

PLSC 503 – Spring 2020

Event Counts

April 21, 2020



Things That Are Not Counts

- Ordinal scales/variables
- Grouped Binary Data
 - $\frac{N \text{ of "successes"}}{N \text{ of "trials"}}$
 - Binomial data
 - = counts only if $\Pr(\text{"success"})$ is small

Count Properties

- Discrete / integer-values
- Non-negative
- “Cumulative”

Count Data: Motivation

$$\text{Arrival Rate} = \lambda$$

$$\Pr(\text{Event})_{t,t+h} = \lambda h$$

$$\Pr(\text{No Event})_{t,t+h} = 1 - \lambda h$$

$$\begin{aligned}\Pr(Y_t = y) &= \frac{\exp(-\lambda h) \lambda h^y}{y!} \\ &= \frac{\exp(-\lambda) \lambda^y}{y!}\end{aligned}$$

Poisson Assumptions

- No Simultaneous Events
- Constant Arrival Rate
- Independent Event Arrivals

Poisson: Other Motivations

For M independent Bernoulli trials with (sufficiently small) probability of success π and where

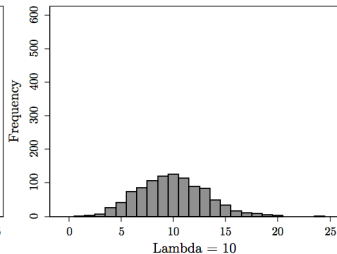
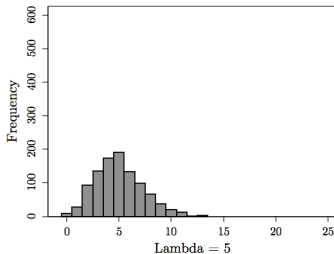
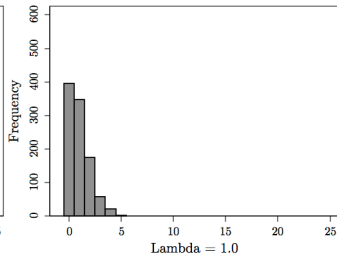
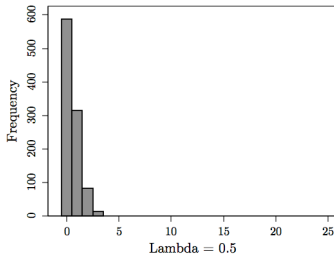
$$M\pi \equiv \lambda > 0,$$

$$\begin{aligned}\Pr(Y_i = y) &= \lim_{M \rightarrow \infty} \left[\binom{M}{y} \left(\frac{\lambda}{M}\right)^y \left(1 - \frac{\lambda}{M}\right)^{M-y} \right] \\ &= \frac{\lambda^y \exp(-\lambda)}{y!}\end{aligned}$$

Poisson: Characteristics

- Discrete
- $E(Y) = \text{Var}(Y) = \lambda$
- Is not preserved under affine transformations...
- For $X \sim \text{Poisson}(\lambda_X)$ and $Y \sim \text{Poisson}(\lambda_Y)$,
 $Z = X + Y \sim \text{Poisson}(\lambda_{X+Y})$ iff X and Y are
independent but
- ...same is not true for differences.
- $\lambda \rightarrow \infty \iff Y \sim N$

Poissons: Examples



Poisson Regression

Suppose

$$E(Y_i) \equiv \lambda_i = \exp(\mathbf{X}_i\boldsymbol{\beta})$$

then

$$\Pr(Y_i = y | \mathbf{X}_i, \boldsymbol{\beta}) = \frac{\exp[-\exp(\mathbf{X}_i\boldsymbol{\beta})][\exp(\mathbf{X}_i\boldsymbol{\beta})]^y}{y!}$$

Poisson Likelihood

$$L = \prod_{i=1}^N \frac{\exp[-\exp(\mathbf{X}_i\boldsymbol{\beta})][\exp(\mathbf{X}_i\boldsymbol{\beta})]^{Y_i}}{Y_i!}$$

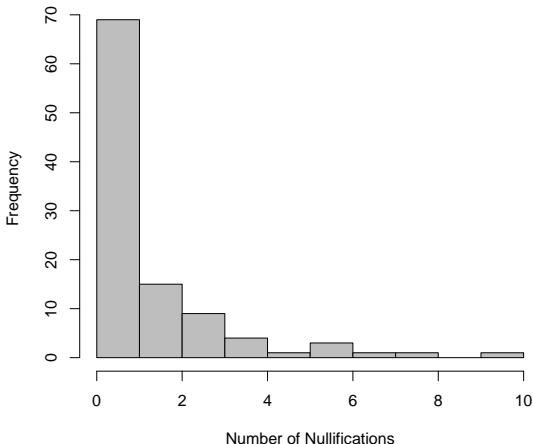
$$\ln L = \sum_{i=1}^N [-\exp(\mathbf{X}_i\boldsymbol{\beta}) + Y_i\mathbf{X}_i\boldsymbol{\beta} - \ln(Y_i!)]$$

