

# PLSC 503 – Spring 2020

## Multiplicative Interactions

March 5, 2020

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

$$Y_i = \beta_0 + \beta_1 D_{1i} + \beta_2 D_{2i} + u_i$$

means:

$$E(Y|D_1 = 1, D_2 = 0) - E(Y|D_1 = 0, D_2 = 0) = E(Y|D_1 = 1, D_2 = 1) - E(Y|D_1 = 0, D_2 = 1) [\equiv \beta_1]$$

and

$$E(Y|D_1 = 0, D_2 = 1) - E(Y|D_1 = 0, D_2 = 0) = E(Y|D_1 = 1, D_2 = 1) - E(Y|D_1 = 1, D_2 = 0) [\equiv \beta_2].$$

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{1i} X_{2i} + u_i$$

$$\begin{aligned} E(Y_i) &= \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{1i} X_{2i} \\ &= \beta_0 + \beta_2 X_{2i} + (\beta_1 + \beta_3 X_{2i}) X_{1i} \\ &= \beta_0 + \beta_2 X_{2i} + \psi_1 X_{1i} \end{aligned}$$

where  $\psi_1 = \beta_1 + \beta_3 X_{2i}$ . This means:

$$\frac{\partial E(Y_i)}{\partial X_1} = \beta_1 + \beta_3 X_{2i}.$$

Similarly:

$$\begin{aligned} E(Y_i) &= \beta_0 + \beta_1 X_{1i} + (\beta_2 + \beta_3 X_{1i}) X_{2i} \\ &= \beta_0 + \beta_1 X_{1i} + \psi_2 X_{2i} \end{aligned}$$

which implies:

$$\frac{\partial E(Y_i)}{\partial X_2} = \beta_2 + \beta_3 X_{1i}.$$

## “Direct Effects”

If  $X_2 = 0$ , then:

$$\begin{aligned} E(Y_i) &= \beta_0 + \beta_1 X_{1i} + \beta_2(0) + \beta_3 X_{1i}(0) \\ &= \beta_0 + \beta_1 X_{1i}. \end{aligned}$$

Similarly, for  $X_1 = 0$ :

$$\begin{aligned} E(Y_i) &= \beta_0 + \beta_1(0) + \beta_2 X_{2i} + \beta_3(0)X_{2i} \\ &= \beta_0 + \beta_2 X_{2i} \end{aligned}$$

**In most instances, the quantities we care about are not  $\beta_1$  and  $\beta_2$ , but rather  $\psi_1$  and  $\psi_2$ .**

Point estimates:

$$\hat{\psi}_1 = \hat{\beta}_1 + \hat{\beta}_3 X_2$$

and

$$\hat{\psi}_2 = \hat{\beta}_2 + \hat{\beta}_3 X_1.$$

For variance, recall that:

$$\text{Var}(a + bZ) = \text{Var}(a) + Z^2 \text{Var}(b) + 2Z \text{Cov}(a, b)$$

Means that:

$$\widehat{\text{Var}}(\hat{\psi}_1) = \widehat{\text{Var}}(\hat{\beta}_1) + X_2^2 \widehat{\text{Var}}(\hat{\beta}_3) + 2X_2 \widehat{\text{Cov}}(\hat{\beta}_1, \hat{\beta}_3).$$

and

$$\widehat{\text{Var}}(\hat{\psi}_2) = \widehat{\text{Var}}(\hat{\beta}_2) + X_1^2 \widehat{\text{Var}}(\hat{\beta}_3) + 2X_1 \widehat{\text{Cov}}(\hat{\beta}_2, \hat{\beta}_3).$$



# Types of Interactions: Dichotomous $X$ s

For

$$Y_i = \beta_0 + \beta_1 D_{1i} + \beta_2 D_{2i} + \beta_3 D_{1i} D_{2i} + u_i$$

we have:

