Problem 1 (30 points). Position fixing using the Kalman Filter Let L_1, L_2, L_3 be three beacons and let P be your position. Let $\mathrm{F_A}$ be a frame and w be a point such that $\vec{r}_{P/w}\big|_{\mathrm{A}} = \begin{bmatrix}0.7212 & 2.4080 & 0\end{bmatrix}^{\mathrm{T}}$ and, for all $i \in \{1, 2, 3\}$,

$$\vec{r}_{L_i/w}\big|_{\mathrm{A}} = \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} \text{ m}, \tag{3}$$

where $x_i, y_i \in \mathbb{R}$. Furthermore, for all $i \in \{1, 2, 3\}$, let the 2D distance from the beacons L_i to P be r_i, such that $r_i = \sqrt{(0.7212 - x_i)^2 + (2.4080 - y_i)^2}$. Suppose that $(x_1, y_1) = (0, 0)$ m, $(x_2, y_2) = (5, 5)$ m, and $(x_3, y_3) = (2.5, 0)$ m, Thus, it follows that $r_1 = 2.5$ m, $r_2 = 5$ m, $r_3 = 3$ m approximately. Furthermore, suppose that, for all $i \in \{1, 2, 3\}$, noisy 2D distance measurements from beacon L_i are available every one second. That is, for all $k > 0$,

$$Y_k = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} + D w_k, \tag{4}$$

where $D = \mathrm{diag}(0.1, 0.1, 0.1)$ and $w_k \sim \mathcal{N}(0, I_3)$.

We will use the Kalman Filter to obtain an estimate of $\vec{r}_{P/w}\big|_{\mathrm{A}}$, which we will refer to as $\vec{r}_{\hat{P}/w}\big|_{\mathrm{A}}$. In order to do so, define, for all $k \geq 0$, $\hat{X}_k = \begin{bmatrix}\hat{x}_k & \hat{y}_k\end{bmatrix}^{\mathrm{T}}$, such that $\vec{r}_{\hat{P}/w,k}\big|_{\mathrm{A}} = \begin{bmatrix}\hat{x}_k & \hat{y}_k & 0\end{bmatrix}^{\mathrm{T}}$. For all $k \geq 0$, the dynamics model that will be used in the Kalman Filter is given by $\hat{X}_{k+1} = \hat{X}_k$. Furthermore, for all $k \geq 0$, the measurement model is given by

$$g(\hat{X}_k) = \begin{bmatrix} \sqrt{(\hat{x}_k - x_1)^2 + (\hat{y}_k - y_1)^2} \\ \sqrt{(\hat{x}_k - x_2)^2 + (\hat{y}_k - y_2)^2} \\ \sqrt{(\hat{x}_k - x_3)^2 + (\hat{y}_k - y_3)^2} \end{bmatrix}. \tag{5}$$

a) Write the Kalman filter equations to obtain an estimate of x_k.

b) Let $Q = 0$ and $R = 0.1$*eye(3), and suppose that $\hat{X}_0 = \begin{bmatrix}4 & 4\end{bmatrix}^{\mathrm{T}}$ m. For all $p_0 \in \{0.01, 0.1, 1, 10\}$ such that $P_{0|0} = p_0$*eye(2), obtain the estimates $\hat{X}_k$ for all $k \in \{0, 1, \ldots, 50\}$. In one figure, plot the 2D trajectory $\hat{X}_k$ for all $k \in \{0, 1, \ldots, 50\}$, for each value of p_0 and place a red point on the position of P. In another figure, use semilogy to plot the frobenius norm of $P_{k|k}$ versus k for all $k \in \{0, 1, \ldots, 50\}$.