Example: Judicial Review

- Y_i = number of Acts of Congress overturned by the Supreme Court in each Congress,
- The *mean tenure* (tenure) of the Supreme Court's justices ($\bar{X} = 10.4, \sigma = 3.4, E(\hat{\beta}) > 0$).
- Whether (1) or not (0) there was *unified government* (unified) ($\bar{X} = 0.83, E(\hat{\beta}) < 0$).

Supreme Court Nullifications, 1789-1996

```
> Nulls<-read.dta("nulls.dta")  
> hist(Nulls$nulls,main="",xlab="Number of Nullifications",  
      col="grey")
```



Estimation

```
> nulls.poisson<-glm(nulls~tenure+unified,family="poisson",data=Nulls)
> summary(nulls.poisson)

Call:
glm(formula = nulls ~ tenure + unified, family = "poisson", data = Nulls)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-2.367  -1.503  -0.623   0.561   4.153

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept)  -0.8776     0.3713   -2.36  0.01809 *
tenure         0.0959     0.0256    3.74  0.00018 ***
unified       0.1435     0.2327    0.62  0.53747
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 251.80  on 103  degrees of freedom
Residual deviance: 237.52  on 101  degrees of freedom
AIC: 385.1

Number of Fisher Scoring iterations: 6
```

Interpretation: Incidence Rate Ratios

$$\begin{aligned}\frac{\hat{\lambda}|X_D = 1}{\hat{\lambda}|X_D = 0} &= \frac{\exp(\hat{\beta}_0 + \bar{\mathbf{X}}\hat{\beta} + (X_D = 1)\hat{\beta}_{X_D})}{\exp(\hat{\beta}_0 + \bar{\mathbf{X}}\hat{\beta} + (X_D = 0)\hat{\beta}_{X_D})} \\ &= \exp(\hat{\beta}_{X_D})\end{aligned}$$

- Like ORs
- unified: $IRR = \exp(0.143) = 1.15$

Incidence Rate Ratios, continued

$$\text{IRR}_{x_k, x_k + \delta} = \exp(\delta \hat{\beta}_k)$$

So, a ten-year difference in tenure:

$$\begin{aligned} \text{IRR} &= \exp(10 \times 0.096) \\ &= \exp(0.96) \\ &= 2.61 \end{aligned}$$

Incidence Rate Ratios

```
> library(mfx)
> nulls.poisson.IRR<-poissonirr(nulls~tenure+unified,
                                data=NULLs)
> nulls.poisson.IRR
```

Call:

```
poissonirr(formula = nulls ~ tenure + unified, data = NULLs)
```

Incidence-Rate Ratio:

	IRR	Std. Err.	z	P> z	
tenure	1.1006	0.0282	3.74	0.00018	***
unified	1.1543	0.2686	0.62	0.53747	

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Predicted Values (\hat{Y} s)

Mean predicted Y :

$$\begin{aligned} E(Y|\bar{\mathbf{X}}_i) &= \exp[-0.878 + (0.096 \times 10) + (0.143 \times 1)] \\ &= \exp(0.225) \\ &= 1.25 \end{aligned}$$

In-Sample

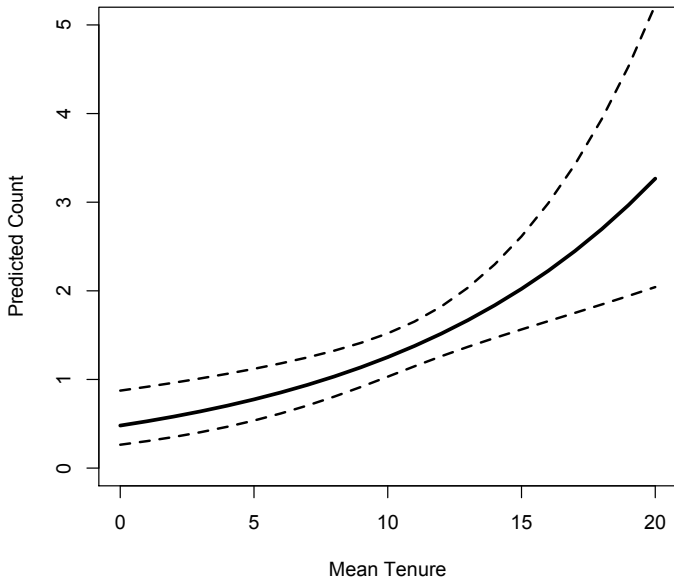
- R : `in $fitted.values`
- Stata : use `predict`

