PLSC 503 – Spring 2020
Bivariate Regression, I

January 16, 2020

Regression...

$$Y_i = \mu + u_i \tag{1}$$

$$\mu_i = \beta_0 + \beta_1 X_i$$

so:

$$Y_i = \beta_0 + \beta_1 X_i + u_i \tag{2}$$

Goals:

- Estimate $\hat{\beta}_0$ and $\hat{\beta}_1$
- Estimate the *variability* $\hat{\beta}_0$ and $\hat{\beta}_1$

If we have $\hat{\beta}_0$ and $\hat{\beta}_1$, then:

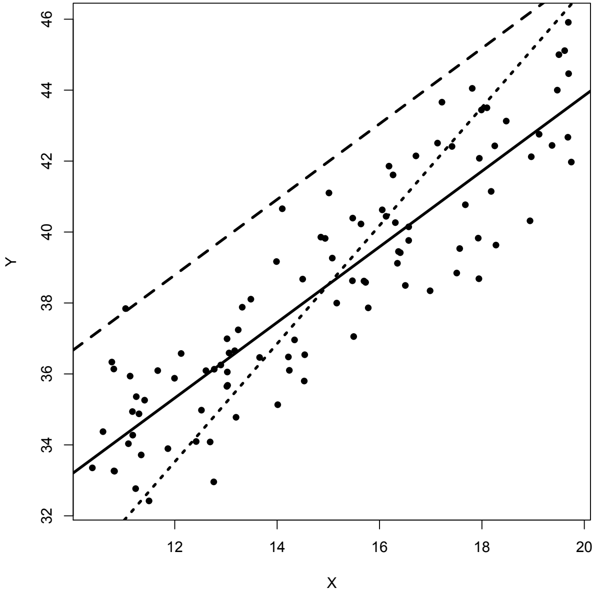
$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i \tag{3}$$

and

$$\begin{aligned} \hat{u}_i &= Y_i - \hat{Y}_i \\ &= Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i \end{aligned} \tag{4}$$

Q: How to estimate $\hat{\beta}_0$ and $\hat{\beta}_1$?

Scatterplot: X and Y (with regression lines)



Ordinary Least Squares

Choose $\hat{\beta}_0$ and $\hat{\beta}_1$ to minimize $\hat{S} = \sum_{i=1}^{N} \hat{u}_i^2$.

$$
\begin{aligned}
\hat{S} &= \sum_{i=1}^{N} (Y_i - \hat{Y}_i)^2 \\
&= \sum_{i=1}^{N} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)^2 \\
&= \sum_{i=1}^{N} (Y_i^2 - 2Y_i\hat{\beta}_0 - 2Y_i\hat{\beta}_1 X_i + \hat{\beta}_0^2 + 2\hat{\beta}_0\hat{\beta}_1 X_i + \hat{\beta}_1^2 X_i^2)
\end{aligned}
$$

Differentiate:

$$\begin{aligned}\frac{\partial \hat{S}}{\partial \hat{\beta}_0} &= \sum_{i=1}^{N}(-2Y_i + 2\hat{\beta}_0 + 2\hat{\beta}_1 X_i) \\ &= -2\sum_{i=1}^{N}(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i) \\ &= -2\sum_{i=1}^{N}\hat{u}_i\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \hat{S}}{\partial \hat{\beta}_1} &= \sum_{i=1}^{N}(-2Y_i X_i + 2\hat{\beta}_0 X_i + 2\hat{\beta}_1 X_i^2) \\ &= -2\sum_{i=1}^{N}(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)X_i \\ &= -2\sum_{i=1}^{N}\hat{u}_i X_i\end{aligned}$$

OLS (continued)

Yields:

$$\sum_{i=1}^{N} Y_i = N\hat{\beta}_0 + \hat{\beta}_1 \sum_{i=1}^{N} X_i$$

and

$$\sum_{i=1}^{N} Y_i X_i = \hat{\beta}_0 \sum_{i=1}^{N} X_i + \hat{\beta}_1 \sum_{i=1}^{N} X_i^2$$

OLS (continued)

Solving yields:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{N}(X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{N}(X_i - \bar{X})^2} = \frac{\text{Covariance of } X \text{ and } Y}{\text{Variance of } X} \quad (5)$$

and

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} \quad (6)$$

Infant Mortality Data

```
> url <- getURL("https://raw.githubusercontent.com/PrisonRodeo/
    PLSC503-2020-git/master/Data/CountryData2000.csv")
> Data <- read.csv(text = url) # read the "votes" data
> rm(url)
>
> # Summary statistics
>
> # install.packages("psych") <- Install psych package, if necessary
> library(psych)

> with(Data, describe(infantmortalityperK))
  vars   n  mean    sd median trimmed   mad min max range skew kurtosis   se
1    1 179 43.83 40.39     29   38.38 34.26 2.9 167 164.1    1     0.06 3.02

> with(Data, describe(DPTpct))
  vars   n  mean    sd median trimmed   mad min max range  skew kurtosis   se
1    1 181 81.71 19.77     90   85.23 11.86  24  99    75 -1.31     0.57 1.47
```

