PLSC 503 - Fall 2020
Event Counts, II

April 23, 2020


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26
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27
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26

Heterogeneity, Contagion, and Dispersion

Cats:

$$
\begin{aligned}
Y_{cats} &= \{0,1,1,0,2,0,1,0,3,1,2,1,0,2\} \\
\bar{Y}_{cats} &= 1.0, \\
\sigma_{cats} &= 0.92.
\end{aligned}
$$

Heterogeneity, Contagion, and Dispersion

$$\mathrm{E}(Y_{cats}) = \lambda_{cats}$$

Assumes:

- $Y = 0$ at $t = 0$,
- Exclusive events
- $t_j = t_k \, \forall \, j \neq k$
- Constant, independent Pr(Event) over t

Antelope

$$\begin{aligned} Y_{antelope} &= \{0,0,0,0,0,0,0,0,0,0,0,0,7,7\} \\ \bar{Y}_{antelope} &= 1.0, \\ \sigma_{antelope} &= 6.46. \end{aligned}$$

Positive contagion $\rightarrow$ *overdispersion*.

$$
\begin{aligned}
Y_{foxes} &= \{1,0,1,1,1,1,1,2,1,1,1,1,1,1\} \\
\bar{Y}_{foxes} &= 1.0, \\
\sigma_{foxes} &= 0.15.
\end{aligned}
$$

Negative contagion $\rightarrow$ *underdispersion*.

Aggregation & Cross-Period Effects

$$
\begin{aligned}
Y_{cats} &= \{1, 1, 2, 1, 4, 3, 2\} \\
Y_{antelope} &= \{0, 0, 0, 0, 0, 0, 14\} \\
Y_{foxes} &= \{1, 2, 2, 3, 2, 2, 2\}
\end{aligned}
$$

- Correct specification
- Correct distribution for ϵ
- Constant $\mathrm{E}(Y|\mathbf{X}, \boldsymbol{\beta})$

$$\lambda_i \equiv \mathrm{E}(Y_i) = f[\mathbf{X}_i\boldsymbol{\beta} + Z_i\theta]$$

Overdispersion: A Test

Examine:

$$\hat{u}_i = \delta\hat{\lambda}_i + \epsilon_i$$

where

$$\hat{u}_i = \frac{(Y_i - \hat{\lambda}_i)^2 - Y_i}{\hat{\lambda}_i\sqrt{2}}$$

- Estimate a Poisson regression of Y_i on $\mathbf{X}_i$, and generate predicted counts $\hat{\lambda}_i$.
- Calculate $\hat{u}_i$ according to the equation above.
- Estimate δ using OLS, and test $H_0 : \hat{\delta} = 0$.

$$\begin{aligned} \mathrm{E}(Y_i) \equiv \lambda_i &= \exp(\mathbf{X}_i\boldsymbol{\beta} + u_i) \\ &= \exp(\mathbf{X}_i\boldsymbol{\beta})\exp(u_i) \\ &= \lambda_i\nu_i \end{aligned}$$

$$\nu_i \sim \text{gamma}\left(1, \frac{1}{\alpha}\right)$$

$$\Pr(Y_i = y|\lambda_i, \alpha) = \left(\frac{\Gamma(\alpha^{-1} + Y_i)}{\Gamma(\alpha^{-1})\Gamma(Y_i + 1)}\right)\left(\frac{\alpha^{-1}}{\alpha^{-1} + \lambda_i}\right)^{\alpha^{-1}}\left(\frac{\lambda_i}{\lambda_i + \alpha^{-1}}\right)^{Y_i}$$

where

$$\Gamma(a) = \int_0^\infty \exp(-t)t^{a-1}dt$$

Negative Binomial

Basis:

$$\lambda_i = \exp(\mathbf{X}_i \boldsymbol{\beta})$$

Model has

$$\mathrm{E}(Y) = \lambda$$

$$\mathrm{Var}(Y) = \lambda(1 + \alpha\lambda), \quad \alpha > 0$$

or (equivalently):

$$\mathrm{Var}(Y) = \lambda + \frac{\lambda^2}{\theta}, \text{ where } \theta = \frac{1}{\alpha}.$$