Out-of-Sample: use `predict`

Example: Out-Of-Sample Predicted Values

```
> tenure<-seq(0,20,1)
> unified<-1
> simdata<-as.data.frame(cbind(tenure,unified))
> nullhats<-predict(nulls.poisson,newdata=simdata,se.fit=TRUE)
>
> # NOTE: These are XBs, not predicted counts.
> # Transforming:
> nullhats$Yhat<-exp(nullhats$fit)
> nullhats$UB<-exp(nullhats$fit + 1.96*(nullhats$se.fit))
> nullhats$LB<-exp(nullhats$fit - 1.96*(nullhats$se.fit))
> plot(simdata$tenure,nullhats$Yhat,t="l",lwd=3,ylim=c(0,5),ylab=
+       "Predicted Count", xlab="Mean Tenure")
> lines(simdata$tenure,nullhats$UB,lwd=2,lty=2)
> lines(simdata$tenure,nullhats$LB,lwd=2,lty=2)
>
> plot(simdata$tenure,nullhats$Yhat,t="l",lwd=3,ylim=c(0,5),ylab=
+       "Predicted Count", xlab="Mean Tenure")
> lines(simdata$tenure,nullhats$UB,lwd=2,lty=2)
> lines(simdata$tenure,nullhats$LB,lwd=2,lty=2)
```

Plotting Out-Of-Sample Predicted Values



Predicted Probabilities

$$\Pr(\widehat{Y_i = y} | \mathbf{X}_i, \hat{\beta}) = \frac{\exp[-\exp(\mathbf{X}_i \hat{\beta})][\exp(\mathbf{X}_i \hat{\beta})]^y}{y!}$$

$$\begin{aligned} \rightarrow \Pr(\widehat{Y_i = 0} | \bar{\mathbf{X}}_i, \hat{\beta}) &= \frac{[\exp(-1.25)](1.25)^0}{0!} \\ &= \frac{(0.287)(1)}{1} \\ &= 0.287 \end{aligned}$$

$$\begin{aligned} \Pr(\widehat{Y_i = 1} | \bar{\mathbf{X}}_i, \hat{\beta}) &= \frac{[\exp(-1.25)](1.25)^1}{1!} \\ &= \frac{(0.287)(1.25)}{1} \\ &= 0.359 \end{aligned}$$

Predicted Probabilities

$$\begin{aligned}\Pr(\widehat{Y_i = 2} | \bar{\mathbf{X}}_i, \hat{\beta}) &= \frac{[\exp(-1.25)](1.25)^2}{2!} \\ &= \frac{(0.287)(1.563)}{2} \\ &= 0.224\end{aligned}$$

$$\begin{aligned}\Pr(\widehat{Y_i = 3} | \bar{\mathbf{X}}_i, \hat{\beta}) &= \frac{[\exp(-1.25)](1.25)^3}{3!} \\ &= \frac{(0.287)(1.953)}{6} \\ &= 0.093\end{aligned}$$

“Exposure” and “Offsets”

$$E(Y_i | \mathbf{X}_i, M_i) = \lambda_i M_i$$

Same as including $\ln(M_i)$ in \mathbf{X} with $\beta_{\ln M} = 1$.

Example: Data on numbers of interstate disputes by country, 1950-1985...

- $N = 102$, but
- N_{dyads} = number of dyad-years which were aggregated to create each observation, ranging from five to 3249
- disputes = number of (interstate) dispute-years that country experienced during 1950-1985
- allies = number of (dyadic) ally-years each country had during 1950-1985
- $\text{openness} = \frac{1}{36} \left(\frac{\text{Imports}_t + \text{Exports}_t}{\text{GDP}_t} \right)$ across all 36 years in the data.

