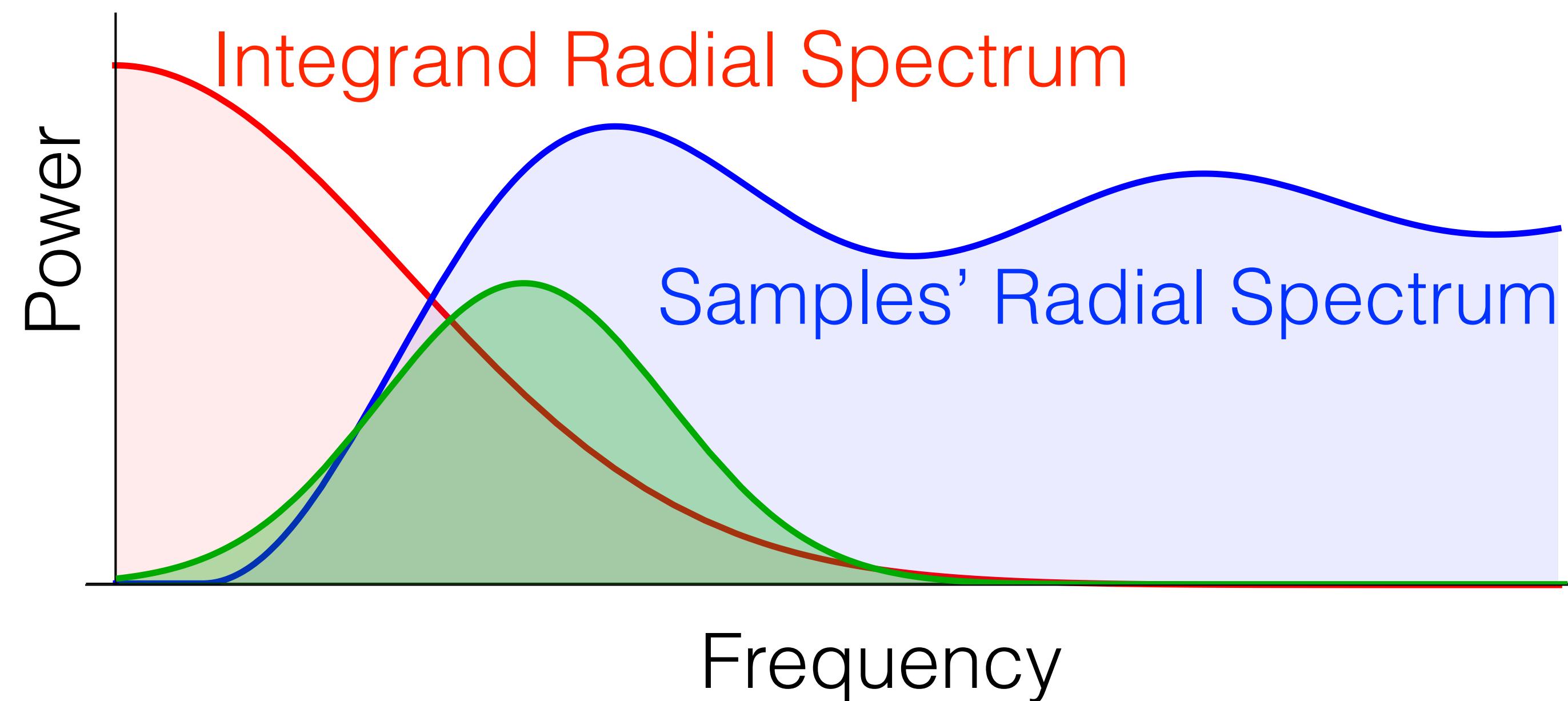
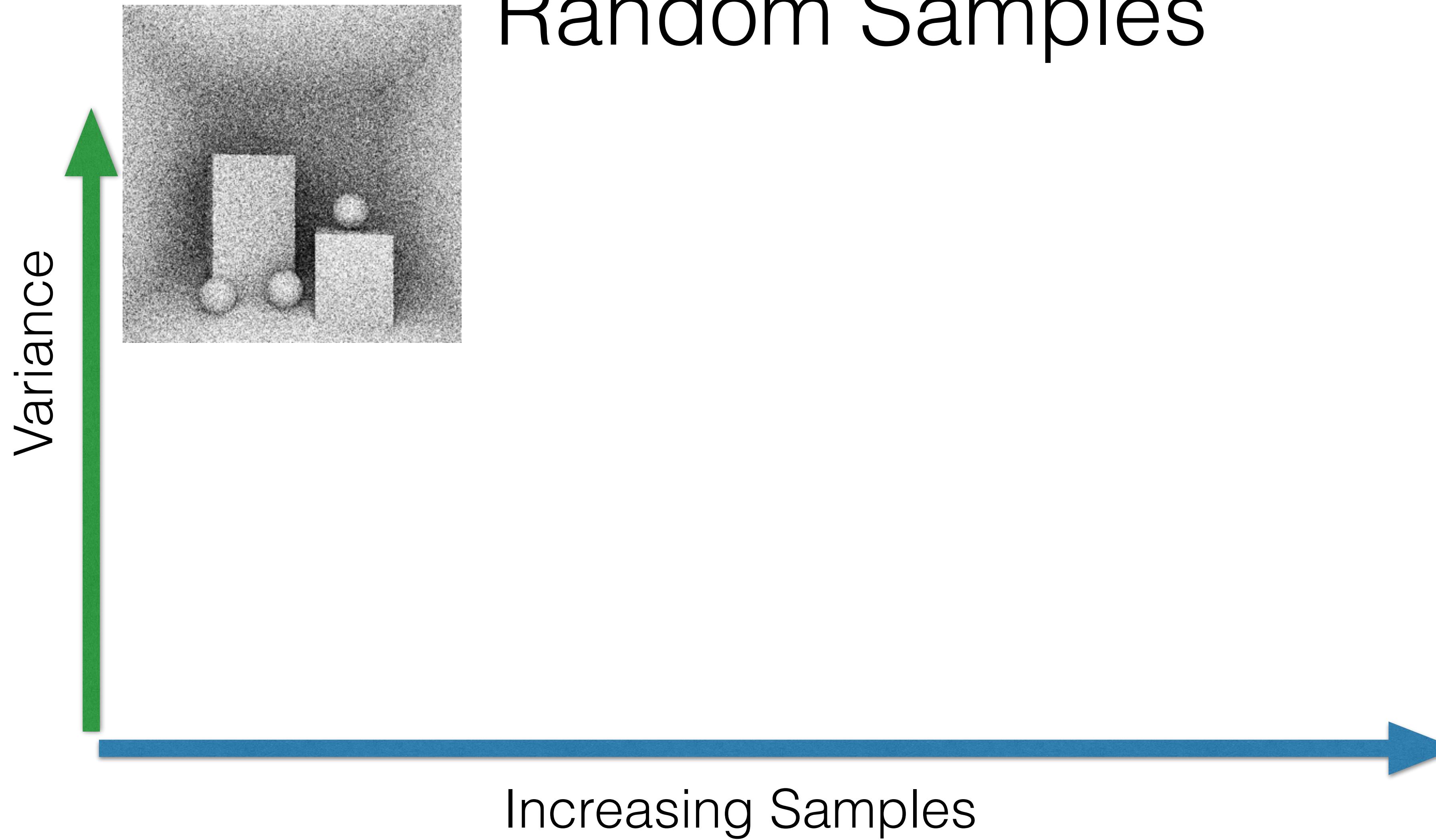