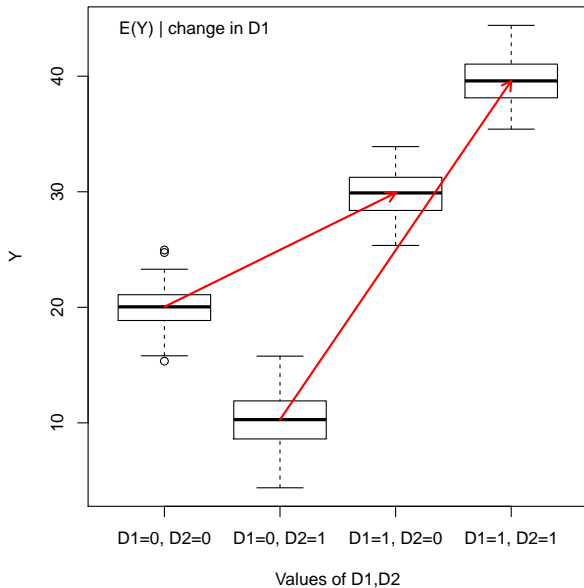
$$E(Y|D_1 = 0, D_2 = 0) = \beta_0$$

$$E(Y|D_1 = 1, D_2 = 0) = \beta_0 + \beta_1$$

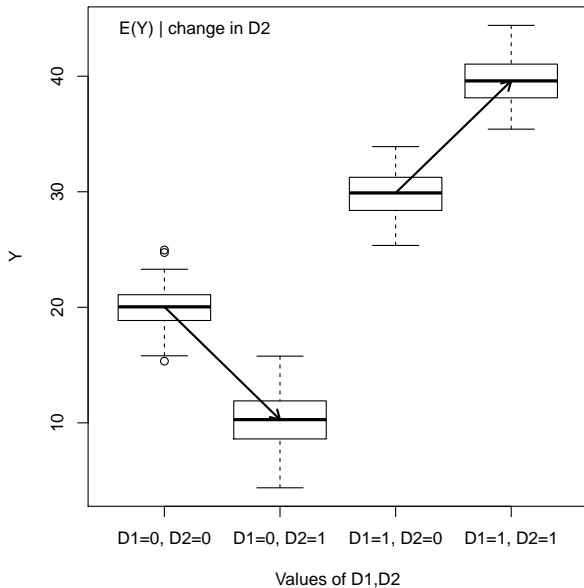
$$E(Y|D_1 = 0, D_2 = 1) = \beta_0 + \beta_2$$

$$E(Y|D_1 = 1, D_2 = 1) = \beta_0 + \beta_1 + \beta_2 + \beta_3$$

# Values of $E(Y)$ for Changes in $D_1$



# Values of $E(Y)$ for Changes in $D_2$



# Dichotomous and Continuous $X$ s

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 D_i + \beta_3 X_i D_i + u_i$$

gives:

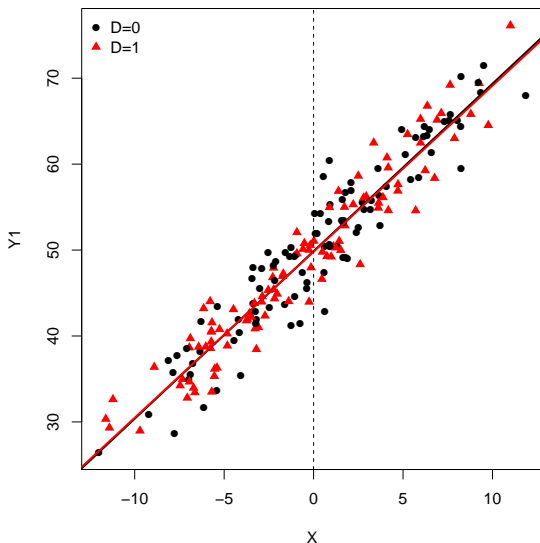
$$E(Y|X, D = 0) = \beta_0 + \beta_1 X$$

$$E(Y|X, D = 1) = (\beta_0 + \beta_2) + (\beta_1 + \beta_3)X$$

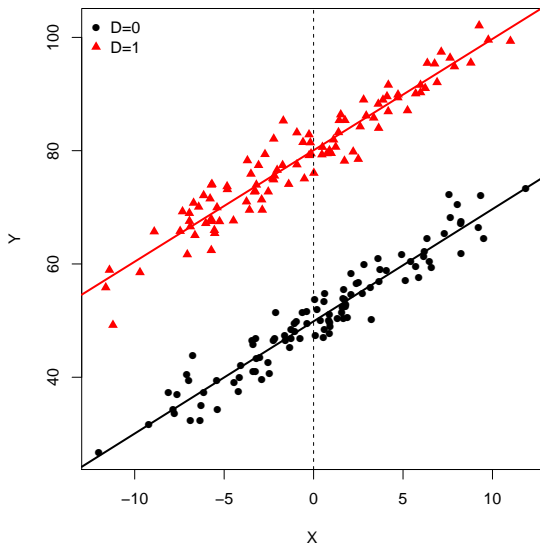
Four possibilities:

- $\beta_2 = \beta_3 = 0$
- $\beta_2 \neq 0$  and  $\beta_3 = 0$
- $\beta_2 = 0$  and  $\beta_3 \neq 0$
- $\beta_2 \neq 0$  and  $\beta_3 \neq 0$

