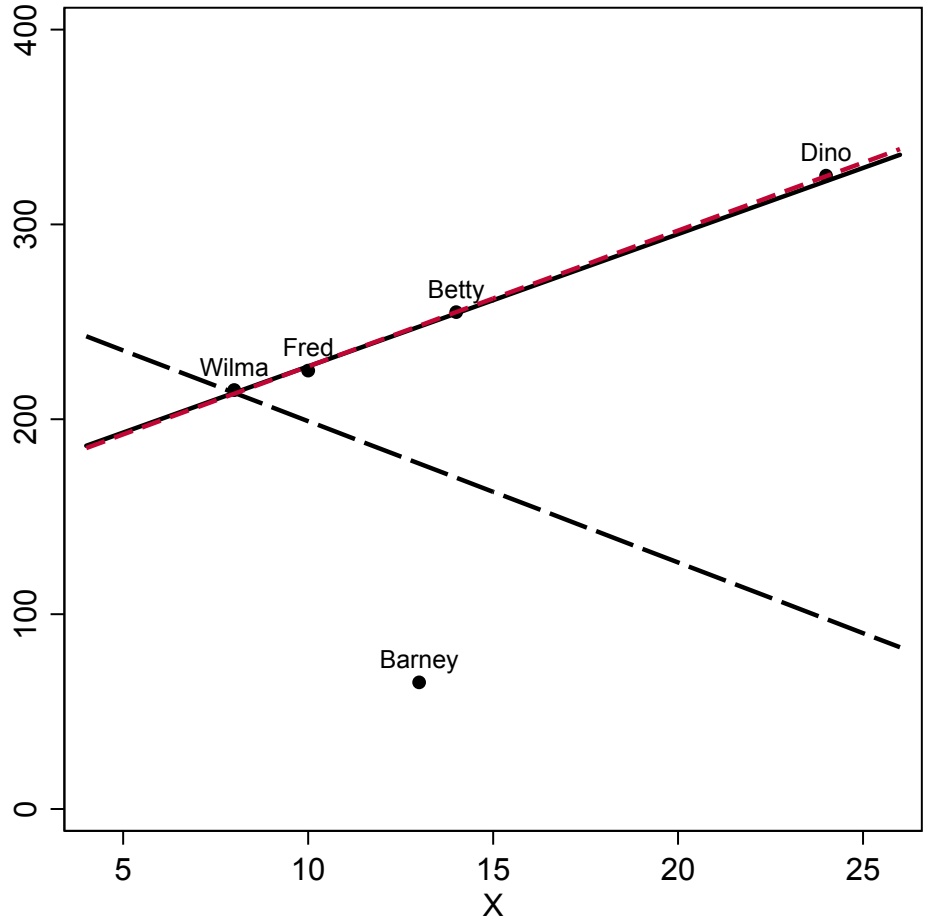
PLSC 503 – Spring 2020
Residuals, Model Fit, and Outliers

February 25, 2020

Discrepancy, Leverage, and Influence



Note: Solid line is the regression fit for Wilma, Fred, and Betty only.
Long-dashed line is the regression for Wilma, Fred, Betty, and Barney.
Short-dashed (red) line is the regression for Wilma, Fred, Betty and Dino.

Discrepancy, Leverage, and Influence

$$\text{Influence} = \text{Leverage} \times \text{Discrepancy}$$

Leverage

$$\begin{aligned} \hat{\mathbf{Y}} &= \mathbf{X}\hat{\boldsymbol{\beta}} \\ &= \mathbf{X}[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}] \\ &= \mathbf{H}\mathbf{Y} \end{aligned}$$

where

$$\mathbf{H} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'.$$

$$h_i = \mathbf{X}_i(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}_i'$$

Residuals

Variation:

$$\widehat{\text{Var}(\hat{u}_i)} = \hat{\sigma}^2[1 - \mathbf{X}_i(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}_i'] \tag{1}$$

$$\begin{aligned} \widehat{\text{s.e.}(\hat{u}_i)} &= \hat{\sigma}\sqrt{[1 - \mathbf{X}_i(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}_i']} \\ &= \hat{\sigma}\sqrt{1 - h_i} \end{aligned} \tag{2}$$

"Standardized":

$$\tilde{u}_i = \frac{\hat{u}_i}{\hat{\sigma}\sqrt{1 - h_i}} \tag{3}$$

"Studentized": define

$$\begin{aligned} \hat{\sigma}^2_{-i} &= \text{Variance for the } N-1 \text{ observations} \neq i \\ &= \frac{\hat{\sigma}^2(N-K)}{N-K-1} - \frac{\hat{u}_i^2}{(N-K-1)(1-h_i)}. \end{aligned} \tag{4}$$

Then:

$$\hat{u}'_i = \frac{\hat{u}_i}{\hat{\sigma}_{-i}\sqrt{1-h_i}} \tag{5}$$

Influence

"DFBETA":

$$D_{ki} = \hat{\beta}_k - \hat{\beta}_{k(-i)} \tag{6}$$

"DFBETAS" (the "S" is for "standardized):

$$D^*_{ki} = \frac{D_{ki}}{\widehat{\text{s.e.}(\hat{\beta}_{k(-i)})}} \tag{7}$$

Cook's D:

$$\begin{aligned} D_i &= \frac{\tilde{u}_i^2}{K} \times \frac{h_i}{1-h_i} \\ &= \frac{h_i \hat{u}_i^2}{K\hat{\sigma}^2(1-h_i)^2} \end{aligned} \tag{8}$$

