

PLSC 503 – Spring 2020

Multiplicative Interactions, I

March 5, 2020

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

$$Y_i = \beta_0 + \beta_1 D_{1i} + \beta_2 D_{2i} + u_i$$

means:

$$E(Y|D_1 = 1, D_2 = 0) - E(Y|D_1 = 0, D_2 = 0) = E(Y|D_1 = 1, D_2 = 1) - E(Y|D_1 = 0, D_2 = 1) [\equiv \beta_1]$$

and

$$E(Y|D_1 = 0, D_2 = 1) - E(Y|D_1 = 0, D_2 = 0) = E(Y|D_1 = 1, D_2 = 1) - E(Y|D_1 = 1, D_2 = 0) [\equiv \beta_2].$$

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{1i} X_{2i} + u_i$$

$$\begin{aligned} E(Y_i) &= \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{1i} X_{2i} \\ &= \beta_0 + \beta_2 X_{2i} + (\beta_1 + \beta_3 X_{2i}) X_{1i} \\ &= \beta_0 + \beta_2 X_{2i} + \psi_1 X_{1i} \end{aligned}$$

where $\psi_1 = \beta_1 + \beta_3 X_{2i}$. This means:

$$\frac{\partial E(Y_i)}{\partial X_1} = \beta_1 + \beta_3 X_{2i}.$$

Similarly:

$$\begin{aligned} E(Y_i) &= \beta_0 + \beta_1 X_{1i} + (\beta_2 + \beta_3 X_{1i}) X_{2i} \\ &= \beta_0 + \beta_1 X_{1i} + \psi_2 X_{2i} \end{aligned}$$

which implies:

$$\frac{\partial E(Y_i)}{\partial X_2} = \beta_2 + \beta_3 X_{1i}.$$

“Direct Effects”

If $X_2 = 0$, then:

$$\begin{aligned} E(Y_i) &= \beta_0 + \beta_1 X_{1i} + \beta_2(0) + \beta_3 X_{1i}(0) \\ &= \beta_0 + \beta_1 X_{1i}. \end{aligned}$$

Similarly, for $X_1 = 0$:

$$\begin{aligned} E(Y_i) &= \beta_0 + \beta_1(0) + \beta_2 X_{2i} + \beta_3(0)X_{2i} \\ &= \beta_0 + \beta_2 X_{2i} \end{aligned}$$

In most instances, the quantities we care about are not β_1 and β_2 , but rather ψ_1 and ψ_2 .

Point estimates:

$$\hat{\psi}_1 = \hat{\beta}_1 + \hat{\beta}_3 X_2$$

and

$$\hat{\psi}_2 = \hat{\beta}_2 + \hat{\beta}_3 X_1.$$

For variance, recall that:

$$\text{Var}(a + bZ) = \text{Var}(a) + Z^2 \text{Var}(b) + 2Z \text{Cov}(a, b)$$

Means that:

$$\widehat{\text{Var}}(\hat{\psi}_1) = \widehat{\text{Var}}(\hat{\beta}_1) + X_2^2 \widehat{\text{Var}}(\hat{\beta}_3) + 2X_2 \widehat{\text{Cov}}(\hat{\beta}_1, \hat{\beta}_3).$$

and

$$\widehat{\text{Var}}(\hat{\psi}_2) = \widehat{\text{Var}}(\hat{\beta}_2) + X_1^2 \widehat{\text{Var}}(\hat{\beta}_3) + 2X_1 \widehat{\text{Cov}}(\hat{\beta}_2, \hat{\beta}_3).$$

Types of Interactions: Dichotomous X s

For

$$Y_i = \beta_0 + \beta_1 D_{1i} + \beta_2 D_{2i} + \beta_3 D_{1i} D_{2i} + u_i$$

we have:

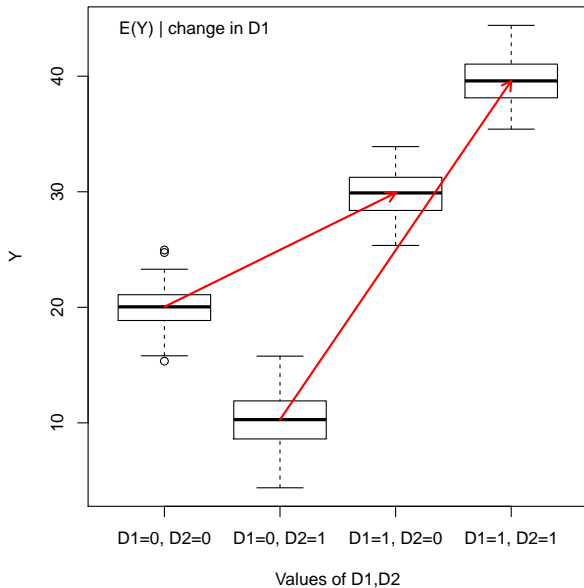
$$E(Y|D_1 = 0, D_2 = 0) = \beta_0$$

$$E(Y|D_1 = 1, D_2 = 0) = \beta_0 + \beta_1$$

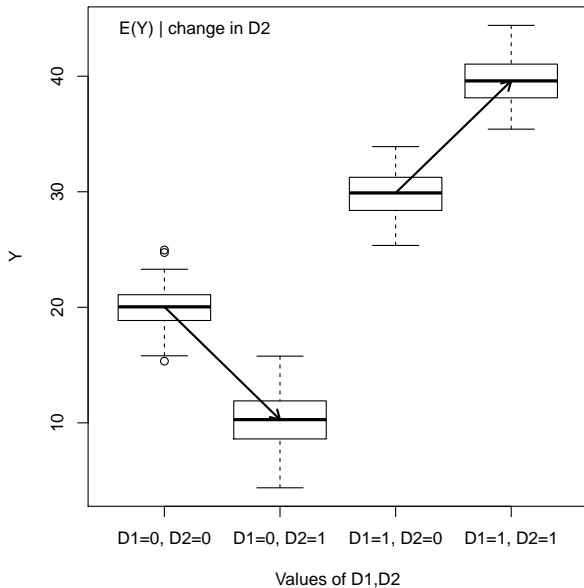
$$E(Y|D_1 = 0, D_2 = 1) = \beta_0 + \beta_2$$

$$E(Y|D_1 = 1, D_2 = 1) = \beta_0 + \beta_1 + \beta_2 + \beta_3$$

Values of $E(Y)$ for Changes in D_1



Values of $E(Y)$ for Changes in D_2



Dichotomous and Continuous X s

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 D_i + \beta_3 X_i D_i + u_i$$

gives:

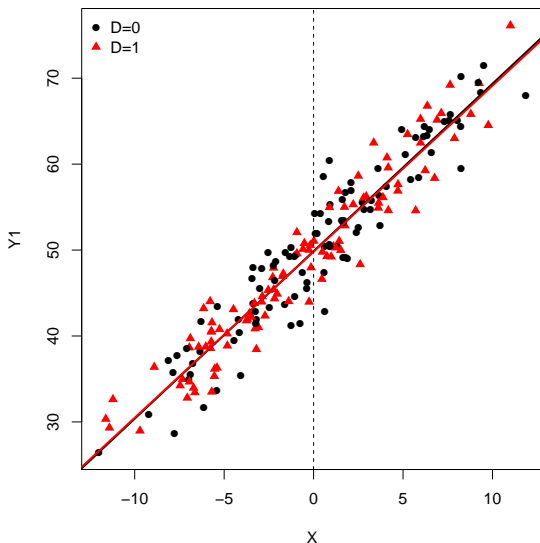
$$E(Y|X, D = 0) = \beta_0 + \beta_1 X$$

$$E(Y|X, D = 1) = (\beta_0 + \beta_2) + (\beta_1 + \beta_3)X$$

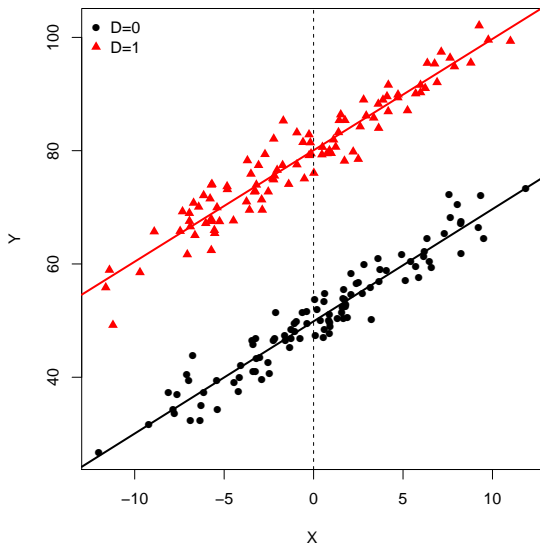
Four possibilities:

- $\beta_2 = \beta_3 = 0$
- $\beta_2 \neq 0$ and $\beta_3 = 0$
- $\beta_2 = 0$ and $\beta_3 \neq 0$
- $\beta_2 \neq 0$ and $\beta_3 \neq 0$

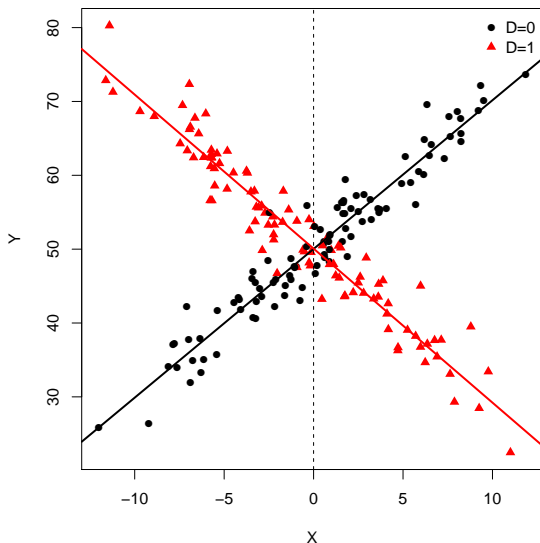
Scatterplot and Regression Lines of Y on X for $D = 0$ and $D = 1$: No Slope or Intercept Differences ($\beta_2 = \beta_3 = 0$)



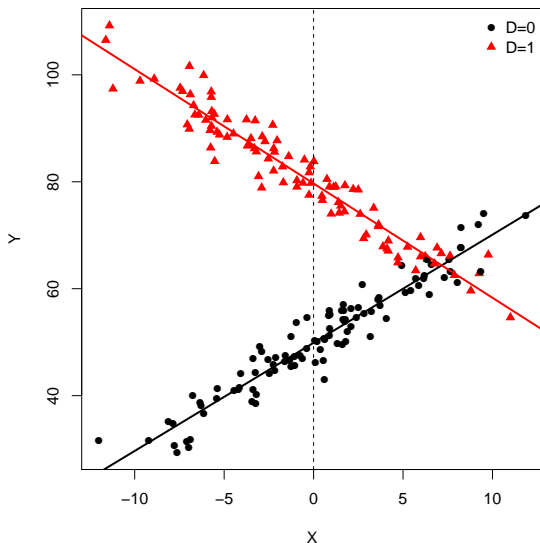
Scatterplot and Regression Lines of Y on X for $D = 0$ and $D = 1$: Intercept Shift ($\beta_2 \neq 0, \beta_3 = 0$)



Scatterplot and Regression Lines of Y on X for $D = 0$ and $D = 1$: Slope Change ($\beta_2 = 0$, $\beta_3 \neq 0$)



Scatterplot and Regression Lines of Y on X for $D = 0$ and $D = 1$: Slope and Intercept Change ($\beta_2 \neq 0$, $\beta_3 \neq 0$)



