PLSC 503 – Spring 2020
Maximum Likelihood, I

March 26, 2020

A Model

$$Y \sim N(\mu, \sigma^2)$$

$$\begin{aligned} \mathrm{E}(Y) &= \mu \\ \mathrm{Var}(Y) &= \sigma^2 \end{aligned}$$

Some Data

$$
\begin{aligned}
Y \quad = \quad & 64 \\
& 63 \\
& 59 \\
& 71 \\
& 68
\end{aligned}
$$

Probabilities, Marginal

$$\Pr(Y_i = y_i) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(Y_i - \mu)^2}{2\sigma^2}\right]$$

So

$$\begin{aligned}
\Pr(Y_1 = 64) &= \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(64 - \mu)^2}{2\sigma^2}\right] \\
\Pr(Y_2 = 63) &= \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(63 - \mu)^2}{2\sigma^2}\right]
\end{aligned}$$

$$\ldots$$

Probabilities, Joint

$$\Pr(A, B \mid A \perp B) = \Pr(A) \times \Pr(B)$$

So:

$$\begin{aligned}\Pr(Y_1 = 64, Y_2 = 63) \quad &= \quad \frac{1}{\sqrt{2\pi\sigma^2}} \text{exp} \left[-\frac{(64 - \mu)^2}{2\sigma^2} \right] \times \\ & \qquad \frac{1}{\sqrt{2\pi\sigma^2}} \text{exp} \left[-\frac{(63 - \mu)^2}{2\sigma^2} \right]\end{aligned}$$

More Generally...

$$
\begin{aligned}
\Pr(Y_i = y_i \; \forall \; i) & \equiv L(Y|\mu, \sigma^2) \\
& = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y_i - \mu)^2}{2\sigma^2}\right]
\end{aligned}
$$

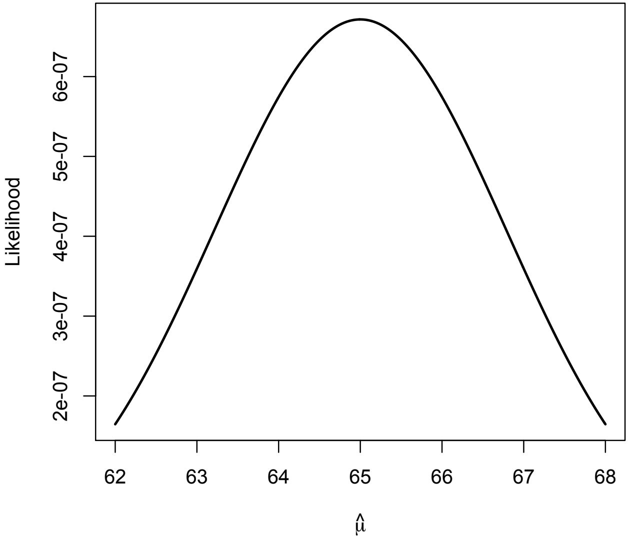
Likelihood

$$L(\hat{\mu}, \hat{\sigma}^2 | Y) \propto \Pr(Y | \hat{\mu}, \hat{\sigma}^2)$$

For $\hat{\mu} = 68$, $\hat{\sigma} = 4$:

$$\begin{aligned} L &= \frac{1}{\sqrt{2\pi 16}} \exp\left[-\frac{(64-68)^2}{32}\right] \times \\ & \quad \frac{1}{\sqrt{2\pi 16}} \exp\left[-\frac{(63-68)^2}{32}\right] \times \\ & \quad \frac{1}{\sqrt{2\pi 16}} \exp\left[-\frac{(59-68)^2}{32}\right] \times \ldots \\ &= \text{some reeeeeally small number...} \end{aligned}$$

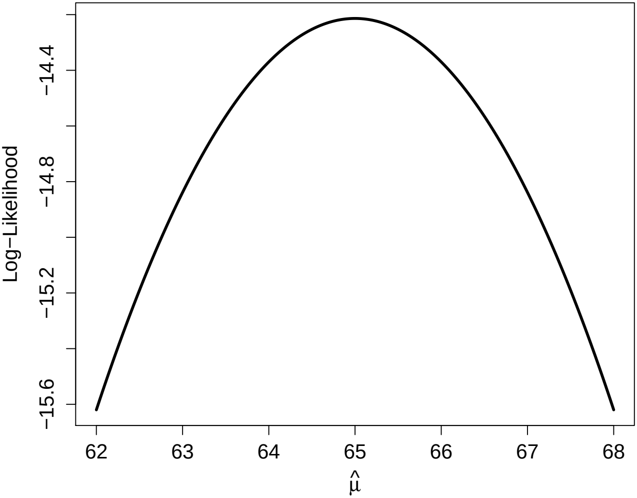
What a Likelihood Looks Like



Log-Likelihood

$$
\begin{aligned}
\ln L(\hat{\mu}, \hat{\sigma}^2 | Y) &= \ln \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(Y_i - \mu)^2}{2\sigma^2}\right] \\
&= \sum_{i=1}^{N} \ln \left\{ \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(Y_i - \mu)^2}{2\sigma^2}\right] \right\} \\
&= -\frac{N}{2} \ln(2\pi) - \left[\sum_{i=1}^{N} \frac{1}{2} \ln\ \sigma^2 + \frac{1}{2\sigma^2} (Y_i - \mu)^2 \right]
\end{aligned}
$$