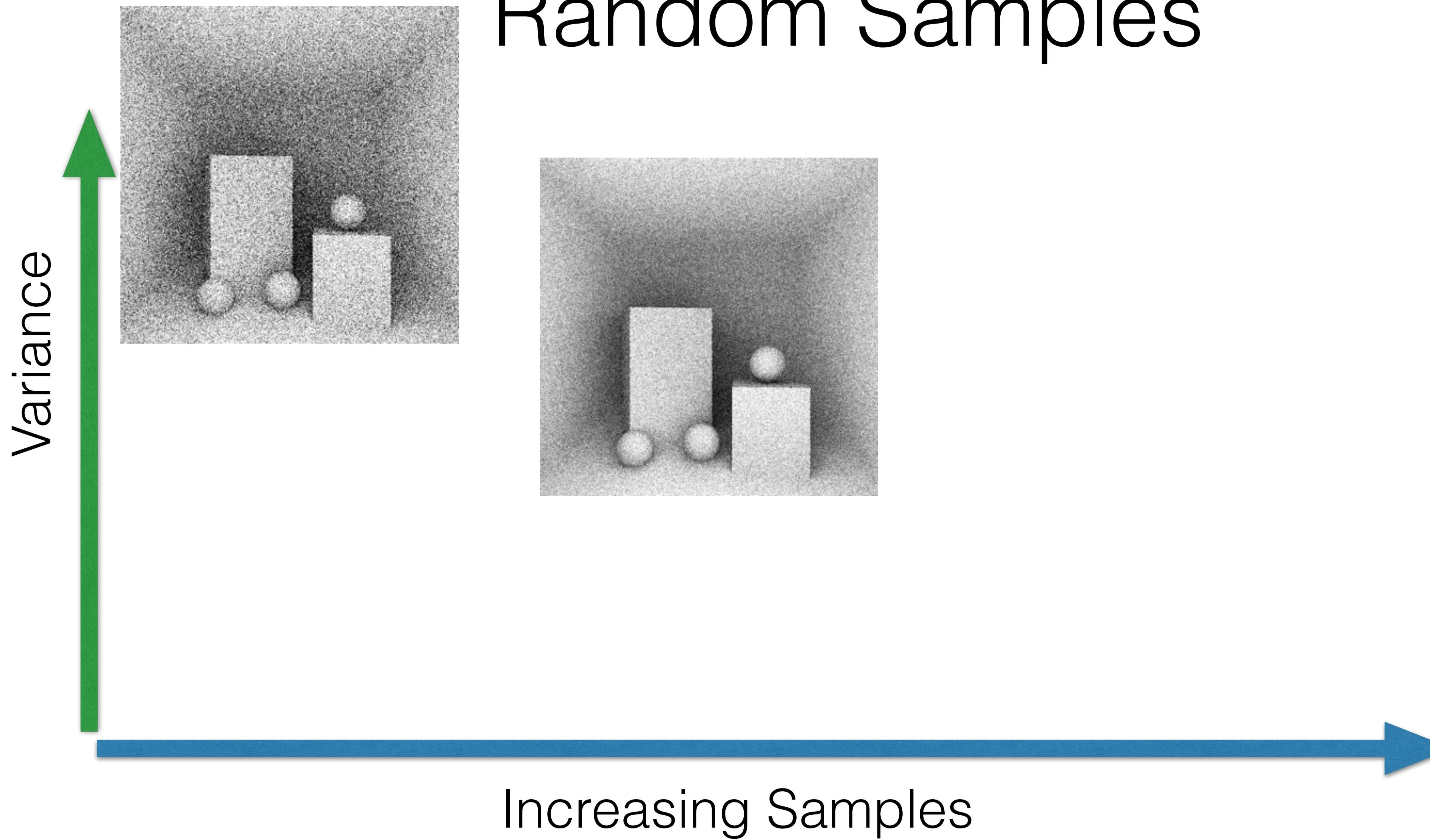
Part-3: Formal Treatment of MSE, Bias and Variance



