C2 Assignment:
PG.P.10.4. Quantum Mechanics-1

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September 1, 2020

Part 1: Short answer questions

2. **Show that linear momentum operator is Hermitian.**

Solution: An operator $\hat{A}$ is said to be Hermitian if

$$\left\langle \phi \middle| \hat{A}\psi \right\rangle = \left\langle \hat{A}\phi \middle| \psi \right\rangle \tag{1}$$

The momentum operator $\hat{p}$, in the position basis, is defined as:

$$\hat{\vec{p}} = -i\hbar\,\vec{\nabla} \tag{2}$$

In one dimension, this reduces to:

$$\vec{p} = p_x = -i\hbar\frac{\partial}{\partial x} \tag{3}$$

Consider two vectors $|\psi\rangle$ and $|\phi\rangle$.

$$\begin{aligned}
\langle\phi|\hat{p}\psi\rangle &= \int_{-\infty}^{\infty} \phi^*(x)\,\hat{p}\,\psi(x)\,dx \\
&= \int_{-\infty}^{\infty} \phi^*(x)\left(-i\hbar\frac{d}{dx}\right)\psi(x)\,dx \\
&= \int_{-\infty}^{\infty} \phi^*(x)\left(-i\hbar\frac{d\psi(x)}{dx}\right)dx \\
&= -i\hbar\int_{-\infty}^{\infty} \phi^*(x)\left(\frac{d\psi(x)}{dx}\right)dx
\end{aligned}$$

The integral is now of the form uv'. Hence this can be integrated by parts:

$$\langle\phi|\hat{p}\psi\rangle = -i\hbar\left(\phi^*(x)\psi(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{d\phi^*(x)}{dx}\psi(x)\,dx\right)$$

Now, the wave functions vanish as $x \to \pm\infty$. Hence the first term becomes zero.

Hence,

$$\begin{aligned}
\langle\phi|\hat{p}\psi\rangle &= i\hbar\int_{-\infty}^{\infty}\frac{d\phi^*(x)}{dx}\psi(x)\,dx \\
&= \int_{-\infty}^{\infty}\left(i\hbar\frac{d\phi^*(x)}{dx}\right)\psi(x)\,dx \\
&= \int_{-\infty}^{\infty}\left(-i\hbar\frac{d\phi(x)}{dx}\right)^*\psi(x)\,dx \\
&= \int_{-\infty}^{\infty}[\hat{p}\,\phi(x)]^*\,\psi(x)\,dx \\
&= \langle\hat{p}\,\phi|\psi\rangle
\end{aligned}$$

$$\therefore \boxed{\langle\phi|\hat{p}\psi\rangle = \langle\hat{p}\,\phi|\psi\rangle}$$

Hence, by equation (1), the linear momentum operator $\hat{p}$ is a Hermitian operator.

4. **Show that Hermitian operators have real eigenvalues.**

Solution: Consider a Hermitian operator $\hat{A}$ which has the eigen-value equation:

$$\hat{A}\,|\psi\rangle = a\,|\psi\rangle \tag{4}$$

Since these vectors are members of a Hilbert's space, they have finite inner products. Hence, we can take the inner product with $|\psi\rangle$ in the above equation:

$$\left\langle\psi\middle|\hat{A}\,\psi\right\rangle = a\,\langle\psi|\psi\rangle \tag{5}$$

Since $\hat{A}$ is Hermitian, by equation (1),

$$\left\langle\psi\middle|\hat{A}\,\psi\right\rangle = \left\langle\hat{A}\,\psi\middle|\psi\right\rangle = a^*\,\langle\psi|\psi\rangle \tag{6}$$

Therefore, from equations (5) and (6),

$$a = a^*$$

This will be true only if a is a real number, *i.e.*, $a \in \mathbb{R}$.

$\therefore$ Hermitian operators have real eigenvalues.

