MAE 560: CFD Course Project 1
Solution of Linear Wave Equation

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Contents

Chapter 1

Introduction

The equations which describe fluid flow, transport mechanism are PDEs with advection terms and it is very difficult to solve these equations numerically. Currently typical interest centers on solving shock waves which are discontinuous and very difficult for numerical schemes to handle. Some part of these equations are similar to linear wave equation, which describes the advection of a conserved scalar due to a velocity field. Solving the linear wave equation numerically using different numerical schemes will help in understanding its behavior for various schemes and it would help when solving complicated equations describing fluid flow, transport phenomena in many applications like physics, engineering, and earth sciences.

In this project the linear wave equation is solved numerically, to understand its behavior, using different discretization techniques like first-order backward difference, second-order backward difference, third order backward difference, Crank-Nicolson, Lax-Wendroff schemes in combination with different integration methods like explicit and implicit methods.

Chapter 2

Numerical Schemes

The linear wave equation describes the advection of a conserved scalar due to a velocity field and it is mathematically expressed as below:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{2.1}$$

where a is the speed of the wave.

This chapter will discuss different schemes to solve the linear wave equation and their behavior. The wave equation is solved using following combinations of discretization and integrations methods:

1) First-order backward difference, explicit integration
2) second-order backward difference, implicit integration
3) Crank-Nicolson, implicit integration
4) Lax-Wendroff scheme, explicit integration
5) Lax scheme, explicit integration
6) First-order backward difference, implicit integration

For all these schemes, the most important mathematical concept used for discretization is Taylor series expansion. The Taylor series expansion is given by:

$$u(x+\Delta x) = u(x) + \frac{u^{'}}{1!}\Delta x + \frac{u^{''}}{2!}\Delta x^2 + \frac{u^{'''}}{3!}\Delta x^3 + \frac{u^{''''}}{4!}\Delta x^4 + \frac{u^{'''''}}{5!}\Delta x^5 + ... \tag{2.2}$$

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2.1 First-order backward difference, explicit integration

The first-order backward difference scheme discretizes the linear wave equation forward in time and backward in space. The backward difference is derived from Taylor series expansion as shown below:

$$u_{i-1}^n = u_i^n + \frac{\partial u_i^n}{\partial x}\frac{-\Delta x}{1!} + \frac{\partial^2 u_i^n}{\partial x^2}\frac{(-\Delta x)^2}{2!} + \frac{\partial^3 u_i^n}{\partial x^3}\frac{(-\Delta x)^3}{3!} + ... \tag{2.3}$$

$$\frac{\partial u_i^n}{\partial x} = \frac{u_i^n - u_{i-1}^n}{\Delta x} + O(\Delta x) \tag{2.4}$$

The order of error in backward difference discretization is first order in space and the leading error term is second derivative which makes the discretization dissipative in space. Similarly the time derivative from the Taylor series expansion is given by:

$$u_i^{n+1} = u_i^n + \frac{\partial u_i^n}{\partial t}\frac{\Delta t}{1!} + \frac{\partial^2 u_i^n}{\partial t^2}\frac{\Delta t^2}{2!} + \frac{\partial^3 u_i^n}{\partial t^3}\frac{\Delta t^3}{3!} + ... \tag{2.5}$$

$$\frac{\partial u_i^n}{\partial t} = \frac{u_i^{n+1} - u_i^n}{\Delta t} + O(\Delta t) \tag{2.6}$$

The final discretization of linear wave equation will be first order in time as well as space and it is given by:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_i^n - u_{i-1}^n}{\Delta x} + O(\Delta t) + O(\Delta x) = 0 \tag{2.7}$$

The discretization can be rearranged into a single equation as shown below:

$$u_i^{n+1} = \nu u_{i-1}^n + (1-\nu)u_i^n \tag{2.8}$$

CFL number is given by:

$$\nu = \frac{a\Delta t}{\Delta x} \tag{2.9}$$

The domain for this problem is a line which is divided into N+2 nodes with i=0 on left boundary to i=N+1 on the right boundary. The value of u at each node point from i=1 to i=N can be expressed as:

$$u_1^{n+1} = \nu u_0^n + (1-\nu)u_1^n \tag{2.10}$$

$$u_2^{n+1} = \nu u_0^n + (1-\nu)u_2^n \tag{2.11}$$

$$u_N^{n+1} = \nu u_{N-1}^n + (1-\nu)u_2^{N+1} \tag{2.12}$$

Rearranging the equations will give a system of matrices

$$\begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ \vdots \\ u_N^{n+1} \end{pmatrix} = \begin{bmatrix} \nu & 0 & 0 & \cdots & 0 \\ \nu & 1-\nu & 0 & \cdots & 0 \\ 0 & \nu & 1-\nu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \nu & 1-\nu \end{bmatrix} \begin{pmatrix} u_1^n \\ u_2^n \\ u_3^n \\ \vdots \\ u_N^n \end{pmatrix} + \begin{pmatrix} \nu u_0^n \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The system of matrices are of the form u=Kx+ub. The matrix ub contains the information about boundary condition.

While solving PDE's using numerical schemes, it is important to consider the numerical stability of the schemes. For linear equations, the round-off error will satisfy the discretization.

$$\frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} + a\frac{\varepsilon_i^n - \varepsilon_{i-1}^n}{\Delta x} = 0 \tag{2.13}$$

For this analysis, John Von-Newman error model is used

$$\varepsilon_i^n = e^{at^n} e^{jK_m X_i} \tag{2.14}$$

$$\varepsilon_i^{n+1} = e^{a(t^n+\Delta t)} e^{jK_m X_i} \tag{2.15}$$

For the scheme to be stable, the magnitude of the amplification factor as to be less than 1

$$G = \frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} = e^{a\Delta t} \tag{2.16}$$

$$|G|^2 \leq 1 \tag{2.17}$$

After substituting the error model into the discretization and some algebra gives the condition for stability as shown below

$$\frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} - 1 + \nu(1 - e^{-jK_m \Delta X}) = 0 \tag{2.18}$$

$$G = 1 - \nu(1 - cos\theta) - \nu sin\theta i \tag{2.19}$$

$$G = 1 - 2\nu sin^2(\theta/2) - \nu sin\theta i \tag{2.20}$$

$$0 < \nu \leq 1 \tag{2.21}$$

2.2 Second-order backward difference, explicit integration

Second order backward difference is given by

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{1.5u_i^n - 2u_{i-1}^n + 0.5u_{i-2}^n}{\Delta x} + O(\Delta t) + O(\Delta x^2) = 0 \tag{2.22}$$

this discretization is second order in space and the leading error term is third derivative, which makes the scheme dispersive. Rearranging the discretization and assembling will give below equations and system of matrices to be solved

$$u_i^{n+1} = -0.5\nu u_{i-2}^n + 2\nu u_{i-1}^n + (1 - 1.5\nu)u_i^n \tag{2.23}$$

$$\begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ u_4^{n+1} \\ \vdots \\ u_N^{n+1} \end{pmatrix} = \begin{bmatrix} 1-2\nu & 0 & 0 & 0 & \cdots & 0 \\ 2\nu & 1-1.5\nu & 0 & 0 & \cdots & 0 \\ -0.5\nu & 2\nu & 1-0.5\nu & 0 & \cdots & 0 \\ 0 & -0.5\nu & 2\nu & 1-0.5\nu & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & -0.5\nu & 2\nu & 1-0.5\nu \end{bmatrix} \begin{pmatrix} u_1^n \\ u_2^n \\ u_3^n \\ \vdots \\ u_N^n \end{pmatrix} + \begin{pmatrix} 2\nu u_0^n \\ -0.5\nu u_0^n \\ 0 \\ o \\ \vdots \\ 0 \end{pmatrix}$$

2.3 Crank-Nicolson, implicit integration

In implicit methods, the spatial derivative is calculated at current time step. Crank-Nicolson method is central difference in space and uses semi implicit integration. In this discretization, some part of the spatial derivative is calculated at current time step and some part at the previous time step.