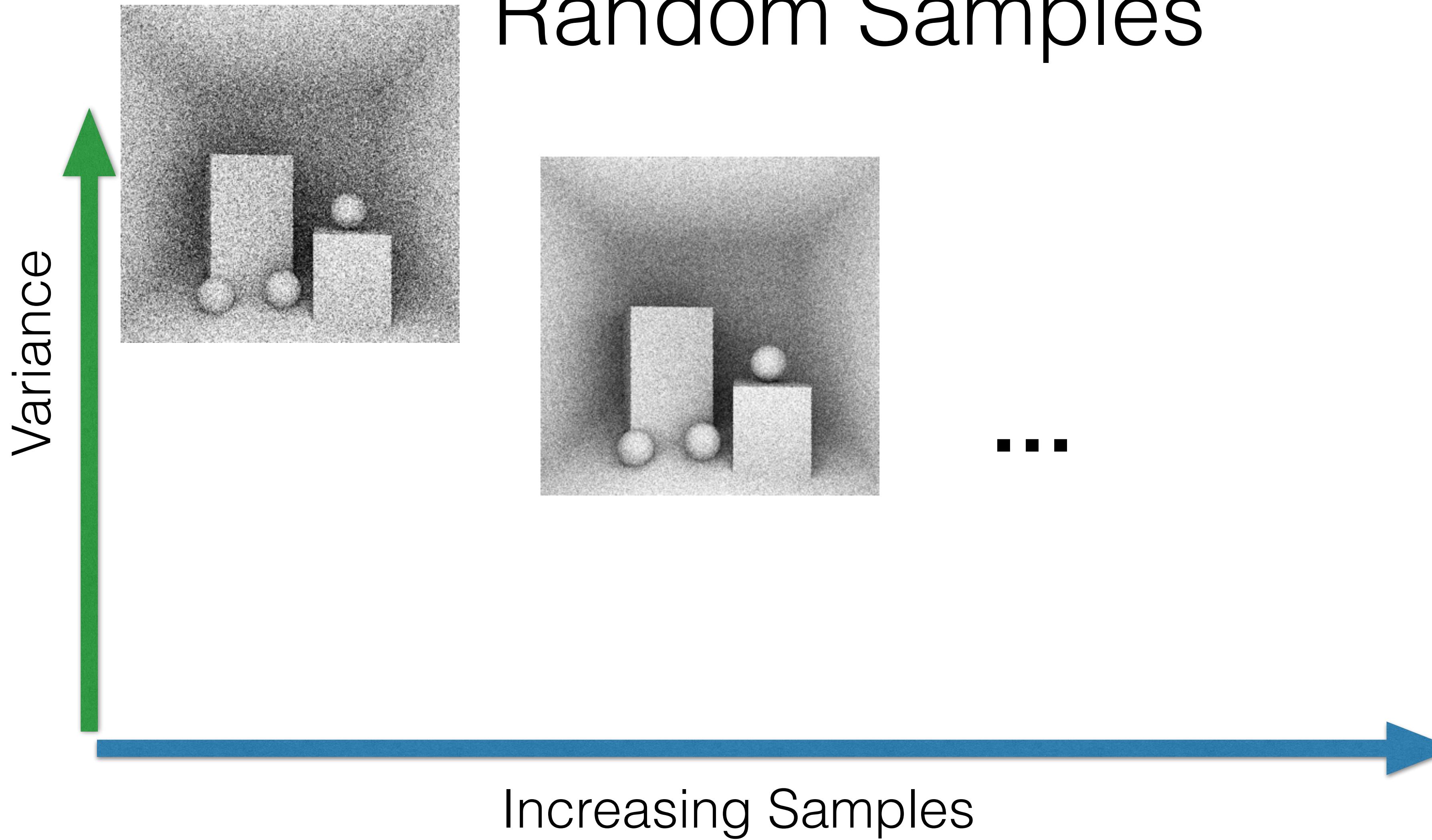
Convergence rate for Random Samples