5. **Find $[x, p^2]$ or $[x^2, p]$ or $[x, p^n]$ or $[x^n, p]$.**

Solution:

$$[x, p] = i\hbar \tag{7}$$

By the properties of commutators:

$$[A, BC] = B\,[A, C] + [A, B]\,C \tag{8}$$

$$\begin{aligned}[x,p^2] &= p\,[x,p] + [x,p]\,p \\ &= i\hbar\,p + i\hbar\,p \\ &= 2i\hbar\,p\end{aligned}$$

$$\begin{aligned}[x^2,p] &= x\,[x,p] + [x,p]\,x \\ &= i\hbar\,x + i\hbar\,x \\ &= 2i\hbar\,x\end{aligned}$$

$$\begin{aligned}[x^3,p] &= [x\,x^2,p] \\ &= x\,[x^2,p] + [x,p]\,x^2 \\ &= x(2i\hbar\,x) + i\hbar\,x^2 \\ &= 3i\hbar\,x^2\end{aligned}$$

$$\begin{aligned}[x,p^3] &= [x,p\,p^2] \\ &= p\,[x,p^2] + [x,p]\,p^2 \\ &= p\,(2i\hbar\,p) + i\hbar\,p^2 \\ &= 3i\hbar\,p^2\end{aligned}$$

So, in general,

$$\boxed{[x^n,p] = ni\hbar\,x^{n-1}} \tag{9}$$

$$\boxed{[x,p^n] = ni\hbar\,p^{n-1}} \tag{10}$$

Long Answer Questions

8. **Derive the expression for general uncertainty relation for non-commuting operators $\hat{A}$, $\hat{B}$ satisfying**

$$[\hat{A}, \hat{B}] = i\hat{C}$$

Solution:

If $|\psi\rangle$ is an eigenstate of $\hat{A}$, then

$$\hat{A}\,|\psi\rangle = \lambda\,|\psi\rangle \tag{11}$$

where λ is real. The uncertainty in the measurement of $\hat{A}$ is

$$\Delta A = \sqrt{\left\langle \left(\hat{A} - \left\langle \hat{A} \right\rangle\right)^2 \right\rangle} = \sqrt{\left\langle \hat{A}^2 \right\rangle - \left\langle \hat{A} \right\rangle^2}$$

Consider the uncertainty in the measurement of $\hat{A}$ in the eigenstate $|\psi\rangle$:

$$\begin{aligned}(\Delta A)^2 &= \left\langle \hat{A}^2 \right\rangle - \left\langle \hat{A} \right\rangle^2 \\ &= \langle\psi|\hat{A}^2|\psi\rangle - \langle\psi|\hat{A}|\psi\rangle^2 \\ &= \lambda^2 - \lambda^2 \\ &= 0\end{aligned} \tag{12}$$

Suppose $\hat{A}$, $\hat{B}$ are non-commuting Hermitian operators satisfying

$$[\hat{A}, \hat{B}] = i\hat{C}$$

$\hat{C}$ is Hermitian:

$$\begin{aligned}\hat{C} &= -i[\hat{A}, \hat{B}] \\ &= -i\left(\hat{A}\hat{B} - \hat{B}\hat{A}\right)\end{aligned}$$

$$\begin{aligned}\hat{C}^\dagger &= \left[-i\left(\hat{A}\hat{B} - \hat{B}\hat{A}\right)\right]^\dagger \\ &= \left(\hat{A}\hat{B} - \hat{B}\hat{A}\right)^\dagger (-i)^\dagger \\ &= i\left(\hat{A}\hat{B} - \hat{B}\hat{A}\right)^\dagger \\ &= i\left[\left(\hat{A}\hat{B}\right)^\dagger - \left(\hat{B}\hat{A}\right)^\dagger\right] \\ &= i\left[\hat{B}\hat{A} - \hat{A}\hat{B}\right] \\ &= -i[\hat{A}, \hat{B}] \\ &= \hat{C}\end{aligned}$$

Suppose the system is in some state $|\phi\rangle$. Define :

$$|\chi\rangle \equiv \left(\hat{A} - \left\langle \hat{A} \right\rangle \hat{I}\right) |\phi\rangle \tag{13}$$

$$|\zeta\rangle \equiv \left(\hat{B} - \left\langle \hat{B} \right\rangle \hat{I}\right) |\phi\rangle \tag{14}$$

By Schwarz inequality,

$$\langle\chi|\chi\rangle \langle\zeta|\zeta\rangle \geq \|\langle\chi|\zeta\rangle\|^2 \tag{15}$$

