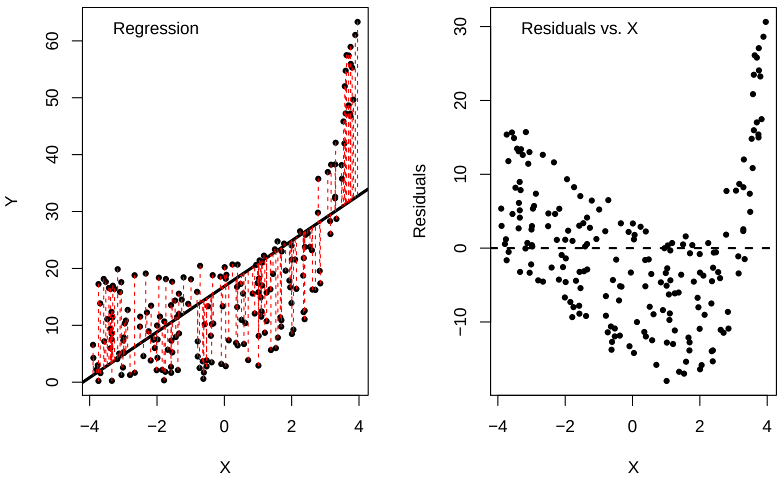
PLSC 503 – Spring 2020
Transformations

February 13, 2020

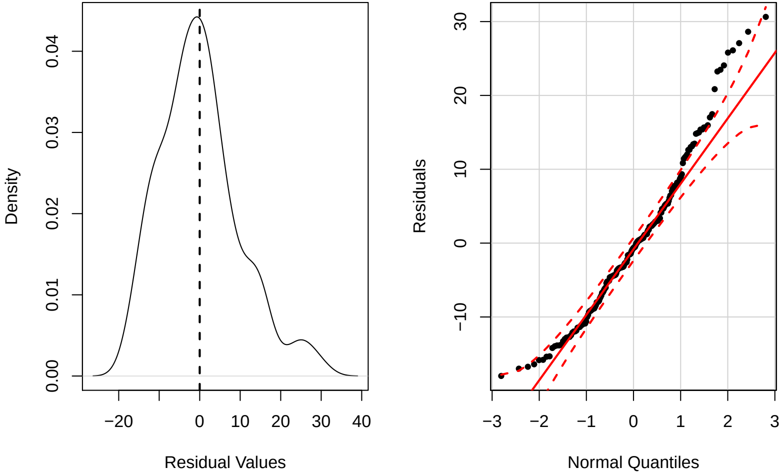
Why Transform?

- Normality (of u_is)
- Linearity
- Additivity
- Interpretation / Model Specification

What Difference Does It Make? (Part I)



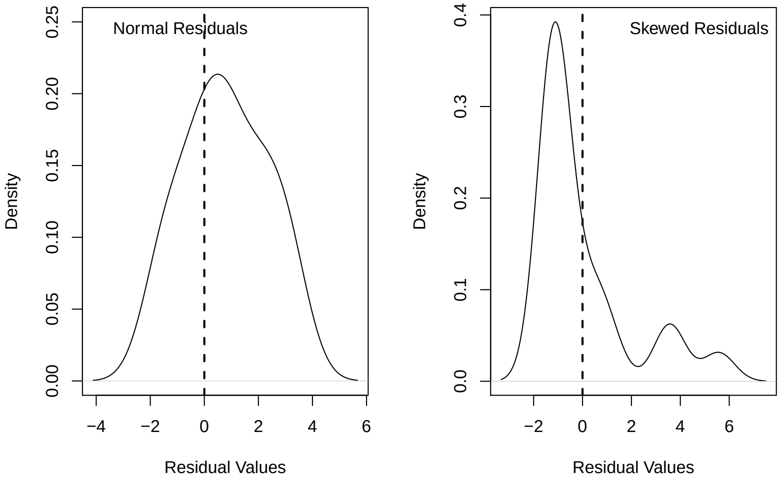
Residuals Are Still (Pretty) Normal...



