MA 202: MATHEMATICS IV

Numerical Analysis of Dynamics of the Foucault Pendulum

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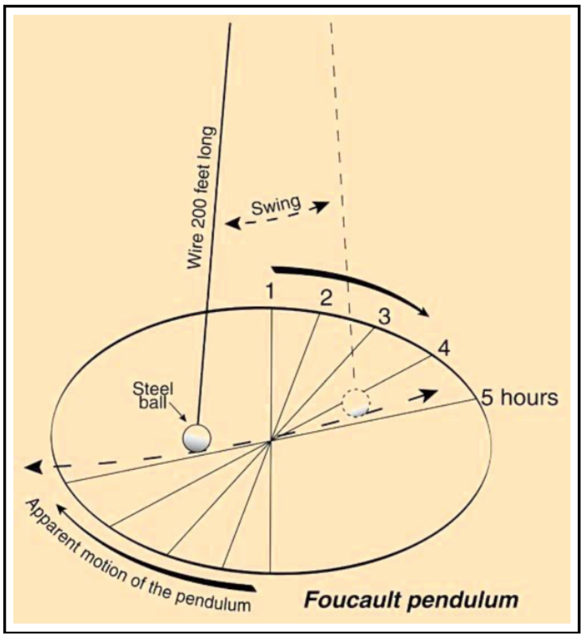
1. Introduction

1.1 Problem Statement

The Foucault Pendulum is a known device in the physical world, used to demonstrate the theory of the Earth Rotation. It is a beneficial but straightforward device. It is a low-cost demonstration of the rotation of the Earth. It was introduced by French physicist "Sir Léon Foucault" in 1851 as part of an experiment to demonstrate the theory of Earth's rotation. The Foucault pendulum is simply a pendulum that can oscillate freely in any plane.It was the first successful demonstration of the earth's rotation in a laboratory setting rather than through astronomical observations.

Foucault found out that the pendulum's plane of oscillation is not affected by the twist of its fixed point. Thus, the pendulum's plane of oscillation can be used as a fixed plane to prove Earth's rotation. The results of the experiment successfully proved the phenomenon of Earth's rotation. Hence, it is crucial for us to understand the dynamics of the Foucault Pendulum.



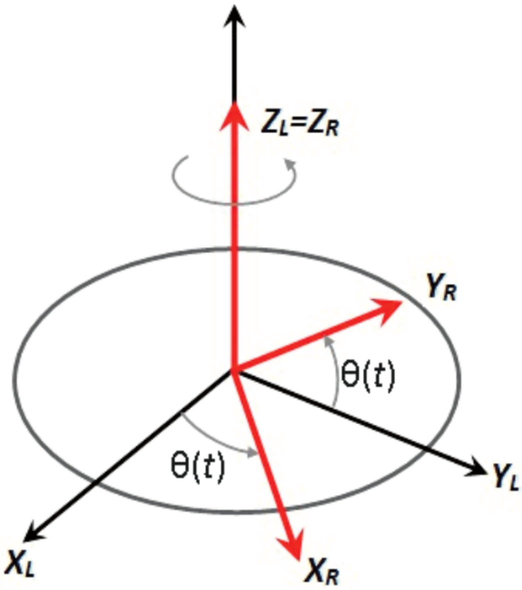


Figure 1: Schematic representation of Focault Pendulum Swinging

The dynamics of the Foucault Pendulum can be modeled mathematically using Newton's laws of motion. Reasonable assumptions that concur with the model in a practical sense are made for simplification of the governing equations. We employ various numerical methods, having different orders of accuracy, to conduct a comprehensive study of the dynamics involved with the pendulum's swinging motion. We will also compare and examine the results obtained by numerical methods with those obtained through analytical methods.

1.2 Physical Model along with Problems

We have to study its physical model before solving it mathematically. From the physical model, we proceed with the construction of the mathematical model. Our aim in this project is to study the dynamics of the Foucault Pendulum, which is subjected to perturbations due to the Coriolis effect. In doing so, we will consider the below-mentioned physical model.

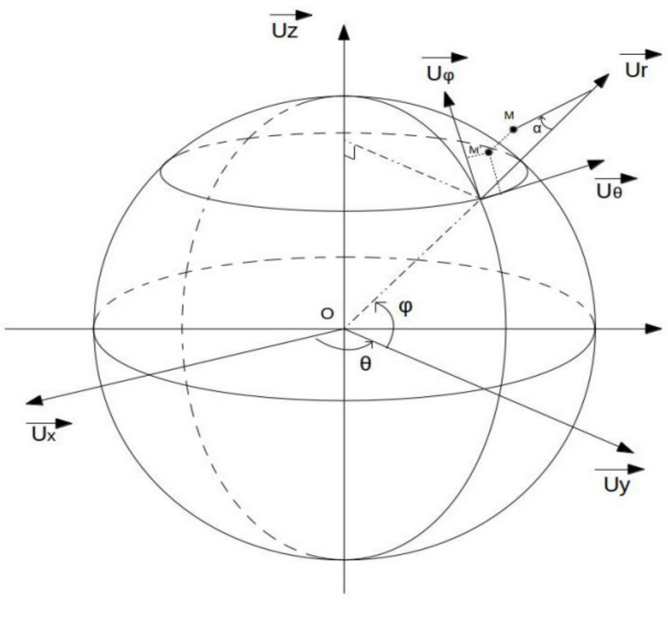


Figure 2: Terrestrial spherical frame of reference

We will consider the terrestrial frame of reference for the entire problem. Let us assume that the pendulum is situated at an altitude of φ concerning the origin (center of the Earth). Let us take the pendulum's position with respect to both the spherical coordinate system and cartesian coordinate system in 3d. The vectors Ux, Uy, and Uz, represent the base of the coordinate system w.r.t x-axis, y-axis, z-axis. On the other hand, the vectors Ur, Uθ, and Uφ are the base of the spherical coordinate system w.r.t radial distance, polar axis, and the azimuthal angle. Here, W is the weight of the pendulum's bob, T is the tension of the wire, and l is the length of the wire.

The pendulum oscillates in the plane, which is considered fixed in space and can be used to compute the rotation of Earth with respect to it. While figuring out the mathematical governing equations for the pendulum dynamics, we will consider the Coriolis force, which is a result of Earth's rotation. The perturbations created by the Coriolis forces reduce the accuracy of the pendulum. This will be considered in the 'State Equations' section.

2. Mathematical Formulation

2.1 Notations:

The below notations are used for the whole problem.

(i) Ui - Position vector of the pendulum

(ii) φ - Pendulum's altitude

(iii) Ω - Rotational vector of Earth

(iv) P - Weight of the bob of the pendulum

(v) T - Tension in the wire of pendulum

(vi) L - Length of the wire of pendulum

(vii) Γ - Bob's acceleration vector

(viii) m - Mass of pendulum bob

(ix) g - Gravitational Acceleration

2.2 Assumptions

We will make the following assumptions to study the dynamics of the Foucault Pendulum.

1. Perturbations due to Coriolis Effect are considered, but permutations due to other forces are not.

2. Earth is considered to be a perfect sphere.

3. Earth is assumed to be an inertial frame of reference.

4. Latitude of the pendulum is constant.

5. Rotation speed of the Earth is constant.

6. Variation in the position vector of bob is negligible compared to the length of the pendulum.

3. Model

3.1 State Equations with Newtonian Method

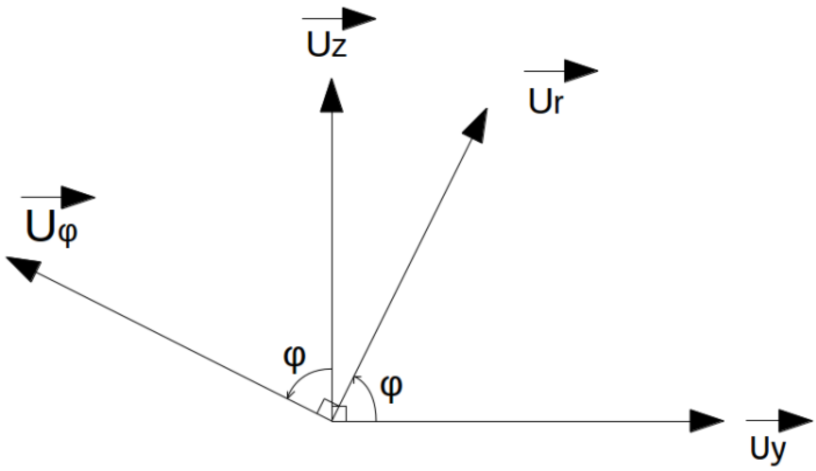


Figure 3: Plane depicting the projection of Earth's rotational vector onto the local frame of reference

As seen before, we have considered that the Earth rotates about an axis $\vec{U}_z$.