What a Log-Likelihood Looks Like



The "Maximum" Part

For $L = f(Y, \theta)$,

- Calculate $\frac{\partial \ln L}{\partial \theta}$,
- Set $\frac{\partial \ln L}{\partial \theta} = 0$, solve for $\hat{\theta}$,
- Calculate $\frac{\partial^2 \ln L}{\partial \theta^2}$,
- Verify $\frac{\partial^2 \ln L}{\partial \theta^2} < 0$.

Example: Normal Y

$$\ln L(\hat{\mu}, \hat{\sigma}^2 | Y) = -\frac{N}{2} \ln(2\pi) - \left[\sum_{i=1}^{N} \frac{1}{2} \ln\ \sigma^2 + \frac{1}{2\sigma^2} (Y_i - \mu)^2 \right]$$

$$\begin{aligned}
\frac{\partial \ln L}{\partial \mu} &= \frac{1}{\sigma^2} \sum_{i=1}^{N} (Y_i - \mu) \\
\frac{\partial \ln L}{\partial \sigma^2} &= \frac{-N}{2\sigma^2} + \frac{1}{2} \sigma^4 \sum_{i=1}^{N} (Y_i - \mu)^2
\end{aligned}$$

Example: Normal Y (continued)

$$
\begin{aligned}
\hat{\mu} &= \frac{1}{N}\sum_{i=1}^{N} Y_i \\
\hat{\sigma}^2 &= \frac{1}{N}\sum_{i=1}^{N}(Y_i - \bar{Y})^2
\end{aligned}
$$

Example: Linear Regression

$$
\begin{aligned}
\mathrm{E}(Y) \equiv \mu &= \beta_0 + \beta_1 X_i \\
\mathrm{Var}(Y) &= \sigma^2
\end{aligned}
$$

$$
L(\beta_0, \beta_1, \sigma^2 | Y) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(Y_i - \beta_0 - \beta_1 X_i)^2}{2\sigma^2}\right]
$$

Linear Regression (continued)

$$\begin{aligned} \ln L(\beta_0, \beta_1, \sigma^2 | Y) &= \ln \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(Y_i - \beta_0 - \beta_1 X_i)^2}{2\sigma^2}\right] \\ &= -\frac{N}{2}\ln(2\pi) - \sum_{i=1}^{N}\left[\frac{1}{2}\ln\sigma^2 + \frac{1}{2\sigma^2}(Y_i - \beta_0 - \beta_1 X_i)^2\right] \end{aligned}$$

Kernel:

$$-\sum_{i=1}^{N}\left[\frac{1}{2}\ln\sigma^2 + \frac{1}{2\sigma^2}(Y_i - \beta_0 - \beta_1 X_i)^2\right]$$

MLE in General

$$\Pr(Y) = f(\mathbf{X}, \theta)$$

$$L = \prod_{i=1}^{N} f(Y_i | \mathbf{X}_i, \theta)$$

$$\ln L = \sum_{i=1}^{N} \ln f(Y_i | \mathbf{X}_i, \theta)$$

$$\ln L(\hat{\theta} | Y, \mathbf{X}) = \max_{\theta} \{\ln L(\theta | Y, \mathbf{X})\}$$

Digression: Taylor Series Approximation

For a $k+1$-times differentiable function $f(x)$, we can approximate the function at a with a *Taylor series approximation*:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

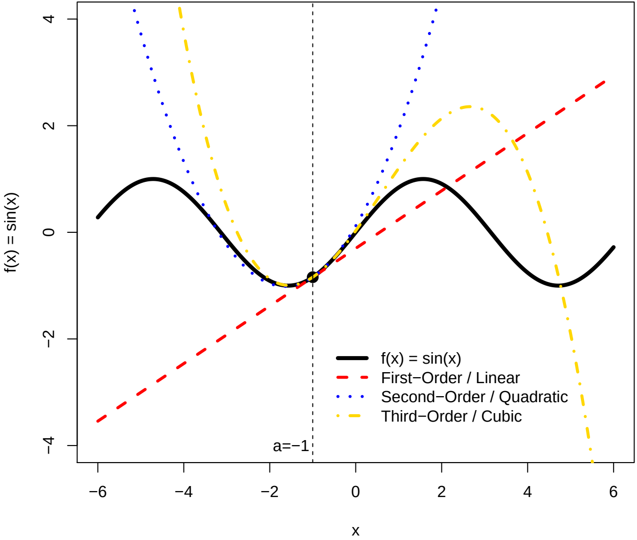
Special cases: First-order / linear:

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a)$$

Second-order / quadratic:

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

Taylor Series, Illustrated



The Gradient

The gradient is:

$$\mathbf{g}(\hat{\theta}) = \frac{\partial \ln L(\hat{\theta})}{\partial \hat{\theta}}$$

First-order Taylor series approximation at θ:

$$\frac{\partial \ln L}{\partial \hat{\theta}} \approx \frac{\partial \ln L}{\partial \theta} + \frac{\partial^2 \ln L}{\partial \theta^2}(\hat{\theta} - \theta)$$

Yields:

$$\begin{aligned} \hat{\theta} - \theta &= \left(-\frac{\partial^2 \ln L}{\partial \theta^2}\right)^{-1} \frac{\partial \ln L}{\partial \theta} \\ &= -\mathbf{H}(\theta)^{-1}\mathbf{g}(\theta) \end{aligned}$$

Consistency

Need

$$\text{plim}(\hat{\theta} - \theta) = 0$$

So:

- *Assume* $\mathbf{H}(\theta) \stackrel{a}{\rightarrow} \mathbf{A} < \infty$
- *Show* $\mathrm{E}[\mathbf{g}(\theta)] \rightarrow \mathbf{0}$ as $N \rightarrow \infty$

Consistency (continued)

$$\begin{aligned}
\mathsf{E}[\mathbf{g}(\theta)] &= \frac{1}{N}\,\mathsf{E}\left(\frac{\partial \ln L_1}{\partial\theta} + \frac{\partial \ln L_2}{\partial\theta} + \ldots + \frac{\partial \ln L_N}{\partial\theta}\right) \\
&= \frac{1}{N}\left[\mathsf{E}\left(\frac{\partial \ln L_1}{\partial\theta}\right) + \mathsf{E}\left(\frac{\partial \ln L_2}{\partial\theta}\right) + \ldots\right] \\
&\overset{a}{=} \mathbf{0}
\end{aligned}$$

Efficiency

Cramer-Rao say:

$$\mathsf{Var}(\hat{\theta}) \geq \left[-\mathsf{E} \left(\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} \right) \right]^{-1}$$

Efficiency, continued

$$
\begin{aligned}
\mathsf{Var}(\hat{\theta}) &= \mathsf{E}[(\hat{\theta}-\theta)(\hat{\theta}-\theta)'] \\
&= \mathsf{E}\left[\left(-\frac{\partial^2 \ln L}{\partial\theta^2}\right)^{-1}\frac{\partial \ln L}{\partial\theta}\frac{\partial \ln L'}{\partial\theta}\left(-\frac{\partial^2 \ln L}{\partial\theta^2}\right)^{-1}\right]
\end{aligned}
$$

For MLE:

$$
\mathsf{E}\left[\frac{\partial \ln L}{\partial\theta}\frac{\partial \ln L'}{\partial\theta}\right] = \mathsf{E}\left[\frac{\partial^2 \ln L}{\partial\theta^2}\right]
$$

So,

$$
\begin{aligned}
\mathsf{Var}(\hat{\theta}) &= \left[-\mathsf{E}\left(\frac{\partial^2 \ln L}{\partial\theta^2}\right)\right]^{-1} \\
&= [\mathbf{I}(\theta)]^{-1}
\end{aligned}
$$

Normality

By the Law of Large Numbers:

$$\frac{\hat{\theta} - \theta}{\sqrt{\mathbf{I}(\theta)^{-1}}} \sim N(\mathbf{0}, \mathbf{1})$$

Or, equivalently:

$$\hat{\theta} \sim N(\theta, \mathbf{I}(\theta)^{-1})$$

Invariance: Parameters

For

$$\gamma = h(\theta)$$

$$\hat{\gamma}_{ML} = h(\hat{\theta}_{ML})$$

Suppose

$$\phi^2 = 1/\sigma^2$$

so that

$$Y \sim N(\mu, \phi^2).$$

Invariance: Example

Then:

$$\ln L(\hat{\mu}, \hat{\phi}^2) = -\left[\sum_{i=1}^{N} \frac{1}{2} \ln\ \phi^2 - \frac{1}{2\phi^2}(Y_i - \mu)^2\right]$$

and:

$$\frac{\partial \ln L}{\partial \phi^2} = \frac{-N}{2\phi^2} + \frac{1}{2}\phi^4 \sum_{i=1}^{N}(Y_i - \mu)^2$$

and:

$$\begin{aligned} \hat{\phi}^2 &= \frac{N}{\sum_{i=1}^{N}(Y_i - \bar{Y})^2} \\ &= \frac{1}{\hat{\sigma}^2} \end{aligned}$$

Summary

MLEs:

- Maximize $L(\theta|Y, \mathbf{X})$
- Are consistent in N
- Are asymptotically efficient
- Are asymptotically Normal
- Are invariant to (injective) transformations and varying sampling methods