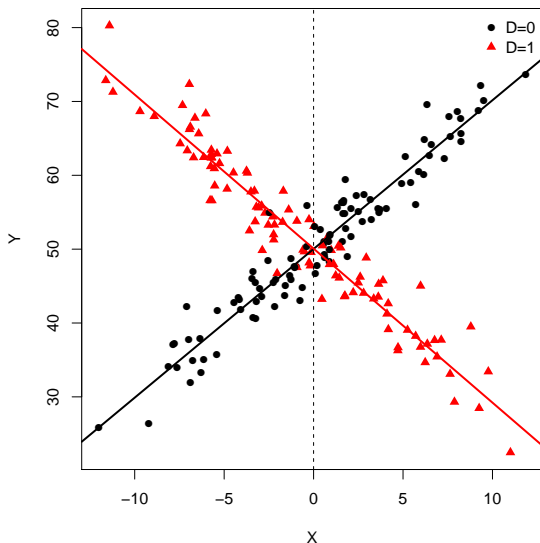
# Scatterplot and Regression Lines of $Y$ on $X$ for $D = 0$ and $D = 1$ : No Slope or Intercept Differences ( $\beta_2 = \beta_3 = 0$ )



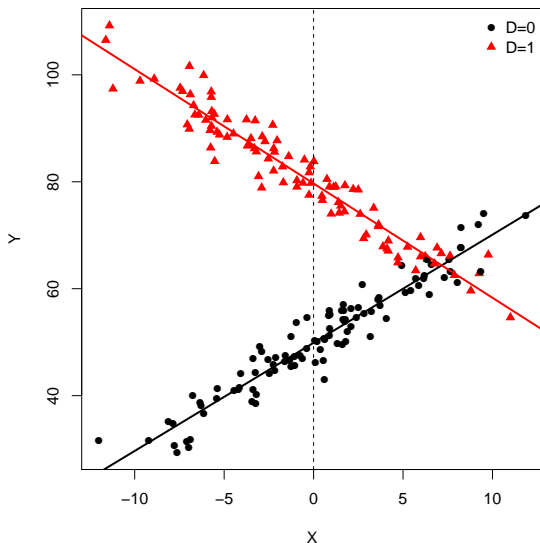
# Scatterplot and Regression Lines of $Y$ on $X$ for $D = 0$ and $D = 1$ : Intercept Shift ( $\beta_2 \neq 0, \beta_3 = 0$ )



# Scatterplot and Regression Lines of $Y$ on $X$ for $D = 0$ and $D = 1$ : Slope Change ( $\beta_2 = 0$ , $\beta_3 \neq 0$ )



# Scatterplot and Regression Lines of $Y$ on $X$ for $D = 0$ and $D = 1$ : Slope and Intercept Change ( $\beta_2 \neq 0$ , $\beta_3 \neq 0$ )





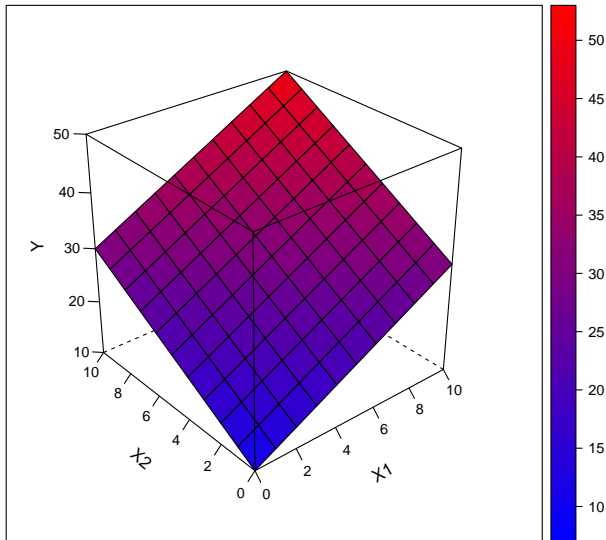
## Two Continuous $X$ s

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{1i} X_{2i} + u_i.$$

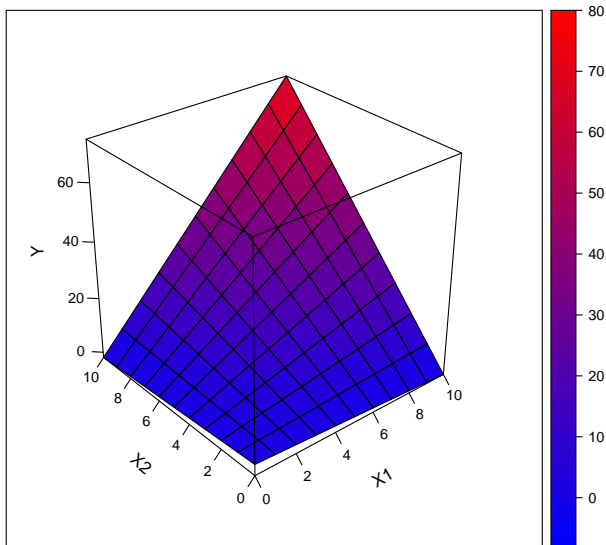
Implies

$$\beta_3 = 0 \rightarrow \frac{\partial E(Y)}{\partial X_1} = \beta_1 \forall X_2 \text{ and } \frac{\partial E(Y)}{\partial X_2} = \beta_2 \forall X_1$$