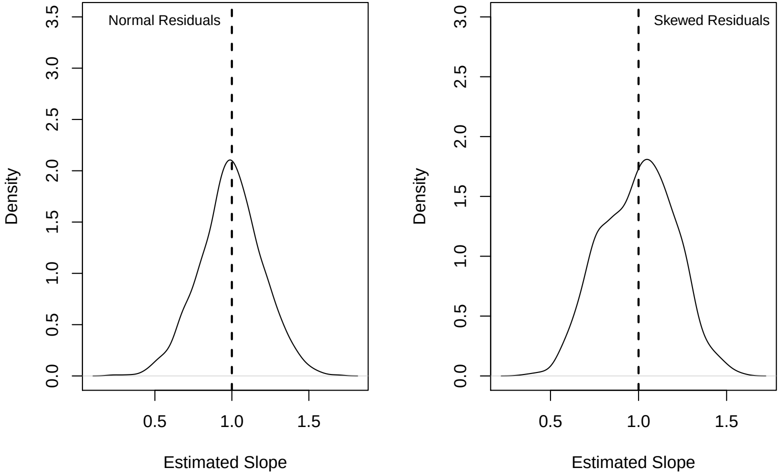
What Difference Does It Make? (Part II)

```
N <- 20 # pretty small sample size
u <- rnorm(N,0,2) # mean zero, s.d = 2

# Exponentiate:
eu <- exp(u)
eu <- eu-mean(eu) # new residuals are mean-zero
eu <- (eu/sd(eu))*2 # and also sd = 2

X <- runif(N,-4,4)
Y1 <- 0 + 1*X + 1*u
Y2 <- 0 + 1*X + 1*eu # same Xs in both
```

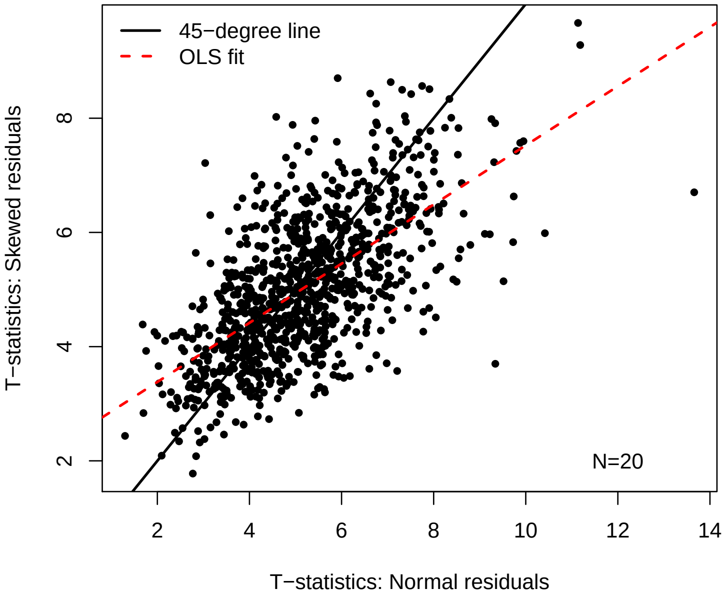
What Difference Does It Make? (Part II)



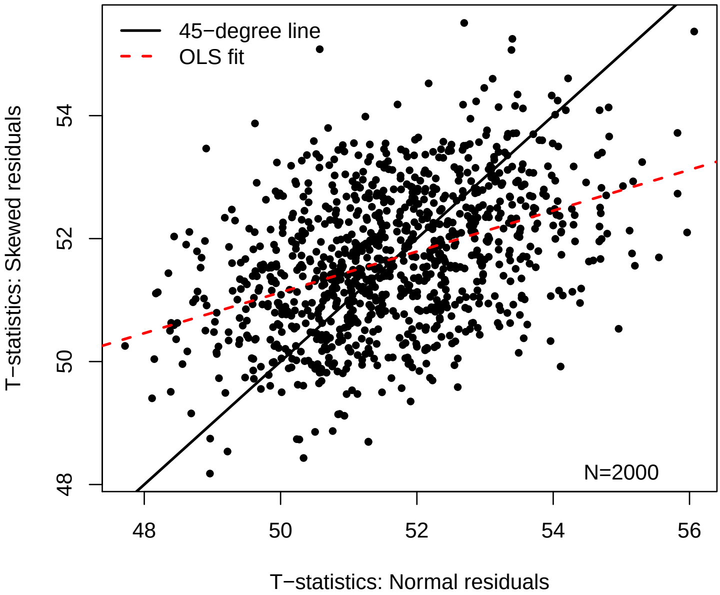
Little Effect On $\hat{\beta}$



Important Differences in Inference



With $N = 2000$? Not So Much...