Now consider,

$$\begin{aligned}
\langle\chi|\chi\rangle &= \left\langle \chi \middle| \left(\hat{A} - \left\langle \hat{A} \right\rangle \hat{I}\right) \middle| \phi \right\rangle \\
&= \left\langle \phi \middle| \left(\hat{A} - \left\langle \hat{A} \right\rangle \hat{I}\right) \left(\hat{A} - \left\langle \hat{A} \right\rangle \hat{I}\right) \middle| \phi \right\rangle \\
&= \left\langle \phi \middle| \left(\hat{A}^2 - 2\left\langle \hat{A} \right\rangle \hat{A} + \left\langle \hat{A} \right\rangle^2\right) \middle| \phi \right\rangle \\
&= \left\langle \phi \middle| \hat{A}^2 \middle| \phi \right\rangle - 2\left\langle \hat{A} \right\rangle^2 + \left\langle \hat{A} \right\rangle^2 \\
&= \left\langle \phi \middle| \hat{A}^2 \middle| \phi \right\rangle - \left\langle \hat{A} \right\rangle^2 \\
&= (\Delta A)^2
\end{aligned} \tag{16}$$

Similarly,

$$\langle\zeta|\zeta\rangle = (\Delta B)^2 \tag{17}$$

Now consider the third term required in equation (15)

$$\begin{aligned}
\langle\chi|\zeta\rangle &= \left\langle \phi \middle| \left(\hat{A} - \left\langle \hat{A} \right\rangle \hat{I}\right) \left(\hat{B} - \left\langle \hat{B} \right\rangle \hat{I}\right) \middle| \phi \right\rangle \\
&= \left\langle \phi \middle| \left(\hat{A}\hat{B} - \left\langle \hat{B} \right\rangle \hat{A} - \left\langle \hat{A} \right\rangle \hat{B} + \left\langle \hat{A} \right\rangle \left\langle \hat{B} \right\rangle\right) \middle| \phi \right\rangle \\
&= \langle\phi|\hat{A}\hat{B}|\phi\rangle - \left\langle \hat{A} \right\rangle \left\langle \hat{B} \right\rangle
\end{aligned} \tag{18}$$

$$\begin{aligned}
\langle\phi|\hat{A}\hat{B}|\phi\rangle &= \left\langle \phi \middle| \left\{ \left(\frac{\hat{A}\hat{B} - \hat{B}\hat{A}}{2}\right) + \left(\frac{\hat{A}\hat{B} + \hat{B}\hat{A}}{2}\right) \right\} \middle| \phi \right\rangle \\
&= \left\langle \phi \middle| \frac{i\hat{C}}{2} \middle| \phi \right\rangle + \frac{1}{2} \left\langle \phi \middle| \hat{A}\hat{B} + \hat{B}\hat{A} \middle| \phi \right\rangle \\
&= \frac{i}{2} \langle\hat{C}\rangle + \frac{1}{2} \left\langle \phi \middle| \hat{A}\hat{B} + \hat{B}\hat{A} \middle| \phi \right\rangle
\end{aligned} \tag{19}$$

Substituting equation (19) in equation (18),

$$\begin{aligned}
\langle\chi|\zeta\rangle &= \frac{i}{2} \langle\hat{C}\rangle + \frac{1}{2} \left\langle \phi \middle| \hat{A}\hat{B} + \hat{B}\hat{A} \middle| \phi \right\rangle - \langle A\rangle \langle B\rangle \\
&= \frac{i}{2} \langle\hat{C}\rangle + \frac{1}{2} \left\langle \phi \middle| \hat{A}\hat{B} + \hat{B}\hat{A} \middle| \phi \right\rangle - \langle A\rangle \langle B\rangle \langle\phi|\phi\rangle \\
&= \frac{i}{2} \langle\hat{C}\rangle + \frac{1}{2} \left\langle \phi \middle| \hat{A}\hat{B} + \hat{B}\hat{A} \middle| \phi \right\rangle - \langle\phi|\langle A\rangle \langle B\rangle|\phi\rangle \\
&= \frac{i}{2} \langle\hat{C}\rangle + \frac{1}{2} \left\langle \phi \middle| \left\{ (\hat{A}\hat{B} + \hat{B}\hat{A}) - 2\langle A\rangle \langle B\rangle \right\} \middle| \phi \right\rangle
\end{aligned}$$

Let $\hat{D} \equiv \left[(\hat{A}\hat{B} + \hat{B}\hat{A}) - 2\langle A\rangle\langle B\rangle\right]$. Note that this is also an operator.
Therefore

$$\langle\chi|\zeta\rangle = \frac{i}{2}\langle\hat{C}\rangle + \frac{1}{2}\left\langle\phi\middle|\hat{D}\middle|\phi\right\rangle \tag{20}$$