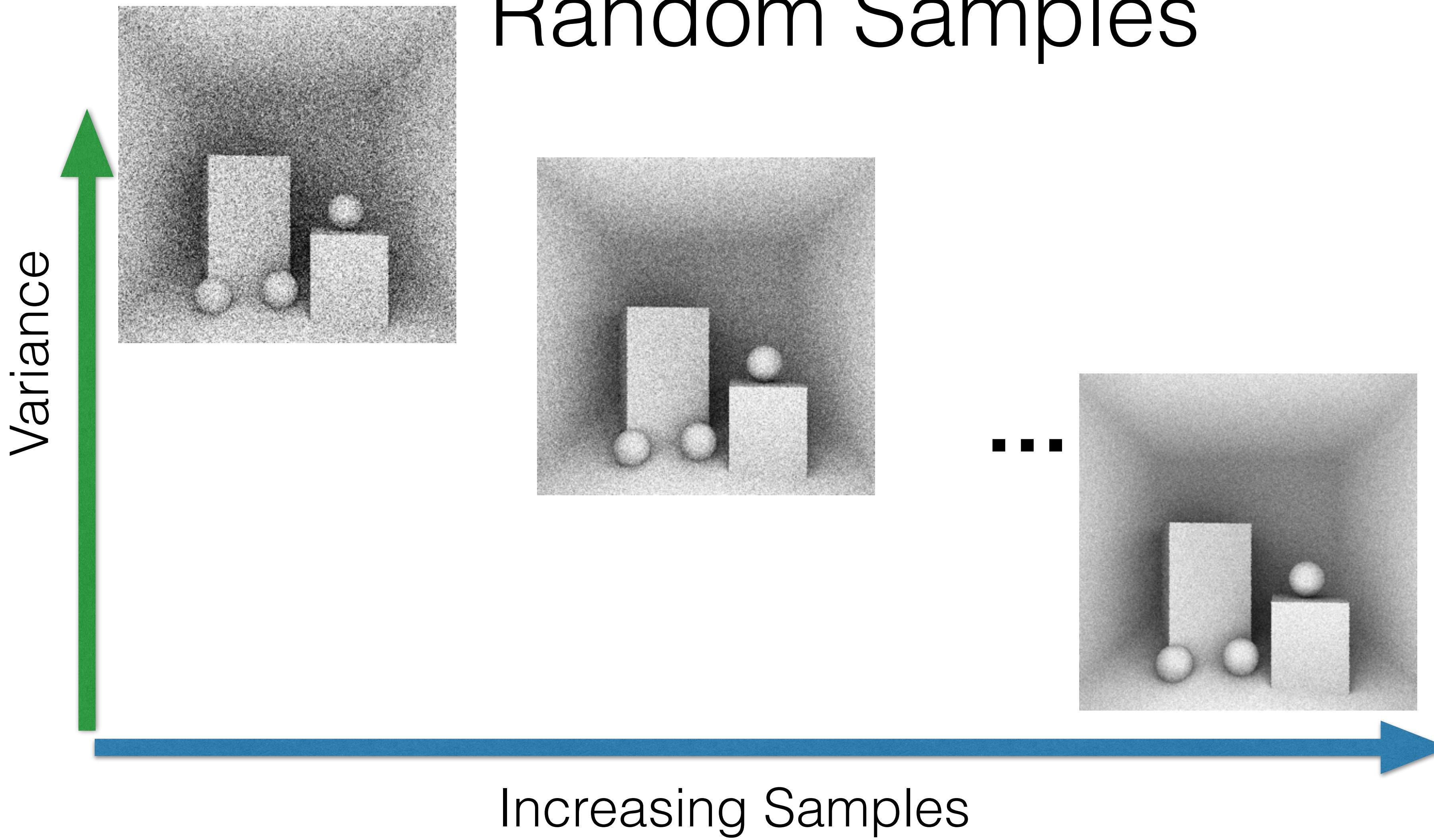
Convergence rate for Random Samples



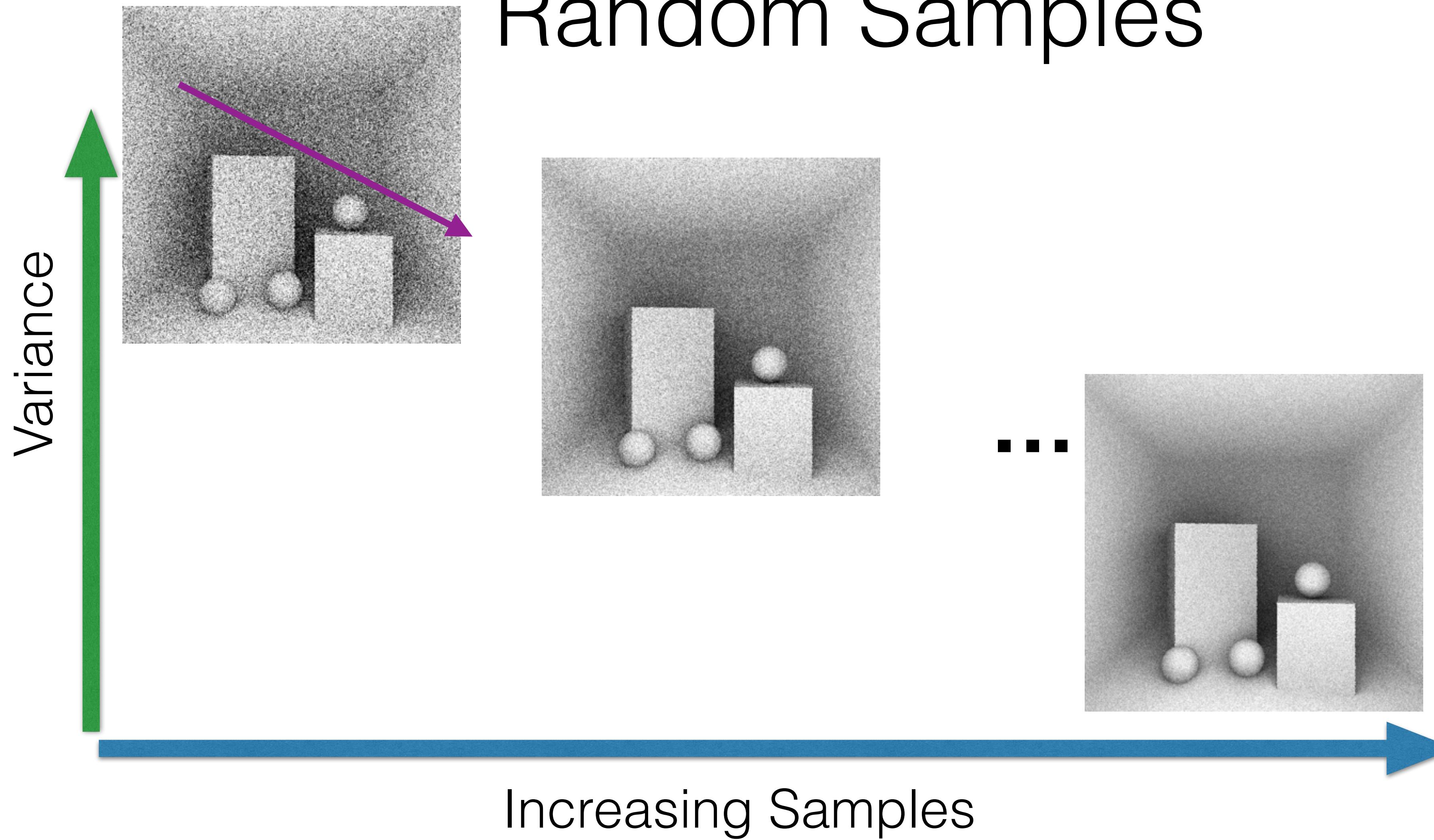