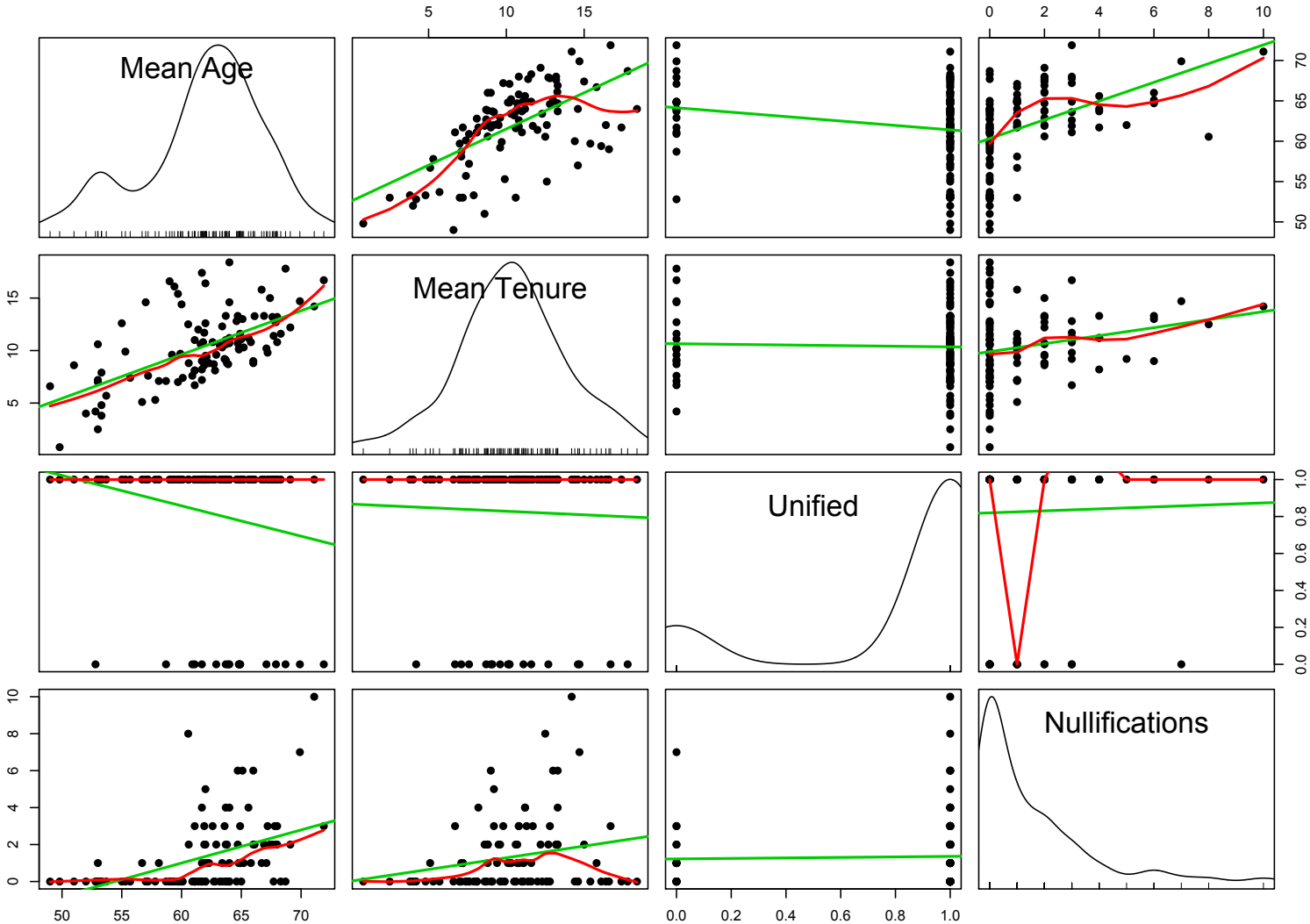
```
> # No Barney OR Dino...
> summary(lm(Y~X,data=subset(flintstones,name!="Dino" & name!="Barney")))

Residuals:
     2      4      5
 0.714 -2.143  1.429

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  159.286      6.776    23.5    0.027 *
X              6.786      0.619    11.0    0.058 .
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 2.67 on 1 degrees of freedom
Multiple R-squared: 0.992,Adjusted R-squared: 0.984
F-statistic:  120 on 1 and 1 DF,  p-value: 0.0579
```

```
> # No Barney (Dino included...)
> summary(lm(Y~X,data=subset(flintstones,name!="Barney")))

Residuals:
        2         3         4         5
-8.88e-16  2.63e-01 -2.11e+00  1.84e+00

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  157.368      2.465    63.8  0.00025 ***
X              6.974      0.161    43.3  0.00053 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 1.99 on 2 degrees of freedom
Multiple R-squared: 0.999,Adjusted R-squared: 0.998
F-statistic: 1.87e+03 on 1 and 2 DF,  p-value: 0.000534
```

"COVRATIO":

$$\text{COVRATIO}_i = \left[(1 - h_i) \left(\frac{N - K - 1 + \hat{u}_i'^2}{N - K} \right)^K \right]^{-1} \tag{9}$$

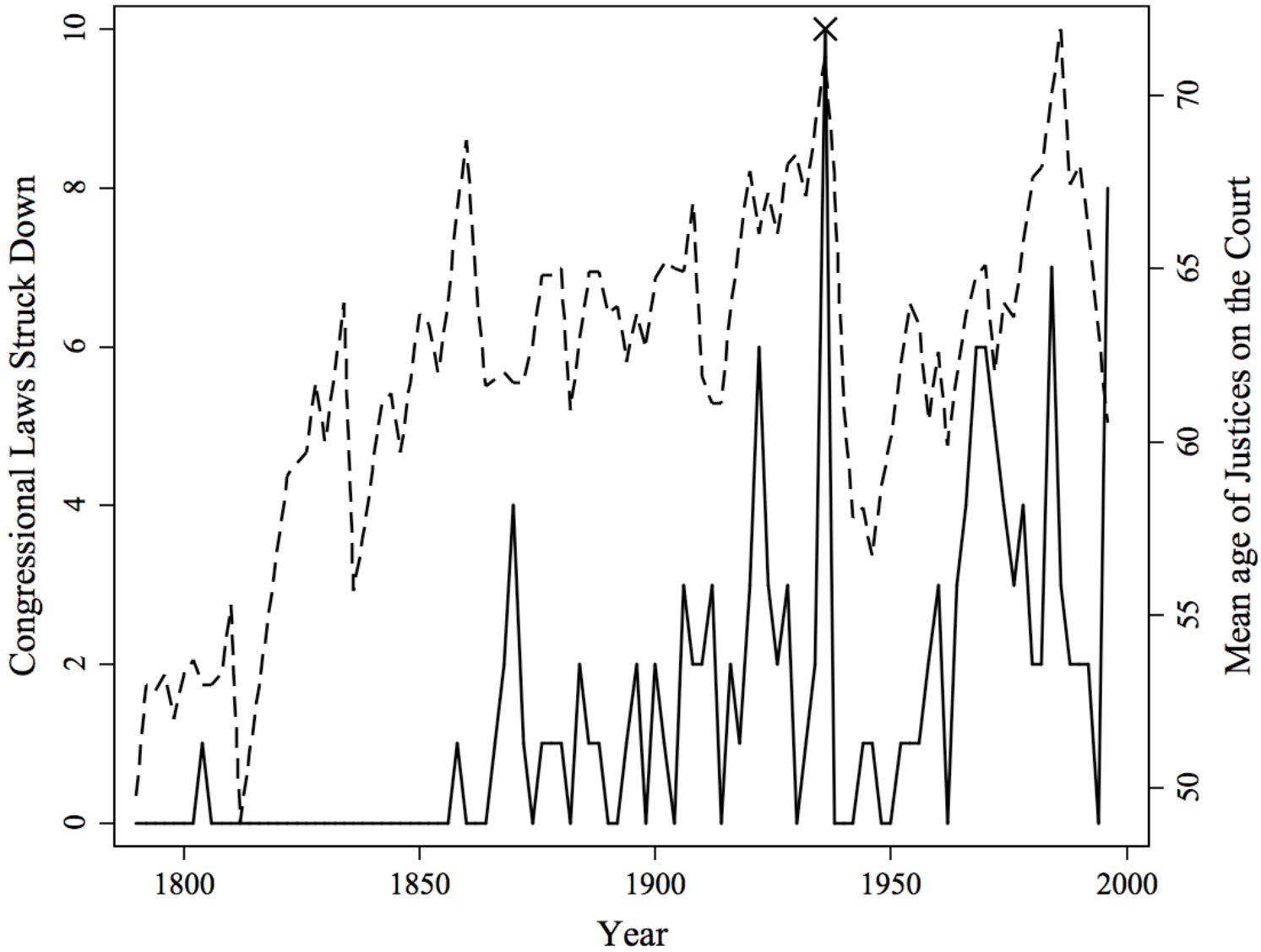
Example: Federal Judicial Review, 1789-1996



Variance

```
> Fit<-lm(nulls~age+tenure+unified)
> summary(Fit)

Residuals:
    Min      1Q  Median      3Q     Max
-2.7857 -1.0773 -0.3634  0.4238  6.9694

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) -12.10340    2.54324  -4.759 6.57e-06 ***
age           0.21886    0.04484   4.881 4.01e-06 ***
tenure       -0.06692    0.06427  -1.041    0.300
unified       0.71760    0.45844   1.565    0.121
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 1.715 on 100 degrees of freedom
Multiple R-squared: 0.2324,Adjusted R-squared: 0.2093
F-statistic: 10.09 on 3 and 100 DF,  p-value: 7.241e-06
```

Federal Judicial Review and Mean SCOTUS Age



Residuals, etc.