Examples

This:

$$Y_i = \beta_0 X_i^{\beta_1} u_i$$

becomes this:

$$\ln(Y_i) = \ln(\beta_0) + \beta_1 X_i + \ln(u_i)$$

And this:

$$\exp(Y_i) = \beta_0 + \beta_1 X_i + u_i$$

becomes this:

$$Y_i = \ln(\beta_0) + \beta_1 \ln(X_i) + \ln(u_i)$$

Monotonic Transformations

The "Ladder of Powers":

Transformation	p	$f(X)$	Fox's $f(X)$
Cube	3	X^3	$\frac{X^3-1}{3}$
Square	2	X^2	$\frac{X^2-1}{2}$
(None/Identity)	(1)	(X)	(X)
Square Root	$\frac{1}{2}$	$\sqrt{X}$	$2(\sqrt{X}-1)$
Cube Root	$\frac{1}{3}$	$\sqrt[3]{X}$	$3(\sqrt[3]{X}-1)$
Log	0 (sort of)	$\ln(X)$	$\ln(X)$
Inverse Cube Root	-$\frac{1}{3}$	$\frac{1}{\sqrt[3]{X}}$	$\frac{\left(\frac{1}{\sqrt[3]{X}}-1\right)}{-\frac{1}{3}}$
Inverse Square Root	-$\frac{1}{2}$	$\frac{1}{\sqrt{X}}$	$\frac{\left(\frac{1}{\sqrt{X}}-1\right)}{-\frac{1}{2}}$
Inverse	-1	$\frac{1}{X}$	$\frac{\left(\frac{1}{X}-1\right)}{-1}$
Inverse Square	-2	$\frac{1}{X^2}$	$\frac{\left(\frac{1}{X^2}-1\right)}{-2}$
Inverse Cube	-3	$\frac{1}{X^3}$	$\frac{\left(\frac{1}{X^3}-1\right)}{-3}$

A General Rule

Using higher-order power transformations (e.g. squares, cubes, etc.) “inflates” large values and “compresses” small ones; conversely, using lower-order power transformations (logs, etc.) “compresses” large values and “inflates” (or “expands”) smaller ones.

Power Transformations: Two Issues

1. X must be *positive*; so:

$$X^* = X + (|X_l| + \epsilon)$$

with (CZ's Rule of Thumb):

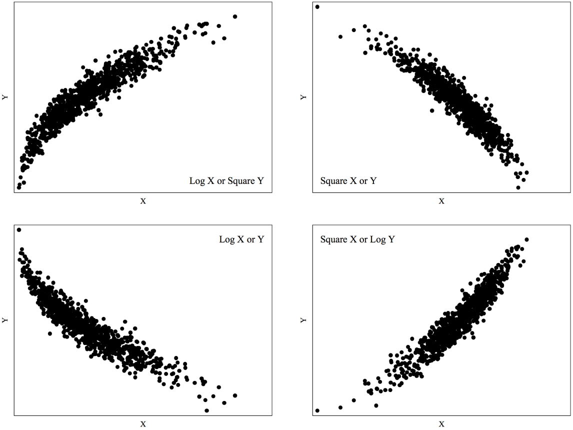
$$\epsilon = \frac{X_{l+1} - X_l}{2}$$

2. Power transformations generally require that:

$$\frac{X_h}{X_l} > 5 \text{ (or so)}$$

Which Transformation?

Mosteller and Tukey's "Bulging Rule":



Nonmonotonicity

Simple solution: Polynomials...