Convergence rate for Random Samples



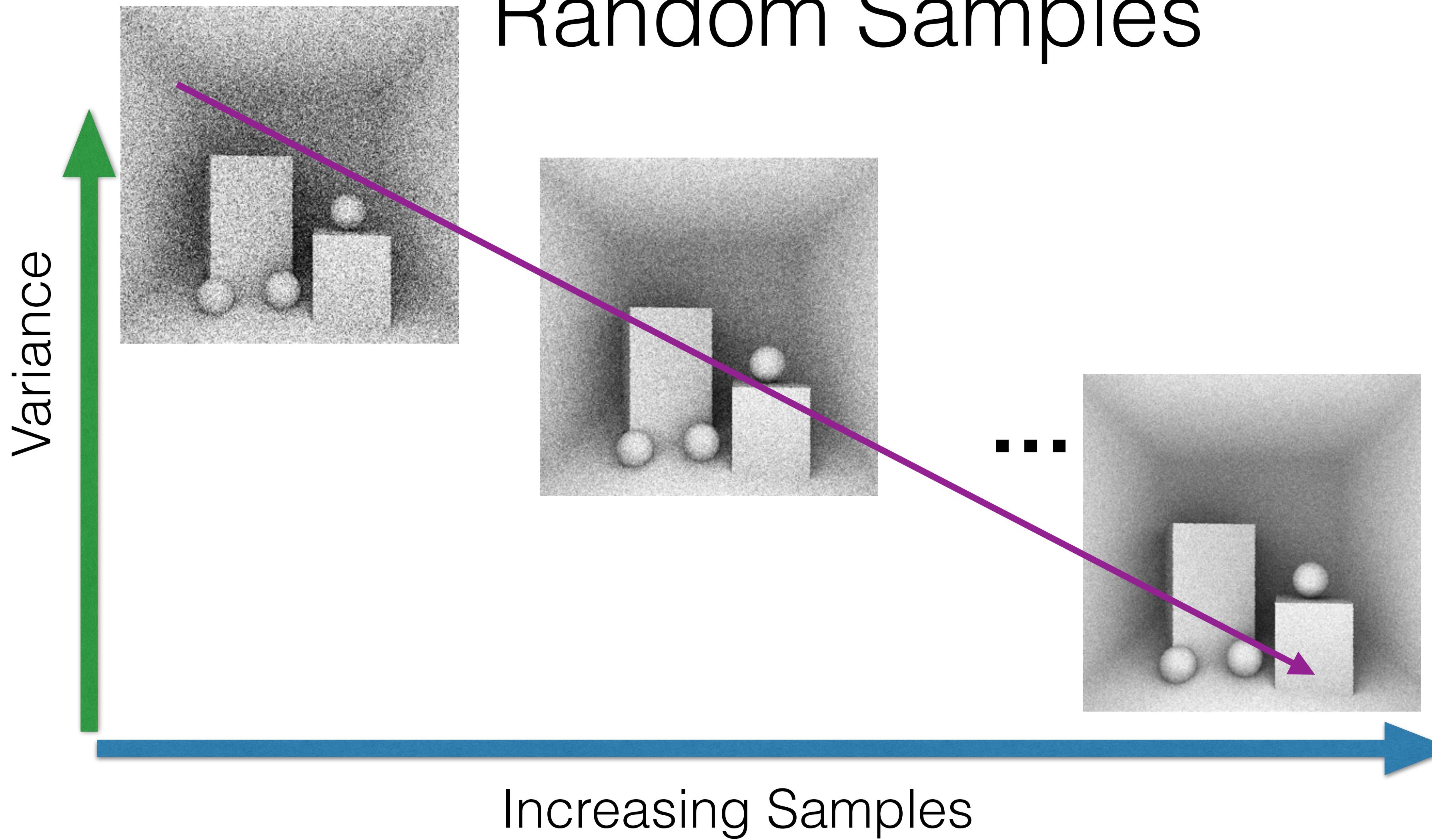