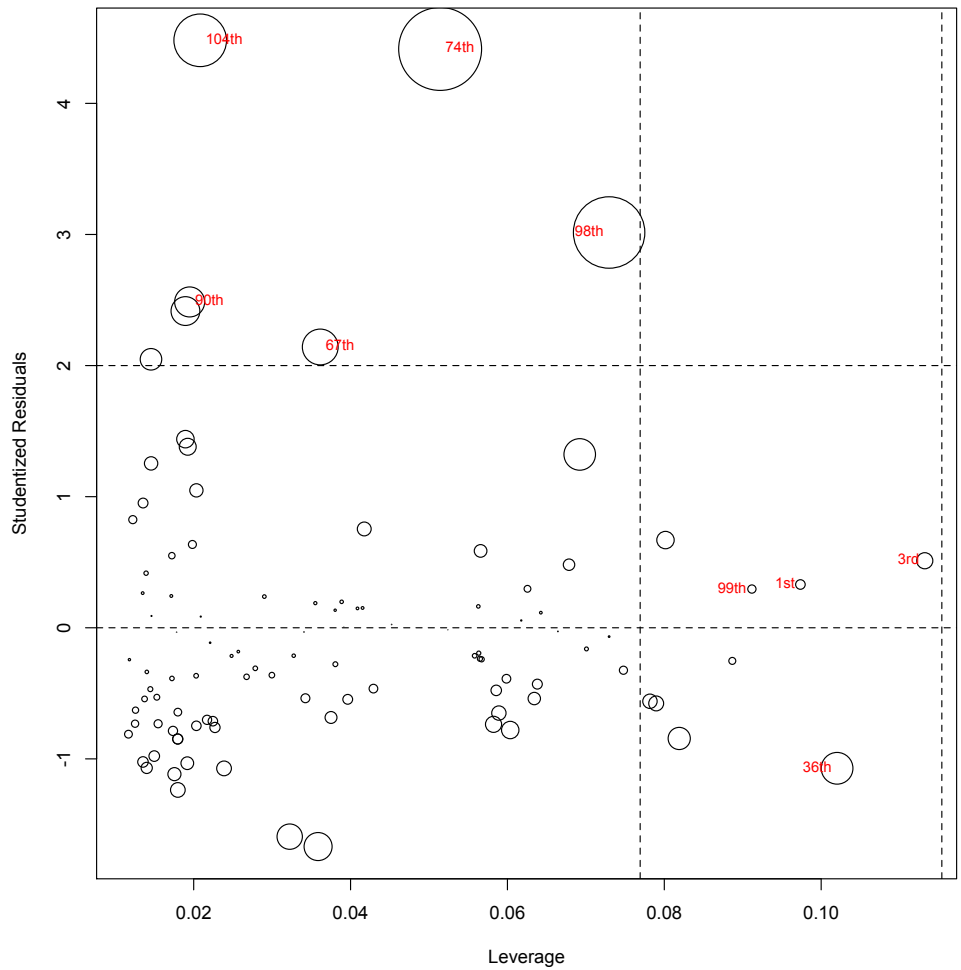
```
> FitResid<-(nulls - predict(Fit)) # residuals
> FitStandard<-rstandard(Fit) # standardized residuals
> FitStudent<-rstudent(Fit) # studentized residuals
> FitCooksD<-cooks.distance(Fit) # Cook's D
> FitDFBeta<-dfbeta(Fit) # DFBeta
> FitDFBetaS<-dfbetas(Fit) # DFBetaS
> FitCOVRATIO<-covratio(Fit) # COVRATIOs
```

Studentized Residuals

```
> FitStudent[74]
      74
4.415151

> Congress74<-rep(0,length=104)
> Congress74[74]<-1

> summary(lm(nulls~age+tenure+unified+Congress74))

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) -10.17290    2.37692  -4.280 4.33e-05 ***
age           0.18820    0.04177   4.505 1.82e-05 ***
tenure       -0.06356    0.05905  -1.076    0.284
unified       0.55159    0.42282   1.305    0.195
Congress74    7.14278    1.61779   4.415 2.58e-05 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 1.576 on 99 degrees of freedom
Multiple R-squared: 0.3586,Adjusted R-squared: 0.3327
F-statistic: 13.84 on 4 and 99 DF,  p-value: 5.304e-09
```

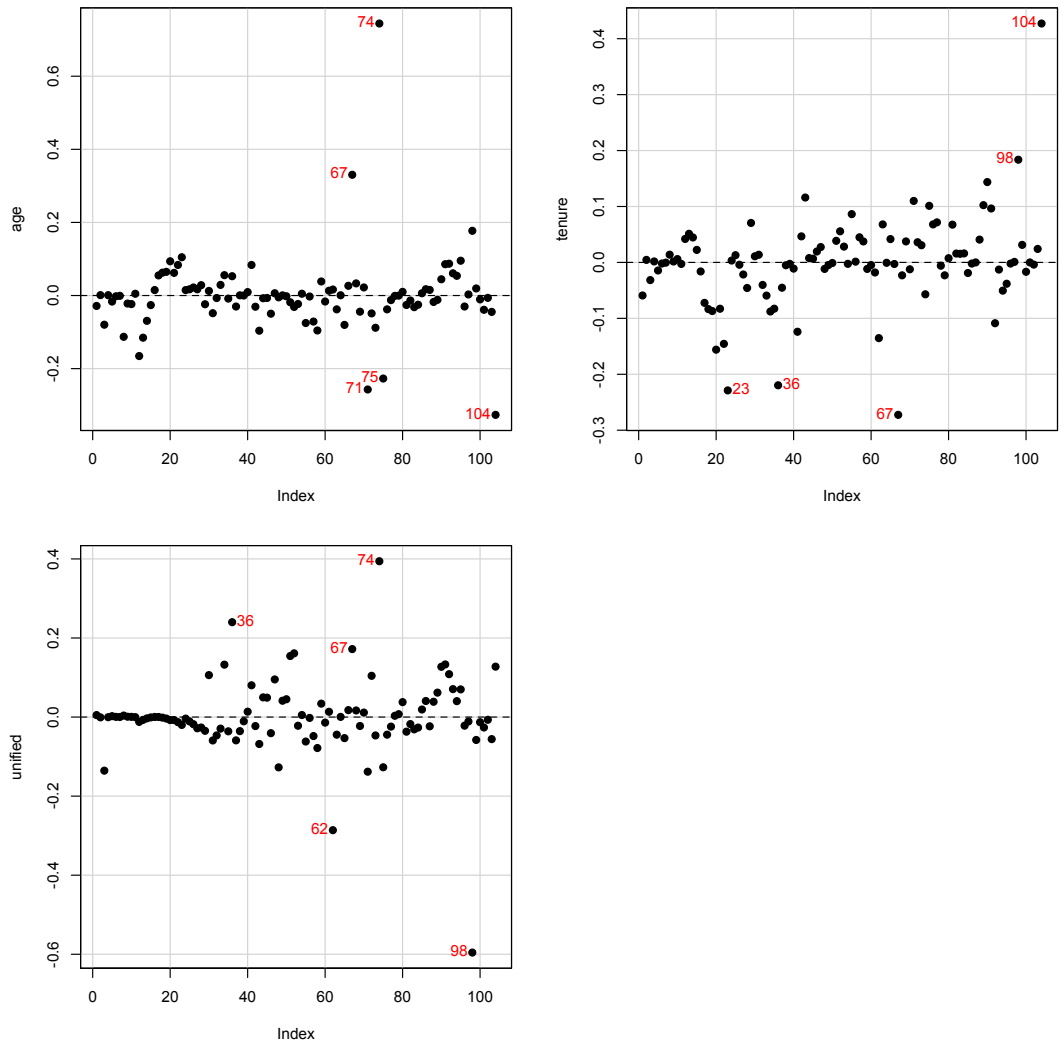
"Bubble Plot"

```
> influencePlot(Fit,id.n=4,labels=Congress,id.cex=0.8,
  id.col="red",xlab="Leverage")
```