Negative Binomial (log-)Likelihood

$$\ln L_{NB} = \sum_{i=1}^{N} \left\{ \left(\sum_{j=0}^{Y_i - 1} \ln(j + \alpha^{-1}) \right) - \ln Y_i! \right.$$
$$\left. - (Y_i - \alpha^{-1}) \ln[1 + \alpha \exp(\mathbf{X}_i \boldsymbol{\beta})] + Y_i \ln \alpha + Y_i \mathbf{X}_i \boldsymbol{\beta} \right\}$$

Note that:

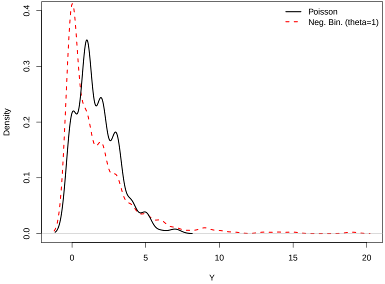
- $\alpha = 0 \iff \mathrm{E}(Y) = \mathrm{Var}(Y)$
- LR test for overdispersion:

$$-2 \times (\widehat{\ln L_{Poisson}} - \widehat{\ln L_{NB}}) \sim \chi^2_1$$

- $\widehat{E(Y_i)} \equiv \hat{\lambda}_i = \exp(\mathbf{X}_i \hat{\boldsymbol{\beta}})$

What Difference Does It Make?

```
> N<-400
> set.seed(7222009)
> X <- runif(N,min=0,max=1)
> YPois <- rpois(N,exp(0+1*X))            # Poisson
> YNB <- rnbinom(N,size=1,mu=exp(0+1*X)) # NB with theta=1.0
>
> describe(cbind(YPois,YNB))
      vars   n mean   sd median trimmed  mad min max range skew kurtosis   se
YPois    1 400 1.72 1.41      1    1.56 1.48   0   7     7 0.92     0.84 0.07
YNB      2 400 1.71 2.44      1    1.22 1.48   0  19    19 2.76    11.15 0.12
```



What Difference Does It Make (cont'd)?

```
> # Regressions:
>
> summary(glm(YPois~X,family="poisson")) # Poisson

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept) -0.009637   0.085337  -0.113     0.91
X            1.030573   0.131992   7.808 5.82e-15 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 516.06  on 399  degrees of freedom
Residual deviance: 453.55  on 398  degrees of freedom
AIC: 1274.4



> summary(glm.nb(YPois~X))               # NB

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept) -0.009629   0.085345  -0.113     0.91
X            1.030557   0.132007   7.807 5.86e-15 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for Negative Binomial(7837.699) family taken to be 1)

    Null deviance: 515.96  on 399  degrees of freedom
Residual deviance: 453.46  on 398  degrees of freedom
AIC: 1276.5

              Theta:  7838
          Std. Err.:  135342
Warning while fitting theta: iteration limit reached

 2 x log-likelihood:  -1270.451
```

What Difference Does It Make (cont'd)?

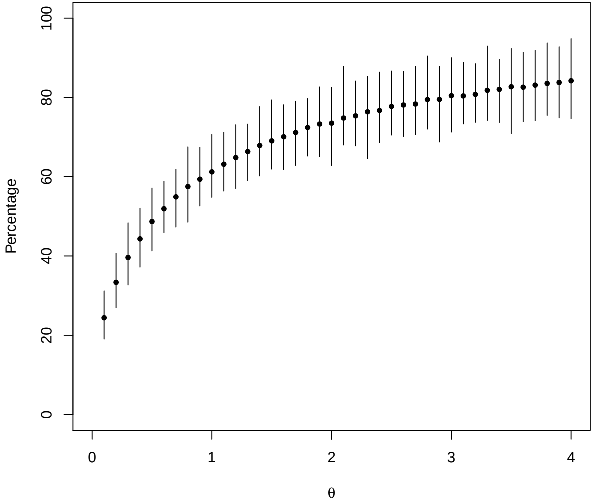
```
> # More regressions:
>
> summary(glm(YNB~X,family="poisson"))  # Poisson

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -0.03170    0.08593  -0.369    0.712
X            1.06109    0.13248   8.009 1.15e-15 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 1118.0  on 399  degrees of freedom
Residual deviance: 1052.1  on 398  degrees of freedom
AIC: 1698.6
```

```
> summary(glm.nb(YNB~X))               # NB

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -0.03525    0.13650  -0.258    0.796
X            1.06773    0.22809   4.681 2.85e-06 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for Negative Binomial(0.8499) family taken to be 1)

    Null deviance: 436.92  on 399  degrees of freedom
Residual deviance: 414.81  on 398  degrees of freedom
AIC: 1407.4

              Theta:  0.850
          Std. Err.:  0.109

 2 x log-likelihood:  -1401.354
```