# Two Continuous Variables: No Interactive Effects



# Two Continuous Variables: Interaction Present



# Quadratic, Cubic, and Other Polynomial Effects

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \beta_3 X_i^3 + \dots + \beta_j X_i^j + u_i.$$

In general:

$$\frac{\partial E(Y)}{\partial X} = \beta_1 + 2\beta_2 X + 3\beta_3 X^2 + \dots + j\beta_j X^{j-1}$$

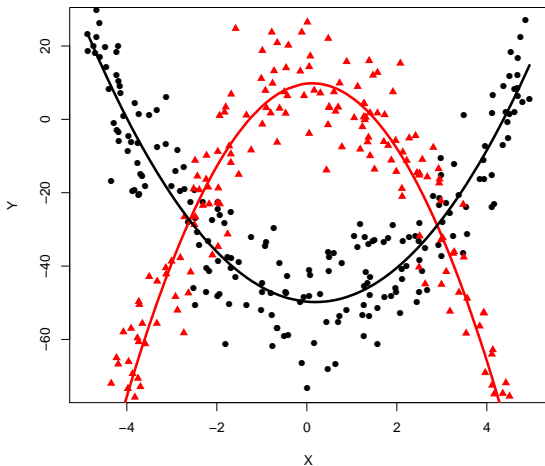
Quadratic case ( $j = 2$ ):

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + u_i.$$

implies

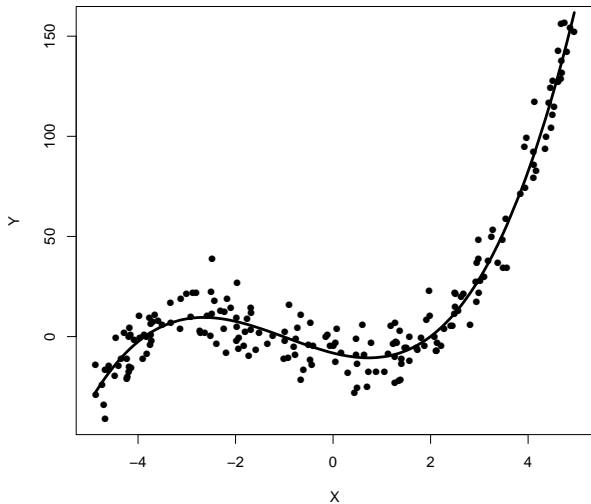
$$\frac{\partial E(Y)}{\partial X} = \beta_1 + 2\beta_2 X$$

# Two Quadratic Relationships



Note: Red line is  $Y_i = 10 + 1X_i - 5X_i^2 + u_i$ ; black line is  $Y_i = -50 - 1X_i + 3X_i^2 + u_i$ .

## Example of a Cubic Relationship



Note: Solid line is  $Y_i = -1 + 1X_i - 8X_i^2 + 5X_i^3 + u_i$ .

# Higher-Order Interactive Models

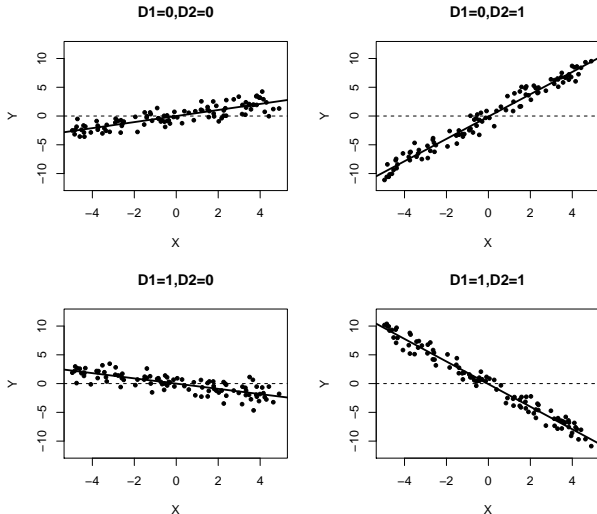
Three-way interaction:

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \\ \beta_4 X_{1i} X_{2i} + \beta_5 X_{1i} X_{3i} + \beta_6 X_{2i} X_{3i} + \beta_7 X_{1i} X_{2i} X_{3i} + u_i$$

Special case of dichotomous  $X_1, X_2$ :

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 D_{1i} + \beta_3 D_{2i} + \\ \beta_4 X_i D_{1i} + \beta_5 X_i D_{2i} + \beta_6 D_{1i} D_{2i} + u_i$$