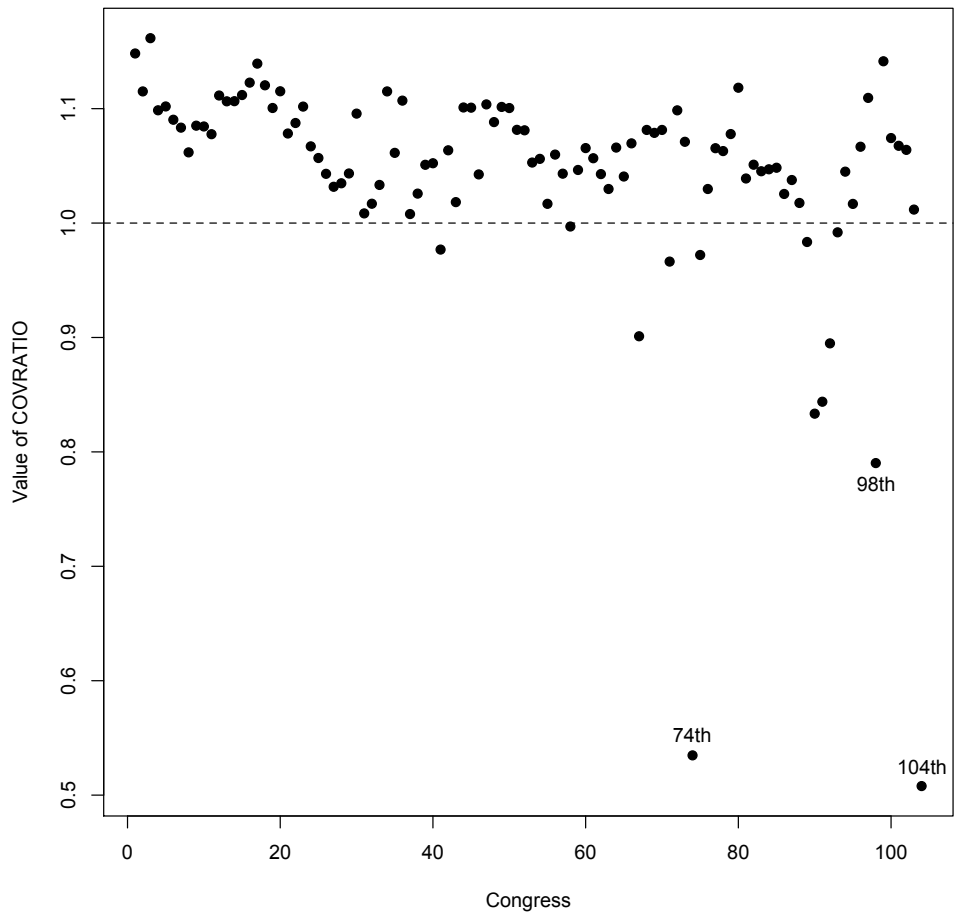
DFBETAS

```
> dfbetasPlots(Fit,id.n=5,id.col="red",main="",pch=19)
```



COVRATIO Plot

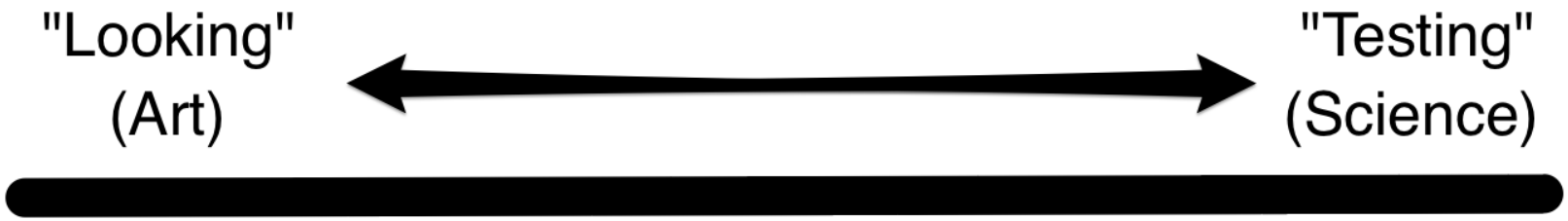
```
> plot(FitCOVRATIO~congress,pch=19,xlab="Congress",ylab="Value of COVRATIO")
> abline(h=1,lty=2)
```



Sensitivity Analyses: Omitting Outliers

```
> Outlier<-rep(0,104)
> Outlier[74]<-1
> Outlier[98]<-1
> Outlier[104]<-1
> DahlSmall<-Dahl[which (Outlier==0),]

> summary(lm(nulls~age+tenure+unified,data=DahlSmall))

Call:
lm(formula = nulls ~ age + tenure + unified, data = DahlSmall)

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) -10.38536    1.99470  -5.206 1.08e-06 ***
age           0.19302    0.03512   5.496 3.13e-07 ***
tenure       -0.10069    0.04974  -2.024   0.0457 *
unified       0.76645    0.36069   2.125   0.0361 *
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 1.319 on 97 degrees of freedom
Multiple R-squared: 0.2578,Adjusted R-squared: 0.2349
F-statistic: 11.23 on 3 and 97 DF,  p-value: 2.167e-06
```

Thinking About Diagnostics



Observational Data
Complex Data
Structure
Informative Missingness
Complex / Uncertain
Causality

Experimental Data
Simple Data Structure
No / Uninformative
Missingness
Simple / Clear Causality

One Approach

Pena, E.A. and E.H. Slate. 2006. "Global Validation of Linear Model Assumptions." *J. American Statistical Association* 101(473):341-354.