Problem 2 (60 points). Let $g = 9.80665$ m/s^2 be the acceleration due to gravity, and let $\phi = \pi/6$ rad. Suppose that a 3-axis accelerometer and a 3-axis rate gyro are attached to a quadcopter following an inclined, circular trajectory. Let $\mathrm{F_A}$ be an inertial frame, let $\mathrm{F_B}$ be a frame fixed to the quadcopter, and suppose that the axes of both the rate gyro and the accelerometer are aligned with $\mathrm{F_B}$. Let c be the center of mass of the quadcopter, let w be a point with zero inertial acceleration, and let $\vec{r}_{c/w}(t)$ and $\mathcal{O}_{\mathrm{B/A}}(t)$ be the position vector of the quadcopter center of mass and the orientation matrix of $\mathrm{F_B}$ relative to $\mathrm{F_A}$ at time t, respectively. Furthermore, suppose that $g_A = \begin{bmatrix} 0 & 0 & -g \end{bmatrix}$ m/s^2 and, for all $k \in \{0, 1, \ldots, 2000\}$, the measurements from the sensors are given by

$$\omega_k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + D_1 w_{1,k} \ \ \text{rad}, \tag{6}$$

$$a_k = \begin{bmatrix} -1 - g\sin\phi \sin kT \\ -g\sin\phi\cos kT \\ -g\cos\phi \end{bmatrix} + D_2 w_{2,k} \ \ \text{m/s}^2, \tag{7}$$

where $T = 0.01$ s is the sampling rate of these sensors, $D_1 = D_2 = \mathrm{diag}(0.1, 0.1, 0.1)$, $w_{1,k} \sim \mathcal{N}(0, I_3)$, and $w_{2,k} \sim \mathcal{N}(0, I_3)$. Furthermore, suppose that

$$\vec{r}_{c/w}\big|_{\mathrm{A}}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{m}, \tag{8}$$

$$\overset{\mathrm{A}\bullet}{\vec{r}}_{c/w}\big|_{\mathrm{A}}(0) = \begin{bmatrix} 0 \\ \cos\phi \\ \sin\phi \end{bmatrix} \text{m/s}, \quad \mathcal{O}_{\mathrm{B/A}}(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}. \tag{9}$$

a) For all $k \in \{0, 1, \ldots, 2000\}$, obtain the estimates $\vec{r}_{c,w}\big|_{\mathrm{A}}(kT)$ via a Kalman filter, using only the rate-gyro and accelerometer measurements. Compare it against the trajectory in the rcwA.mat file by plotting both this reference trajectory and the estimated trajectory in a 3D plot.

b) For all $k \in \{0, 1, \ldots, 2000\}$, let $y_{\mathrm{meas},k} \in \mathbb{R}^3$ be the position vectors from the rcwA.mat file and suppose that these are sampled every T_{MOCAP} seconds to obtain noisy position measurements $y_k \in \mathbb{R}^3$, such that, for all $k \in \{0, 1, \ldots, 2000\}$ such that $\mathrm{mod}(kT, T_{\mathrm{MOCAP}}) = 0$,

$$y_k = y_{\mathrm{meas},k} + D_3 w_{3,k}, \tag{10}$$

where $D_3 = \mathrm{diag}(0.005, 0.005, 0.005)$, and $w_{3,k} \sim \mathcal{N}(0, I_3)$. Use the position measurements in your Kalman filter with $R = 0.001 I_3, Q = 10 I_6$, and $P_{0|0} = 10 I_6$. Do this for $T_{\mathrm{MOCAP}} = 1$ and $T_{\mathrm{MOCAP}} = 0.1$ s. In one figure, use a 3D plot to plot the trajectory in the rcwA.mat file and the estimated trajectories obtained for $T_{\mathrm{MOCAP}} = 1$ and $T_{\mathrm{MOCAP}} = 0.1$ s. In another figure, plot the 3 components of the trajectory in the rcwA.mat file and the estimated trajectories obtained for $T_{\mathrm{MOCAP}} = 1$ and $T_{\mathrm{MOCAP}} = 0.1$ s versus time in a 3-by-1 figure grid using the subplot function.

NOTE: You will find the rcwA.mat file in the folder “Matlab Material” posted on Canvas.