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1} + u_{i+1}^n - u_{i-1}^n}{4\Delta x} = 0 \tag{2.24}$$

this method is second order in space and the leading error term is third derivative, which makes the scheme dispersive. Rearranging the discretization and assembling will give below equations and system of matrices to be solved

$$-\frac{\nu}{4}u_{i-1}^{n+1} + u_i^{n+1} + \frac{\nu}{4}u_{i+1}^{n+1} = \frac{\nu}{4}u_{i-1}^n + u_i^n - \frac{\nu}{4}u_{i+1}^n \tag{2.25}$$

$$\begin{bmatrix} 1 & \frac{\nu}{4} & 0 & \cdots & \cdots & \cdots \\ -\frac{\nu}{4} & 1 & \frac{\nu}{4} & 0 & \cdots & \cdots \\ 0 & -\frac{\nu}{4} & 1 & \frac{\nu}{4} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & -\frac{\nu}{4} & 1 & \frac{\nu}{4} \\ 0 & \cdots & \cdots & 0 & -\frac{\nu}{4} & 1 \end{bmatrix} \begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ \vdots \\ u_{N-1}^{n+1} \\ u_N^{n+1} \end{pmatrix} = \begin{bmatrix} 1 & -\frac{\nu}{4} & 0 & \cdots & \cdots & \cdots \\ \frac{\nu}{4} & 1 & -\frac{\nu}{4} & 0 & \cdots & \cdots \\ 0 & \frac{\nu}{4} & 1 & -\frac{\nu}{4} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \frac{\nu}{4} & 1 & -\frac{\nu}{4} \\ 0 & \cdots & \cdots & 0 & \frac{\nu}{4} & 1 \end{bmatrix} \begin{pmatrix} u_1^n \\ u_2^n \\ u_3^n \\ \vdots \\ u_{N-1}^n \\ u_N^n \end{pmatrix} + \begin{pmatrix} u_b \end{pmatrix}$$

the system of matrices in Crank-Nicolson method looks like Kx=b. These system of equations are solved using Thomas algorithm, which is used for solving tri diagonal system of matrices.

Von-Newmann error analysis will give below condition on stability for the scheme

$$\frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} + a\frac{\varepsilon_{i+1}^{n+1} - \varepsilon_{i-1}^{n+1} + \varepsilon_{i+1}^n - \varepsilon_{i-1}^n}{4\Delta x} = 0 \tag{2.26}$$

$$G - 1 + \frac{\nu}{4}(Ge^{j\theta} - Ge^{-j\theta} + e^{j\theta} - e^{-j\theta}) = 0 \tag{2.27}$$

$$G(1 + \frac{\nu}{2}sin\theta j) = (1 - \frac{\nu}{2}sin\theta j) \tag{2.28}$$

$$G = \frac{(1 - \frac{\nu}{2} sin\theta j)}{(1 + \frac{\nu}{2} sin\theta j)} \tag{2.29}$$

$$|G|^2 \leq 1 \tag{2.30}$$

the above condition is satisfied for any value of CFL number and this makes the scheme unconditionally stable

2.4 Lax-Wendroff scheme, explicit integration

Lax-Wendroff scheme, explicit integration method is given by

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{a}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) - \frac{\Delta t a^2}{2\Delta x^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) = 0 \tag{2.31}$$

this discretization is second order in space and time. The leading error term is third derivative, which makes the scheme dispersive. Rearranging the discretization and assembling will give below equations and system of matrices to be solved

$$u_i^{n+1} = \frac{\nu}{2}(1+\nu)u_{i-1}^n + (1-\nu^2)u_i^n + \frac{\nu}{2}(-1+\nu)u_{i+1}^n \tag{2.32}$$

$$a = \frac{\nu}{2}(1+\nu), b = (1-\nu^2), c = \frac{\nu}{2}(-1+\nu) \tag{2.33}$$

$$\begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ \vdots \\ u_{N-1}^{n+1} \\ u_N^{n+1} \end{pmatrix} = \begin{bmatrix} b & c & 0 & \cdots & \cdots & \cdots \\ a & b & c & 0 & \cdots & \cdots \\ 0 & a & b & c & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & a & b & c \\ 0 & \cdots & \cdots & 0 & a & b \end{bmatrix} \begin{pmatrix} u_1^n \\ u_2^n \\ u_3^n \\ \vdots \\ u_{N-1}^n \\ u_N^n \end{pmatrix} + \begin{pmatrix} au_1^n \\ 0 \\ 0 \\ \vdots \\ 0 \\ au_N^n \end{pmatrix}$$

Von-Newmann error analysis will give below condition on stability for the scheme

$$\frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} + \frac{a}{2\Delta x}(\varepsilon_{i+1}^n - \varepsilon_{i-1}^n) - \frac{\Delta t a^2}{2\Delta x^2}(\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n) = 0 \tag{2.34}$$

$$G = 1 - 2\nu^2 sin^2(\theta/2) - \nu sin(\theta) i \tag{2.35}$$

$$|G|^2 \leq 1 \tag{2.36}$$

$$0 < \nu \leq 1 \tag{2.37}$$

2.5 Lax Scheme, explicit integration

Lax-Scheme, explicit integration method is given by

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{a}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) - \frac{1}{2\Delta t}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) = 0 \tag{2.38}$$

The first spatial discretization term in the scheme is second order and it is dispersive. The last discretization term is dissipative which counteracts dispersion and makes the scheme stable. Rearranging the discretization and assembling will give below equations and system of matrices to be solved.

$$u_i^{n+1} = \frac{\nu + 1}{2} u_{i-1}^n + \frac{1 - \nu}{2} u_{i+1}^n \tag{2.39}$$

$$a = \frac{\nu + 1}{2}, c = \frac{1 - \nu}{2} \tag{2.40}$$

$$\begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ \vdots \\ u_{N-1}^{n+1} \\ u_N^{n+1} \end{pmatrix} = \begin{bmatrix} 0 & c & 0 & \cdots & \cdots & \cdots \\ a & 0 & c & 0 & \cdots & \cdots \\ 0 & a & 0 & c & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & a & 0 & c \\ 0 & \cdots & \cdots & 0 & a & 0 \end{bmatrix} \begin{pmatrix} u_1^n \\ u_2^n \\ u_3^n \\ \vdots \\ u_{N-1}^n \\ u_N^n \end{pmatrix}$$

2.6 First-order backward difference, implicit integration

The first order backward difference scheme using implicit integration is expressed as

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_i^{n+1} - u_{i-1}^{n+1}}{\Delta x} = 0 \tag{2.41}$$

Rearranging the discretization and assembling will give below equations and system of matrices to be solved

$$-\nu u_{i-1}^{n+1} + (1 + \nu) u_i^{n+1} = u_i^n \tag{2.42}$$

$$\begin{bmatrix} 1+\nu & 0 & 0 & \cdots & \cdots & \cdots \\ -\nu & 1+\nu & 0 & 0 & \cdots & \cdots \\ 0 & -\nu & 1+\nu & 0 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & -\nu & 1+\nu & 0 \\ 0 & \cdots & \cdots & 0 & -\nu & 1+\nu \end{bmatrix} \begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ \vdots \\ u_{N-1}^{n+1} \\ u_N^{n+1} \end{pmatrix} = \begin{pmatrix} u_1^n \\ u_2^n \\ u_3^n \\ \vdots \\ u_{N-1}^n \\ u_N^n \end{pmatrix}$$

the system of matrices using this method is of the form Kx=b. The first row in the K matrix contains only one element and this makes it to easily solve the system of matrices by first calculating u-value at node i=1 and propagating towards node i=N using a loop.

Chapter 3

Results

The linear wave equation is solved with different numerical schemes using a MATLAB script and the results are discussed in this chapter. The boundary conditions used in this problem is fixed on both the ends and the value of speed of wave(a) is taken as one.