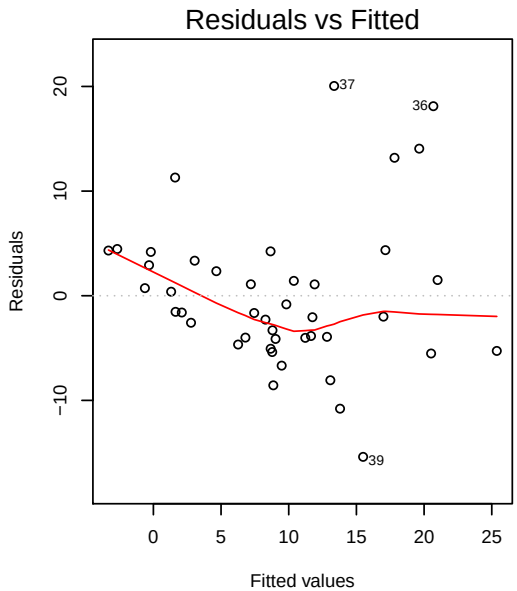
Tests for:

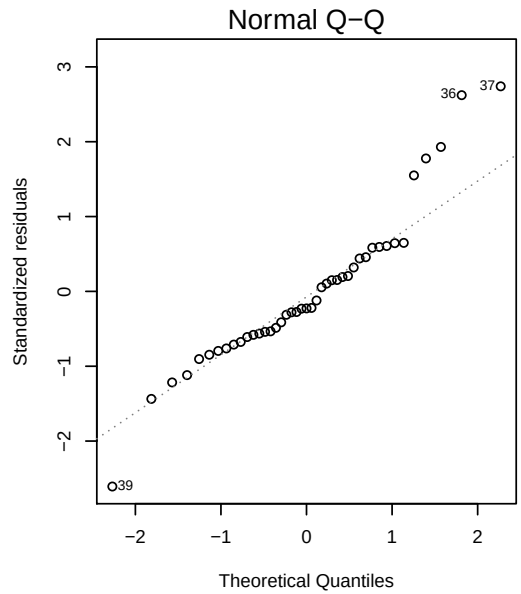
- Normality in $\hat{u}$s (via skewness & kurtosis tests)
- "Link function" (linearity / additivity)
- Constant variance and uncorrelatedness in $\hat{u}$s ("heteroskedasticity" test)

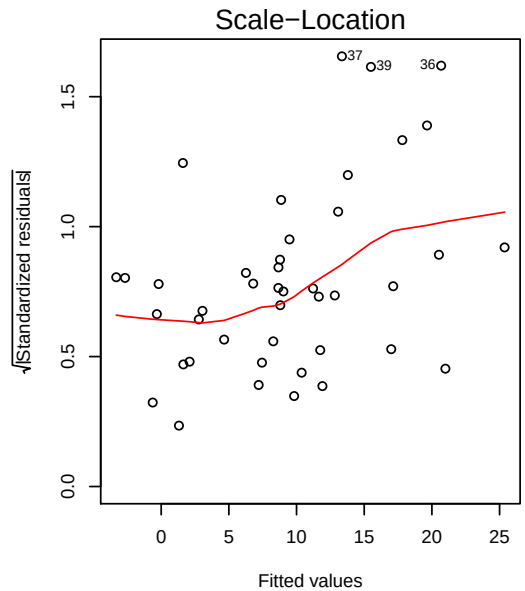
In Action

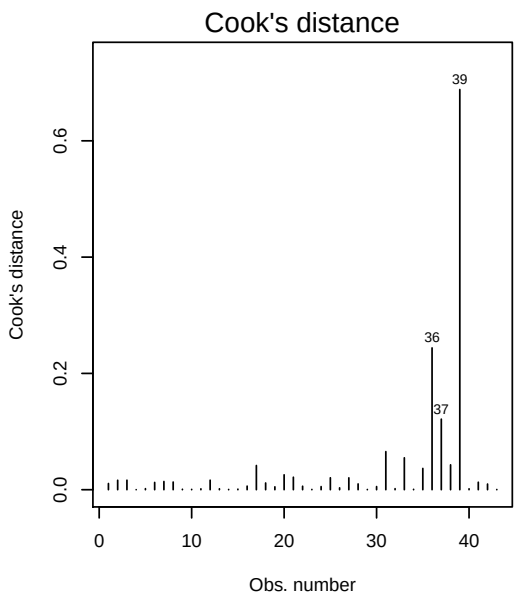
```
> Fit <- with(Africa, lm(adrate~gdppppd+muslperc+subsaharan+healthexp+
                literacy+internalwar))

> library(gvlma)
> Nope <- gvlma(Fit)
> display.gvlmatests(Nope)

ASSESSMENT OF THE LINEAR MODEL ASSUMPTIONS
USING THE GLOBAL TEST ON 4 DEGREES-OF-FREEDOM:
Level of Significance =  0.05

Call:
 gvlma(x = Fit)

                     Value   p-value                   Decision
Global Stat         21.442 0.0002587 Assumptions NOT satisfied!
Skewness             5.720 0.0167698 Assumptions NOT satisfied!
Kurtosis             2.345 0.1256876    Assumptions acceptable.
Link Function        5.892 0.0152059 Assumptions NOT satisfied!
Heteroscedasticity   7.485 0.0062227 Assumptions NOT satisfied!
```