Poisson Regression Underestimates N.B. Variances

```
Sims <- 250 # (250 sims each)
theta <- seq(0.1,4,by=0.1) # values of theta
diffs<-matrix(nrow=Sims,ncol=length(theta))

set.seed(7222009)
for(j in 1:length(theta)) {
  for(i in 1:Sims) {
    X<-runif(N,min=0,max=1)
    Y<-rnbinom(N,size=theta[j],mu=exp(0+1*X))
    p<-glm(Y~X,family="poisson")
    nb<-glm.nb(Y~X)
    diffs[i,j]<- ((sqrt(vcov(p))[2,2]) / sqrt(vcov(nb))[2,2])*100
  }
}
```

Percentage of True (Negative Binomial) S.E. From Fitted Poisson Model, by $\theta = \frac{1}{\alpha}$



Negative Binomial In Practice

Model fitting (in R):

- `glm.nb` (in `MASS`)
- `negbinomial` (in `VGAM`)
- `negbin` (in `aod`)
- `glmnb.fit` (in `statmod`)
- probably others...

Model interpretation + diagnostics:

- `fitNBP` (in `statmod`) (dispersion parameter estimation)
- `negbinirr` (in `mfx`) (IRRs)
- `negbinmfx` (in `mfx`) (marginal effects)
- Predicted values / probabilities via `predict`

Negative Binomial In Practice

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- Probably others...

Model interpretation + diagnostics:

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Underdispersion / CPB

"Continuous parameter binomial":

$$\Pr(Y_i = y|\lambda_i, \alpha) = \frac{\frac{\Gamma\left(\frac{-\lambda_i}{\alpha-1}+1\right)}{Y_i!\Gamma\left(\frac{-\lambda_i}{\alpha-1}-Y_i+1\right)}(1-\alpha)^{Y_i}(\alpha)^{\frac{-\lambda_i}{\alpha-1}-Y_i}}{D_i}$$

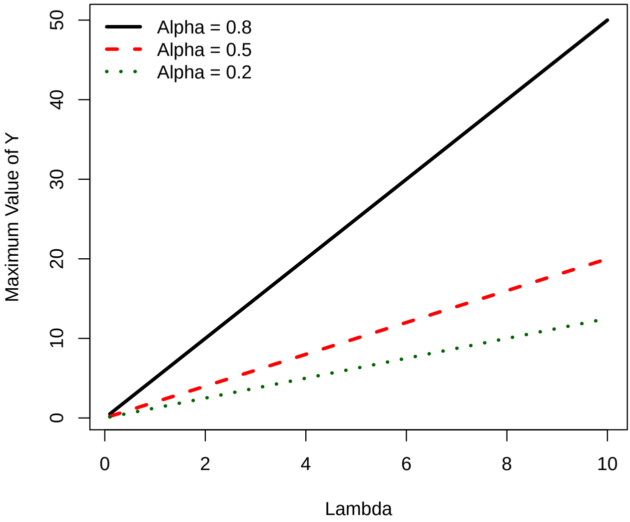
where $D_i = \sum_0^{\frac{-\lambda_i}{\alpha-1}+1}$ of the binomial distribution...

Are You Down With The CPB?

CPB:

- ...also has $\mathrm{E}(Y_i) = \lambda_i$ [with $\mu_i = \exp(\mathbf{X}_i\boldsymbol{\beta})$]
- ...has $\mathrm{Var}(Y) = \lambda_i\alpha$ with $0 < \alpha < 1$
- ... reduces to the standard Poisson when $\alpha = 1$
- ...imposes a theoretical "upper limit" on the count variable. In particular,

$$\max(Y_i) = \frac{-\lambda_i}{\alpha - 1}.$$



Alpha = 0.8
Alpha = 0.5
Alpha = 0.2
Maximum Value of Y
Lambda
0
10
20
30
40
50
0
2
4
6
8
10

CPB (log-)Likelihood

$$\begin{aligned} \ln L_{CPB} &= \sum_{i=1}^{N} \Big\{ \ln\Gamma\left(\frac{-\lambda_i}{\alpha-1}+1\right) - \ln\Gamma\left(\frac{-\lambda_i}{\alpha-1} - Y_i + 1\right) \\ &\quad + Y_i \ln(1-\alpha) + \left(\frac{-\lambda_i}{\alpha-1} - Y_i\right)\ln(\alpha) - \ln(D_i) \Big\} \end{aligned}$$

Example: SCOTUS amicus curiae (1953-85)

"Friend of the Court" briefs...