We will explore the dynamics of the Foucault Pendulum on a local scale for our convenience. We will project the Earth's rotation on the local reference as shown in figure 3.

$\vec{U}_z$ can be expressed as

$$\vec{U}_z = sin\phi\vec{U}_r \;+\; cos\phi\vec{U}_\phi \tag{1}$$

In local frame of reference, Earth rotational vector $\vec{\Omega}$ can be expressed as:

$$\begin{Bmatrix} 0 \\ 0 \\ \Omega \end{Bmatrix}_{(x,y,z)} = \begin{Bmatrix} \Omega\sin\phi \\ 0 \\ \Omega\cos\phi \end{Bmatrix}_{(r,\theta,\phi)} \tag{2}$$

The local frame of reference where the pendulum will swing is depicted as follows:

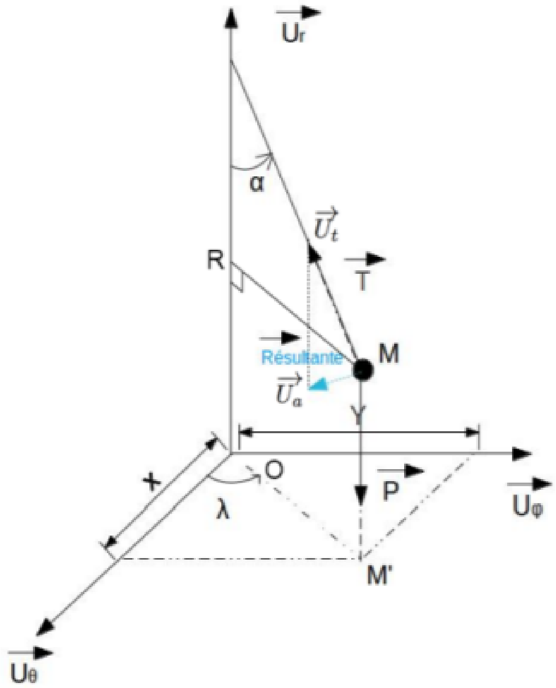


Figure 4: Local frame of reference of the pendulum

Let M denote the instantaneous position of the bob during its swinging motion.

Let R denote the projection of point M on the axis $\overline{Ur}$.

$$r = Radius\ of\ Earth + |\vec{OR}| \tag{3}$$

x , y - positions in projection to vectors $\vec{U}_\theta$ and $\vec{U}_\phi$ respectively.

$$\vec{OM} = \vec{OR} + \vec{RM} = r\vec{U}_r + x\vec{U}_\theta + y\vec{U}_\phi \tag{4}$$

Given that the position vector of the bob is $\vec{OM}$

Velocity of the bob,

$$\vec{V}(\overrightarrow{OM})_R = \frac{d(\overrightarrow{OM})}{dt_{(R)}} = \frac{d}{dt_{(R)}}(r\vec{U_r}+x\vec{U_\theta}+y\vec{U_\phi}) = \dot{r}\vec{U_r}+\dot{x}\vec{U_\theta}+\dot{y}\vec{U_\phi}+r\frac{d(\vec{U_r})}{dt_{(R)}}+x\frac{d(\vec{U_\theta})}{dt_{(R)}}+y\frac{d(\vec{U_\phi})}{dt_{(R)}} \tag{5}$$

The temporal derivatives of the vectors $\vec{U}_r, \vec{U}_\theta$ *and* $\vec{U}_\phi$ are dependent on the inertial frame of reference because they are vectors of the noninertial local frame of reference.

The Earth is supposed to be immovable or stationary in our experiment. As a result, Earth is the reference frame of inertia $\vec{U}_i$ is a vector's time derivative in a non-inertial frame of reference

In terms of the inertial frame of reference, reference is represented as:

$$\frac{d(\overrightarrow{U_i})}{dt_{(R)}} = \frac{d(\overrightarrow{U_r})}{dt_{(R')}} + \overrightarrow{\Omega_{(R'/R)}} \times \overrightarrow{U_i} \tag{6}$$

where $\vec{U}_i$ is the position vector of the pendulum, R is the terrestrial frame of reference or Earth, R' is the local frame of reference. As the local frame of reference does not accelerate with respect to its own reference

$$\frac{d(\overrightarrow{U_i})}{dt_{(R')}} = 0 \tag{7}$$

The time derivative of the local frame vector with respect to its own frame will be zero. Hence, the derivatives of the local plane vectors $\vec{U}_r$, $\vec{U}_\theta$ *and* $\vec{U}_\phi$ with respect to time are as follows:

$$\frac{d(\overrightarrow{U_r})}{dt_{(R)}} = \frac{d(\overrightarrow{U_r})}{dt_{(R')}} + \begin{bmatrix} \Omega\sin\phi \\ 0 \\ \Omega\cos\phi \end{bmatrix}_{(r,\theta,\phi)} \times \begin{bmatrix} U_r \\ 0 \\ 0 \end{bmatrix}_{(r,\theta,\phi)} = \begin{bmatrix} 0 \\ \Omega\cos\phi \\ 0 \end{bmatrix}_{(r,\theta,\phi)}$$

$$\frac{d(\overrightarrow{U_\theta})}{dt_{(R)}} = \frac{d(\overrightarrow{U_\theta})}{dt_{(R')}} + \begin{bmatrix} \Omega\sin\phi \\ 0 \\ \Omega\cos\phi \end{bmatrix}_{(r,\theta,\phi)} \times \begin{bmatrix} 0 \\ U_\theta \\ 0 \end{bmatrix}_{(r,\theta,\phi)} = \begin{bmatrix} -\Omega\cos\phi \\ 0 \\ \Omega\sin\phi \end{bmatrix}_{(r,\theta,\phi)}$$

$$\frac{d(\overrightarrow{U_\phi})}{dt_{(R)}} = \frac{d(\overrightarrow{U_\phi})}{dt_{(R')}} + \begin{bmatrix} \Omega\sin\phi \\ 0 \\ \Omega\cos\phi \end{bmatrix}_{(r,\theta,\phi)} \times \begin{bmatrix} 0 \\ 0 \\ U_\phi \end{bmatrix}_{(r,\theta,\phi)} = \begin{bmatrix} 0 \\ -\Omega\sin\phi \\ 0 \end{bmatrix}_{(r,\theta,\phi)} \qquad (8)$$

where $\vec{U}_r, \vec{U}_\theta$ *and* $\vec{U}_\phi$ are unit vectors.

The variation in ‘r’ with respect to the pendulum length and oscillation can be neglected. Hence,

$$\frac{dr}{dt} = \dot{r} = 0 \qquad (9)$$

The velocity is given by the bob is :

$$\begin{aligned}\vec{V}(\overrightarrow{OM})_R &= 0 + r\Omega\cos\phi\overrightarrow{U_\theta} + \dot{x}\overrightarrow{U_\theta} + x\Omega(\sin\phi\overrightarrow{U_\phi} - \cos\phi\overrightarrow{U_r}) + \dot{y}\overrightarrow{U_\phi} - y\Omega\sin\phi\overrightarrow{U_\theta} \\ &= (-x\Omega\cos\phi)\overrightarrow{U_r} + (r\Omega\cos\phi - y\Omega\sin\phi + \dot{x})\overrightarrow{U_\theta} + (x\Omega\sin\phi + \dot{y})\overrightarrow{U_\phi}\end{aligned} \qquad (10)$$

We will also consider some key assumptions in this model.

