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Numerical Analysis Formula Sheet

Based on the lectures by
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8th Semester
Numerical Analysis
Formula Sheet

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Contents

1 Interpolation

1.1 Weierstrass' Approximation Theorem

Theorem 1. *Let f be a continuous function defined on $[a, b]$. Then, given any $\varepsilon > 0$, $\exists$ a polynomial $p(x)$, such that*

$$|f(x) - p(x)| < \varepsilon$$

$\forall\ x \in [a, b]$.

1.2 Newton-Gregory Forward Difference Formula

Consider a set of equally-spaced finite number of data points:

x_0	y_0
x_1	y_1
x_2	y_2
$\vdots$	$\vdots$
x_n	y_n

where $x_i - x_{i-1} = h\ \forall\ i = 1, 2, \ldots, n$
The differences $y_1 - y_0, y_2 - y_1 \ldots, y_n - y_{n-1}$, when denoted by $\Delta y_1, \Delta y_2, \ldots, \Delta y_n$ respectively are called the *first forward differences* and Δ is called the **forward difference operator**.

$$\Delta y_i^r = y_{i+1}^{r-1} - y_i^{r-1} \tag{1}$$

For example, $\Delta^1 y_0 = y_1 - y_0$, $\Delta^2 y_0 = \Delta^1 y_1 - \Delta^1 y_0$, $\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$ and so on.

x-value	y-value	1st diff.	2nd diff.	3rd diff.	4th diff
x_0	y_0				
		Δy_0			
$x_0 + h$	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
$x_0 + 2h$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
$x_0 + 3h$	y_3		$\Delta^2 y_2$		
		Δy_3			
$x_0 + 4h$	y_4				

Let

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1) \ldots (x - x_{n-1})$$

be a polynomial that satisfies the $n + 1$ tabulated data. Then, the expression becomes:

$$y(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!\, h^2}(x - x_0)(x - x_1) + \cdots + \frac{\Delta^n y_0}{n!\, h^n}(x - x_0)(x - x_1) \cdots (x - x_{n-1}) \tag{2}$$

Now, using the substitution $\dfrac{x - x_0}{h} = u$, the equation 2 becomes:

$$\boxed{y(u) = y_0 + \Delta y_0 u + \frac{\Delta^2 y_0}{2!} u(u - 1) + \cdots + \frac{\Delta^n y_0}{n!} u(u - 1)(u - 2) \cdots (u - (n - 1))} \tag{3}$$

1.3 Newton-Gregory Backward Difference Formula

The differences $y_1 - y_0, y_2 - y_1 \ldots, y_n - y_{n-1}$, when denoted by $\nabla y_1, \nabla y_2, \ldots, \nabla y_n$ respectively are called the *first backward differences* and ∇ is called the **backward difference operator**.

$$\nabla y_i^r = y_{i+1}^{r-1} - y_i^{r-1} \tag{4}$$

For example, $\nabla^1 y_0 = y_1 - y_0$, $\nabla^2 y_0 = \nabla^1 y_1 - \nabla^1 y_0$, $\nabla^3 y_0 = \nabla^2 y_1 - \nabla^2 y_0$ and so on.

x-value	y-value	1st diff.	2nd diff.	3rd diff.	4th diff
x_0	y_0				
		∇y_0			
x_0+h	y_1		$\nabla^2 y_0$		
		∇y_1		$\nabla^3 y_0$	
x_0+2h	y_2		$\nabla^2 y_1$		$\nabla^4 y_0$
		∇y_2		$\nabla^3 y_1$	
x_0+3h	y_3		$\nabla^2 y_2$		
		∇y_3			
x_0+4h	y_4				

Let

$$y(x) = a_0 + a_1(x-x_n) + a_2(x-x_n)(x-x_{n-1}) + \cdots + a_n(x-x_n)(x-x_{n-1})\ldots(x-x_0)$$

be a polynomial that satisfies the $n+1$ tabulated data. Then, the expression becomes:

$$y(x) = y_0 + \frac{\nabla y_n}{h}(x-x_n) + \frac{\nabla^2 y_n}{2!\,h^2}(x-x_n)(x-x_{n-1}) + \cdots + \frac{\nabla^n y_n}{n!\,h^n}(x-x_n)(x-x_{n-1})\cdots(x-x_0) \tag{5}$$

Now, using the substitution $\dfrac{x-x_n}{h} = p$, the equation 5 becomes:

$$\boxed{y(p) = y_n + \nabla y_n p + \frac{\nabla^2 y_n}{2!}p(p+1) + \cdots + \frac{\nabla^n y_n}{n!}p(p+1)(p+2)\cdots(p+n-1)} \tag{6}$$