- $N = 7157$
- `namici` is the number of amicus curiae briefs filed in each case,
- `term` is the term (i.e., year) of the Court,
- `civlibs` is whether (=1) or not (=0) the case involved a civil rights and liberties issue.

```
> describe(amici)
        vars    n  mean   sd median trimmed   mad min max range  skew kurtosis   se
namici     1 7157  1.03 2.54      0    0.42  0.00   0  53    53  5.63    56.36 0.03
term       2 7157 71.12 9.19     72   71.46 11.86  53  85    32 -0.24    -1.07 0.11
civlibs    3 7157  0.50 0.50      1    0.50  0.00   0   1     1  0.00    -2.00 0.01
```

Amicus Example: Poisson

```
> amici.poisson<-glm(namici~term+civlibs,data=amici,family="poisson")
> summary(amici.poisson)

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -4.51196    0.11190  -40.32   <2e-16 ***
term         0.06361    0.00147   43.27   <2e-16 ***
civlibs     -0.29797    0.02350  -12.68   <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 22875  on 7156  degrees of freedom
Residual deviance: 20675  on 7154  degrees of freedom
  (4 observations deleted due to missingness)
AIC: 26862

Number of Fisher Scoring iterations: 6
```

Overdispersion Test: "By Hand"

```
> Phats<-fitted.values(amici.poisson)
> Uhats<-((amici$namici-Phats)^2 - amici$namici) / (Phats * sqrt(2))
> summary(lm(Uhats~Phats))

Call:
lm(formula = Uhats ~ Phats)

Residuals:
   Min     1Q Median     3Q    Max
  -5.9   -3.0   -2.3   -1.9 1707.0

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)    1.579      0.693    2.28    0.023 *
Phats          1.466      0.591    2.48    0.013 *
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 28.4 on 7155 degrees of freedom
Multiple R-squared:  0.000858,Adjusted R-squared:  0.000718
F-statistic: 6.14 on 1 and 7155 DF,  p-value: 0.0132
```

Negative Binomial Regression

```
> library(MASS)
> amici.NB<-glm.nb(namici~term+civlibs,data=amici)
> summary(amici.NB)

Call:
glm.nb(formula = namici ~ term + civlibs, data = amici, init.theta = 0.256657474,
    link = log)

Coefficients:
            Estimate Std. Error z value             Pr(>|z|)
(Intercept) -4.68314    0.22058  -21.23 < 0.0000000000000002 ***
term         0.06573    0.00304   21.60 < 0.0000000000000002 ***
civlibs     -0.26777    0.05403   -4.96           0.00000072 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for Negative Binomial(0.2567) family taken to be 1)

    Null deviance: 5442  on 7156  degrees of freedom
Residual deviance: 4968  on 7154  degrees of freedom
AIC: 17378

Number of Fisher Scoring iterations: 1
              Theta:  0.25666
          Std. Err.:  0.00838
```

Poisson vs. NB

```
> # alpha:
>
> 1 / amici.NB$theta
[1] 3.896


> # Coefficient estimates:
>
> cbind(amici.poisson$coefficients,coef(amici.NB))
                [,1]     [,2]
(Intercept) -4.51196 -4.68314
term         0.06361  0.06573
civlibs     -0.29797 -0.26777


> # Estimated standard errors:
>
> cbind(diag(sqrt(vcov(amici.poisson))),diag(sqrt(vcov(amici.NB))))
               [,1]     [,2]
(Intercept) 0.11190 0.220582
term        0.00147 0.003043
civlibs     0.02350 0.054032
```

Predicted Values: Poisson and NB

```
> plot(amici.poisson$fitted.values,amici.NB$fitted.values,pch=20,
    xlab="Poisson",ylab="Negative Binomial",main="",
    xlim=c(0,3),ylim=c(0,3))
> abline(a=0,b=1,lwd=1,lty=2)
```