# Three-Way Interaction: Two Dummy and One Continuous Covariates





```
> ClintonTherm<-read.csv("ClintonTherm.csv")
> summary(ClintonTherm)
```

caseid	ClintonTherm	RConserv	ClintonConserv
Min. :1001	Min. : 0	Min. :1.000	Min. :1.000
1st Qu.:1440	1st Qu.: 30	1st Qu.:3.000	1st Qu.:2.000
Median :1854	Median : 60	Median :4.000	Median :3.000
Mean :2001	Mean : 57	Mean :4.323	Mean :2.985
3rd Qu.:2262	3rd Qu.: 85	3rd Qu.:5.000	3rd Qu.:4.000
Max. :3403	Max. :100	Max. :7.000	Max. :7.000

PID	GOP
Min. :1.000	Min. :0.0000
1st Qu.:1.000	1st Qu.:0.0000
Median :2.000	Median :0.0000
Mean :2.059	Mean :0.3161
3rd Qu.:3.000	3rd Qu.:1.0000
Max. :5.000	Max. :1.0000

# A Basic Regression

```
> summary(with(ClintonTherm, lm(ClintonTherm~RConserv+GOP)))
```

Call:

```
lm(formula = ClintonTherm ~ RConserv + GOP)
```

Coefficients:

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	93.4756	2.2278	41.96	<2e-16 ***
RConserv	-6.4866	0.5373	-12.07	<2e-16 ***
GOP	-26.6699	1.6056	-16.61	<2e-16 ***

---

Signif. codes: 0 \*\*\* 0.001 \*\* 0.01 \* 0.05 . 0.1 1

Residual standard error: 23.65 on 1294 degrees of freedom

Multiple R-squared: 0.3795, Adjusted R-squared: 0.3786

F-statistic: 395.7 on 2 and 1294 DF, p-value: < 2.2e-16

# An Interactive Model

```
> fit1<-with(ClintonTherm, lm(ClintonTherm~RConserv+GOP+
                             RConserv*GOP))
> summary(fit1)
```

Call:

```
lm(formula = ClintonTherm ~ RConserv + GOP + RConserv * GOP)
```

Coefficients:

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	89.9271	2.4866	36.165	< 2e-16 ***
RConserv	-5.5705	0.6085	-9.154	< 2e-16 ***
GOP	-6.4840	6.5690	-0.987	0.32379
RConserv:GOP	-4.0581	1.2808	-3.168	0.00157 **

---

Signif. codes: 0 \*\*\* 0.001 \*\* 0.01 \* 0.05 . 0.1 1

Residual standard error: 23.57 on 1293 degrees of freedom

Multiple R-squared: 0.3843, Adjusted R-squared: 0.3829

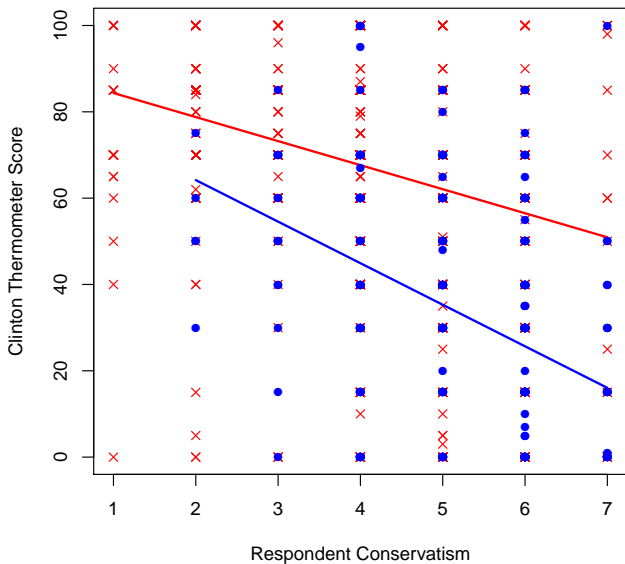
F-statistic: 269 on 3 and 1293 DF, p-value: < 2.2e-16

## Two Regressions, Sort Of

$$\begin{aligned}E(\text{Thermometer} \mid \text{Non-GOP})_i &= 89.9 - 6.5(0) - 5.6(\text{R's Conservatism}_i) \\&\quad - 4.0(0 \times \text{R's Conservatism}_i) \\&= \mathbf{89.9 - 5.6(\text{R's Conservatism}_i)}\end{aligned}$$

$$\begin{aligned}E(\text{Thermometer} \mid \text{GOP})_i &= [89.9 - 6.5(1)] + [-5.6 - 4.0(1 \times \text{R's Conservatism}_i)] \\&= \mathbf{83.4 - 9.6(\text{R's Conservatism}_i)}\end{aligned}$$

# Thermometer Scores by Conservatism, GOP and Non-GOP



# Interactive Results are (Almost) Identical to Separate Regressions

```
> NonReps<-subset(ClintonTherm,GOP==0)
> summary(with(NonReps, lm(ClintonTherm~RConserv)))
```

Call:

```
lm(formula = ClintonTherm ~ RConserv, data = NonReps)
```

Coefficients:

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	89.9271	2.4695	36.416	<2e-16 ***
RConserv	-5.5705	0.6043	-9.217	<2e-16 ***