1. The latitude of the pendulum is constant. ($\dot{\phi} = 0$)
2. The rotation speed of Earth is constant. ($\dot{\Omega} = 0$)

Using these assumptions, the expression for the final acceleration of the bob can be obtained as follows :

$$\vec{\Gamma}(\overrightarrow{OM}) = \frac{\vec{V}(\overrightarrow{OM})}{dt_{(R)}}$$
$$= (\ddot{x} - 2\dot{y}\Omega\sin\phi - x\Omega^2)\overrightarrow{U_\theta} + (\ddot{y} + 2\dot{x}\Omega\sin\phi - y\Omega^2\sin^2\phi + r\Omega^2\cos\phi\sin\phi)\overrightarrow{U_\phi} \tag{11}$$

Now, let us consider the forces which are acting on the pendulum's bob.
The weight of the bob 'P'.
The tension in wire 'T'.
The net force acting on the bob is given by the following expression:

$$\Sigma(\overrightarrow{F^{ext/b}}) = \vec{T} + \vec{P}$$
$$= T\overrightarrow{U_t} - P\cos\alpha\overrightarrow{U_t} + P\sin\alpha\overrightarrow{U_a} \tag{12}$$

We project the force vectors onto vectors that are parallel and orthogonal to the pendulum's swinging movement.
α- Instantaneous angle of pendulum's movement The projection onto vector −→Ut gives

$$T - P\,cos\alpha = 0 \tag{13}$$

Therefore, the only term of the net force acting on the bob is P sinα$\vec{U}_a$.

Consider the projection of the net force vector $\vec{U}_a$ onto the vectors $\vec{U}_\Theta$ and $\vec{U}_\Phi$.

P = mg
m - mass of the bob
g - gravitational acceleration at location where pendulum is suspended
M - instantaneous position of bob
R - projection of point M onto axis U_r
l - length of pendulum's wire

$$\Sigma(\overrightarrow{F^{ext/b}}) = mg\sin\alpha\overrightarrow{U_a}$$
$$= mg\frac{RM}{l}\overrightarrow{U_a}$$
$$= \frac{mg}{l}(x\overrightarrow{U_\theta} + y\overrightarrow{U_\phi}) \quad (14)$$

Employing Newton's Second Law of motion,

$$\Sigma(\overrightarrow{F^{ext/b}}) = m\vec{\Gamma} \quad (15)$$

Using earlier expression of net force, we can rewrite the second law as,

$$\frac{mg}{l}(x\overrightarrow{U_\theta} + y\overrightarrow{U_\phi}) = m\Gamma\overrightarrow{U_a} \quad (16)$$

The pendulum's period is given by the expression $w_o = \sqrt{g/l}$

By replacing the acceleration using the expression given by Eq 11.

$$m\omega_o^2(x\overrightarrow{U_\theta}+y\overrightarrow{U_\phi}) = m(\ddot{x}-2\Omega\sin\phi\dot{y}-x\Omega^2)\overrightarrow{U_\theta}+m(\ddot{y}+2\Omega\sin\phi\dot{x}-y\Omega^2\sin^2\phi+r\Omega^2\cos\phi\sin\phi)\overrightarrow{U_\phi} \quad (17)$$

To simplify the expression, we will project the acceleration on their respective vectors

$$\overrightarrow{U_\theta} : \ddot{x} - 2\Omega\sin\phi\dot{y} - x\Omega^2 = -\omega_o^2 x$$
$$\overrightarrow{U_\phi} : \ddot{y} + 2\Omega\sin\phi\dot{x} - y\Omega^2\sin^2\phi + r\Omega^2\cos\phi\sin\phi = -\omega_o^2 y \quad (18)$$

We may ignore the terms $x\Omega^2$ and $y\Omega^2 sin^2\varphi$ as they have a value of the order 10^{-10} m/s^2 and can be neglected with respect to other terms in the equation. Hence, the final equations which describe the dynamics of the Foucault Pendulum have been derived.

$$\ddot{x} - 2\Omega\sin\phi\dot{y} + \omega_o^2 x = 0$$
$$\ddot{y} + 2\Omega\sin\phi\dot{x} + \omega_o^2 y = -r\Omega^2\cos\phi\sin\phi \quad (19)$$

3.2 Analytical Solution:→

A set of non-homogeneous second-order linear equations is derived as the governing equations

$$\omega = \sqrt{g/l}$$

where $g = 9.81 m/s^2$ and the pendulum's point of suspension is at altitude $l = 8\ m$.

r = (Earth's radius + laboratory's altitude).

The laboratory's latitude(ϕ) is *0.872665 rad,* and the Earth's angular rotation speed(Ω) is *7.2921150 x 105 rad/s.*

r = I + Earth's radius = 6.371397×10^6 m.

Assume a system of linear homogeneous differential equations to solve analytically. All of these measurements are based on the laboratory's present location. To solve the non-homogeneous system, first, analyze the linear homogeneous equation system shown below.

$$\ddot{x}_0 - 2\Omega \sin\varphi\, \dot{y}_0 + \omega_0^2 x_0 = 0$$
$$\ddot{y}_0 + 2\Omega \sin\varphi\, \dot{x}_0 + \omega_0^2 y_0 = 0 \tag{20}$$

Multiplying the second equation and combining it with the first provides the single differential equation $zo = xo + iyo$, where z is a complex variable.

$$\ddot{z}_0 + 2i\Omega \sin\varphi\, \dot{z}_0 + \omega_0^2\, z_0 = 0 \tag{21}$$

To solve the differential equation, one must first solve the characteristic equation for r:

$$r^2 + 2i\Omega \sin\varphi\, r + \omega_0^2 = 0$$
$$\implies \quad r_{1,2} = -i\Omega \sin\varphi \pm i\sqrt{\Omega^2 \sin^2\varphi + \omega_0^2} \tag{22}$$

The solution for z is well-known and may be written as:

$$z_o(t) = C_1 e^{r1t} + C_2 e^{r2t} \tag{23}$$

The initial circumstances at time t=0 can be used to derive C1 and C2. The following two conditions are used to create the starting conditions:

$$z_o(t=0) = C_1 + C_2$$
$$\dot{z}_o(t=0) = C_1 r_1 + C_2 r_2 \tag{24}$$

$$C_{1,2} = 0.5 \pm \frac{\Omega \sin\phi}{\sqrt{(\Omega \sin\phi)^2 + \omega_o^2}} \tag{25}$$

The calculated values are:

ω_o=1.10736173

r1=1.107304141

r2= -1.107415862

C1=0.5000504451

C2=0.4999495549

In this case, the homogeneous solution has the following form:

$$z_o(t) = C_1 e^{r1t} + C_2 e^{r2t} \tag{26}$$

Now we'll figure out how to solve the overall equation analytically. To derive the specific solution, we start with the equilibrium position, where acceleration and velocity in both the x and y directions are zero therefore we obtain

X_{eq}=0 and $Y_{eq} =- r\Omega^2 cos\phi sin\phi/\omega_o$ (Y_{eq}=-0.01360454653m).

Y_{eq} is a specific solution in this case.

$$z_o(t) = C_1 e^{r1t} + C_2 e^{r2t} + iy_{eq} \quad (27)$$

As a result, the complete solution may be written as:

$$x(t) = C_1 cos(r_1 t) + C_2 cos(r_2 t)$$

$$y(t) = C_1 sin(r_1 t) + C_2 sin(r_2 t) + Y_{eq} \quad (28)$$

Because Y_{eq} alters the apparent vertical changes by an angle,

α = sin α = $Y_{eq}/l =- 1.700568316 \times 10^{-3}$rad,

Now, the new points of equilibrium are Y_{eq} and X_{eq}.

