peterson graph?
    Ans: 3. In the given below graph, 1, 2, 3 represents three different colors.

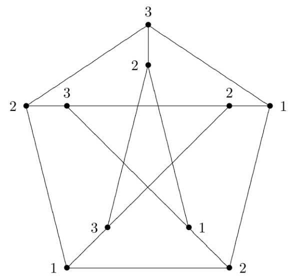


Fig. 7. Coloring of a Peterson graph

11. `Claim:` If G contains K_4 as a subgraph, then G is at least 4-colorable. Is this true? Are there 4-colorable graphs without K_4. Are there triangle free graphs with chromatic number 4? Above claim is true. In the 4-colorable graph shown below, G_1 has no K_3 and G_2 has no K_4 as a subgraph.

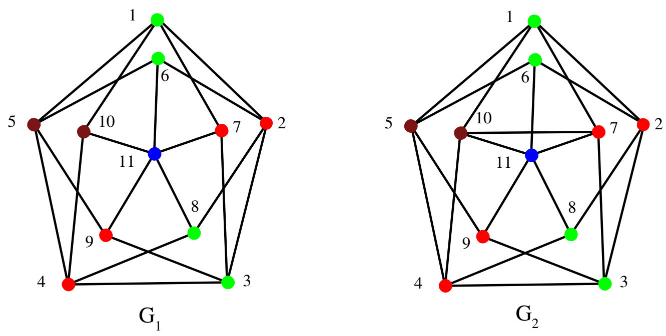


Fig. 8. 4-colorable Graphs

Acknowledgements: Lecture contents presented in this module and subsequent modules are based on the text books mentioned at the reference and most importantly, lectures by discrete mathematics exponents affiliated to IIT Madras; Prof P.Sreenivasa Kumar, Prof Kamala Krithivasan, Prof N.S.Narayanaswamy, Prof S.A.Choudum, Prof Arindama Singh, and Prof R.Rama. Author sincerely acknowledges all of them. Special thanks to Teaching Assistants Mr.Renjith.P and Ms.Dhanalakshmi.S for their sincere and dedicated effort and making this scribe possible. This lecture scribe is based on the course 'Discrete Structures for Computing' offered to B.Tech COE 2014 batch during Aug-Nov 2015. The author greatly benefited by the class room interactions and wishes to appreciate the following students: Mr.Vignesh Sairaj, Ms.Kritika Prakash, and Ms.Lalitha. Finally, author expresses sincere gratitude to Mr.Anoop Krishnan for thorough proof reading and valuable suggestions for improving the presentation of this article. His valuable comments have resulted in a better article.

References:

1. K.H.Rosen, Discrete Mathematics and its Applications, McGraw Hill, 6th Edition, 2007
2. D.F.Stanat and D.F.McAllister, Discrete Mathematics in Computer Science, Prentice Hall, 1977.

3. C.L.Liu, Elements of Discrete Mathematics, Tata McGraw Hill, 1995

Reading assignment
[1]. Shakuntala Devi: "Puzzles to Puzzle you"
"More Puzzles",
"Figuring: The Joy Of Numbers"
[2]. George J. Summers: "The Great Book of Mind Teasers and Mind Puzzlers"

A Matter of Time [1]

Fifty minutes ago if it was four times as many minutes past three o'clock. How many minutes is it to six o'clock