---

Signif. codes: 0 \*\*\* 0.001 \*\* 0.01 \* 0.05 . 0.1 1

Residual standard error: 23.41 on 885 degrees of freedom

Multiple R-squared: 0.08759, Adjusted R-squared: 0.08656

F-statistic: 84.96 on 1 and 885 DF, p-value: < 2.2e-16

# Interactive Results are (Almost) Identical to Separate Regressions

```
> Reps<-subset(ClintonTherm,GOP==1)
> summary(with(Reps, lm(ClintonTherm~RConserv)))
```

Call:

```
lm(formula = ClintonTherm ~ RConserv, data = Reps)
```

Coefficients:

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	83.443	6.170	13.524	< 2e-16 ***
RConserv	-9.629	1.144	-8.419	6.52e-16 ***

---

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 23.92 on 408 degrees of freedom

Multiple R-squared: 0.148, Adjusted R-squared: 0.1459

F-statistic: 70.88 on 1 and 408 DF, p-value: 6.518e-16

# Discovering $\hat{\psi}_1$ and $\hat{\psi}_2$

For RConserv:

$$\begin{aligned}\text{Clinton Thermometer}_i &= \beta_0 + (\beta_1 + \beta_3 \text{GOP}_i) \text{R's Conservatism}_i + \\ &\quad \beta_2 \text{GOP}_i + u_i \\ &= \beta_0 + \psi_{1i} \text{R's Conservatism}_i + \beta_2 \text{GOP}_i + u_i.\end{aligned}$$

So:

$$\hat{\psi}_{1i} = \hat{\beta}_1 + \hat{\beta}_3 \times \text{GOP}_i$$

and

$$\hat{\sigma}_{\psi_1} = \sqrt{\widehat{\text{Var}}(\hat{\beta}_1) + (\text{GOP})^2 \widehat{\text{Var}}(\hat{\beta}_3) + 2(\text{GOP}) \widehat{\text{Cov}}(\hat{\beta}_1, \hat{\beta}_3)}.$$



# Discovering $\hat{\psi}_1$ and $\hat{\psi}_2$

For GOP:

$$\begin{aligned}\text{Clinton Thermometer}_i &= \beta_0 + (\beta_2 + \beta_3 \times \text{R's Conservatism}_i)\text{GOP}_i + \\ &\quad \beta_1(\text{R's Conservatism}_i) + u_i \\ &= \beta_0 + \psi_{2i}\text{GOP}_i + \beta_1(\text{R's Conservatism}_i) + u_i.\end{aligned}$$

So:

$$\hat{\psi}_{2i} = \hat{\beta}_2 + \hat{\beta}_3 \times (\text{R's Conservatism}_i).$$

and

$$\hat{\sigma}_{\psi_2} = \sqrt{\widehat{\text{Var}}(\hat{\beta}_2) + (\text{R's Conservatism}_i)^2 \widehat{\text{Var}}(\hat{\beta}_3) + 2k \widehat{\text{Cov}}(\hat{\beta}_2, \hat{\beta}_3)}.$$

# Discovering $\hat{\psi}_1$ and $\hat{\psi}_2$

```
> Psi1<-fit1$coeff[2]+fit1$coeff[4]
```

```
> Psi1  
RConserv  
-9.628577
```

```
> SPsi1<-sqrt(vcov(fit1)[2,2] + (1)^2*vcov(fit1)[4,4] + 2*1*vcov(fit1)[2,4])  
> SPsi1  
[1] 1.127016
```

```
> Psi1 / SPsi1 # <-- t-statistic  
RConserv  
-8.543422
```

# Discovering $\hat{\psi}_1$ and $\hat{\psi}_2$

```
> # psi_2 | RConserv = 1
> fit1$coeff[3]+(1 * fit1$coeff[4])
      GOP
-10.54208

> sqrt(vcov(fit1)[3,3] + (1)^2*vcov(fit1)[4,4] + 2*1*vcov(fit1)[3,4])
[1] 5.335847

# Implies t is approximately 2

> # psi_2 | RConserv = 7
> fit1$coeff[3]+(7 * fit1$coeff[4])
      GOP
-34.89045

> sqrt(vcov(fit1)[3,3] + (7)^2*vcov(fit1)[4,4] + 2*7*vcov(fit1)[3,4])
[1] 3.048302

# t is approximately 11
```

# An Easier Way: linearHypothesis()

```
> library(car)
> linearHypothesis(fit1,"RConserv+RConserv:GOP")
Linear hypothesis test
```

Hypothesis:

$\text{RConserv} + \text{RConserv:GOP} = 0$

Model 1: restricted model

Model 2:  $\text{ClintonTherm} \sim \text{RConserv} + \text{GOP} + \text{RConserv} * \text{GOP}$