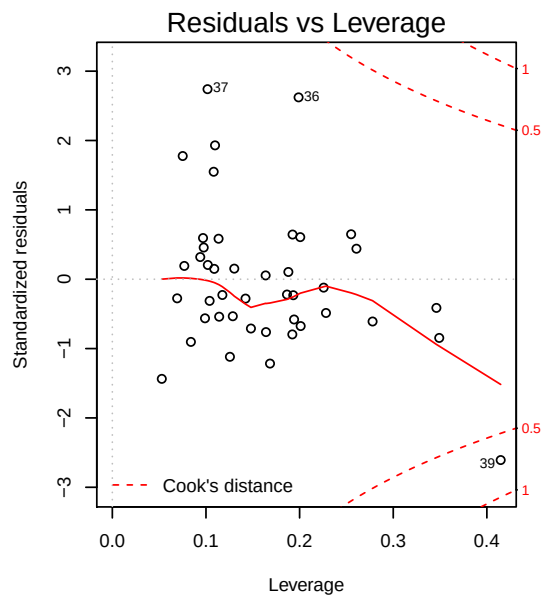
Another Approach: `plot(fit)`

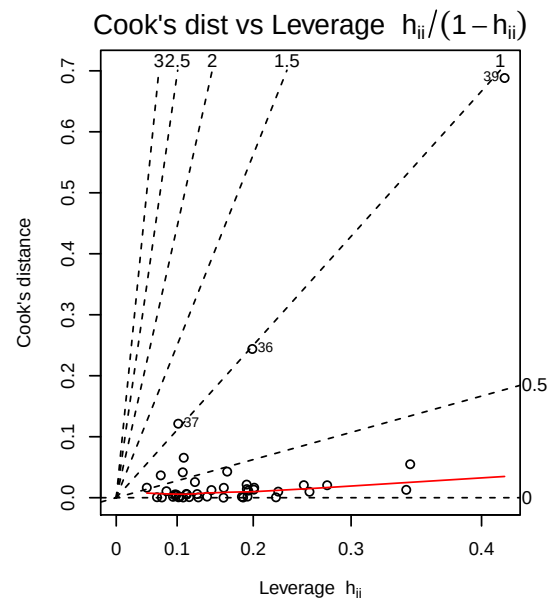




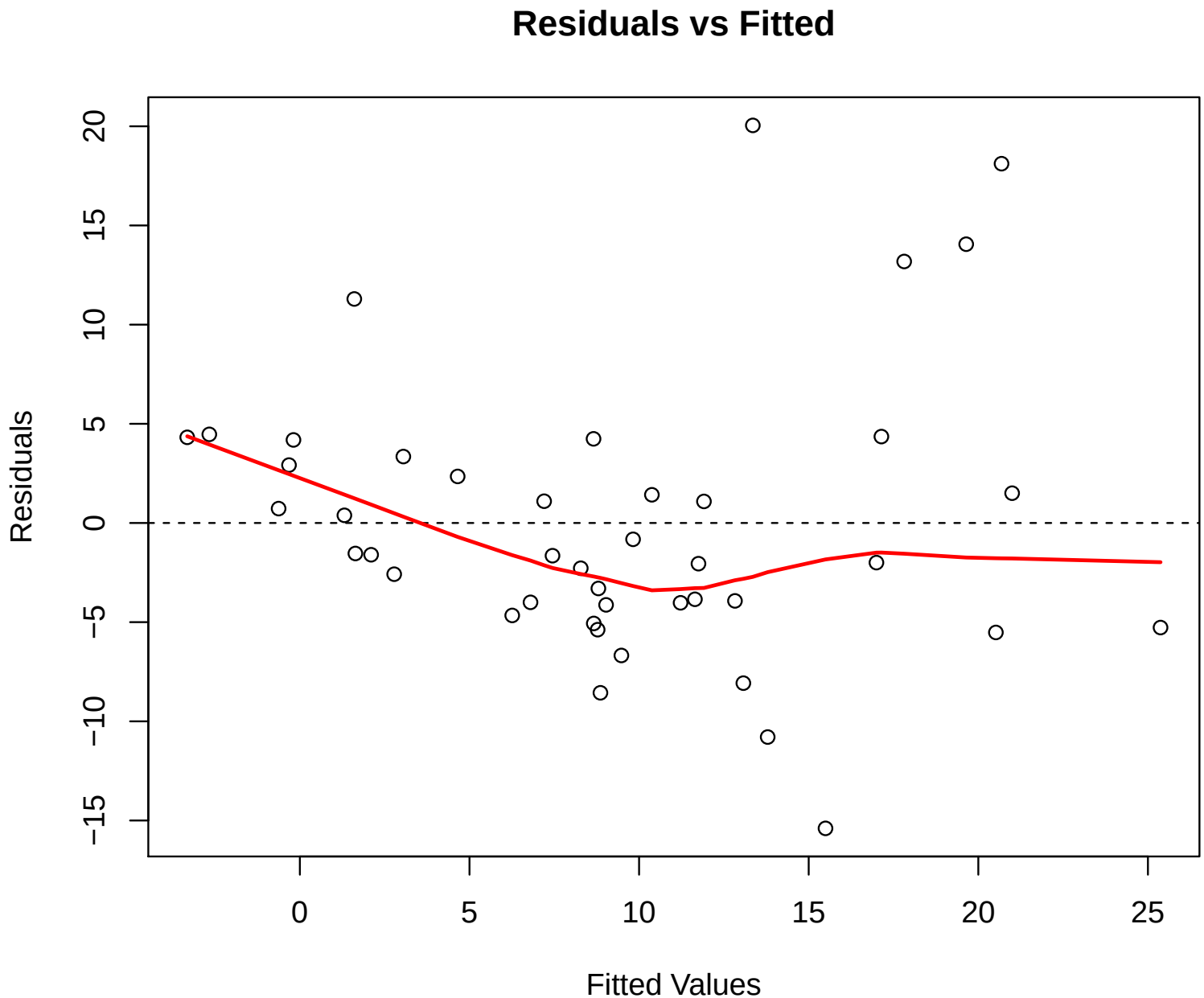




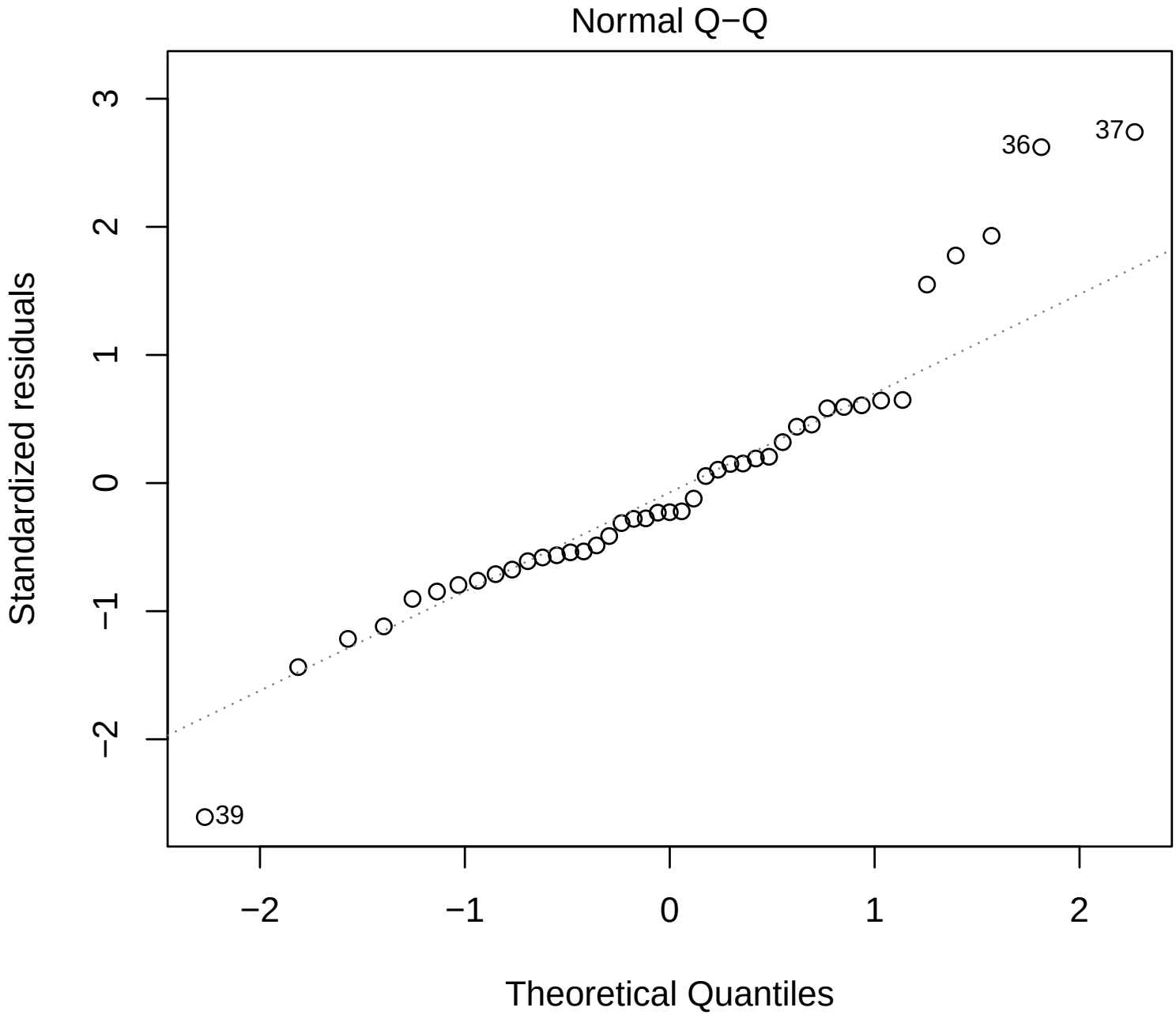