Now,

$$\|\langle\chi|\zeta\rangle\|^2 = \frac{1}{4}\left\langle\hat{C}\right\rangle^2 + \frac{1}{4}\left\langle\phi\middle|\hat{D}\middle|\phi\right\rangle^2 \tag{21}$$

$(\because |(a+ib)|^2 = (a+ib)(a-ib) = a^2 + b^2)$
Since all the terms in the above equation are real,

$$\|\langle\chi|\zeta\rangle\|^2 \geq \frac{1}{4}\left\langle\hat{C}\right\rangle^2 \tag{22}$$

Combining equation(15) with the above equation,

$$\langle\chi|\chi\rangle\langle\zeta|\zeta\rangle \geq \|\langle\chi|\zeta\rangle\|^2 \geq \frac{1}{4}\left\langle\hat{C}\right\rangle^2$$

Now, using the results from (16) and (17),

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}\left\langle\hat{C}\right\rangle^2 \tag{23}$$

$$\boxed{(\Delta A)(\Delta B) \geq \frac{1}{2}|\langle\hat{C}\rangle|} \tag{24}$$

The above gives the general uncertainty principle for any two non-commuting Hermitian operators:
If $\hat{A}$ and $\hat{B}$ are two non-commuting Hermitian operators, then the product of the uncertainties in $\hat{A}$ and $\hat{B}$ is always greater than $\frac{|\langle\hat{C}\rangle|}{2}$.

12. **Solve the Schrödinger equation for the oscillator using the operator method and obtain the energy eigenvalues.**

Solution: The time-independent Schrödinger equation is given by:

$$\hat{H}\psi = E\psi \tag{25}$$

By using the fact that any potential $V(x)$ is approximately parabolic in the neighbourhood of a local minimum,

$$V(x) = \frac{1}{2}k\hat{x}^2 = \frac{1}{2}m\omega^2\hat{x}^2$$

where $\omega = \sqrt{k/m}$.
Substituting the equation for the Hamiltonian operator as

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$$

Equation 25 becomes

$$\frac{1}{2m}\left[\hat{p}^2 + (m\omega\hat{x})^2\right]\psi(x) = E\psi(x) \tag{26}$$

(*Note that p and x are operators.*)

We define two operators:

$$\begin{aligned}
\hat{a} &= \frac{1}{\sqrt{2\hbar m\omega}}(i\hat{p} + m\omega\hat{x}) && (27)\\
\hat{a}^\dagger &= \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega\hat{x}) && (28)
\end{aligned}$$

Now, consider

$$\begin{aligned}
\hat{a}\hat{a}^\dagger &= \frac{1}{2\hbar m\omega}(i\hat{p} + m\omega\hat{x})(-i\hat{p} + m\omega\hat{x})\\
&= \frac{1}{2\hbar m\omega}\{\hat{p}^2 + (m\omega\hat{x})^2 - im\omega(\hat{x}\hat{p} - \hat{p}\hat{x})\}\\
&= \frac{1}{2\hbar m\omega}\{\hat{p}^2 + (m\omega\hat{x})^2 - im\omega[\hat{x}, \hat{p}]\}
\end{aligned}$$

Now,

$$[x, p] = i\hbar$$

$$\therefore \hat{a}\hat{a}^\dagger = \frac{1}{2\hbar m\omega}\{\hat{p}^2 + (m\omega\hat{x})^2 + m\omega\hbar\}$$

Using the equation for Hamiltonian as in equation (26),

$$\hat{a}\hat{a}^\dagger = \frac{1}{\hbar\omega}\left(\hat{H} + \frac{\hbar\omega}{2}\right)$$

$$\therefore \hat{a}\hat{a}^\dagger = \left(\frac{\hat{H}}{\hbar\omega} + \frac{1}{2}\right) \tag{29}$$

Now, consider the following commutator relations:

$$\begin{aligned}[\hat{a}, \hat{a}^\dagger] &= \frac{1}{2m\hbar\omega}\left(i\hat{p} + m\omega\hat{x}, -i\hat{p} + m\omega\hat{x}\right) \\ &= \frac{1}{2m\hbar\omega}\left(im\omega[\hat{p}, \hat{x}] - im\omega[\hat{x}, \hat{p}]\right) \\ &= \frac{1}{2m\hbar\omega}\left(2m\hbar\omega\right) \\ &= 1\end{aligned} \tag{30}$$