3.1 First-order backward difference, explicit integration

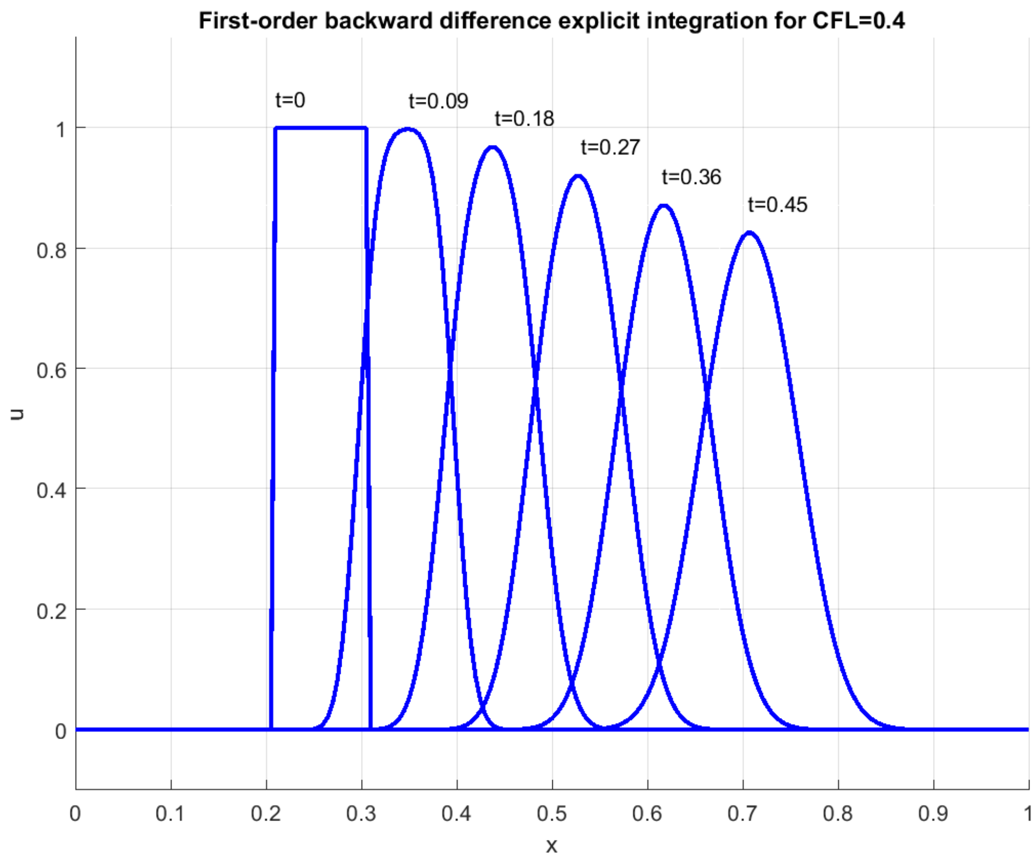

Figure 3.1: First-order backward difference explicit integration for CFL= 0.4

Figure 3.1 shows the solution using first-order backward difference explicit integration method for CFL=0.4. The solution shows that high gradients at the initial disturbance is damped down to a smooth one with time and the reason is because the leading error term in the scheme is second derivative, which is dissipative in nature.

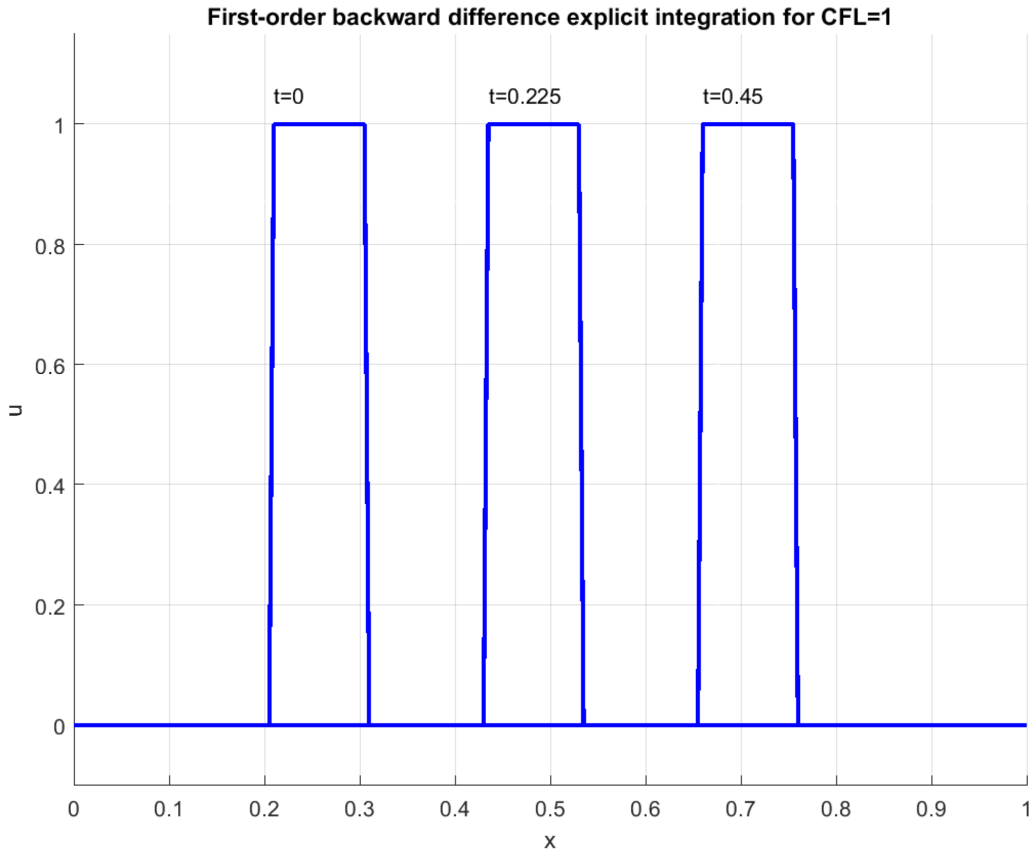

Figure 3.2: First-order backward difference explicit integration for CFL= 1.0

Figure 3.2 shows that the solution for CFL=1 is a perfect square disturbance propagates without any damping. This is because at CFL=1, the discretization becomes exact differential as the leading error term goes to zero.

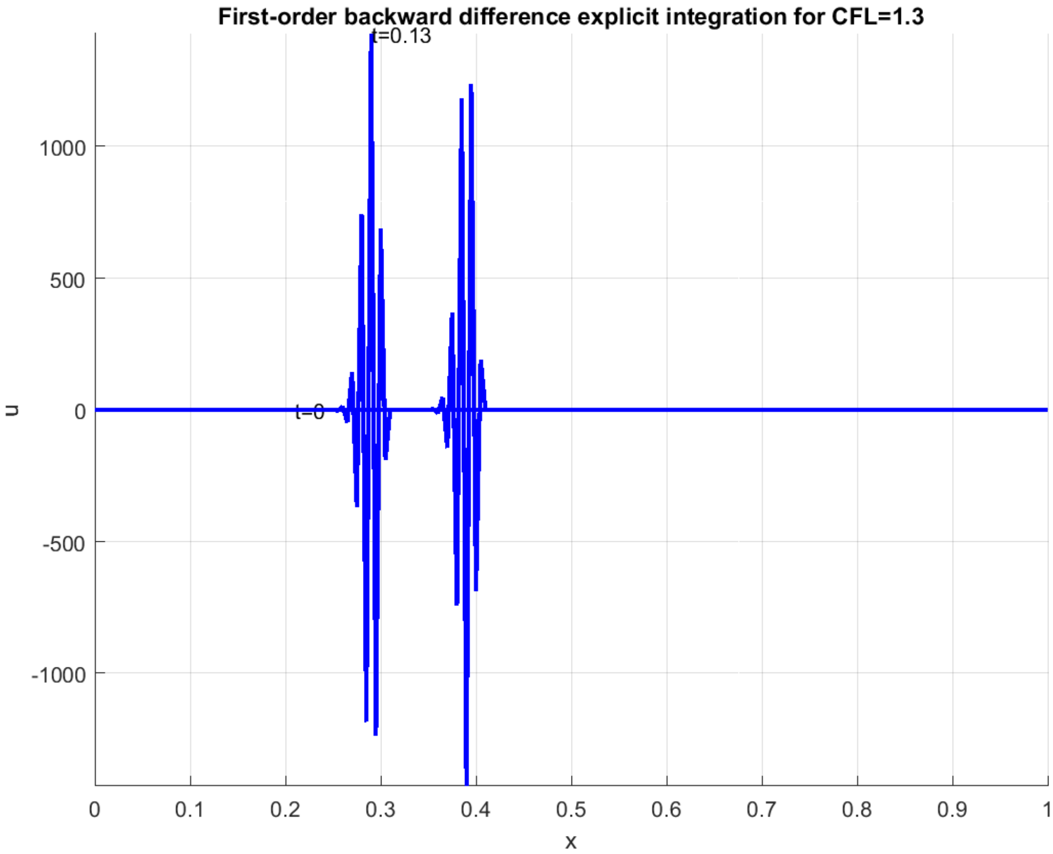

Figure 3.3: First-order backward difference explicit integration for CFL= 1.3

Figure 3.3 shows that the solution for CFL=1.3 is unstable. This is because the CFL number is out of limits for the scheme to be stable (Eq. 2.21).