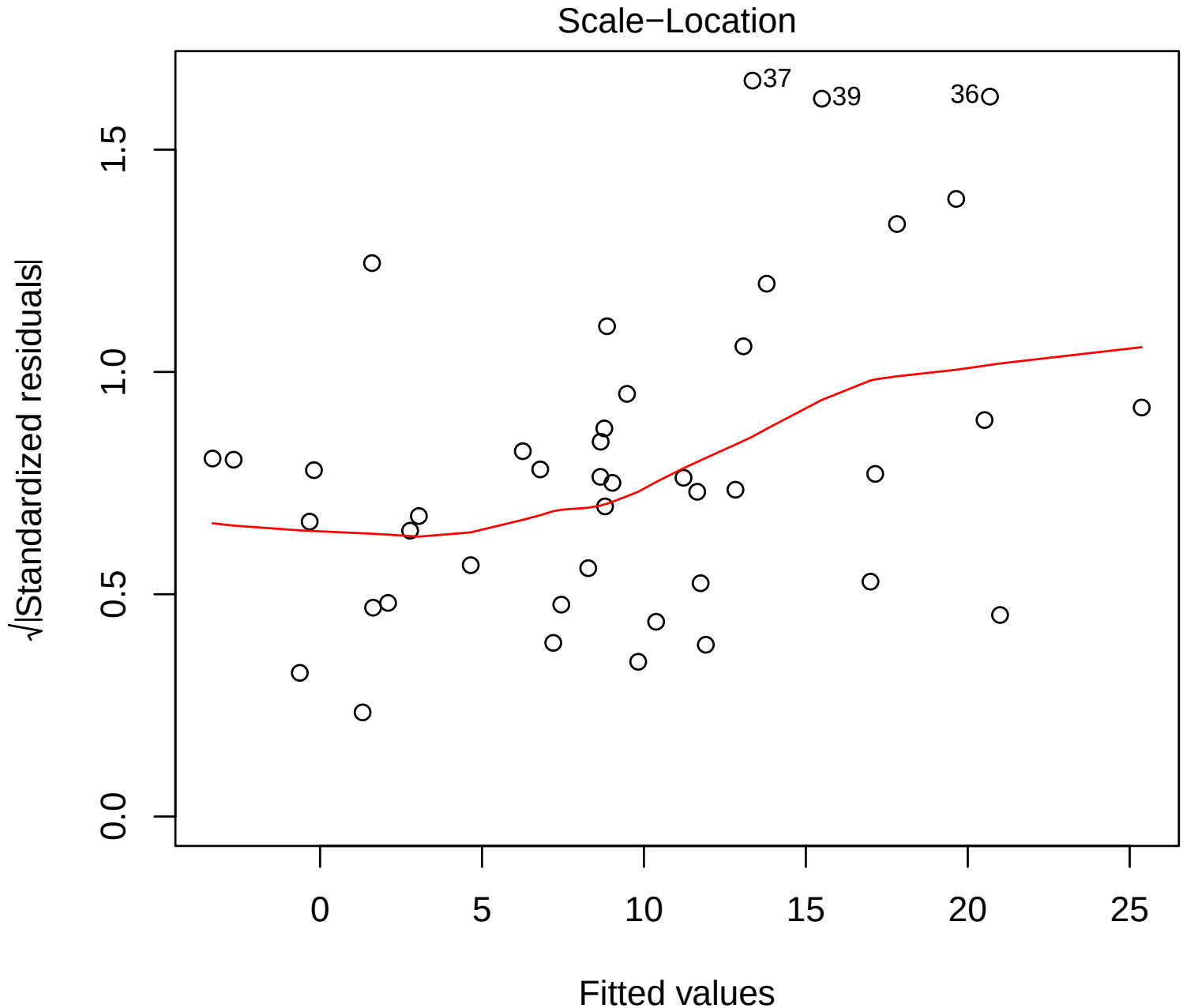
#1: Residuals vs. Fitted Values



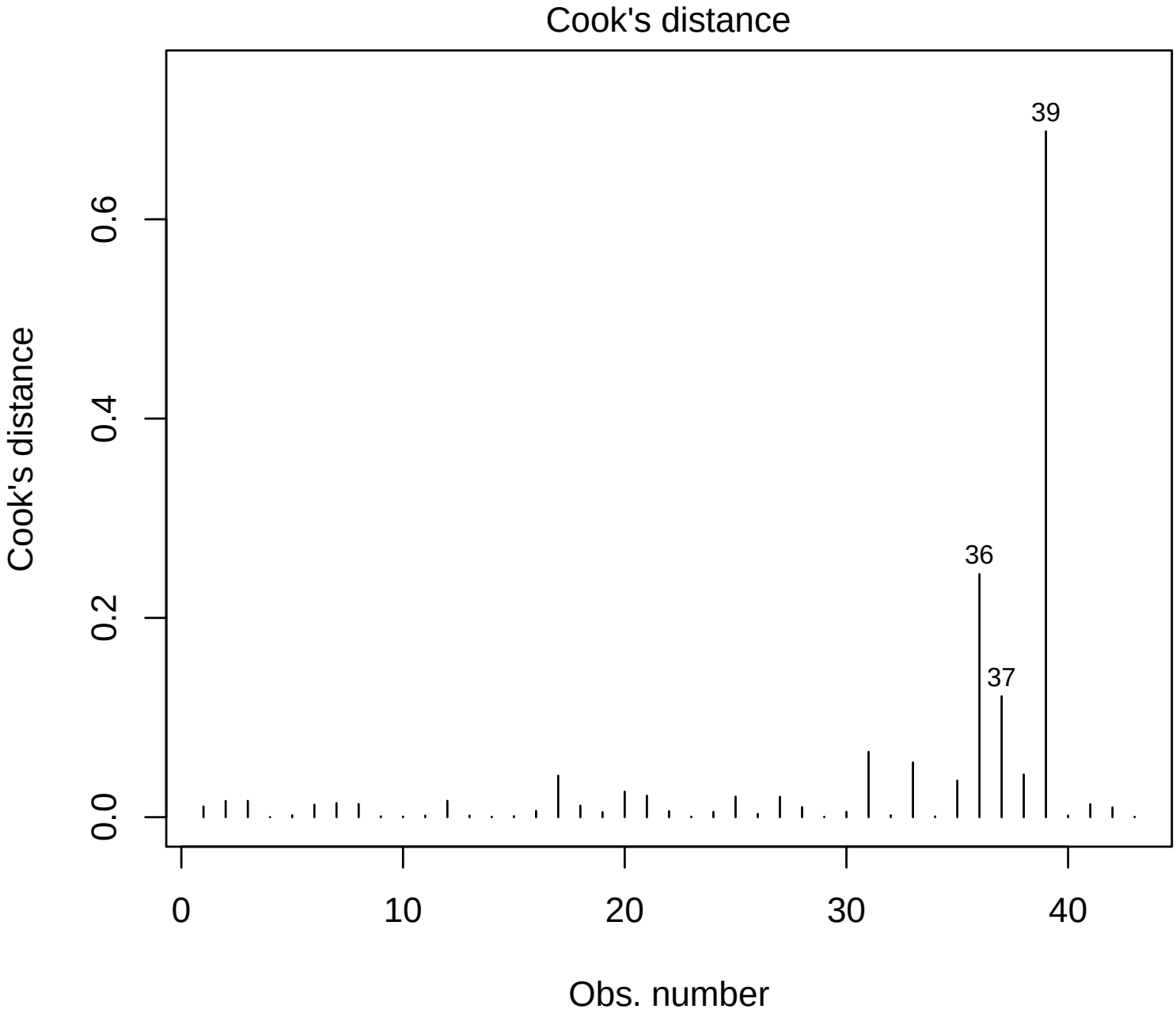
#2: Q-Q Plot of $\hat{u}$s



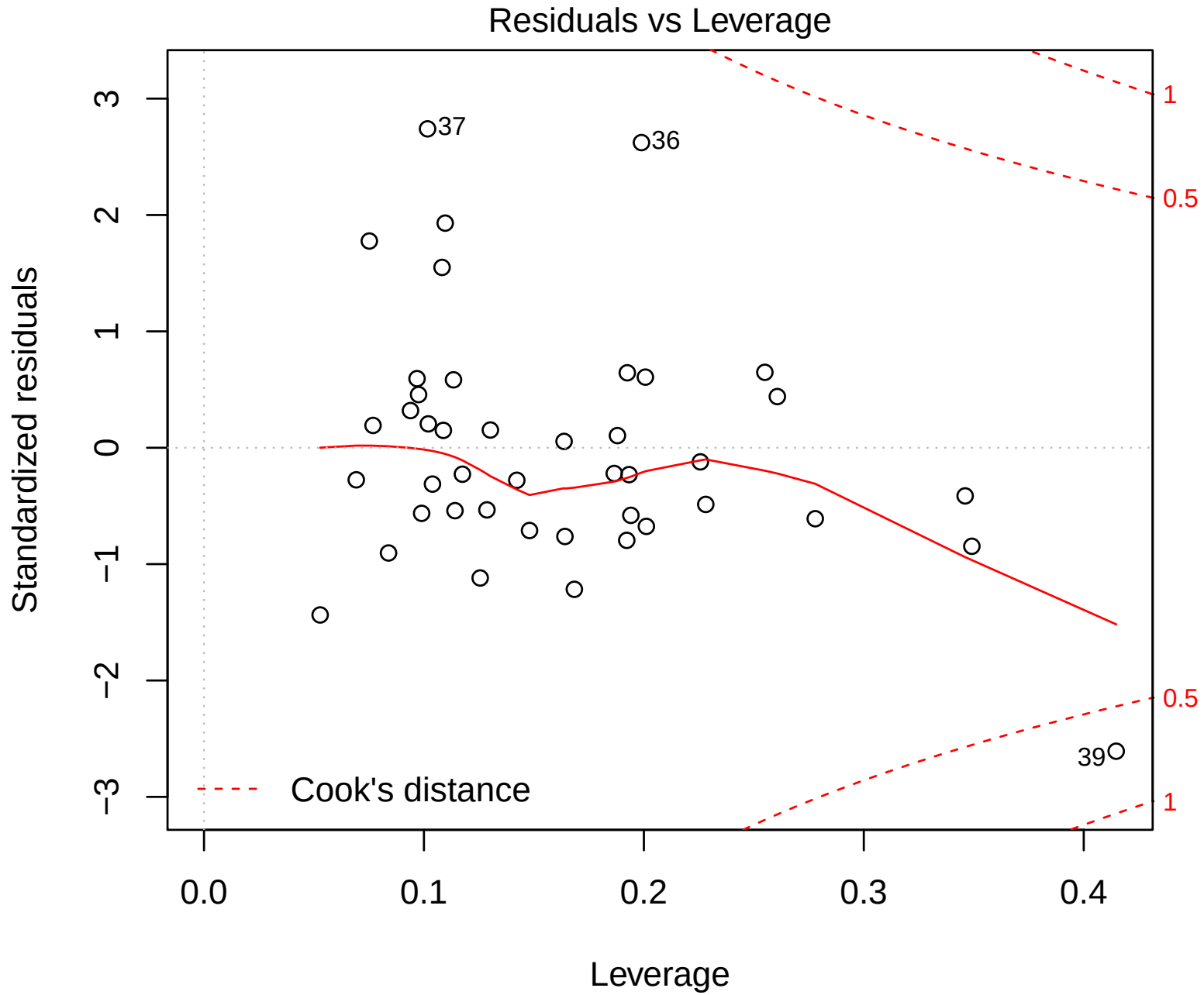
"Scale-Location" Plot