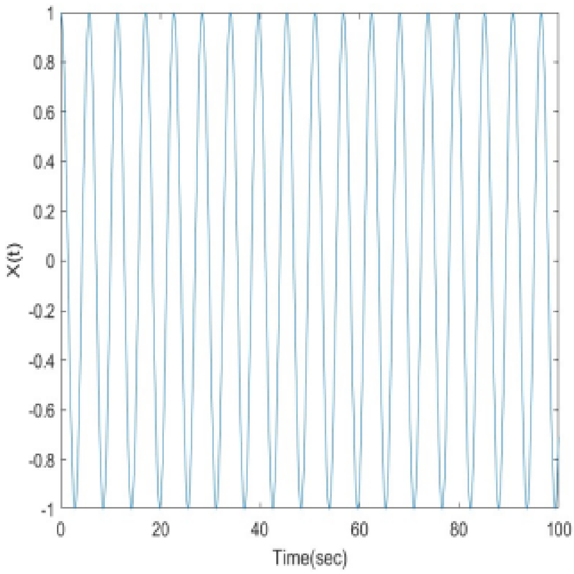
$$\Longrightarrow \qquad r_{1,2} = -i\Omega\sin\varphi \pm i\sqrt{\Omega^2\sin^2\varphi + \omega_0^2}$$

Here we assume that the angular frequency of earth is much much smaller than the natural frequency of pendulum that means $\Omega << \omega_o$

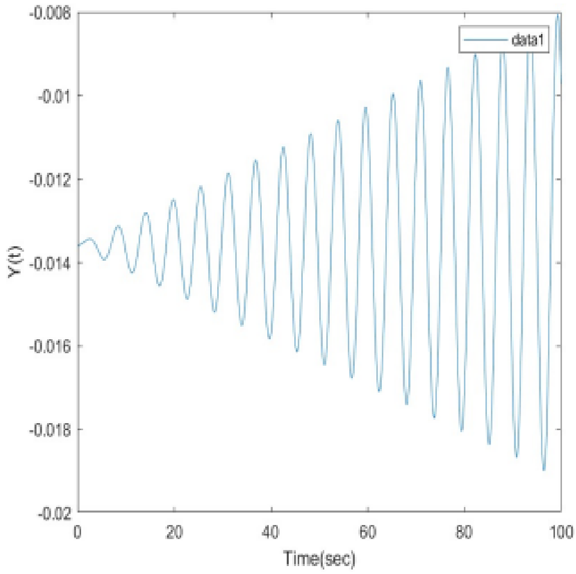
$$z_o(t) = e^{-i\Omega t} Acos(w_o t)$$

Here $e^{-i\Omega t}$ this term define the direction the pendulum and $Acos(w_o t)$ give the overall magnitude

So, using this equation, we can determine the precise magnitude and direction of the pendulum's displacement, which is exactly what we're looking for. We conclude that the earth rotates and can even estimate the rotation's rate.



(a) Scale for x-axis is 1 sec



(b) Scale for x-axis is 1 sec

Figure 5: Plots obtained from Analytical Solution

4. Numerical Methods

The length of the string of the pendulum we have considered is l = *10 m.* The Earth's radius + the height of the pendulum bob from the ground, r = *6.371397* * 10^6. *m.*

Let us assume the angular velocity of Earth's rotation to be **Ω** = *7.292 x 10−5 rad/s.*

Let us assume the latitude of the place where this experiment was performed to be φ = *0.872665 rad* (50°). Let us assume that the acceleration due to gravity has a value

of g = *9.81* m/sec^2. We know that ω = $\sqrt{(g/l)}$. We performed the simulation for a duration of 2 hrs. We are assuming a step size of h = 0.1 sec.

We convert Eq (19) to a system of first-order ODE's.

First we consider,

$$\frac{dx}{dt} = \mathrm{m}$$

$$\frac{dy}{dt} = \mathrm{n} \tag{29}$$

Substituting the above we get the following system of first-order ODEs,

$$\frac{dm}{dt} = 2(\Omega)(n)(sin(\phi)) - \omega^2 x$$

$$\frac{dn}{dt} = -2(\Omega)(n)(sin(\phi) - \omega^2 y - r(\Omega^2)\ sin(\phi)\ cos(\phi) \tag{30}$$

Runge Kutta Method

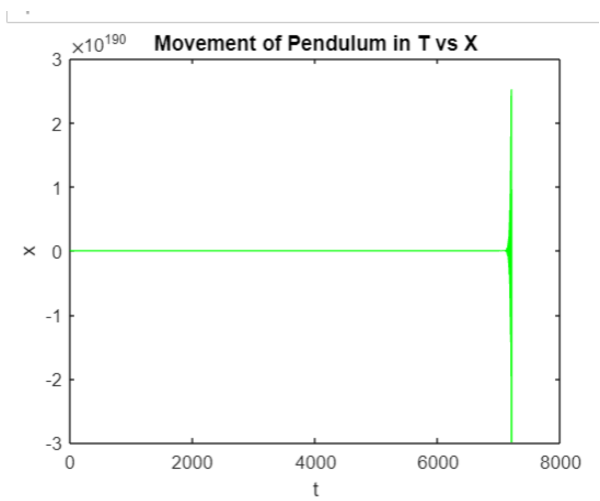
Runge-Kutta (RK) methods achieve the accuracy of a Taylor series approach without requiring the calculation of higher derivatives. Many variations exist but all can be cast in the generalized form of

New value = old value + slope * step size

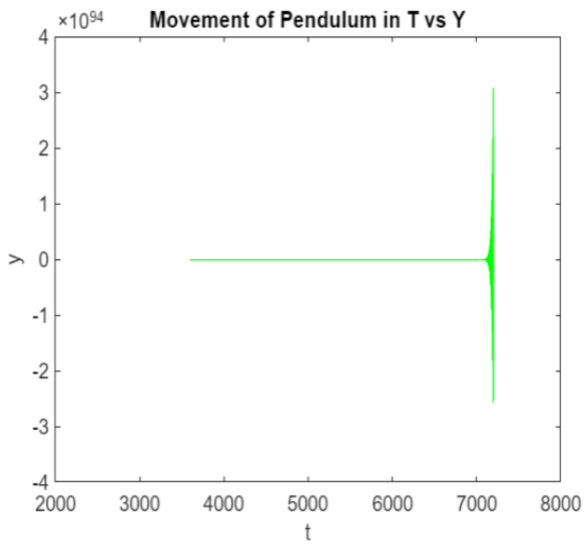
$$y_{i+1} = y_i + f(x_i, y_i)h$$

4.1 Euler's Method

It is a first-order Runge Kutta numerical method, which is used to solve the differential equations of the form $\frac{dy}{dx}$ = f(x,y). It is used for approximating solutions to the differential equations and works by approximating a solution curve with line segments. So, we tried to solve it with Euler's Method.



Scale: 1 sec for x-axis



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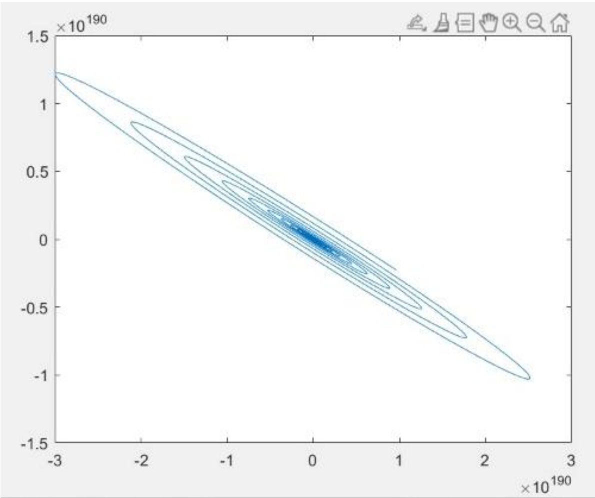
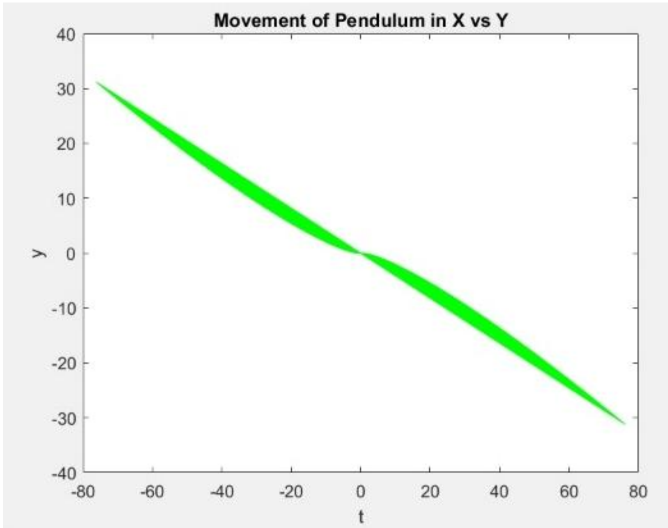