3.2 Second-order backward difference, explicit integration

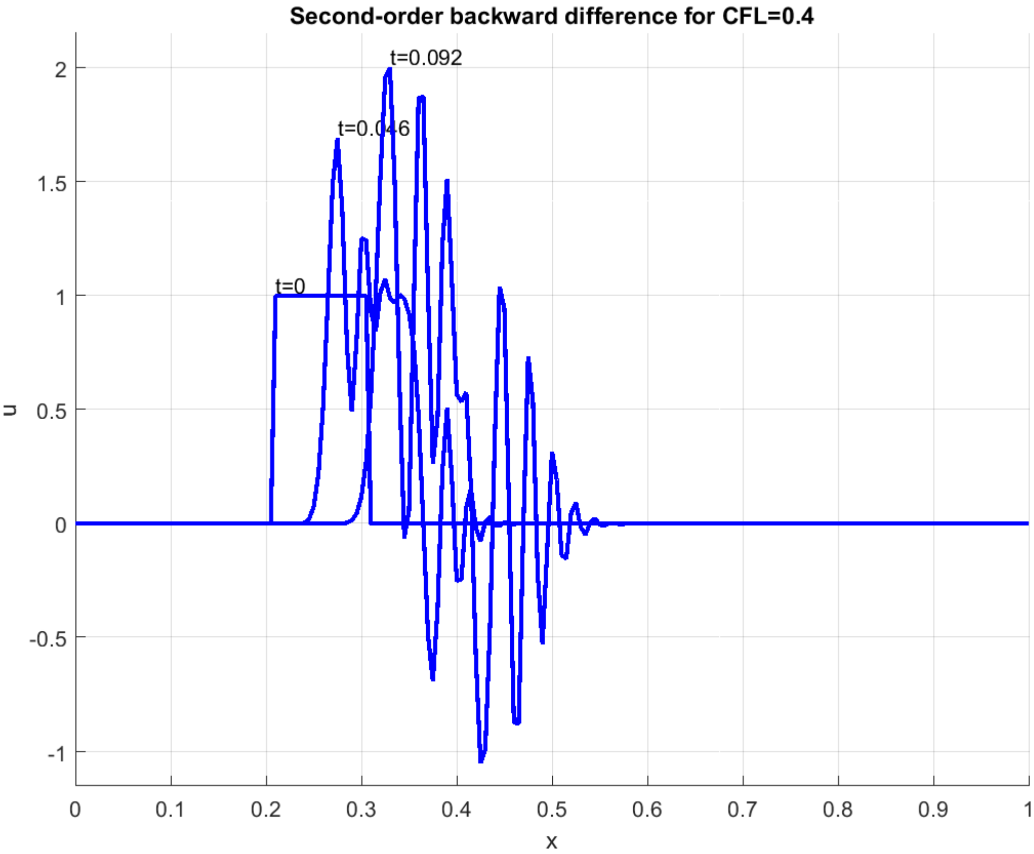

Figure 3.4: Second-order backward difference explicit integration for CFL= 0.4

Figure 3.4 shows that there are spurious oscillations in the solution. This is because the leading error term in second order backward difference is a third derivative and odd derivatives are dispersive.

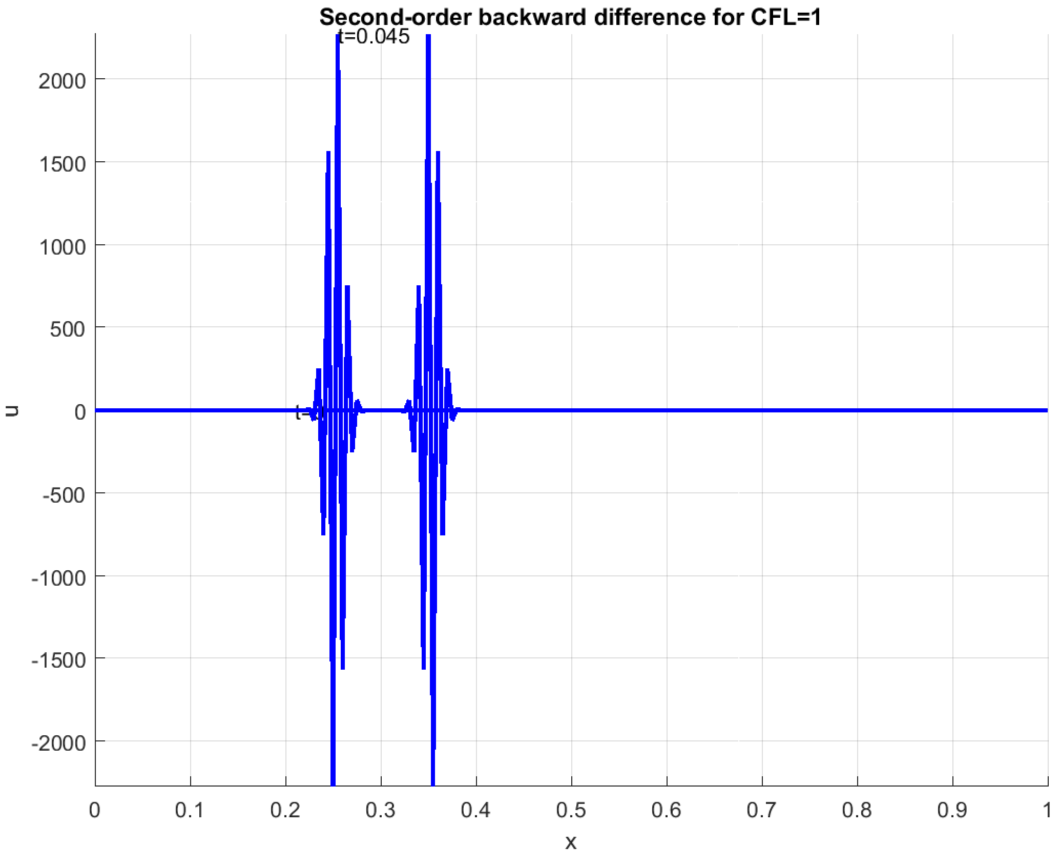

Figure 3.5: Second-order backward difference explicit integration for CFL= 1.0

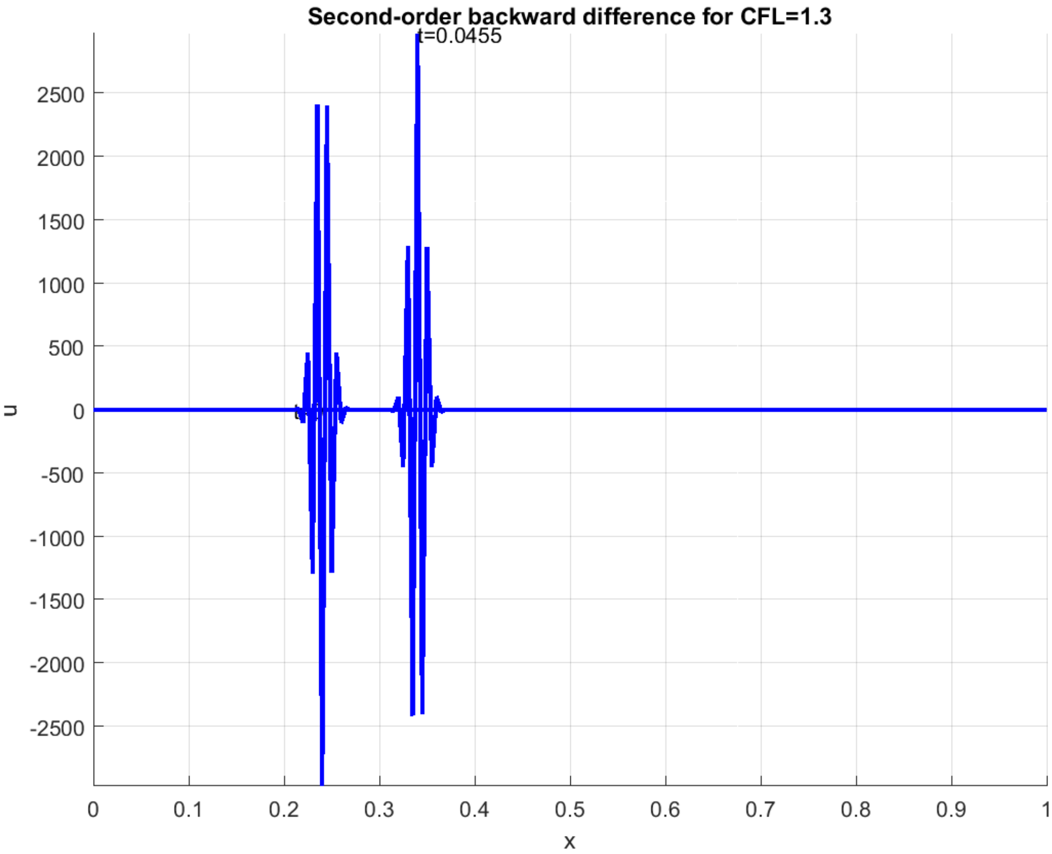

Figure 3.6: Second-order backward difference explicit integration for CFL= 1.3

Figure 3.5 and Figure 3.6 shows that the solution is unphysical for CFL number 1 and 1.3. The reason for the behavior is due to the scheme is unstable for these CFL numbers

