1)

a) $A = LL^T$

$$A = \begin{pmatrix} A_{00} & a_{01} \\ a_{10}^T & \alpha_{11} \end{pmatrix} \quad L = \begin{pmatrix} L_{00} & 0 \\ l_{10}^T & \lambda_{11} \end{pmatrix}$$

$$A = \begin{pmatrix} L_{00} & 0 \\ l_{10}^T & \lambda_{11} \end{pmatrix} \begin{pmatrix} L_{00} & 0 \\ l_{10}^T & \lambda_{11} \end{pmatrix}^T$$

$$= \begin{pmatrix} L_{00} L_{00}^T & L_{00} l_{10} \\ l_{10}^T L_{00}^T & l_{10}^T l_{10} + \lambda_{11} \lambda_{11} \end{pmatrix}$$

as $a_{01} = a_{10}^T$ (SPD)

$$A_{00} = L_{00} L_{00}^T$$

$$a_{10}^T = l_{10}^T L_{00}^T$$

$$\alpha_{11} = l_{10}^T l_{10} + \lambda_{11}^2$$

Cholesky factorization

$\Rightarrow$ $A_{00} = L_{00} L_{00}^T$

$a_{10}^T = l_{10}^T L_{00}^T$

$\Rightarrow$ $l_{10}^T = a_{10}^T L_{00}^{-T}$

$\alpha_{11} = l_{10}^T l_{10} + \lambda_{11}^2$

$\lambda_{11} = \sqrt{\alpha_{11} - l_{10}^T l_{10}}$

Algo :-

$$A \rightarrow \left(\begin{array}{c|c} A_{TL} & \\ \hline A_{BL} & A_{BR} \end{array} \right)$$

While $n(A_{TL}) < n(A)$ do

$$A \rightarrow \left(\begin{array}{c|cc} A_{00} & & \\ \hline a_{10}^T & \alpha_{11} & \\ A_{20} & a_{21} & A_{22} \end{array} \right)$$

$a_{10}^T := \lambda_{10}^T := a_{10}^T A_{00}^{-T}$

$\alpha_{11} := \lambda_{11}$

$\sqrt{\alpha_{11} - a_{10}^T a_{10}}$

Continue

$$\left(\begin{array}{cc|c} A_{00} & & \\ a_{10}^T & \alpha_{11} & \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$$

$$\left(\begin{array}{c|c} ATL & \\ \hline ABL & ABR \end{array}\right)$$

End While

b)

→ Proof by induction

$n = 1$

A = 1×1 matrix.

$A = [a_{11}]$

$l_{11}^2 = a_{11} \Rightarrow l_{11} = \pm\sqrt{a_{11}}$

a_{11} is real and positive

if l_{11} is positive

$l_{11} = \sqrt{a_{11}} \Rightarrow$ unique.

⇒ Inductive hypothesis:

if $A \in R^{n \times n} \Rightarrow A \in R^{(n+1)}$

A is SPD

(1)

$A_{00} = L_{00} L_{00}^T$ Prof 1.a

$\Rightarrow l_{10}^T = a_{10}^T L_{00}^{-T}$

L_{00} is nonsingular if diagnol element are Positive

$L_{00}^T \Rightarrow$ non singular

$L_{00}^{T^{-1}}$ Exists

$l_{10}^T = a_{10}^T L_{00}^{-T}$

$(l_{10}^T)^T = (a_{10}^T L_{00}^{-T})^T$

$l_{10} = L_{00}^{-1} a_{10}$

$L_{00} L_{10} = a_{10}$

$L_{00} l_{10} = a_{10}$ has solution & unique.

$(L_{00} l_{10})^T = (a_{10})^T$

$\Rightarrow l_{10}^T L_{00}^T = a_{10}^T$

$\Rightarrow l_{10}^T = a_{10}^T L_{00}^{-T}$

is well defined & unique.

ii) $\lambda_{11} = \sqrt{\alpha_{11} - l_{10}^T l_{10}}$

$\alpha_{11} \in \mathbb{R}$ & +ve.

l_{10}^T is well defined & unique

$\alpha_{11} - l_{10}^T l_{10}$ Exists & Unique.

hence λ_{11} Exists

Thus $A \in \mathbb{R}^{(n+1) \times (n+1)}$ has unique Cholesky factorization