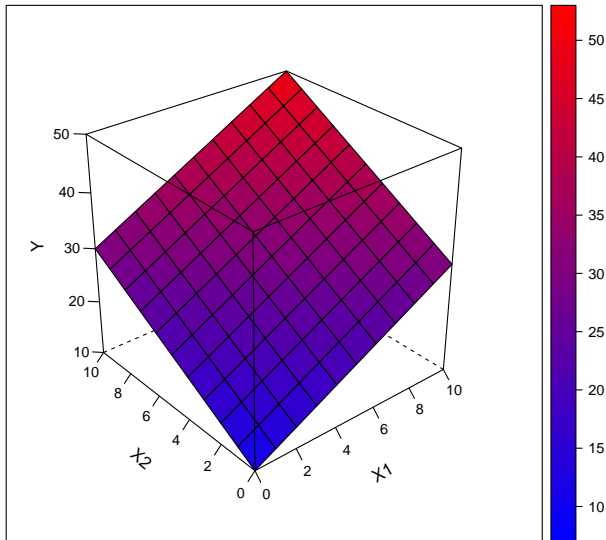
Two Continuous X s

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{1i} X_{2i} + u_i.$$

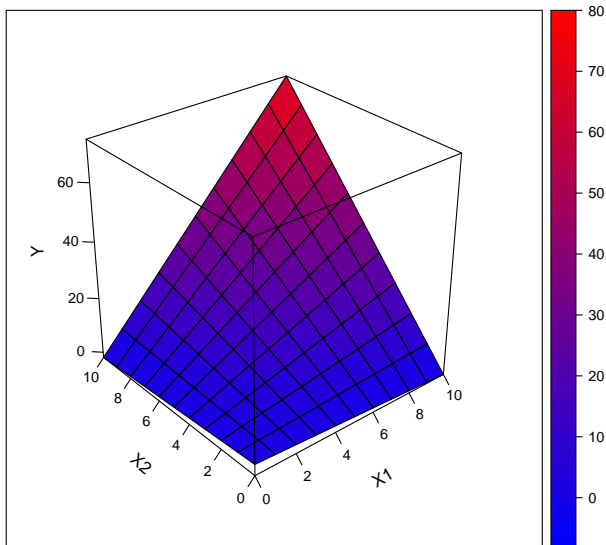
Implies

$$\beta_3 = 0 \rightarrow \frac{\partial E(Y)}{\partial X_1} = \beta_1 \forall X_2 \text{ and } \frac{\partial E(Y)}{\partial X_2} = \beta_2 \forall X_1$$

Two Continuous Variables: No Interactive Effects



Two Continuous Variables: Interaction Present



Quadratic, Cubic, and Other Polynomial Effects

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \beta_3 X_i^3 + \dots + \beta_j X_i^j + u_i.$$

In general:

$$\frac{\partial E(Y)}{\partial X} = \beta_1 + 2\beta_2 X + 3\beta_3 X^2 + \dots + j\beta_j X^{j-1}$$

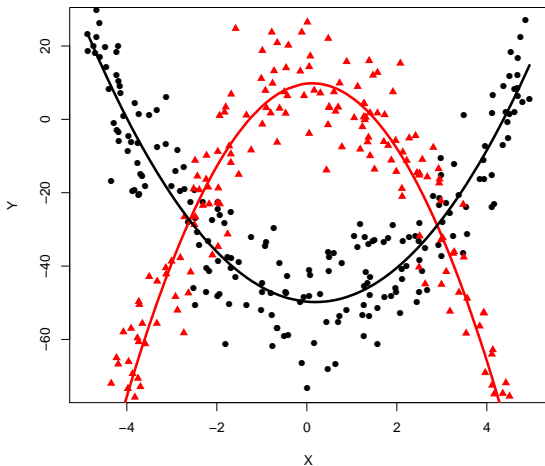
Quadratic case ($j = 2$):

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + u_i.$$

implies

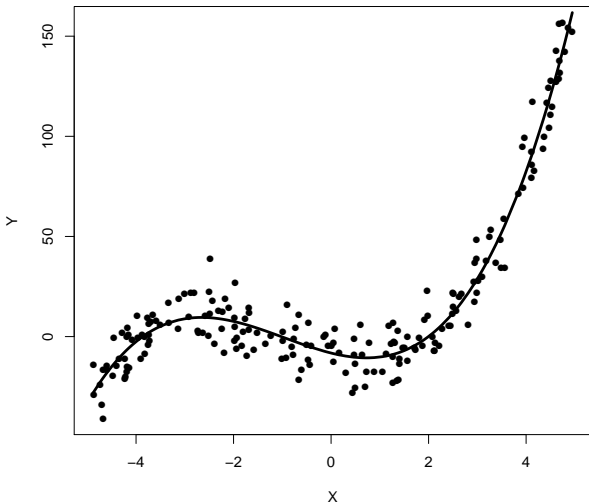
$$\frac{\partial E(Y)}{\partial X} = \beta_1 + 2\beta_2 X$$

Two Quadratic Relationships



Note: Red line is $Y_i = 10 + 1X_i - 5X_i^2 + u_i$; black line is $Y_i = -50 - 1X_i + 3X_i^2 + u_i$.

Example of a Cubic Relationship



Note: Solid line is $Y_i = -1 + 1X_i - 8X_i^2 + 5X_i^3 + u_i$.

Higher-Order Interactive Models

Three-way interaction:

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \\ \beta_4 X_{1i} X_{2i} + \beta_5 X_{1i} X_{3i} + \beta_6 X_{2i} X_{3i} + \beta_7 X_{1i} X_{2i} X_{3i} + u_i$$

Special case of dichotomous X_1, X_2 :

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 D_{1i} + \beta_3 D_{2i} + \\ \beta_4 X_i D_{1i} + \beta_5 X_i D_{2i} + \beta_6 D_{1i} D_{2i} + \beta_7 X_i D_{1i} D_{2i} + u_i$$

Three-Way Interaction: Two Dummy and One Continuous Covariates