3.3 Crank-Nicolson, implicit integration

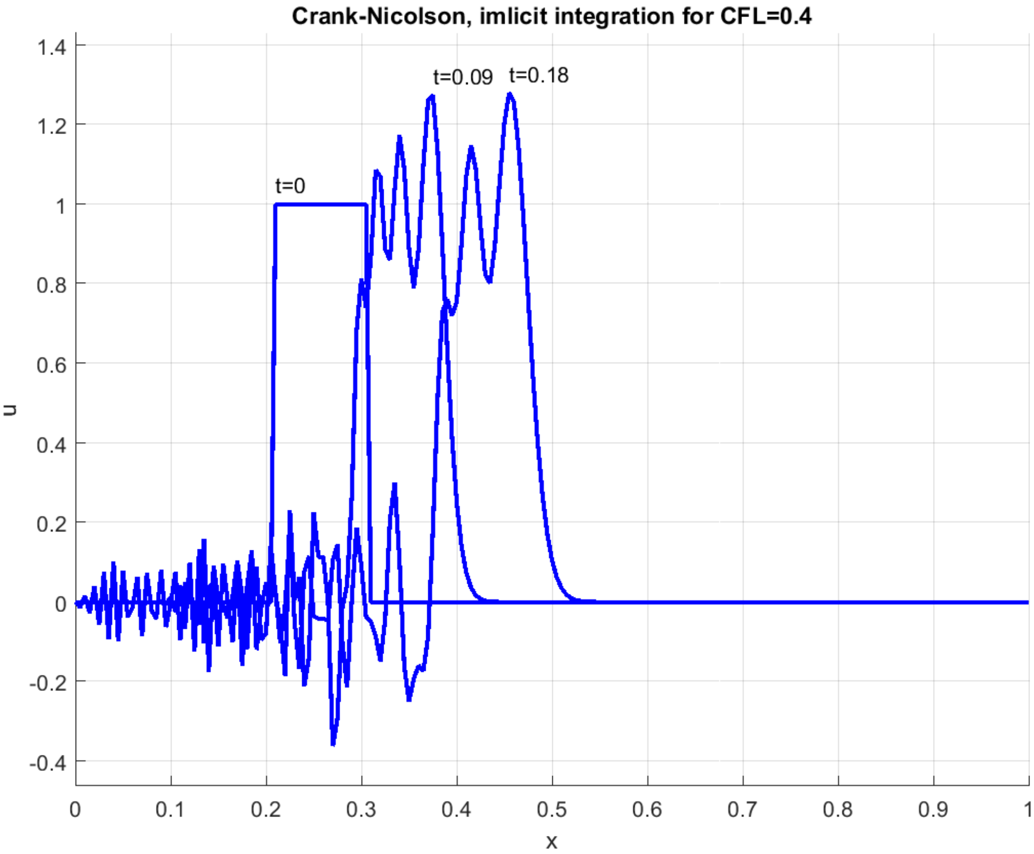

Figure 3.7: Crank-Nicolson implicit integration for CFL= 0.4

The results using Crank-Nicolson implicit method shows spurious oscillations in the solution. This is because the leading error term contains third derivative and the odd derivatives are dispersive. The results also shows that the scheme behaves similar for all CFL numbers because of its unconditionally stable nature.