Convergence rate for Random Samples



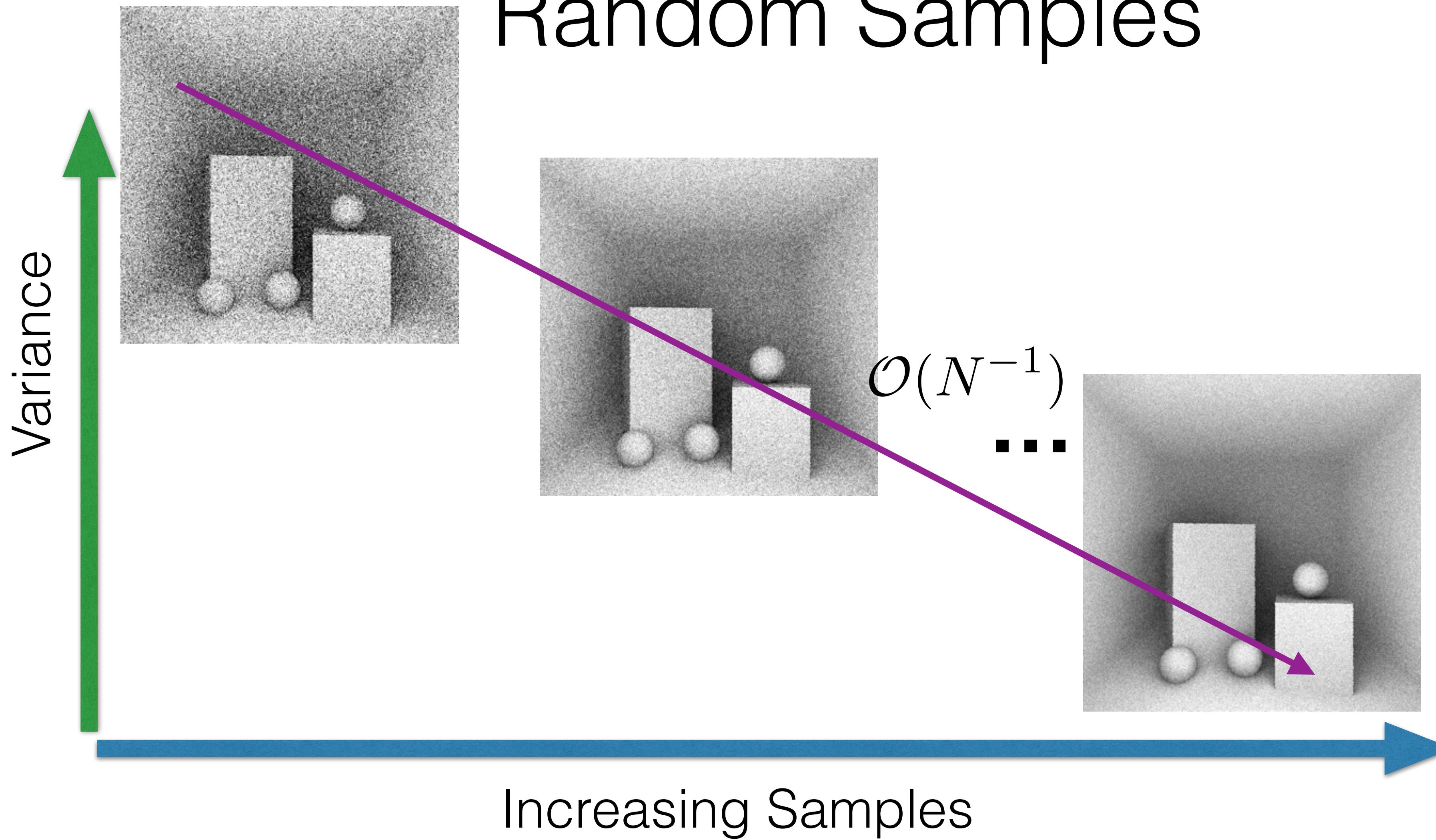
Convergence