1.4 Lagrange's Interpolation Polynomial

$$y(x) = a_0(x-x_1)(x-x_2)\ldots(x-x_n) + a_1(x-x_0)(x-x_2)\ldots(x-x_n) + a_2(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n) + \cdots + a_n(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})$$

On simplification:

$$\boxed{y(x) = y_0\left(\frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}\right) + y_1\left(\frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}\right) + \cdots + y_n\left(\frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}\right)} \tag{7}$$

1.5 Newton's Divided Difference formula

$$\begin{aligned} F[x] = {} & F[x_0] + (x - x_0)F[x_0, x_1] + (x - x_0)(x - x_1)F[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)F[x_0, x_1, x_2, x_3] \\ & + \cdots + (x - x_0)(x - x_1)(x - x_2)\cdots(x - x_{n-1})F[x_0, x_1, \ldots, x_n] \end{aligned} \tag{8}$$

where $F[x_i, x_{i+1}] = \dfrac{F[x_{i+1}] - F[x_i]}{x_{i+1} - x_i}$

2 Numerical Differentiation

2.1 Using forward difference

By using the Newton's forward difference equation given by 3,

$$\begin{aligned} y(u) = {} & y_0 + \Delta y_0 u + \frac{\Delta^2 y_0}{2!}u(u-1) + \frac{\Delta^3 y_0}{3!}u(u-1)(u-2) + \frac{\Delta^4 y_0}{4!}u(u-1)(u-2)(u-3) + \frac{\Delta^5 y_0}{5!}u(u-1)(u-2)(u-3)(u-4) \\ & + \cdots + \frac{\Delta^n y_0}{n!}u(u-1)(u-2)\cdots(u-(n-1)) \end{aligned}$$

$$\begin{aligned} y(u) = {} & y_0 + \Delta y_0 u + \frac{\Delta^2 y_0}{2!}(u^2 - u) + \frac{\Delta^3 y_0}{3!}(u^3 - 3u^2 + 2u) + \frac{\Delta^4 y_0}{4!}(u^4 - 6u^3 + 11u^2 - 6u) + \frac{\Delta^5 y_0}{5!}(u^5 - 10u^4 + 35u^3 - 50u^2 + 24u) \\ & + \frac{\Delta^6 y_0}{6!}(u^6 - 15u^5 + 85u^4 - 225u^3 + 250u^2 - 120u) + \cdots + \frac{\Delta^n y_0}{n!}u(u-1)(u-2)\cdots(u-(n-1)) \end{aligned}$$

Now, since $u = \left(\dfrac{x - x_0}{h}\right)$,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{dy}{du}\left(\frac{1}{h}\right) \\ &= \frac{1}{h} \cdot \frac{dy}{du} \end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{1}{h} \cdot \frac{dy}{du}$$

On simplification after differentiating each term of equation 3,

$$\boxed{\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{\Delta^2 y_0}{2!}(2u-1) + \frac{\Delta^3 y_0}{3!}(3u^2-6u+2) + \frac{\Delta^4 y_0}{4!}(4u^3-18u^2+22u-6) + \cdots\right]} \tag{9}$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{\Delta^2 y_0}{2!}(2u-1) + \frac{\Delta^3 y_0}{3!}(3u^2-6u+2) + \frac{\Delta^4 y_0}{4!}(4u^3-18u^2+22u-6) + \frac{\Delta^5 y_0}{5!}(5u^4-40u^3+105u^2-100u+24) + \frac{\Delta^6 y_0}{6!}(6u^5-75u^4+340u^3-675u^2+548u-120)\right] \tag{10}$$

On differentiating again,

$$\boxed{\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{\Delta^3 y_0}{3!}(6u-6) + \frac{\Delta^4 y_0}{4!}(12u^2-36u+22) + \cdots +\right]} \tag{11}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{\Delta^3 y_0}{3!}(6u-6u) + \frac{\Delta^4 y_0}{4!}(12u^2-36u+22) + \frac{\Delta^5 y_0}{5!}(20u^3-120u^2+210u-100) + \frac{\Delta^6 y_0}{6!}(30u^4-300u^3+1020u^2-1350u+548)\right] \tag{12}$$

2.2 Using backward difference

$$\boxed{\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2!}(2p+1) + \frac{\nabla^3 y_n}{3!}(3p^2+6p+2) + \frac{\nabla^4 y_n}{4!}(4p^3+18p^2+22p+6) + \cdots\right]} \tag{13}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{\nabla^3 y_n}{3!}(6p+6) + \frac{\nabla^4 y_n}{4!}(12p^2+36p+22) + \cdots +\right]} \tag{14}$$