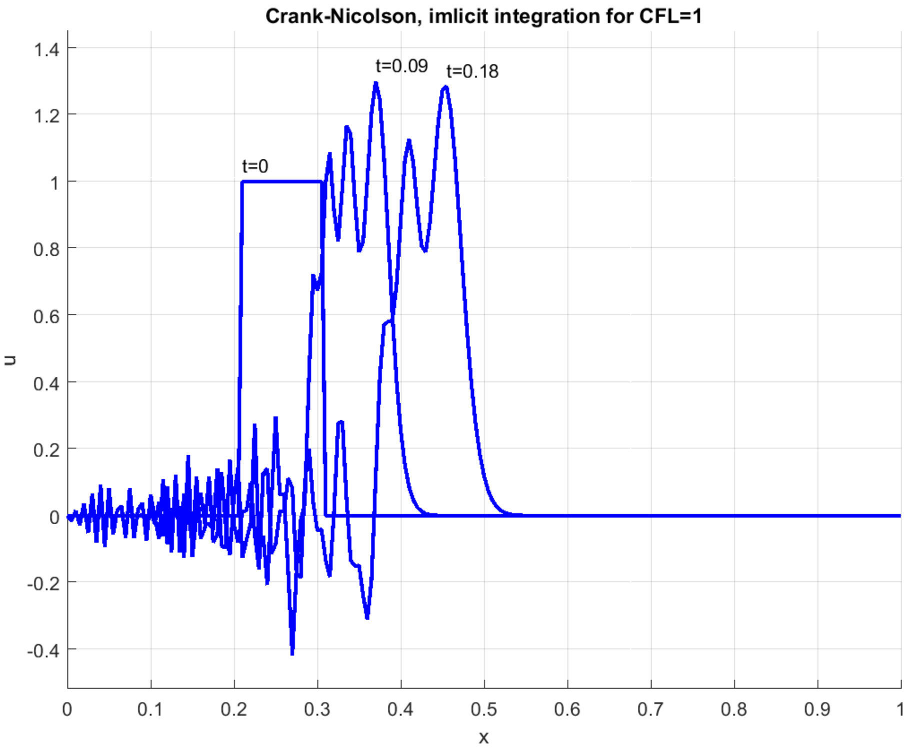

Figure 3.8: Crank-Nicolson implicit integration for CFL= 1.0

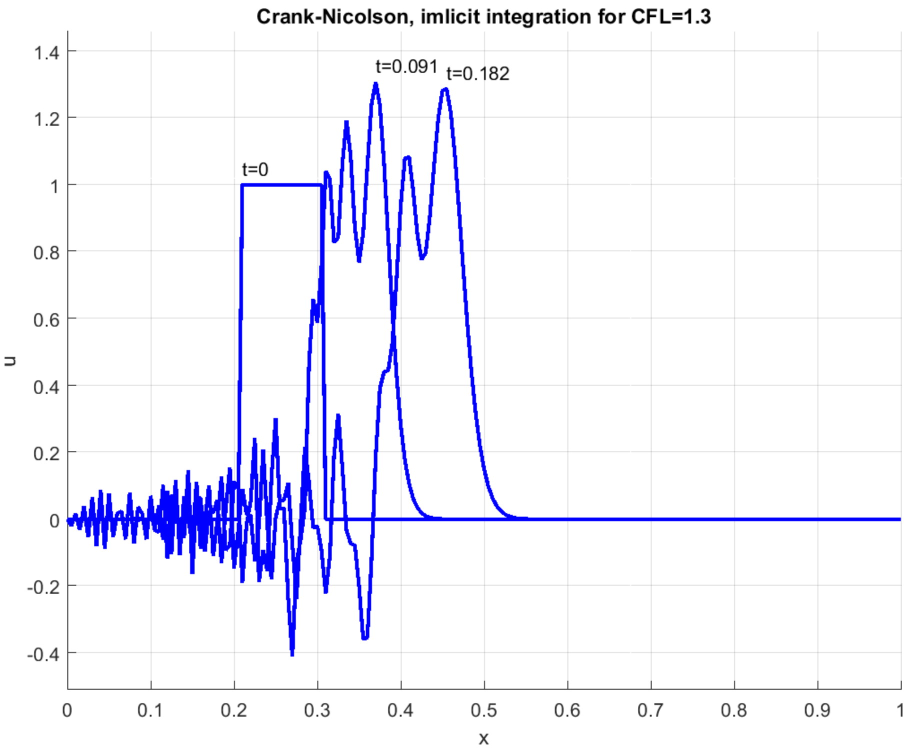

Figure 3.9: Crank-Nicolson implicit integration for CFL= 1.3

3.4 Lax-Wendroff scheme, explicit integration

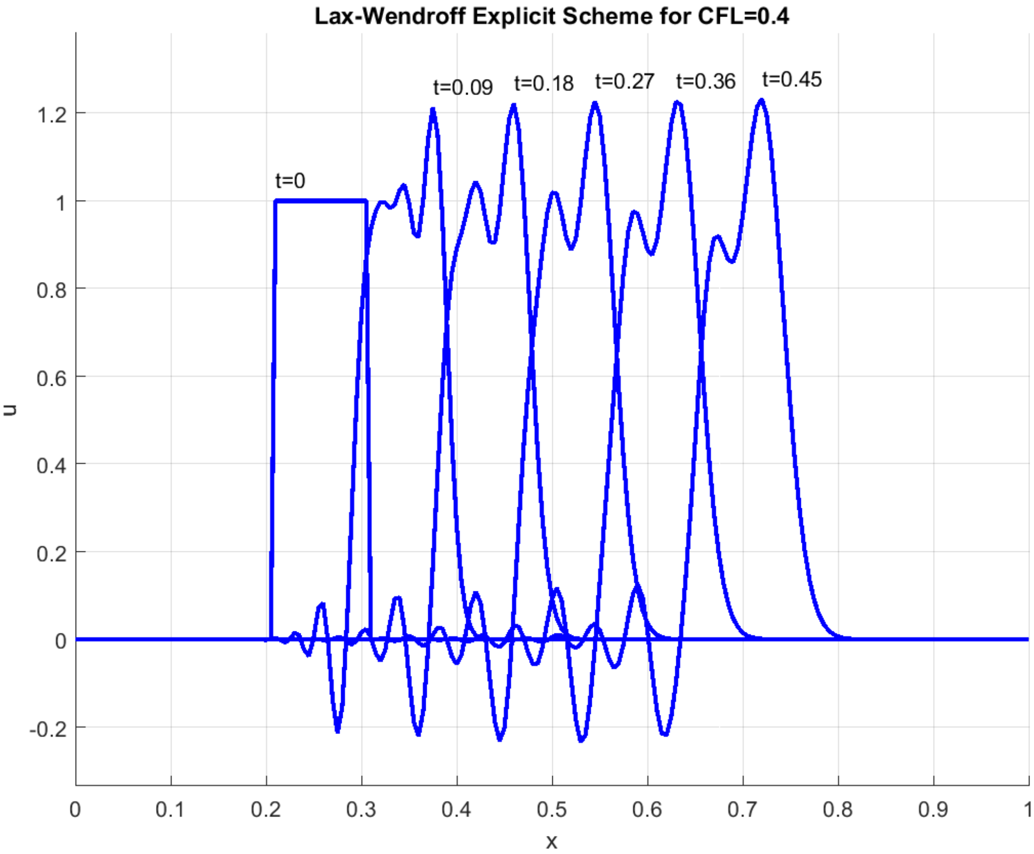

Figure 3.10: Lax-Wendroff implicit integration for CFL= 0.4

Figure 3.10 shows the solution using Lax-Wendroff scheme, explicit integration for CFL=0.4. The result shows that the initial disturbance is propagated with few oscillations. These oscillations are very small compared to Crank-Nicolson method and it can also be observed that there is small damping of the disturbance. This is because the dispersion caused by the central difference term (Eq. 2.31) in the scheme is neutralized by the dissipative nature of the central difference of second order derivative.

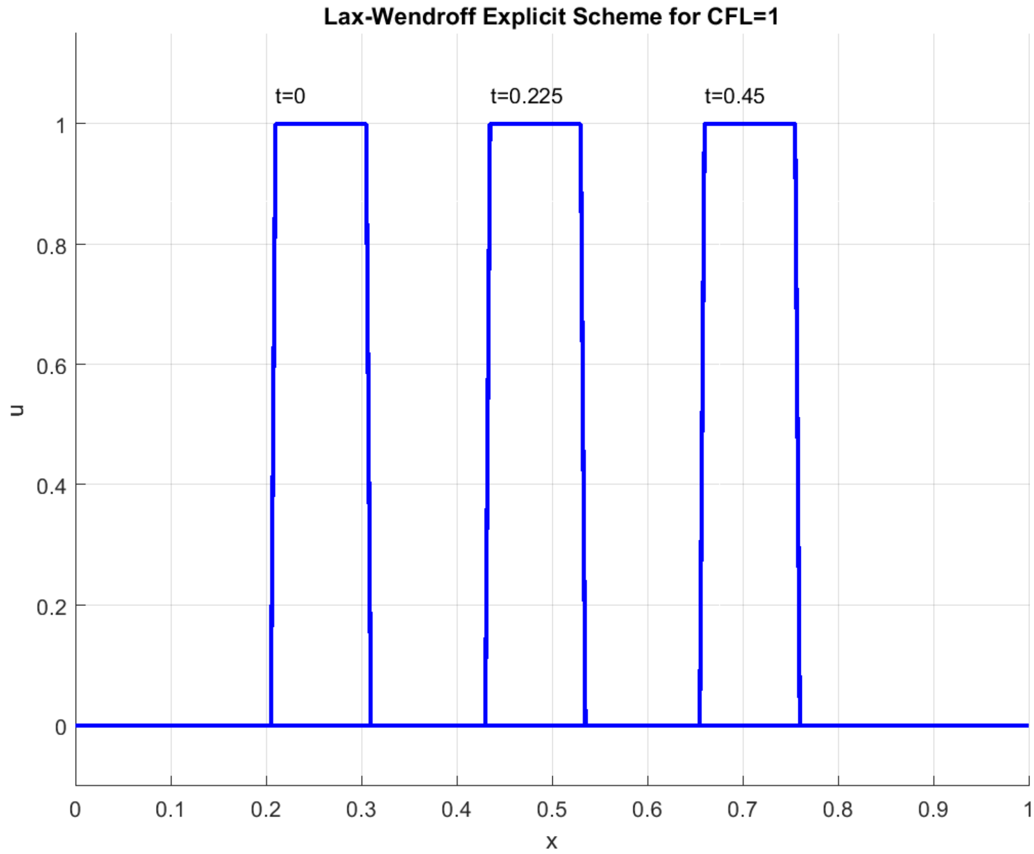

Figure 3.11: Lax-Wendroff implicit integration for CFL= 1.0

Figure 3.11 shows that the solution for CFL=1 is a perfect square disturbance propagates without any damping. This is because at CFL=1, the discretization becomes exact differential and the leading error goes to zero.

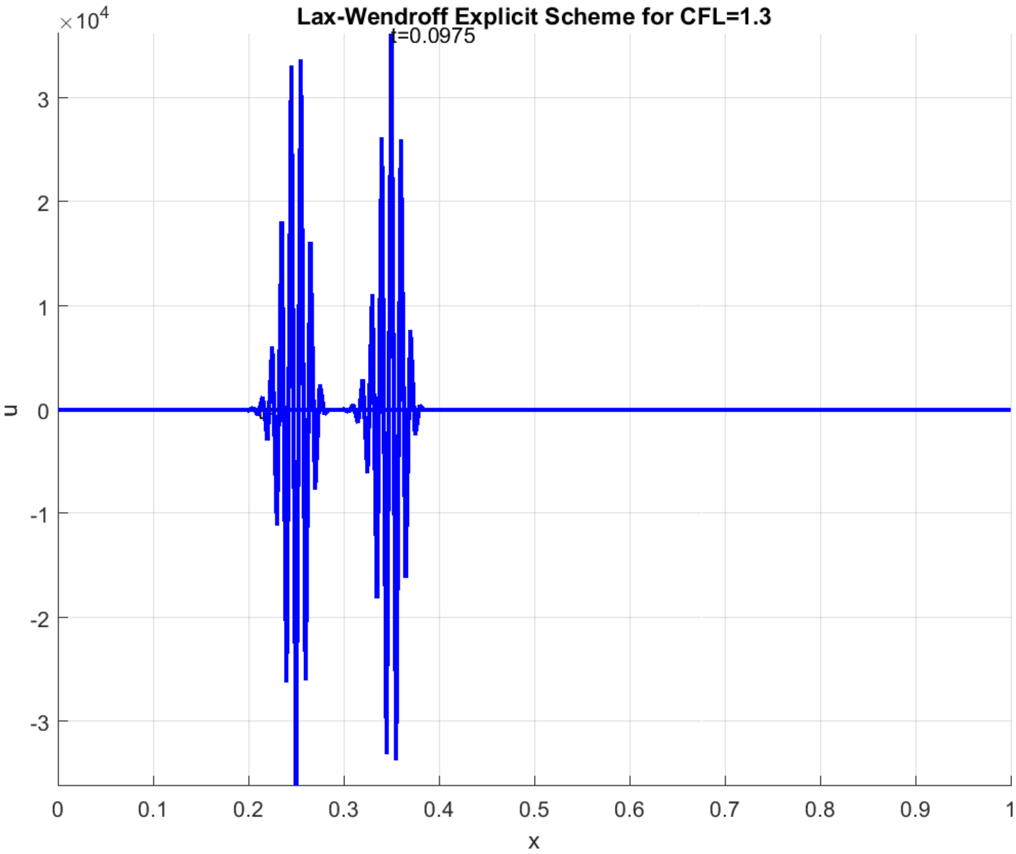

Figure 3.12: Lax-Wendroff implicit integration for CFL= 1.3

Figure 3.12 shows that the solution for CFL=1.3 is unstable. This is because the CFL number is out of limits for the scheme to be stable (Eq. 2.37).