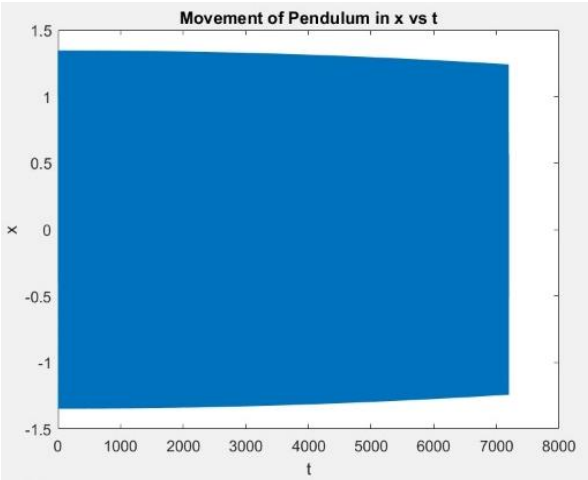


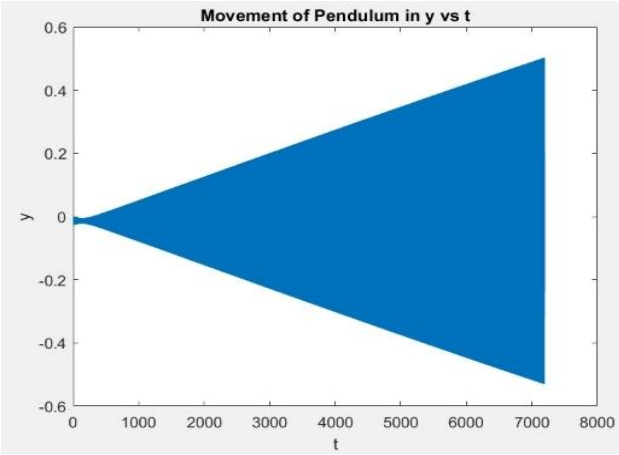
Movement of Pendulum in X vs Y

X vs y plots for Euler's Method

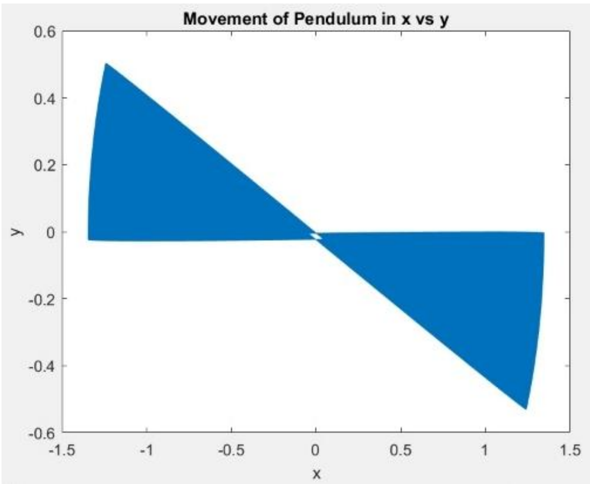
By using Euler's Method we obtained the curves as above. We can see that the results obtained are absurd with Y-axis being of order $10^{94}/10^{190}$. This scale is practically impossible for a pendulum's movement. Hence we tried the simulation of x vs y by reducing the step size h to 0.001 sec. The result obtained by this reduction can be seen in figure 7. b. The Further reduction in step size did not seem feasible as the number of iterations shot up to extremely large numbers. Thus we observe that Euler's method is not appropriate for studying the dynamics of Foucault's pendulum numerically.

4.2 4th order Runge Kutta Method (RK4):





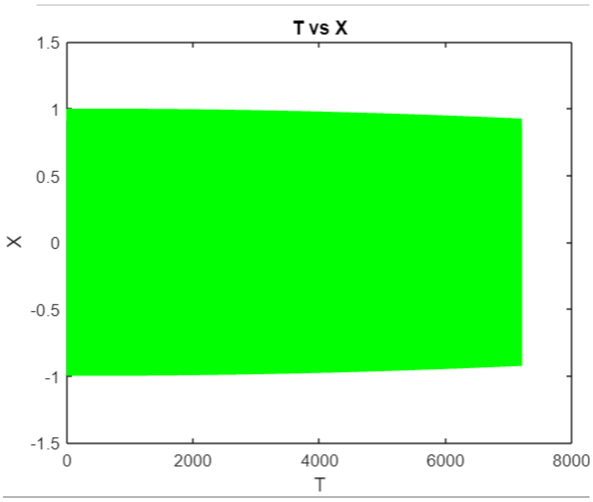
Scale: 1 sec for X-axis, X vs T and Y vs T plots for 4th order RK method

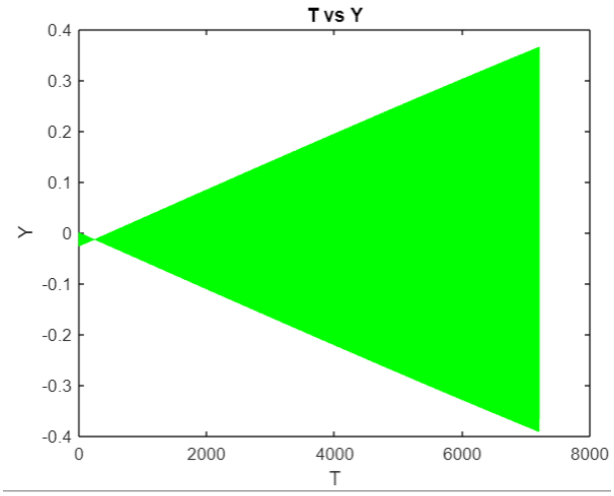


X vs Y plot for 4th order RK method

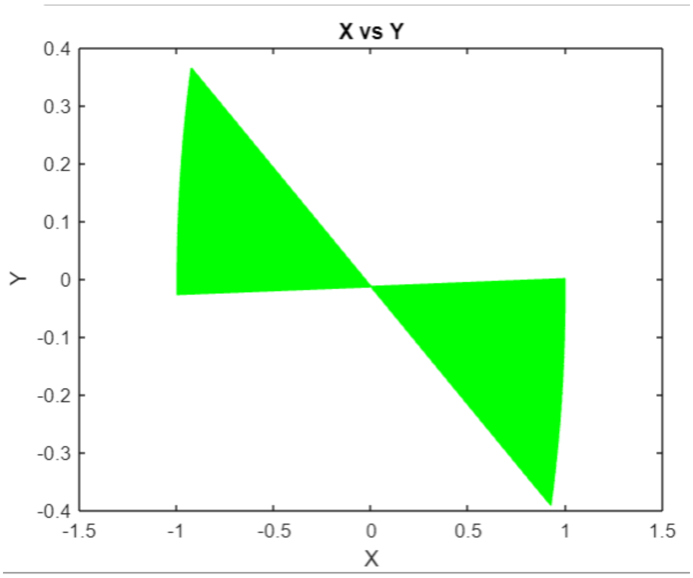
By using the fourth-order Runge-Kutta method we obtained the curves as seen in Figures 12 and 13. Figure 13 clearly indicates that the accuracy of this method is much better than the RK3 method but at the same time, the implementation complexity of this method is also very high. In fact, it is higher than that of all the previously used methods. The results obtained resemble the analytical solutions to a very good extent.