	Res.Df	RSS	Df	Sum of Sq	F	Pr(>F)
1	1294	758714				
2	1293	718173	1	40541	72.99	< 2.2e-16 ***
---						

Signif. codes: 0 \*\*\* 0.001 \*\* 0.01 \* 0.05 . 0.1 1

```
> # Note: Same as t-test:
```

```
> sqrt(72.99)
```

```
[1] 8.543419
```

## An Easier Way: linearHypothesis()

```
> # psi_2 | RConserv = 7:  
> linearHypothesis(fit1,"GOP+7*RConserv:GOP")  
Linear hypothesis test
```

Hypothesis:

$\text{GOP} + 7 \text{ RConserv:GOP} = 0$

Model 1: restricted model

Model 2: ClintonTherm ~ RConserv + GOP + RConserv \* GOP

	Res.Df	RSS	Df	Sum of Sq	F	Pr(>F)
1	1294	790938				
2	1293	718173	1	72766	131.01	< 2.2e-16 ***

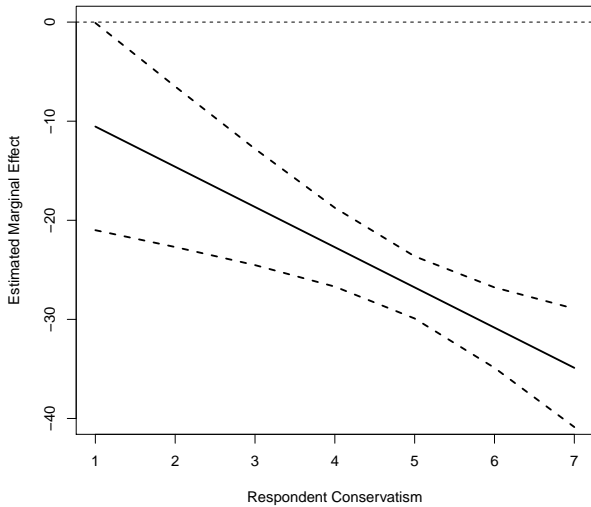
---

Signif. codes: 0 \*\*\* 0.001 \*\* 0.01 \* 0.05 . 0.1 1

## Marginal Effects Plots, I

```
> ConsSim<-seq(1,7,1)
> psis<-fit1$coeff[3]+(ConsSim * fit1$coeff[4])
> psis.ses<-sqrt(vcov(fit1)[3,3] +
  (ConsSim)^2*vcov(fit1)[4,4] + 2*ConsSim*vcov(fit1)[3,4])

> plot(ConsSim,psis,t="l",lwd=2,xlab="Respondent Conservatism",
  ylab="Estimated Marginal Effect",ylim=c(-40,0))
> lines(ConsSim,psis+(1.96*psis.ses),lty=2,lwd=2)
> lines(ConsSim,psis-(1.96*psis.ses),lty=2,lwd=2)
> abline(h=0,lwd=1,lty=2)
```



# Interacting Two Continuous Covariates

```
> fit2<-with(ClintonTherm,  
+           lm(ClintonTherm~RConserv+ClintonConserv+RConserv*ClintonConserv))  
> summary(fit2)
```

Call:

```
lm(formula = ClintonTherm ~ RConserv + ClintonConserv + RConserv *  
    ClintonConserv)
```

Coefficients:

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	119.3515	5.1634	23.115	< 2e-16 ***
RConserv	-19.5673	1.0362	-18.884	< 2e-16 ***
ClintonConserv	-7.9311	1.6477	-4.813	1.66e-06 ***
RConserv:ClintonConserv	3.6293	0.3394	10.695	< 2e-16 ***

---

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 22.03 on 1293 degrees of freedom

Multiple R-squared: 0.4619, Adjusted R-squared: 0.4606

F-statistic: 370 on 3 and 1293 DF, p-value: < 2.2e-16



# Hypothesis Tests

```
> fit2$coef[2]+(1*fit2$coef[4])
```

```
RConserv  
-15.93803
```

```
> sqrt(vcov(fit2)[2,2] + (1)^2*vcov(fit2)[4,4] + 2*1*vcov(fit2)[2,4])  
[1] 0.7439696
```

```
> linearHypothesis(fit2,"RConserv+1*RConserv:ClintonConserv")  
Linear hypothesis test
```

Hypothesis:

```
RConserv + RConserv:ClintonConserv = 0
```

Model 1: restricted model

Model 2: ClintonTherm ~ RConserv + ClintonConserv + RConserv \* ClintonConserv

	Res.Df	RSS	Df	Sum of Sq	F	Pr(>F)
1	1294	850442				
2	1293	627658	1	222784	458.94	< 2.2e-16 ***
---						