3.5 Lax Scheme, explicit integration

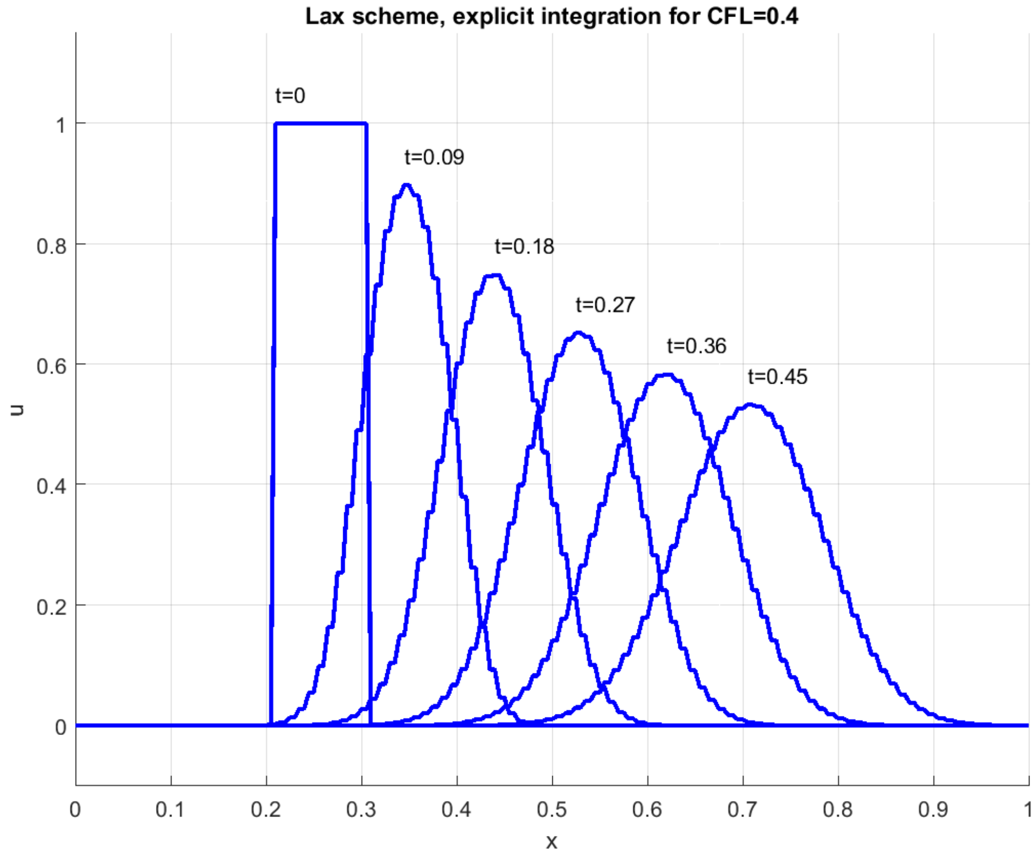

Figure 3.13: Lax scheme explicit integration for CFL= 0.4

Figure 3.13 plots the solution using Lax-Scheme, explicit integration for CFL=0.4. The result shows that the initial disturbance is damped with time and the disturbance is not smooth. The reason for unsmooth profile of the propagating wave is because of the dispersive nature of central difference term (Eq. 2.38).

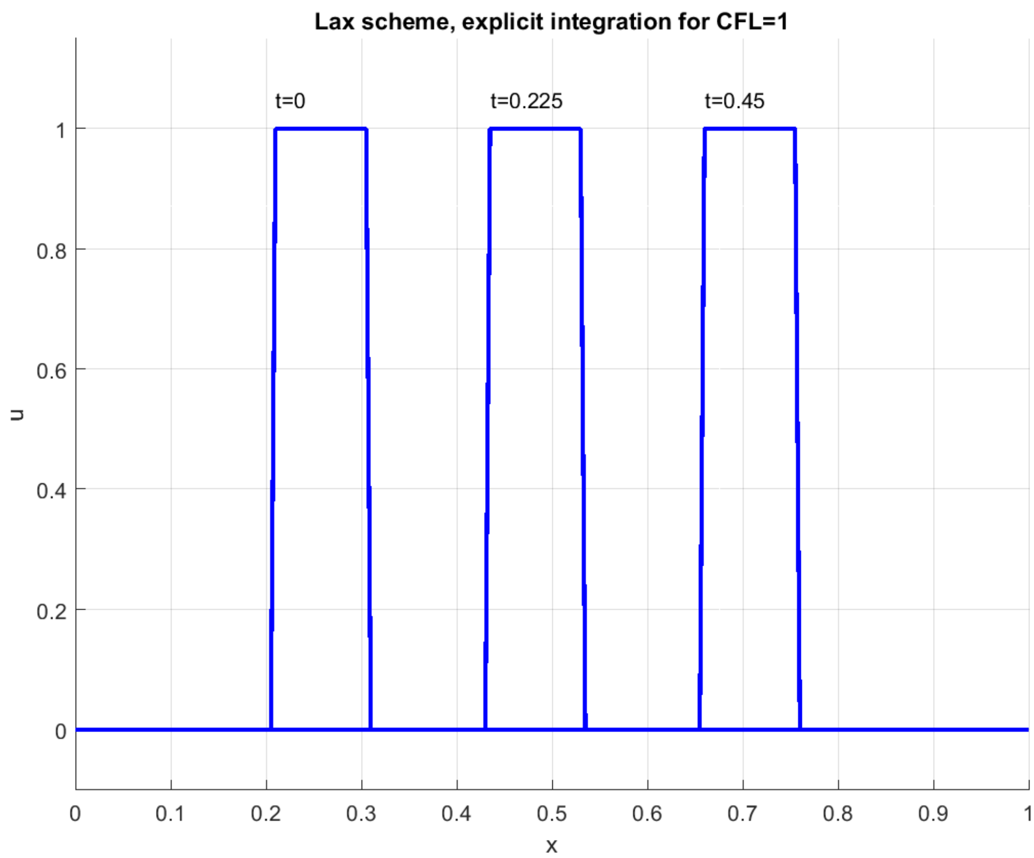

Figure 3.14: Lax scheme explicit integration for CFL= 1.0

Figure 3.14 shows that the solution for CFL=1 is a perfect square disturbance propagates without any damping. This is because at CFL=1, the discretization becomes exact differential and the leading error goes to zero.

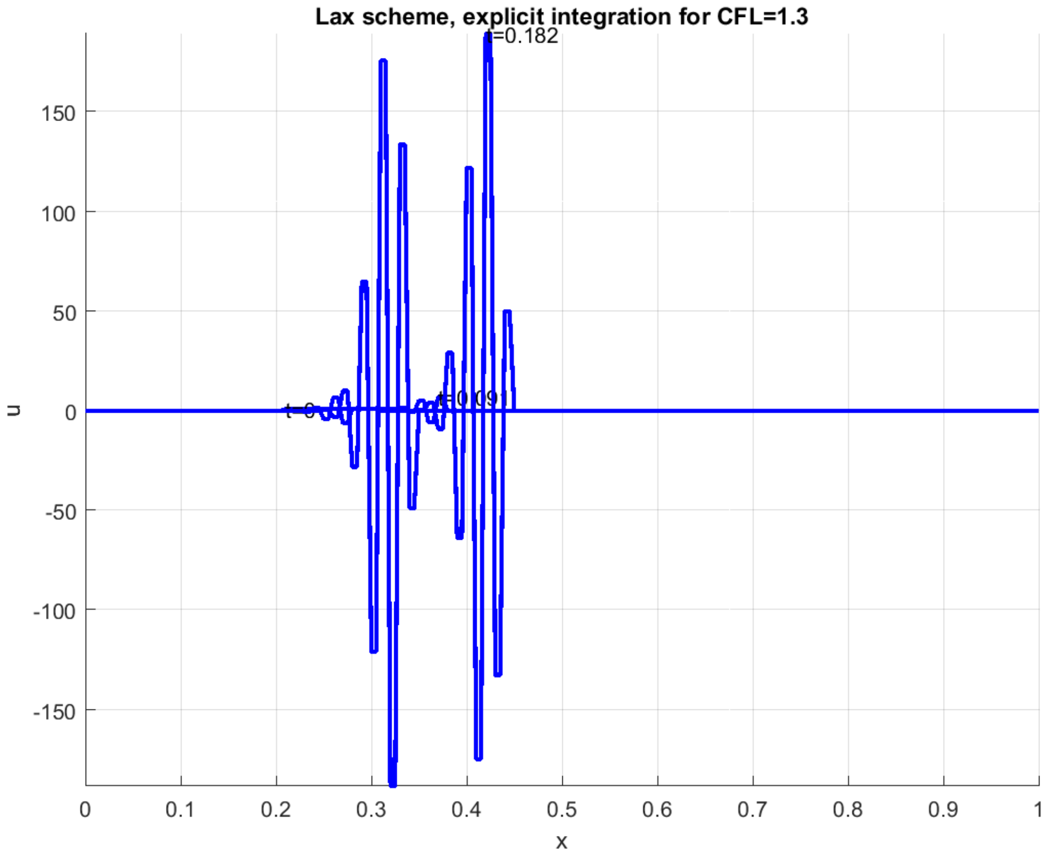

Figure 3.15: Lax scheme explicit integration for CFL= 1.3

Figure 3.15 shows that the solution for CFL=1.3 is unstable. This is because the CFL number is out of limits for the scheme to be stable.