4.3 5th order Runge Kutta Method (RK5):





Scale: 1 sec for X-axis, X vs T and Y vs T plots for 5th order RK method



X vs Y plot for 5th order RK method

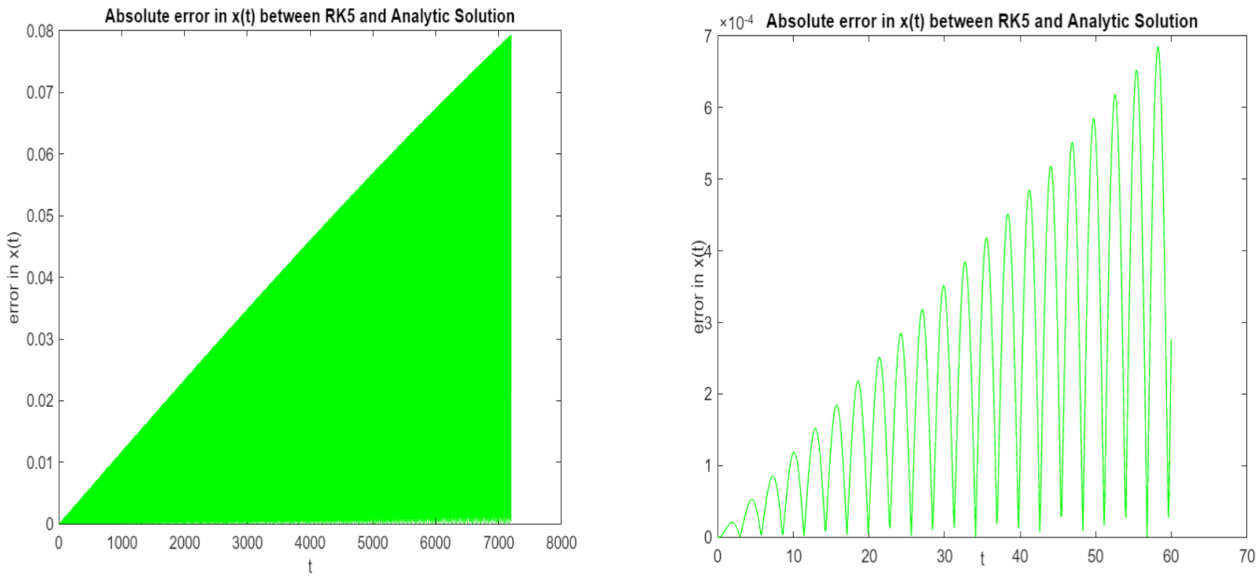
By using the fifth-order Runge-Kutta method we obtained the curves as seen in Figures 14 and 15. Figure 15 clearly states that the accuracy of this method is the best among all the analytical methods used. But at the same time, the implementation complexity of this method is also the highest. The results obtained resemble the analytical solutions to the best extent. Hence it is safe to assume that the most appropriate method to numerically analyze the above ODEs is the fifth-order Runge-Kutta Method. Since, in our experiment, a great deal of precision is required in the analysis of the dynamics of the pendulum, we shall prefer the

accuracy of the numerical method over its implementation complexity. Hence we shall finally consider the fifth-order Runge-Kutta method for our study.

4.4 Error Analysis:

calculated the absolute difference of x(t) and y(t) vs time from the solutions obtained

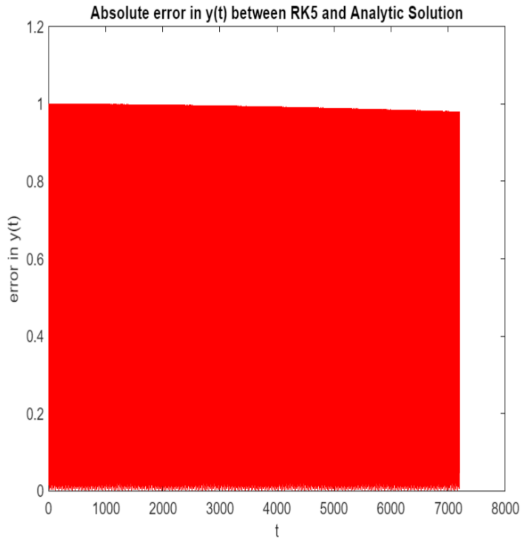
from the 5th order RK method and the analytical solution.

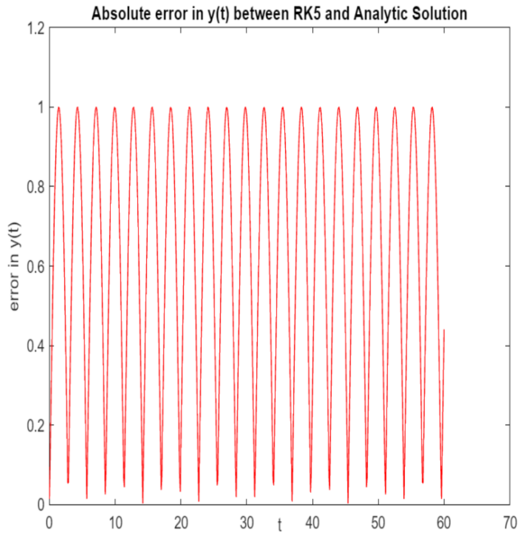


Scale: X-axis is 1-sec

Figure: Absolute error in numerical and analytical x vs time

We can observe that the amplitude of the absolute error of x(t) increases over time





Scale: X-axis is 1-sec

Figure: Absolute error in numerical and analytical x vs time

There is a periodic variation of the error but amplitude is almost constant.

5. Description of Algorithm

As all are Range Kutta Method, it is a similar algorithm. We need to use the following algorithm by converting it to MATLAB code to solve the given ODEs numerically,

1. Converting the second-order ODE's into four first-order derivatives by denoting the first derivative of x and y by two new variables.
2. Provide the initial conditions for the four first-order ODEs.
3. Initialise a loop that runs until the simulation time of two hours is reached. Within the loop, there will be six different constants for each of the four variables whose values we have to determine.
4. These constants will be the function of either the four variables alone or will also include
   the previously calculated constants.
5. Let the variable be 'x' then: for RK5,

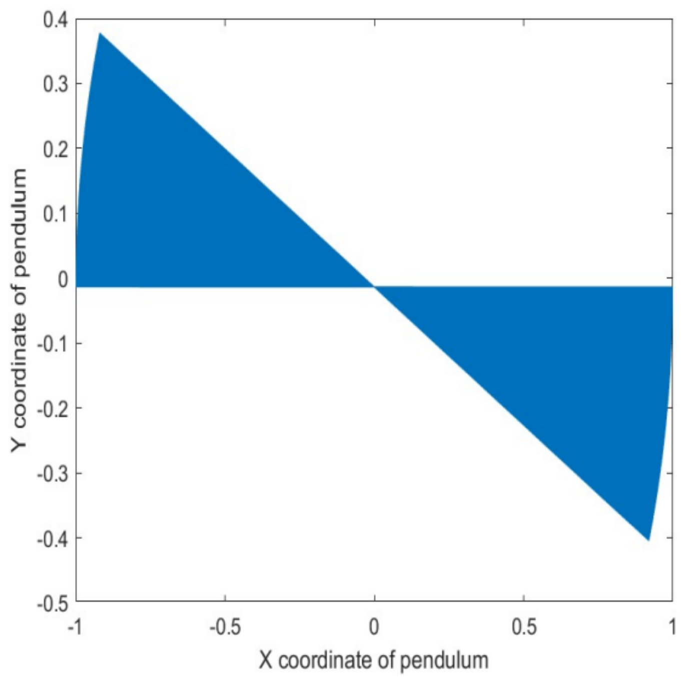
$$x_{i+1} = x_i + \frac{h}{90}(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6) \tag{31}$$

$$\begin{aligned}
k_1 &= f(t_i, z_i) \\
k_2 &= f(t_i + h/4, z_i + h/4) \\
k_3 &= f(t_i + h/4, z_i + k_1 h/4 + k_2 h/8) \\
k_4 &= f(t_i + h/2, z_i - k_2 h/2 + k_3 h) \\
k_5 &= f(t_i + 3h/4, z_i + 3k_1 h/16 + 9k_4 h/16) \\
k_6 &= f(t_i + \mathrm{h},\ z_i - 3k_1 h/7 + 2k_2 h/7, z_i + 12k_3 h/7 - 12k_4 h/7 + 8k_5 h/7)
\end{aligned} \tag{32}$$