- Second-order / quadratic:

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + u_i$$

- Third-order / cubic:

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \beta_3 X_i^3 + u_i$$

- pth-order:

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \beta_3 X_i^3 + ... + \beta_p X_i^p + u_i$$

Transformed Xs: Interpretation

For:

$$\ln(Y_i) = \beta_0 + \beta_1 X_i + u_i,$$

then:

$$\mathrm{E}(Y) = \exp(\beta_0 + \beta_1 X_i)$$

and so:

$$\frac{\partial \mathrm{E}(Y)}{\partial X} = \exp(\beta_1).$$

Transformed Xs: Interpretation

Similarly, for:

$$Y_i = \beta_0 + \beta_1 \ln(X_i) + u_i$$

we have:

$$\frac{\partial \mathrm{E}(Y)}{\partial \ln(X)} = \beta_1.$$

So doubling X (say, from X_ℓ to $2X_\ell$):

$$\begin{aligned}
\Delta \mathrm{E}(Y) &= \mathrm{E}(Y|X = 2X_\ell) - \mathrm{E}(Y|X = X_\ell) \\
&= [\beta_0 + \beta_1 \ln(2X_\ell)] - [\beta_0 + \beta_1 \ln(X_\ell)] \\
&= \beta_1[\ln(2X_\ell) - \ln(X_\ell)] \\
&= \beta_1 \ln(2)
\end{aligned}$$

Log-Log Regressions

Specifying:

$$\ln(Y_i) = \beta_0 + \beta_1 \ln(X_i) + ... + u_i$$

means:

$$\text{Elasticity}_{YX} \equiv \frac{\%\Delta Y}{\%\Delta X} = \beta_1.$$

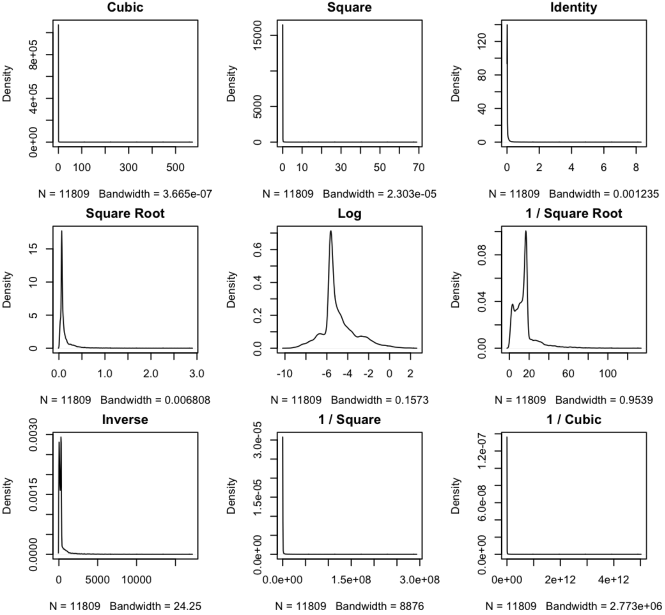
IOW, a one-percent change in X leads to a $\hat{\beta}_1$-percent change in Y.

An Example: Military Spending and GDP

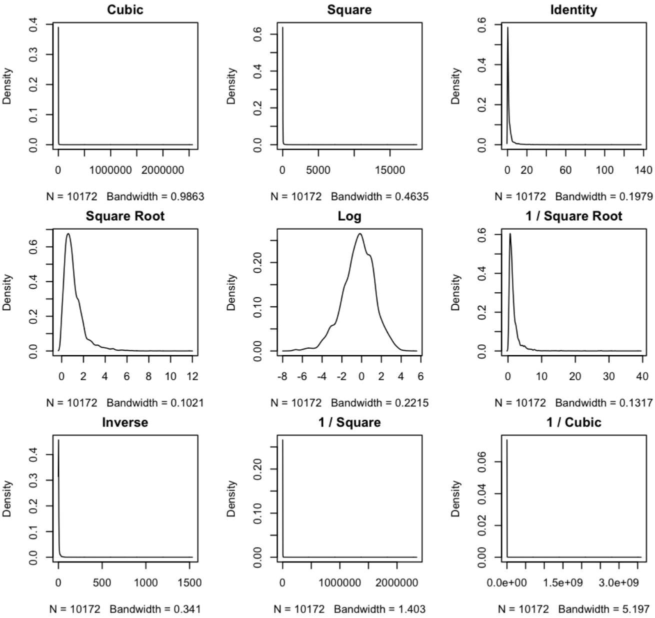
Data are from Fordham and Walker...

```
> with(Data, summary(milgdp))
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.    NA's
  0.000   0.238   0.749   2.115   2.104 136.900    4327

> with(Data, summary(gdp))
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.    NA's
 0.0001  0.0033  0.0047  0.0534  0.0153  8.3010    2690
```