3.6 First-order backward difference, implicit integration

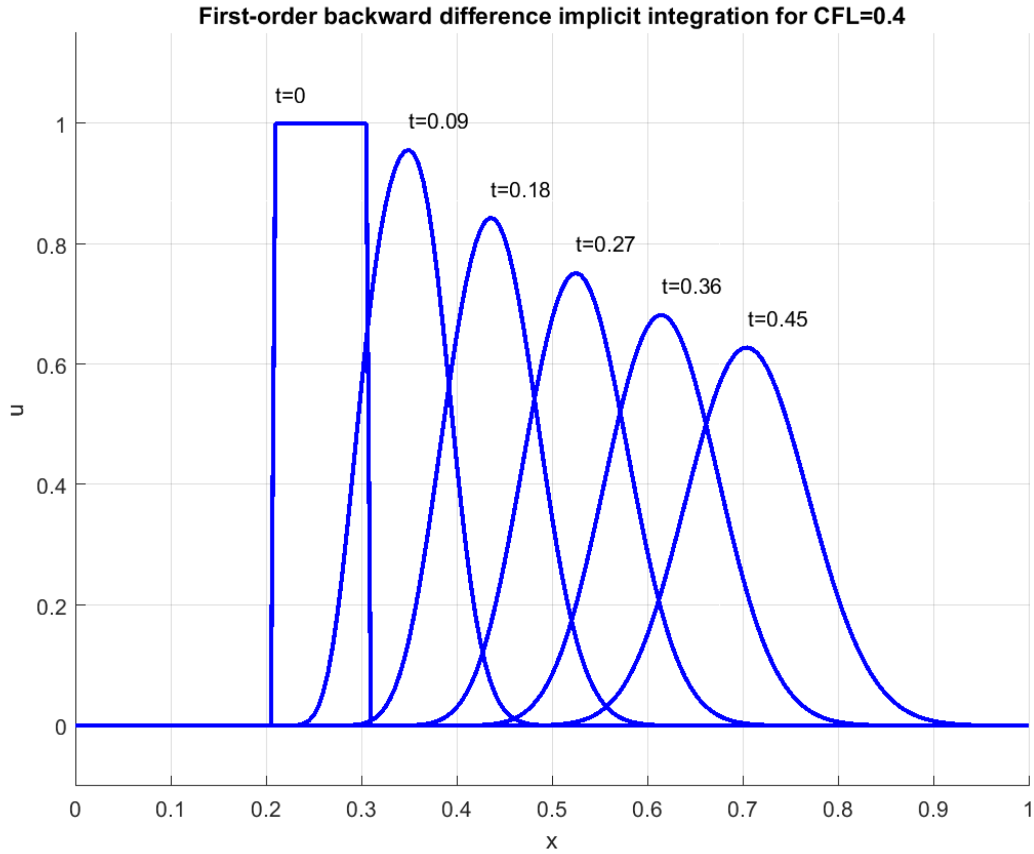

Figure 3.16: First-order backward difference implicit integration for CFL= 0.4

The results of first-order backward difference, implicit integrations shows that the behavior of the scheme is dissipative and the solution is stable for all CFL numbers because it is unconditionally stable. Further it can be inferred from the solutions that both explicit and implicit schemes are dissipative however the implicit scheme is more dissipative compared to explicit.

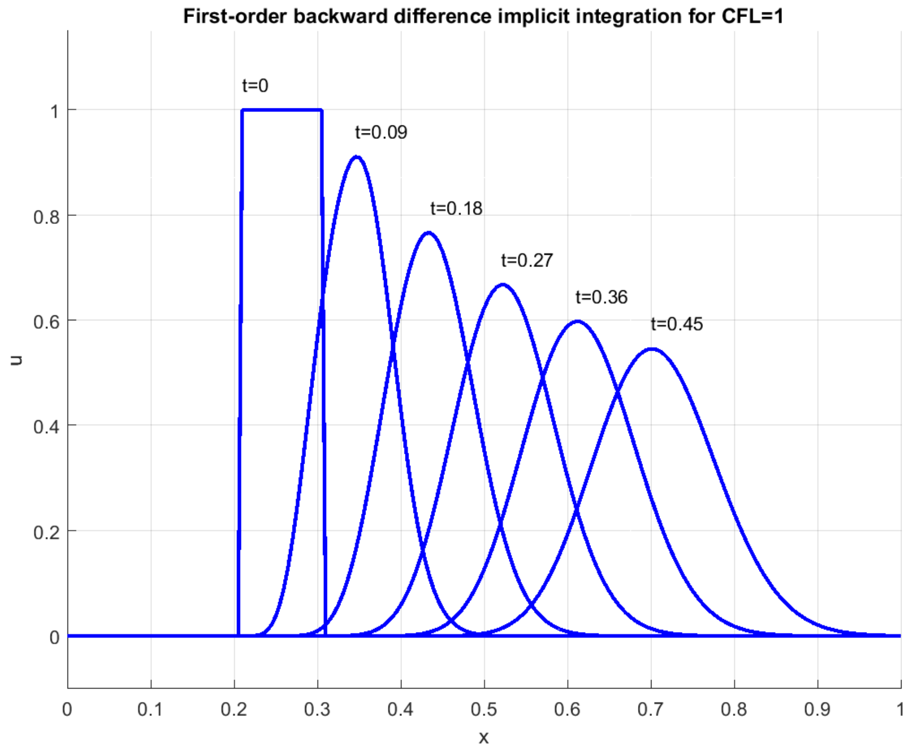

Figure 3.17: First-order backward difference implicit integration for CFL= 1.0

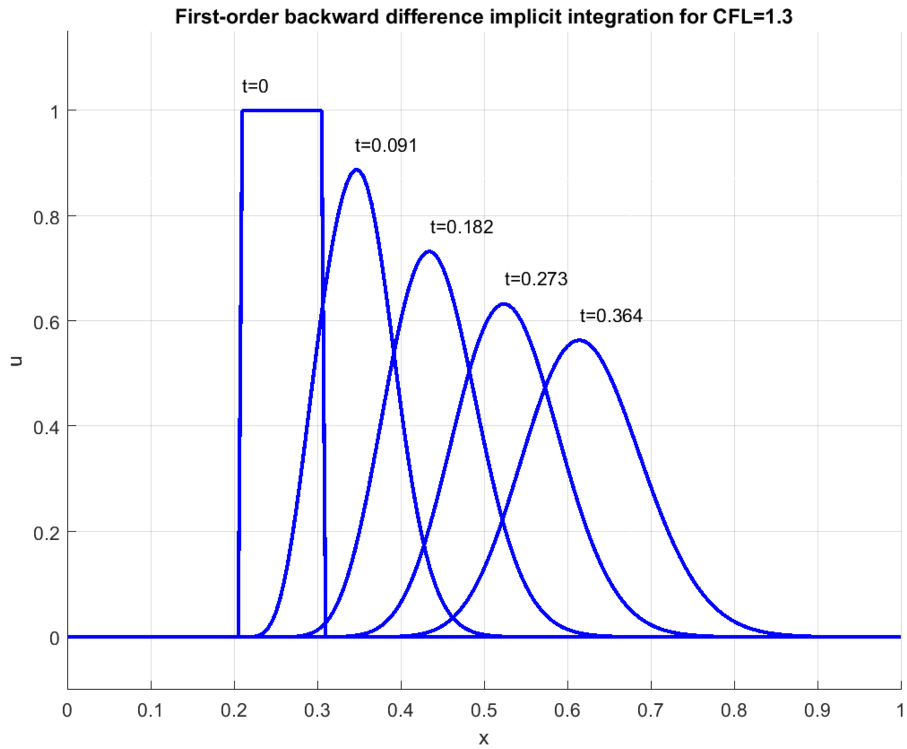

Figure 3.18: First-order backward difference implicit integration for CFL= 1.3

Chapter 4

Appendix