Now, using the result in (30) and commutator property,

$$[\hat{a}^\dagger, \hat{a}] = -1 \tag{31}$$

Now, consider

$$\begin{aligned}[\hat{a}^\dagger\hat{a}\,,\, \hat{a}] &= \hat{a}^\dagger[\hat{a}, \hat{a}] + [\hat{a}^\dagger, \hat{a}]\hat{a} \\ &= -\hat{a}\end{aligned}$$

$$\begin{aligned}[\hat{a}^\dagger\hat{a}\,,\, \hat{a}^\dagger] &= \hat{a}^\dagger[\hat{a}, \hat{a}^\dagger] + [\hat{a}^\dagger, \hat{a}^\dagger]\hat{a} \\ &= \hat{a}^\dagger\end{aligned}$$

Now, define

$$\hat{N} \equiv \hat{a}^\dagger\hat{a} \tag{32}$$

Note that $\hat{N}$ is Hermitian.

Let us rewrite the commutator relations we have obtained so far, using the definition (32) for $\hat{a}^\dagger\hat{a}$,

- $[\hat{a}\,,\, \hat{a}^\dagger] = 1$
- $[N, \hat{a}] = -\hat{a}$
- $[N, \hat{a}^\dagger] = \hat{a}^\dagger$

Now, from equation (29), using equations (31) and (32), we can write

$$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right) \tag{33}$$

$\hat{N}$ possesses eigen states with real eigenvalues. Let $|n\rangle$ be an eigenstate of $\hat{N}$, with eigenvalue n.

$$\hat{N}\,|n\rangle = n\,|n\rangle \tag{34}$$

Let $\left|\chi\right\rangle \equiv \hat{a}\left|n\right\rangle$. Operating $\hat{N}$ on $\left|\chi\right\rangle$, we see that:

$$\begin{aligned}
\hat{N}\left|\chi\right\rangle &= \hat{N}\hat{a}\left|n\right\rangle \\
&= (\hat{a}\hat{N} - \hat{a})\left|n\right\rangle \quad \text{(from commutator relation } [\hat{N}, \hat{a}]) \\
&= \hat{a}\hat{N}\left|n\right\rangle - \hat{a}\left|n\right\rangle \\
&= n\,\hat{a}\left|n\right\rangle - \hat{a}\left|n\right\rangle \quad \text{(from the eigenvalue equation (34))} \\
&= (n-1)\hat{a}\left|n\right\rangle \\
&= (n-1)\left|\chi\right\rangle
\end{aligned} \tag{35}$$

which means, $\left|\chi\right\rangle$ is also an eigen state of $\hat{N}$ with eigenvalue $(n-1)$. Similarly, $(\hat{a})^2\left|n\right\rangle$ is also an eigenstate of $\hat{N}$ with eigenvalue $(n-2)$.

Now, let $\left|\xi\right\rangle \equiv \hat{a}^\dagger\left|n\right\rangle$. Similar to equation (35),

$$\begin{aligned}
\hat{N}\left|\xi\right\rangle &= \hat{N}\hat{a}^\dagger\left|n\right\rangle \\
&= (\hat{a}^\dagger + \hat{a}^\dagger\hat{N})\left|n\right\rangle \quad \text{(from commutator relation } [\hat{N}, \hat{a}^\dagger]) \\
&= \hat{a}^\dagger\left|n\right\rangle + \hat{a}^\dagger\hat{N}\left|n\right\rangle \\
&= \hat{a}^\dagger\left|n\right\rangle + n\hat{a}^\dagger\left|n\right\rangle \quad \text{(from the eigenvalue equation (34))} \\
&= (n+1)\hat{a}^\dagger\left|n\right\rangle \\
&= (n+1)\left|\xi\right\rangle
\end{aligned} \tag{36}$$

which means, $\left|\xi\right\rangle$ is also an eigen state of $\hat{N}$ with eigenvalue $(n+1)$.

Similarly, $(\hat{a}^\dagger)^2\left|n\right\rangle$ is also an eigenstate of $\hat{N}$ with eigenvalue $(n+2)$.

Due to these properties, the operators $\hat{a}$ and $\hat{a}^\dagger$ are known as *annihilation* and *creation* operators respectively. This is because, given a particular state, one can apply $\hat{a}^\dagger$ and $\hat{a}$ on that state to get new states with higher and lower eigenvalues respectively.

$$\left\langle\chi|\chi\right\rangle = \left\langle n\middle|\hat{a}^\dagger\hat{a}\middle|n\right\rangle = \left\langle n\middle|\hat{N}\middle|n\right\rangle = n\left\langle n|n\right\rangle \geq 0$$

So n is non-negative and real.