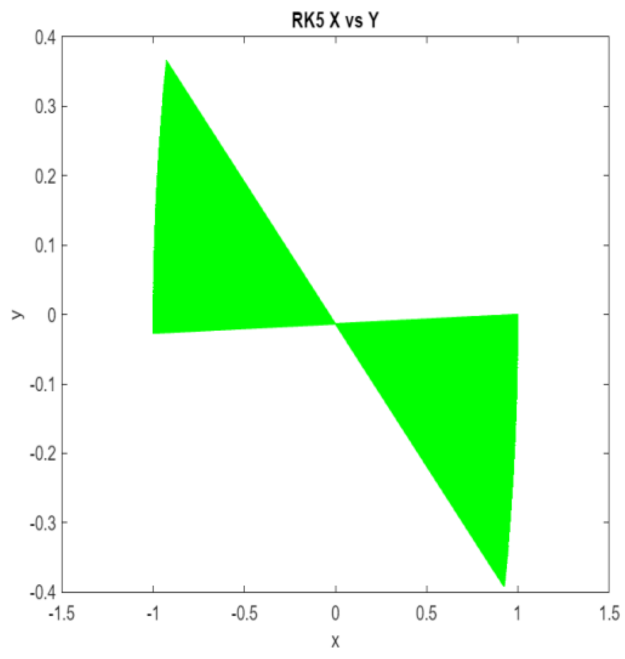
   Here t is a variable, z is dependent on
6. Update the above constants and the variables in each ith iteration and at the same time also update the time taken by the step size. Finally, use the 'plot' function of MATLAB to obtain the various plots required

6. Results and Discussions

We have successfully analyzed the dynamics of Foucault's pendulum for a simulation time of 2 hrs. The motion of the pendulum in the domain is obtained both analytically and numerically and is shown in the following graphs.



Analytical



RK 5

We can see the difference between the motion of the pendulum in the analytical method and the RK5 method.

7. Conclusions

As we can see from the plots of the motion of the pendulum, the plane of oscillation of the pendulum changes its position with time. The obtained analytical solution is complicated. So, the numerical analysis is easier to apply than analytical and also time-saving, but we know there will be errors from numerical methods, Therefore, for reducing/minimizing the errors, an optimized numerical method should be used which is fifth-order Runge Kutta Method. Even here, the error varies periodically with time but it never increases beyond a limit of 0.001 %, which is a negligible quantity and practically very less.

Thus, we prove the rotation of the Earth by using the concept of Foucault's Pendulum.

8. References

1.http://www.legi.grenoble-inp.fr/people/Achim.Wirth/final_version.pdf

Images:

2. https://www.ias.ac.in/article/fulltext/reso/024/06/0661-0679

Computer Programs:

Euler's method:

```
omega = 7.2921150E-5;
phi = 0.872665;
g = 9.81;
l = 8;
wo = sqrt(g/l);
r = 6371397; %m

time = 2*60*60; %2 hours in sec
h = 0.1;

t= 0;

x = zeros(time/h+1,0);
y = zeros(time/h+1,0);
m = zeros(time/h+1,0);
n = zeros(time/h+1,0);
t_vector = zeros(time/h+1,0);
x(1) = 1;
y(1) = 0;
m(1) = 0;
n(1) = 0;
t_vector(1) = 0;
i=1;

while t<time
    a1 = dx_dt(x(i),y(i),m(i),n(i),omega,phi,wo,r);
    b1 = dy_dt(x(i),y(i),m(i),n(i),omega,phi,wo,r);
    c1 = dm_dt(x(i),y(i),m(i),n(i),omega,phi,wo,r);
    d1 = dn_dt(x(i),y(i),m(i),n(i),omega,phi,wo,r);

    x(i+1) = x(i) + a1*h;
    y(i+1) = y(i) + b1*h;
    m(i+1) = m(i) + c1*h;
    n(i+1) = n(i) + d1*h;
    t_vector(i+1) = t_vector(i)+h;
    t = t+ h;
    i = i+1;
```

```
end

plot(t_vector,x,'g')
xlabel('t')
ylabel('x')
title('Movement of Pendulum in T vs X')




function a = dx_dt(x,y,m,n,omega,phi,wo,r)
    a = m;
end

function b = dy_dt(x,y,m,n,omega,phi,wo,r)
    b = n;
end

function c = dm_dt(x,y,m,n,omega,phi,wo,r)
    c = 2*omega*n*sin(phi) - wo*wo*x;
end

function d = dn_dt(x,y,m,n,omega,phi,wo,r)
    d = -2*omega*m*sin(phi) - wo*wo*y- r*omega*omega*cos(phi)*sin(phi);
end
```

RK3:

```
clear all;
omega = 7.292E-5;
phi = 50;
g = 9.81;
l = 8;
wo = sqrt(g/l);
r = 6371397; %m
time = 2*60*60; %2 hours in sec
h = 0.1;
t = 0;
x = zeros(time/h+1,0);
y = zeros(time/h+1,0);
```

```
m = zeros(time/h+1,0);
n = zeros(time/h+1,0);
T = zeros(time/h +1,0);
x(1) = 1;
y(1) = 0;
m(1) = 1;
n(1) = 0;
i=1;
T(i) = t;
while t<time
    a1 = dx_dt(x(i),y(i),m(i),n(i),omega,phi,wo,r);
    b1 = dy_dt(x(i),y(i),m(i),n(i),omega,phi,wo,r);
    c1 = dm_dt(x(i),y(i),m(i),n(i),omega,phi,wo,r);
    d1 = dn_dt(x(i),y(i),m(i),n(i),omega,phi,wo,r);
    a2 = dx_dt(x(i)+a1*h/2,y(i)+b1*h/2,m(i)+c1*h/2,n(i)+d1*h/2,omega,phi,wo,r);
    b2 = dy_dt(x(i)+a1*h/2,y(i)+b1*h/2,m(i)+c1*h/2,n(i)+d1*h/2,omega,phi,wo,r);
    c2 = dm_dt(x(i)+a1*h/2,y(i)+b1*h/2,m(i)+c1*h/2,n(i)+d1*h/2,omega,phi,wo,r);
    d2 = dn_dt(x(i)+a1*h/2,y(i)+b1*h/2,m(i)+c1*h/2,n(i)+d1*h/2,omega,phi,wo,r);
    a3 = dx_dt(x(i)-a1*h+2*a2*h,y(i)-b1*h+2*b2*h,m(i)-c1*h+2*c2*h,n(i)-d1*h+2*d2*h,omega,phi,wo,r);
    b3 = dy_dt(x(i)-a1*h+2*a2*h,y(i)-b1*h+2*b2*h,m(i)-c1*h+2*c2*h,n(i)-d1*h+2*d2*h,omega,phi,wo,r);
    c3 = dm_dt(x(i)-a1*h+2*a2*h,y(i)-b1*h+2*b2*h,m(i)-c1*h+2*c2*h,n(i)-d1*h+2*d2*h,omega,phi,wo,r);
    d3 = dn_dt(x(i)-a1*h+2*a2*h,y(i)-b1*h+2*b2*h,m(i)-c1*h+2*c2*h,n(i)-d1*h+2*d2*h,omega,phi,wo,r);
    x(i+1) = x(i) + (1/6)*(a1+4*a2+a3)*h;
    y(i+1) = y(i) + (1/6)*(b1+4*b2+b3)*h;
    m(i+1) = m(i) + (1/6)*(c1+4*c2+c3)*h;
    n(i+1) = n(i) + (1/6)*(d1+4*d2+d3)*h;
    t = t+ h;
    i = i+1;
    T(i) = t;
end
plot(T,x)
xlabel('t')
ylabel('x')
title('Movement of Pendulum in x vs t')
function a = dx_dt(x,y,m,n,omega,phi,wo,r)
    a = m;
end
function b = dy_dt(x,y,m,n,omega,phi,wo,r)
    b = n;
end
```

```
function c = dm_dt(x,y,m,n,omega,phi,wo,r)
    c = 2*omega*n*sind(phi) - wo*wo*x;
end
function d = dn_dt(x,y,m,n,omega,phi,wo,r)
    d = -2*omega*m*sind(phi) - wo*wo*y - r*(omega.^2)*cosd(phi)*sind(phi);
end
```