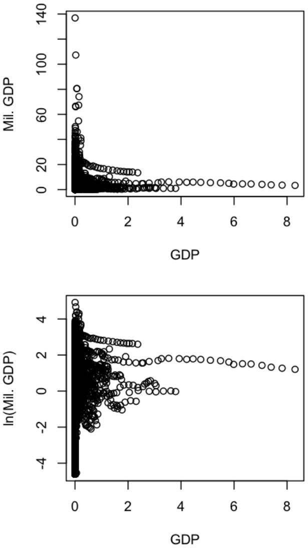
"Ladder of Powers": GDP



“Ladder of Powers”: Military Spending



Scatterplots



Some Regressions

Untransformed:

```
> with(Data, summary(lm(milgdp~gdp)))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   2.0538     0.0481  42.696  < 2e-16 ***
gdp           1.0038     0.1540   6.518 7.45e-11 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 4.757 on 10170 degrees of freedom
  (4327 observations deleted due to missingness)
Multiple R-squared: 0.00416,Adjusted R-squared: 0.004062
F-statistic: 42.49 on 1 and 10170 DF,  p-value: 7.454e-11
```

Logging X:

```
> with(Data, summary(lm(milgdp~log(gdp))))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  4.60137    0.13969   32.94   <2e-16 ***
log(gdp)     0.52196    0.02766   18.87   <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 4.686 on 10170 degrees of freedom
  (4327 observations deleted due to missingness)
Multiple R-squared: 0.03384,Adjusted R-squared: 0.03374
F-statistic: 356.2 on 1 and 10170 DF,  p-value: < 2.2e-16
```

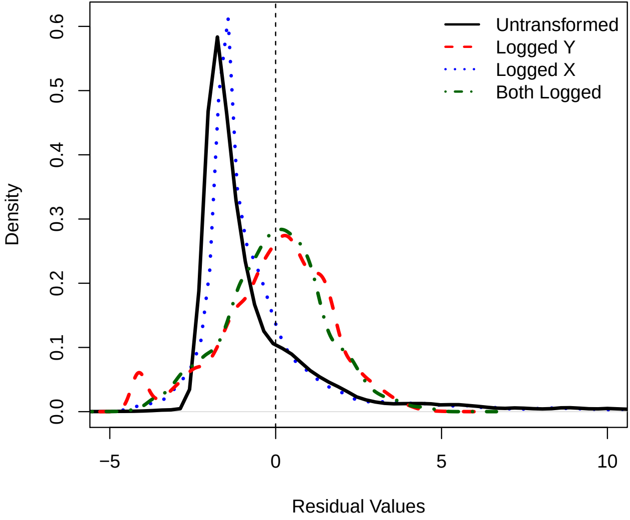
Logging Y:

```
> with(Data, summary(lm(log(milgdp+0.01)~gdp)))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.45918    0.01669  -27.51   <2e-16 ***
gdp          0.75794    0.05343   14.18   <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 1.651 on 10170 degrees of freedom
  (4327 observations deleted due to missingness)
Multiple R-squared: 0.0194,Adjusted R-squared: 0.0193
F-statistic: 201.2 on 1 and 10170 DF,  p-value: < 2.2e-16
```

Logging X and Y:

```
> with(Data, summary(lm(log(milgdp+0.01)~log(gdp))))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 1.644270   0.044736   36.76   <2e-16 ***
log(gdp)    0.431875   0.008858   48.76   <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 1.501 on 10170 degrees of freedom
  (4327 observations deleted due to missingness)
Multiple R-squared: 0.1895,Adjusted R-squared: 0.1894
F-statistic:  2377 on 1 and 10170 DF,  p-value: < 2.2e-16
```

Density Plots of $\hat{u}_i$s



- **Theory is valuable**.
- **Try different things**.
- **Look at plots**.
- **It takes practice**.