rate for Random Samples



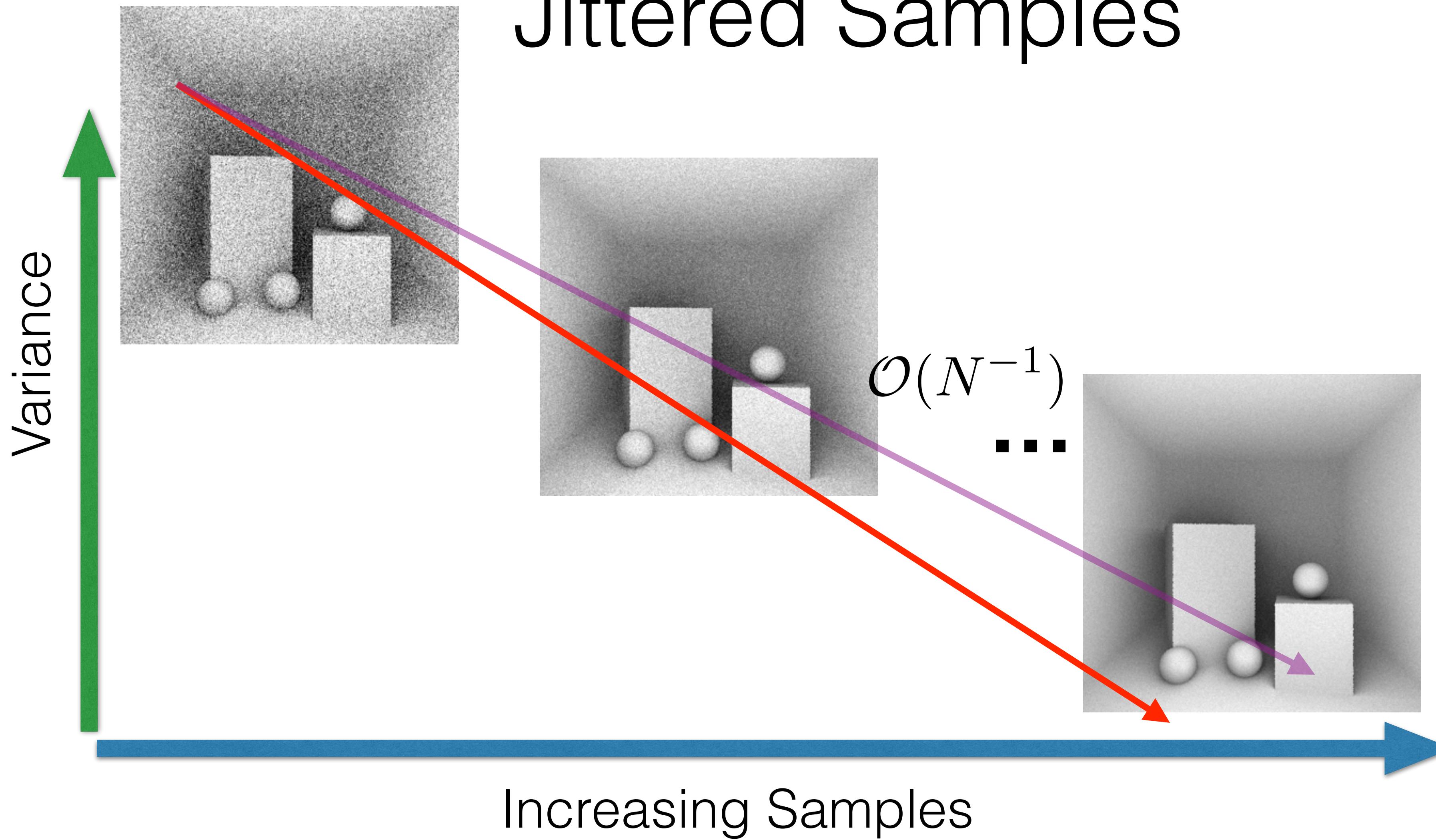
Convergence rate for Random Samples