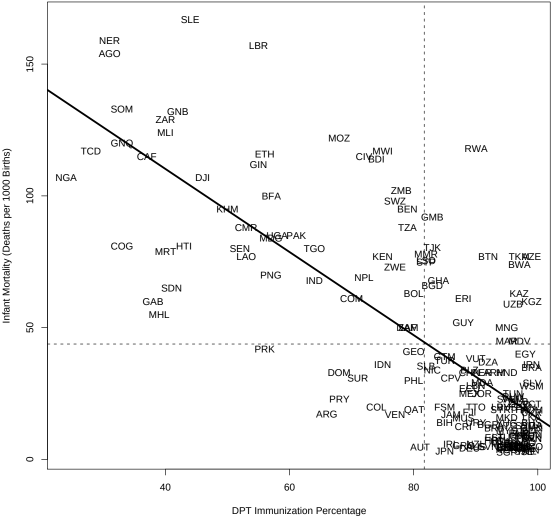
OLS Regression

```
> IMDPT<-lm(infantmortalityperK~DPTpct,data=Data,na.action=na.exclude)
> summary.lm(IMDPT)

Call:
lm(formula = infantmortalityperK ~ DPTpct, data = Data)

Residuals:
    Min      1Q  Median      3Q     Max
-56.801 -16.328  -5.105  11.777  86.590

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 173.2771     8.4893   20.41   <2e-16 ***
DPTpct       -1.5763     0.1009  -15.62   <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 26.19 on 175 degrees of freedom
  (14 observations deleted due to missingness)
Multiple R-squared:  0.5824,Adjusted R-squared:   0.58
F-statistic: 244.1 on 1 and 175 DF,  p-value: < 2.2e-16
```

Analysis of Variance

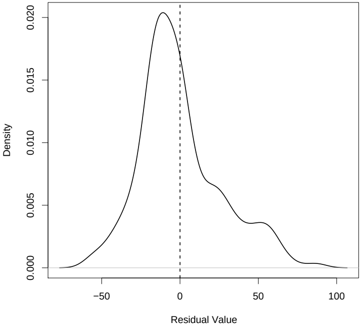
```
> anova(IMDPT)
Analysis of Variance Table

Response: infantmortalityperK
           Df Sum Sq Mean Sq F value    Pr(>F)
DPTpct      1 167423  167423  244.09 < 2.2e-16 ***
Residuals 175 120033     686
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

Regression of Infant Mortality on DPT Immunization Rates



Fitted Values, Residuals, etc.

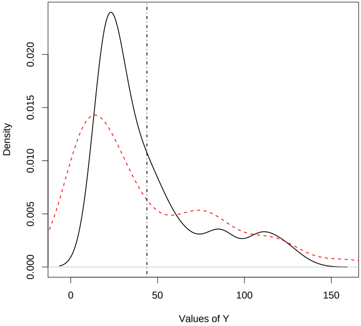
```
> # Residuals (u):
> Data$IMDPTres <- with(Data, residuals(IMDPT))
> describe(Data$IMDPTres)

  var   n mean    sd median   mad   min   max range skew kurtosis   se
1   1 177    0 26.12   -5.1 19.42 -56.8 86.59 143.4 0.75     0.44 1.96
```



Fitted Values

```
> # Fitted Values:
> Data$IMDPThat<-fitted.values(IMDPT)
> describe(Data$IMDPThat)

  var   n  mean    sd median  mad   min   max range skew kurtosis   se
1   1 177 44.26 30.84  31.41 18.7 17.22 135.4 118.2  1.3     0.59 2.32
```

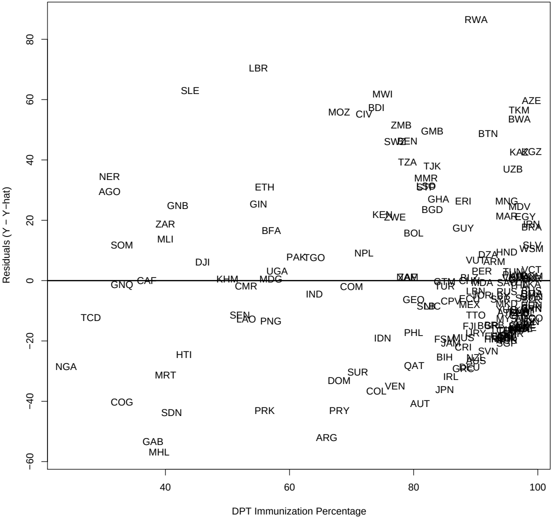
Figure: Density Plot: Actual (Y) and Fitted Values ($\hat{Y}$)



Some Correlations

```
> with(Data, cor(infantmortalityperK,DPTpct,use="complete.obs"))
[1] -0.7632

> with(Data, cor(IMDPTres,infantmortalityperK,use="complete.obs"))
[1] 0.6462

> with(Data, cor(IMDPTres,DPTpct,use="complete.obs"))
[1] 9.573e-17

> with(Data, cor(IMDPThat,infantmortalityperK,use="complete.obs"))
[1] 0.7632

> with(Data, cor(IMDPThat,DPTpct,use="complete.obs"))
[1] -1

> with(Data, cor(IMDPTres,IMDPThat,use="complete.obs"))
[1] 5.302e-17
```

Regression Residuals ($\hat{u}$) vs. DPT Percentage



Squared Residuals vs. DPT Percentage