"Exposure" and "Offsets": Data

```
# Data are aggregated dyadic data, 1950-1985...
```

```
> IR<-read.csv("Data/offsetIR.csv")
```

```
> summary(IR)
```

ccode		Ndyads		disputes		allies		openness		exposure	
Min.	: 2	Min.	: 5	Min.	: 0.00	Min.	: 0.0	Min.	:0.032	Min.	:1.61
1st Qu.:	214	1st Qu.:	44	1st Qu.:	0.00	1st Qu.:	0.0	1st Qu.:	0.185	1st Qu.:	3.79
Median	:436	Median	: 92	Median	: 1.00	Median	: 26.0	Median	:0.296	Median	:4.52
Mean	:418	Mean	: 179	Mean	: 3.55	Mean	: 63.9	Mean	:0.392	Mean	:4.42
3rd Qu.:	598	3rd Qu.:	146	3rd Qu.:	4.00	3rd Qu.:	81.0	3rd Qu.:	0.535	3rd Qu.:	4.98
Max.	:900	Max.	:3249	Max.	:52.00	Max.	:1283.0	Max.	:1.659	Max.	:8.09
								NA's	:12		

```
> cor(IR,use="complete.obs")
```

	ccode	Ndyads	disputes	allies	openness	exposure
ccode	1.00000	-0.29623	-0.1399	-0.3983	0.02744	-0.6544
Ndyads	-0.29623	1.00000	0.8626	0.9200	-0.07511	0.6988
disputes	-0.13989	0.86257	1.0000	0.8255	-0.16819	0.6335
allies	-0.39826	0.92004	0.8255	1.0000	-0.12548	0.7003
openness	0.02744	-0.07511	-0.1682	-0.1255	1.00000	-0.1433
exposure	-0.65442	0.69878	0.6335	0.7003	-0.14325	1.0000

Ignoring Exposure

```
> IR.fit1<-glm(disputes~allies+openness,data=IR,family="poisson")  
> summary(IR.fit1)
```

Coefficients:

	Estimate	Std. Error	z value	Pr(> z)
(Intercept)	1.1559498	0.1117581	10.343	< 2e-16 ***
allies	0.0025184	0.0001159	21.734	< 2e-16 ***
openness	-1.1144132	0.2773631	-4.018	5.87e-05 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for poisson family taken to be 1)

Null deviance: 732.68 on 101 degrees of freedom
Residual deviance: 392.97 on 99 degrees of freedom
(12 observations deleted due to missingness)
AIC: 588.29

Number of Fisher Scoring iterations: 6

Correcting for Exposure

```
> IR.fit2<-glm(disputes~allies+openness,data=IR,family="poisson",  
  offset=log(Ndyads))  
> summary(IR.fit2)
```

Coefficients:

	Estimate	Std. Error	z value	Pr(> z)
(Intercept)	-3.2906055	0.1194616	-27.545	< 2e-16 ***
allies	-0.0006058	0.0001333	-4.544	5.52e-06 ***
openness	-1.6040587	0.3167415	-5.064	4.10e-07 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Null deviance: 320.19 on 101 degrees of freedom
Residual deviance: 277.79 on 99 degrees of freedom
(12 observations deleted due to missingness)
AIC: 473.11

Number of Fisher Scoring iterations: 5

Correcting for Exposure (continued)

```
> IR.fit3<-glm(disputes~allies+openness+log(Ndyads),data=IR,  
+             family="poisson")  
> summary(IR.fit3)
```

Call:

```
glm(formula = disputes ~ allies + openness + log(Ndyads), family = "poisson",  
    data = IR)
```

Deviance Residuals:

Min	1Q	Median	3Q	Max
-2.838	-1.390	-0.758	0.605	4.731

Coefficients:

	Estimate	Std. Error	z value	Pr(> z)
(Intercept)	-2.42656676	0.34345252	-7.07	0.00000000000016 ***
allies	-0.00000948	0.00025687	-0.04	0.97
openness	-1.44462460	0.31193821	-4.63	0.0000036368547 ***
log(Ndyads)	0.81097748	0.07095243	11.43	< 0.0000000000000002 ***

Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

(Dispersion parameter for poisson family taken to be 1)

Null deviance: 732.68 on 101 degrees of freedom
Residual deviance: 270.59 on 98 degrees of freedom
(12 observations deleted due to missingness)
AIC: 467.9

Number of Fisher Scoring iterations: 5

Test $\beta_{\text{exposure}} = 1.0$

```
> # z-test:
```

```
> 2*pnorm((0.811-1)/.071)
[1] 0.007768438
```

```
> # Wald test:
```

```
> wald.test(b=coef(IR.fit3),Sigma=vcov(IR.fit3),Terms=4,H0=1)
```

```
Wald test:
```

```
-----
```

```
Chi-squared test:
```

```
X2 = 7.1, df = 1, P(> X2) = 0.0077
```

- Overdispersion \rightarrow Negative Binomial
- Underdispersion \rightarrow CPB
- “Zero-Inflated” and “Hurdle” Count Models
- Time Series of Counts
- Spatial Count Models
- etc...