$\hat{a}$ cannot be applied indefinitely on a given state. At some point, on repeated application of $\hat{a}$, a *lowest energy state* is obtained. To satisfy this condition, n must be an integer. Therefore, n is a non-negative integer.

The lowest energy state, $\left|0\right\rangle$ has eigenvalue zero:

$$N\left|0\right\rangle = 0 \tag{37}$$

$$\implies \hat{a}\left|0\right\rangle = 0$$

Call $|0\rangle$ as $\psi_0(x)$

$$
\begin{aligned}
\implies \frac{1}{\sqrt{2\hbar m\omega}}\left(i\hat{p}+m\omega\hat{x}\right)\psi_0(x) = &\quad 0\\
\frac{1}{\sqrt{2\hbar m\omega}}\left(i\left(-i\hbar\frac{d}{dx}\right)+m\omega\hat{x}\right)\psi_0(x) = &\quad 0\\
\implies \frac{d\psi_0(x)}{dx}+\frac{m\omega}{\hbar}x\,\psi_0(x) = &\quad 0\\
\text{Solving,}\quad \psi_0(x) = A\,e^{-\left(\frac{m\omega x^2}{2\hbar}\right)}&
\end{aligned}
$$

To get the constant of integration A, apply the normalization condition:

$$
\begin{aligned}
\int_{-\infty}^{\infty}\psi_0^*(x)\,\psi_0(x)\,dx = &\quad 1\\
\implies A^2 = &\quad \sqrt{\frac{m\omega}{\pi\hbar}}\\
\implies A = &\quad \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}
\end{aligned}
$$

Therefore, the time-independent wave function of the lowest energy state is:

$$
\boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\,e^{-\left(\frac{m\omega x^2}{2\hbar}\right)}} \tag{38}
$$

The time dependent wave function is therefore

$$
\Psi_0(x,t) = \psi_0(x)\;\; e^{\frac{-iE_0t}{\hbar}} \tag{39}
$$

Now, from equation (33) it is clear that,

$$
[\hat{H},\hat{N}] = 0
$$

In other words, the operators $\hat{H}$ and $\hat{N}$, commute. Since, they commute, the eigenstates of $\hat{H}$ are also eigenstates of $\hat{N}$ (since they are Hermitian).

This means that, since $|n\rangle$, $|\chi\rangle$ and $|\xi\rangle$ are eigenstates of $\hat{N}$, they must be eigenstates of $\hat{H}$ also.

$$
\begin{aligned}
\hat{H}\,|n\rangle &= \hbar\omega\left(\hat{N}+\frac{1}{2}\right)|n\rangle\\
&= \hbar\omega\left(\hat{N}\,|n\rangle+\frac{1}{2}\,|n\rangle\right)\\
&= \hbar\omega\left(n\,|n\rangle+\frac{1}{2}\,|n\rangle\right)\\
&= \left(n+\frac{1}{2}\right)\hbar\omega\,|n\rangle
\end{aligned} \tag{40}
$$

Using $\hat{H}\,|n\rangle = E_n\,|n\rangle$, we see that

$$\boxed{E_n = \left(n + \frac{1}{2}\right)\hbar\omega} \tag{41}$$

Hence, even when $n = 0$, the energy is non-zero. In other words, there is no zero energy state. From (38) and (41),

$$\begin{aligned}
\Psi_0(x,t) &= \psi_0(x)\; e^{\frac{-iE_0 t}{\hbar}} \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\left(\frac{m\omega x^2}{2\hbar}\right)} e^{\frac{-i\left(\frac{1}{2}\hbar\omega\right)t}{\hbar}} \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}} e^{-\frac{i\omega t}{2}}
\end{aligned} \tag{42}$$

Any higher state can be obtained by applying the creation operator on the ground state $\psi_0(x)$ and continuing the same procedure discussed above.

The general wave function of the harmonic oscillator can be written in terms of the Hermite polynomials as:

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(\xi)\, e^{-\xi^2/2} \tag{43}$$

where $H_n(x)$ are the Hermite polynomials and $\xi = \sqrt{\frac{m\omega}{\hbar}}x$