RK4:

```
clear all;
omega = 7.292E-5;
phi = 50;
g = 9.81;
l = 8;
wo = sqrt(g/l);
r = 6371397; %m
time = 2*60*60; %2 hours in sec
h = 0.1;
t = 0;
x = zeros(time/h+1,0);
y = zeros(time/h+1,0);
m = zeros(time/h+1,0);
n = zeros(time/h+1,0);
T = zeros(time/h+1,0);
x(1) = 1;
y(1) = 0;
m(1) = 1;
n(1) = 0;
i=1;
T(i) = t;
while t<time
    a1 = dx_dt(x(i),y(i),m(i),n(i),omega,phi,wo,r);
    b1 = dy_dt(x(i),y(i),m(i),n(i),omega,phi,wo,r);
    c1 = dm_dt(x(i),y(i),m(i),n(i),omega,phi,wo,r);
    d1 = dn_dt(x(i),y(i),m(i),n(i),omega,phi,wo,r);

    a2 = dx_dt(x(i)+a1*h/2,y(i)+b1*h/2,m(i)+c1*h/2,n(i)+d1*h/2,omega,phi,wo,r);
    b2 = dy_dt(x(i)+a1*h/2,y(i)+b1*h/2,m(i)+c1*h/2,n(i)+d1*h/2,omega,phi,wo,r);
    c2 = dm_dt(x(i)+a1*h/2,y(i)+b1*h/2,m(i)+c1*h/2,n(i)+d1*h/2,omega,phi,wo,r);
    d2 = dn_dt(x(i)+a1*h/2,y(i)+b1*h/2,m(i)+c1*h/2,n(i)+d1*h/2,omega,phi,wo,r);
```

```
    a3 = dx_dt(x(i)+a2*h/2,y(i)+b2*h/2,m(i)+c2*h/2,n(i)+d2*h/2,omega,phi,wo,r);
    b3 = dy_dt(x(i)+a2*h/2,y(i)+b2*h/2,m(i)+c2*h/2,n(i)+d2*h/2,omega,phi,wo,r);
    c3 = dm_dt(x(i)+a2*h/2,y(i)+b2*h/2,m(i)+c2*h/2,n(i)+d2*h/2,omega,phi,wo,r);
    d3 = dn_dt(x(i)+a2*h/2,y(i)+b2*h/2,m(i)+c2*h/2,n(i)+d2*h/2,omega,phi,wo,r);

    a4 = dx_dt(x(i)+a3*h,y(i)+b3*h,m(i)+c3*h,n(i)+d3*h,omega,phi,wo,r);
    b4 = dy_dt(x(i)+a3*h,y(i)+b3*h,m(i)+c3*h,n(i)+d3*h,omega,phi,wo,r);
    c4 = dm_dt(x(i)+a3*h,y(i)+b3*h,m(i)+c3*h,n(i)+d3*h,omega,phi,wo,r);
    d4 = dn_dt(x(i)+a3*h,y(i)+b3*h,m(i)+c3*h,n(i)+d3*h,omega,phi,wo,r);



    x(i+1) = x(i) + (1/6)*(a1+2*a2+2*a3+a4)*h;
    y(i+1) = y(i) + (1/6)*(b1+2*b2+2*b3+b4)*h;
    m(i+1) = m(i) + (1/6)*(c1+2*c2+2*c3+c4)*h;
    n(i+1) = n(i) + (1/6)*(d1+2*d2+2*d3+d4)*h;

    t = t+ h;
    i = i+1;
    T(i) = t;
end

plot(T,x)
xlabel('t')
ylabel('x')
title('Movement of Pendulum in x vs t')



function a = dx_dt(x,y,m,n,omega,phi,wo,r)
    a = m;
end

function b = dy_dt(x,y,m,n,omega,phi,wo,r)
    b = n;
end

function c = dm_dt(x,y,m,n,omega,phi,wo,r)
    c = 2*omega*n*sind(phi) - wo*wo*x;
end
```

```
function d = dn_dt(x,y,m,n,omega,phi,wo,r)
    d = -2*omega*m*sind(phi) - wo*wo*y - r*(omega.^2)*cosd(phi)*sind(phi);
end
```

RK5:

```
clear all;

omega = 7.292E-5;
phi = 50;
g = 9.81;
l = 8;
wo = sqrt(g/l);
r = 6371397; %m

time = 2*60*60; %2 hours in sec
h = 0.1;

t = 0;

x = zeros(time/h+1,0);
y = zeros(time/h+1,0);
m = zeros(time/h+1,0);
n = zeros(time/h+1,0);
T = zeros(time/h+1,0);

x(1) = 1;
y(1) = 0;
m(1) = 1;
n(1) = 0;

i=1;
T(i) = t;

while t<time
    a1 = dx_dt(x(i),y(i),m(i),n(i),omega,phi,wo,r);
    b1 = dy_dt(x(i),y(i),m(i),n(i),omega,phi,wo,r);
    c1 = dm_dt(x(i),y(i),m(i),n(i),omega,phi,wo,r);
    d1 = dn_dt(x(i),y(i),m(i),n(i),omega,phi,wo,r);

    a2 = dx_dt(x(i)+a1*h/2,y(i)+b1*h/2,m(i)+c1*h/2,n(i)+d1*h/2,omega,phi,wo,r);
```

```
    b2 = dy_dt(x(i)+a1*h/2,y(i)+b1*h/2,m(i)+c1*h/2,n(i)+d1*h/2,omega,phi,wo,r);
    c2 = dm_dt(x(i)+a1*h/2,y(i)+b1*h/2,m(i)+c1*h/2,n(i)+d1*h/2,omega,phi,wo,r);
    d2 = dn_dt(x(i)+a1*h/2,y(i)+b1*h/2,m(i)+c1*h/2,n(i)+d1*h/2,omega,phi,wo,r);

    a3 = dx_dt(x(i)+a2*h/2,y(i)+b2*h/2,m(i)+c2*h/2,n(i)+d2*h/2,omega,phi,wo,r);
    b3 = dy_dt(x(i)+a2*h/2,y(i)+b2*h/2,m(i)+c2*h/2,n(i)+d2*h/2,omega,phi,wo,r);
    c3 = dm_dt(x(i)+a2*h/2,y(i)+b2*h/2,m(i)+c2*h/2,n(i)+d2*h/2,omega,phi,wo,r);
    d3 = dn_dt(x(i)+a2*h/2,y(i)+b2*h/2,m(i)+c2*h/2,n(i)+d2*h/2,omega,phi,wo,r);

    a4 = dx_dt(x(i)+a3*h,y(i)+b3*h,m(i)+c3*h,n(i)+d3*h,omega,phi,wo,r);
    b4 = dy_dt(x(i)+a3*h,y(i)+b3*h,m(i)+c3*h,n(i)+d3*h,omega,phi,wo,r);
    c4 = dm_dt(x(i)+a3*h,y(i)+b3*h,m(i)+c3*h,n(i)+d3*h,omega,phi,wo,r);
    d4 = dn_dt(x(i)+a3*h,y(i)+b3*h,m(i)+c3*h,n(i)+d3*h,omega,phi,wo,r);



    x(i+1) = x(i) + (1/6)*(a1+2*a2+2*a3+a4)*h;
    y(i+1) = y(i) + (1/6)*(b1+2*b2+2*b3+b4)*h;
    m(i+1) = m(i) + (1/6)*(c1+2*c2+2*c3+c4)*h;
    n(i+1) = n(i) + (1/6)*(d1+2*d2+2*d3+d4)*h;

    t = t+ h;
    i = i+1;
    T(i) = t;
end

plot(T,x)
xlabel('t')
ylabel('x')
title('Movement of Pendulum in x vs t')



function a = dx_dt(x,y,m,n,omega,phi,wo,r)
    a = m;
end

function b = dy_dt(x,y,m,n,omega,phi,wo,r)
    b = n;
end

function c = dm_dt(x,y,m,n,omega,phi,wo,r)
```

```
    c = 2*omega*n*sind(phi) - wo*wo*x;
end

function d = dn_dt(x,y,m,n,omega,phi,wo,r)
    d = -2*omega*m*sind(phi) - wo*wo*y - r*(omega.^2)*cosd(phi)*sind(phi);
end
```