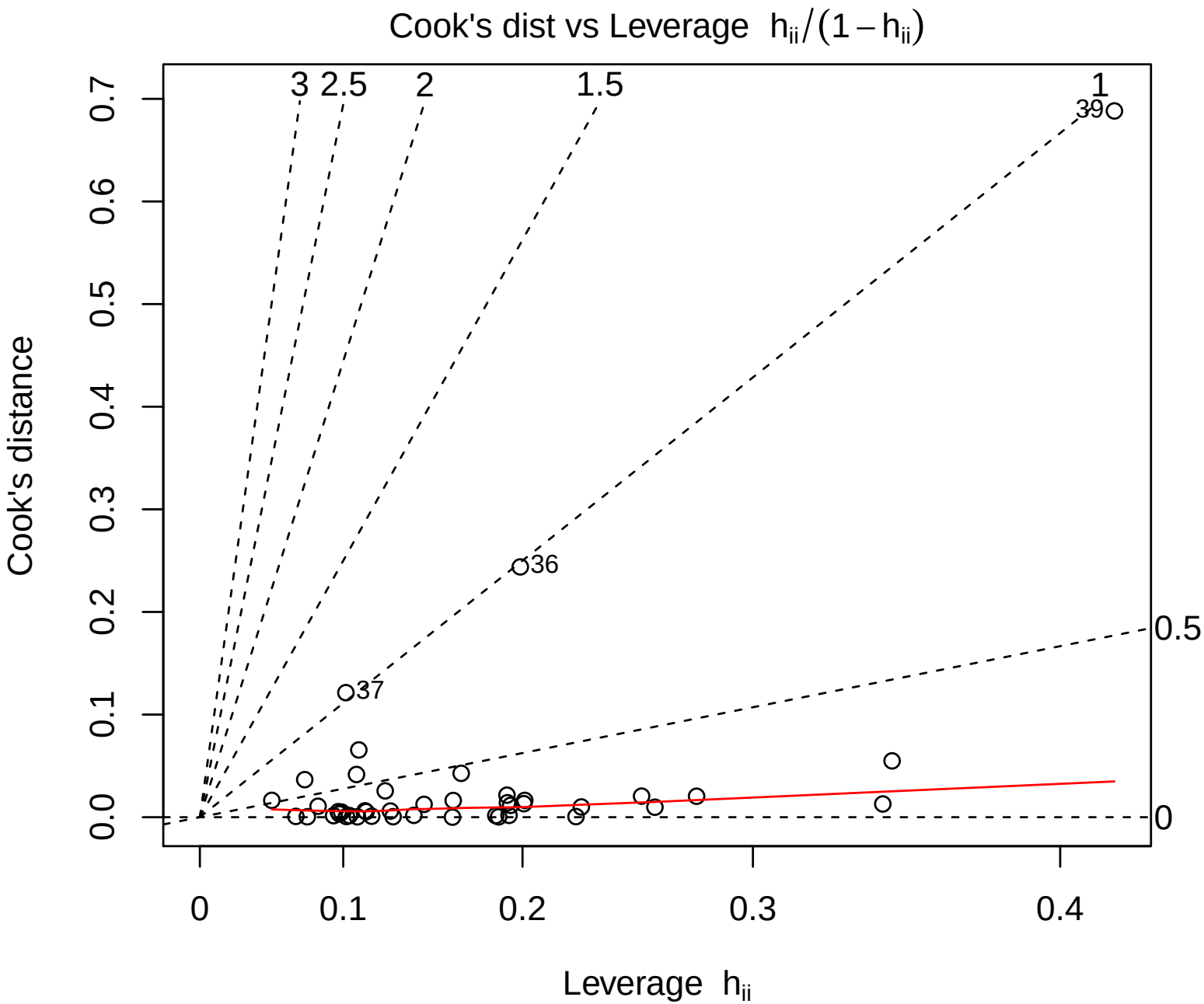
Cook's D


Cook's distance
Cook's distance
0.0
0.2
0.4
0.6
39
36
37
0
10
20
30
40
Obs. number

Residuals vs. Leverage



Cook's D vs. Leverage



Stock-taking...