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

# More Hypothesis Tests

```
> # psi_1 | ClintonConserv = mean
> fit2$coef[2]+((mean(ClintonTherm$ClintonConserv))*fit2$coef[4])

RConserv
-8.735424

> sqrt(vcov(fit2)[2,2] + (mean(ClintonTherm$ClintonConserv)^2*vcov(fit2)[4,4] +
+ 2*(mean(ClintonTherm$ClintonConserv))*vcov(fit2)[2,4]))

[1] 0.4507971

> pt(((fit2$coef[2]+(2.985*fit2$coef[4])) / sqrt(vcov(fit2)[2,2] +
+ (2.985)^2*vcov(fit2)[4,4] + 2*2.985*vcov(fit2)[2,4])),df=1293)

RConserv
6.483788e-74

> # psi_2 | RConserv = 1
> fit2$coef[3]+(1*fit2$coef[4])

ClintonConserv
-4.301803

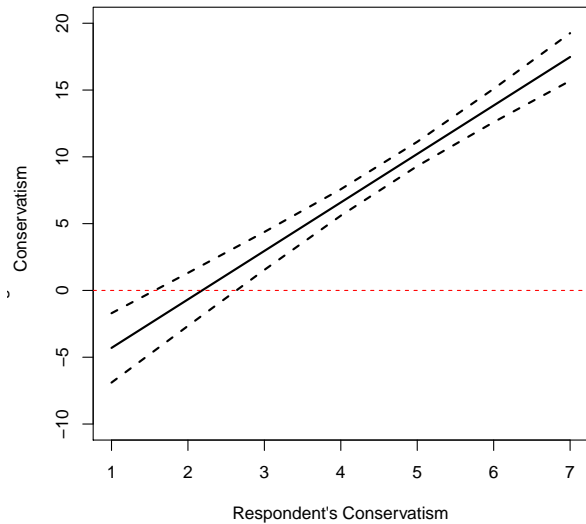
> # psi_2 | RConserv = 6
> fit2$coef[3]+(6*fit2$coef[4])

ClintonConserv
13.84463
```

## Marginal Effect Plot, II

```
> psis2<-fit2$coef[3]+(ConsSim*fit2$coef[4])
> psis2.ses<-sqrt(vcov(fit2)[3,3] + (ConsSim)^2*vcov(fit2)[4,4]
+ 2*ConsSim*vcov(fit2)[3,4])

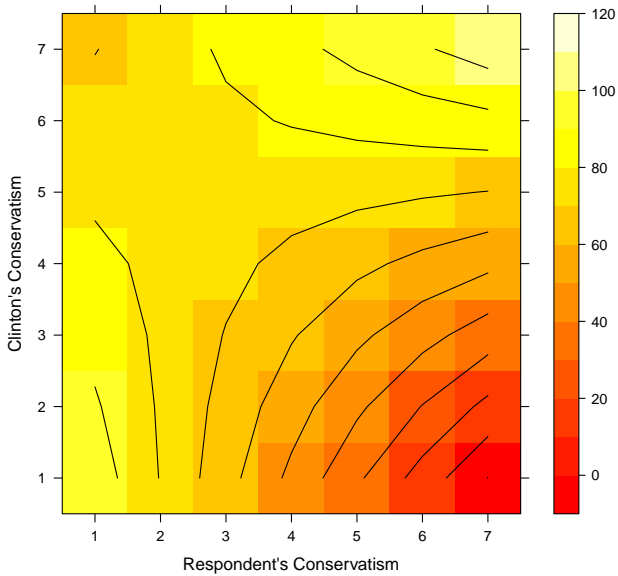
> plot(ConsSim,psis2,t="l",lwd=2,xlab="Respondent's
  Conservatism",ylab="Marginal Effect of Clinton's
  Conservatism",ylim=c(-10,20))
> lines(ConsSim,psis2+(1.96*psis2.ses),lty=2,lwd=2)
> lines(ConsSim,psis2-(1.96*psis2.ses),lty=2,lwd=2)
> abline(h=0,lty=2,lwd=1,col="red")
```



## Predicted Values: A Contour Plot

```
> library(lattice)
> grid<-expand.grid(RConserv=seq(1,7,1),
  ClintonConserv=seq(1,7,1))
> hats<-predict(fit2,newdata=grid)

> levelplot(hats~grid$RConserv*grid$ClintonConserv,
  contour=TRUE,
  cuts=12,pretty=TRUE,xlab="Respondent's Conservatism",
  ylab="Clinton's Conservatism",
  col.regions=heat.colors)
```



## Predicted Values: A Wireframe Plot

```
> trellis.par.set("axis.line",list(col="transparent"))  
  
> wireframe(hats~grid$RConserv*grid$ClintonConserv,  
  drape=TRUE,  
  xlab=list("Respondent's Conservatism",rot=30),  
  ylab=list("Clinton's Conservatism",  
    rot=-40),zlab=list("Predictions",rot=90),  
  scales=list(arrows=FALSE,col="black"),  
  zoom=0.85,pretty=TRUE),  
  col.regions=colorRampPalette(c("blue","red"))(100))
```