3 Numerical Integration

Using the Newton's forward difference formula given by equation 3,

$$y(u) = y_0 + \Delta y_0 u + \frac{\Delta^2 y_0}{2!} u(u-1) + \cdots + \frac{\Delta^n y_0}{n!} u(u-1)(u-2)\cdots(u-(n-1))$$

where $u = \dfrac{x - x_0}{h}$. Then, $du = \dfrac{dx}{h}$. $\implies dx = h \cdot du$.
Now, $x = x_0 \implies u = 0$, $x = x_n \implies u = n$.

$$\therefore \int_{x_0}^{x_n} f(x)dx = h \int_0^n f(u)du$$

3.1 General Quadrature formula

On evaluating the above integral using the expression for $f(u)$ given by equation 3, the **general quadrature formula** is obtained.

$$\int_{x_0}^{x_n} f(x)dx = h\left[y_0 n + \Delta y_0 \frac{n^2}{2} + \frac{\Delta^2 y_0}{2!}\left(\frac{n^3}{3} - \frac{n^2}{2}\right) + \frac{\Delta^3 y_0}{3!}\left(\frac{n^4}{4} - n^3 + n^2\right) + \frac{\Delta^4 y_0}{4!}\left(\frac{n^5}{5} - \frac{3n^2}{2} + \frac{11n^3}{3} - 3n^2\right) + \cdots\right] \tag{15}$$

3.2 Trapezoidal Rule for Numerical Integration

Putting $n = 1$ in equation 15 and taking the curve through (x_0, y_0) as a straight line,

$$\int_{x_1}^{x_2} f(x)dx = \frac{h}{2}(y_1 + y_2)$$

$$\int_{x_{k-1}}^{x_k} f(x)dx = \frac{h}{2}(y_{k-1} + y_k) \ \forall \ k = 1, 2, \ldots, n$$

$$\boxed{\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}(y_0 + y_n) + h(y_1 + y_2 + \cdots + y_n)} \tag{16}$$

3.3 Simpson's one-third Rule

Putting $n = 2$ in equation 15, and taking the curve through $(x_0, y_0), (x_1, y_1)$ and (x_2, y_2) as a parabola,

$$\boxed{\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})]} \tag{17}$$

3.4 Simpson's three-eighth Rule

Putting $n = 2$ in equation 15, and taking the curve through $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ and (x_3, y_3) as a polynomial of third degree,

$$\boxed{\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + y_7 \ldots + y_{n-1}) + 2(y_3 + y_6 + \ldots + y_{n-3})]} \tag{18}$$

3.5 Boole's Rule

$$\int_{x_0}^{x_4} y \; dx = \frac{2h}{45}[7y_0 + 32y_1 + 32y_3 + 7y_4] \tag{19}$$

3.6 Weddle's Rule

$$\int_{x_0}^{x_6} y \; dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6] \tag{20}$$

4 Numerical Solutions of Algebraic and Transcendental equations

4.1 Iteration Method

For finding the root of an equation of the form $f(x) = 0$, the equation should be reduced to the form $x = \phi(x)$ such that $\phi(x)$ is continuously differentiable and $|\phi(x)| < 1$.

4.2 *Regula-Falsi Method* or Method of false position

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

4.3 Newton-Ralphson Method

The Taylor series for a continuous function $f(x)$ is given by

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + \cdots \tag{21}$$

For a continuous function which is differentiable infinitely many times, for small values of h, (where $h = x_1 - x_0$) $f(x)$ can be approximated as:

$$f(x) = f(x_0) + h\ f'(x_0)$$

On rearranging, $h = -\frac{f(x_0)}{f'(x_0)}$. On further simplification, and eliminating h from the expression,

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \tag{22}$$

5 Numerical solutions of first order linear differential equations

5.1 Picard's method

$$\frac{dy}{dx} = f(x, y)$$

The first approximation is given by

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\ dx \tag{23}$$

The second approximation is given by:

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\ dx$$

Similarly, third approximation is

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\ dx$$

5.2 Euler-Cauchy method

$$y_1 = y_0 + h\ f(x_0, y_0)$$
$$y_2 = y_1 + h\ f(x_0 + h, y_1)$$
$$y_3 = y_2 + h\ f(x_0 + 2h, y_2)$$
$$y_4 = y_3 + h\ f(x_0 + 3h, y_3)$$

i.e.,

$$y_n = y_{n-1} + h\ f\left(x_0 + (n-1)h, y_n - 1\right) \tag{24}$$

5.3 Runge-Kutta fourth order method

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Calculate successively:

$$k_1 = hf\left(x_0, y_0\right) \tag{25}$$
$$k_2 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1\right) \tag{26}$$
$$k_3 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2\right) \tag{27}$$
$$k_4 = hf\left(x_0 + h, y_0 + k_3\right) \tag{28}$$

Finally compute:

$$k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \tag{29}$$

References

Dr. B S Grewal, *Higher Engineering Mathematics*, Khanna Publishers