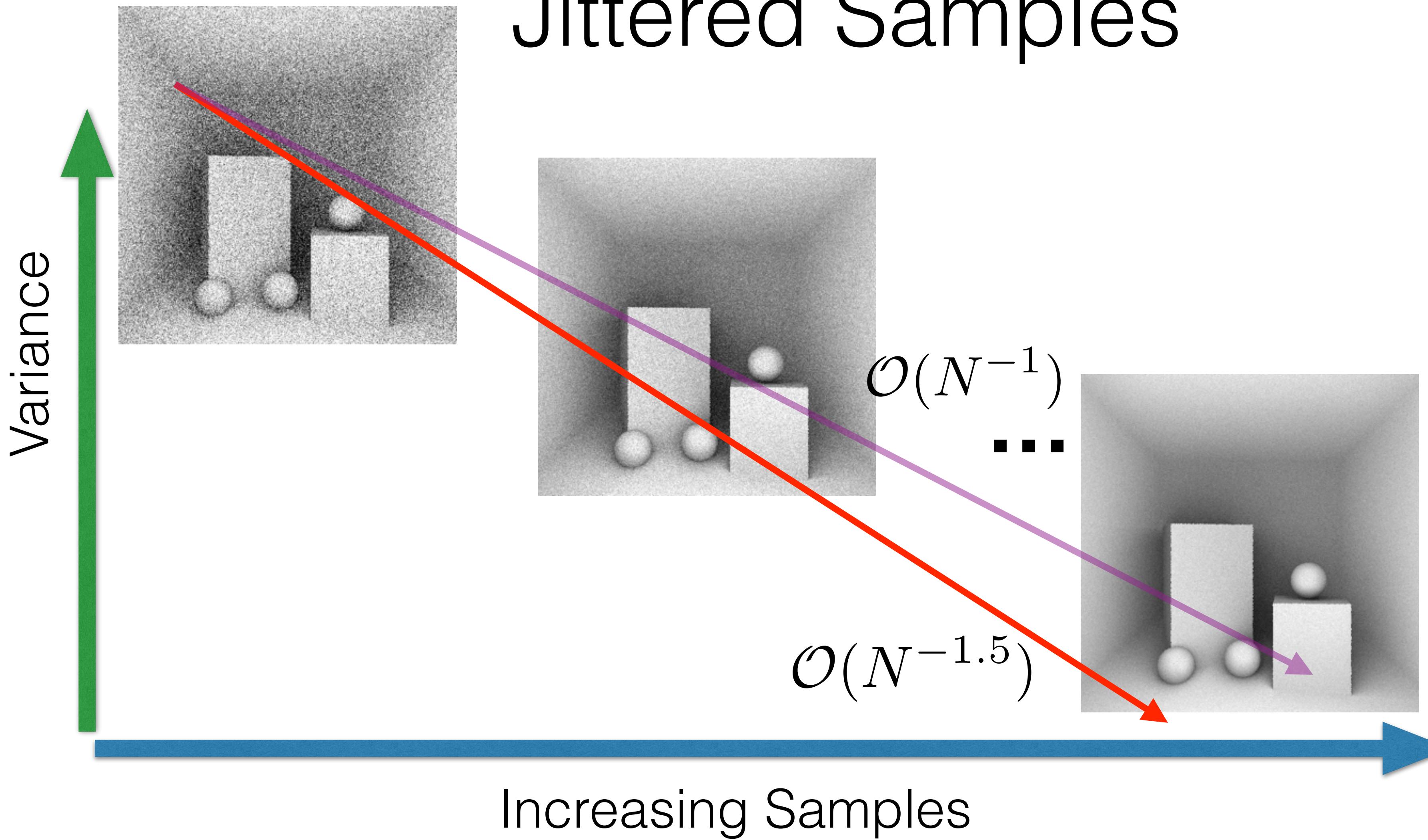
Convergence rate for Random Samples



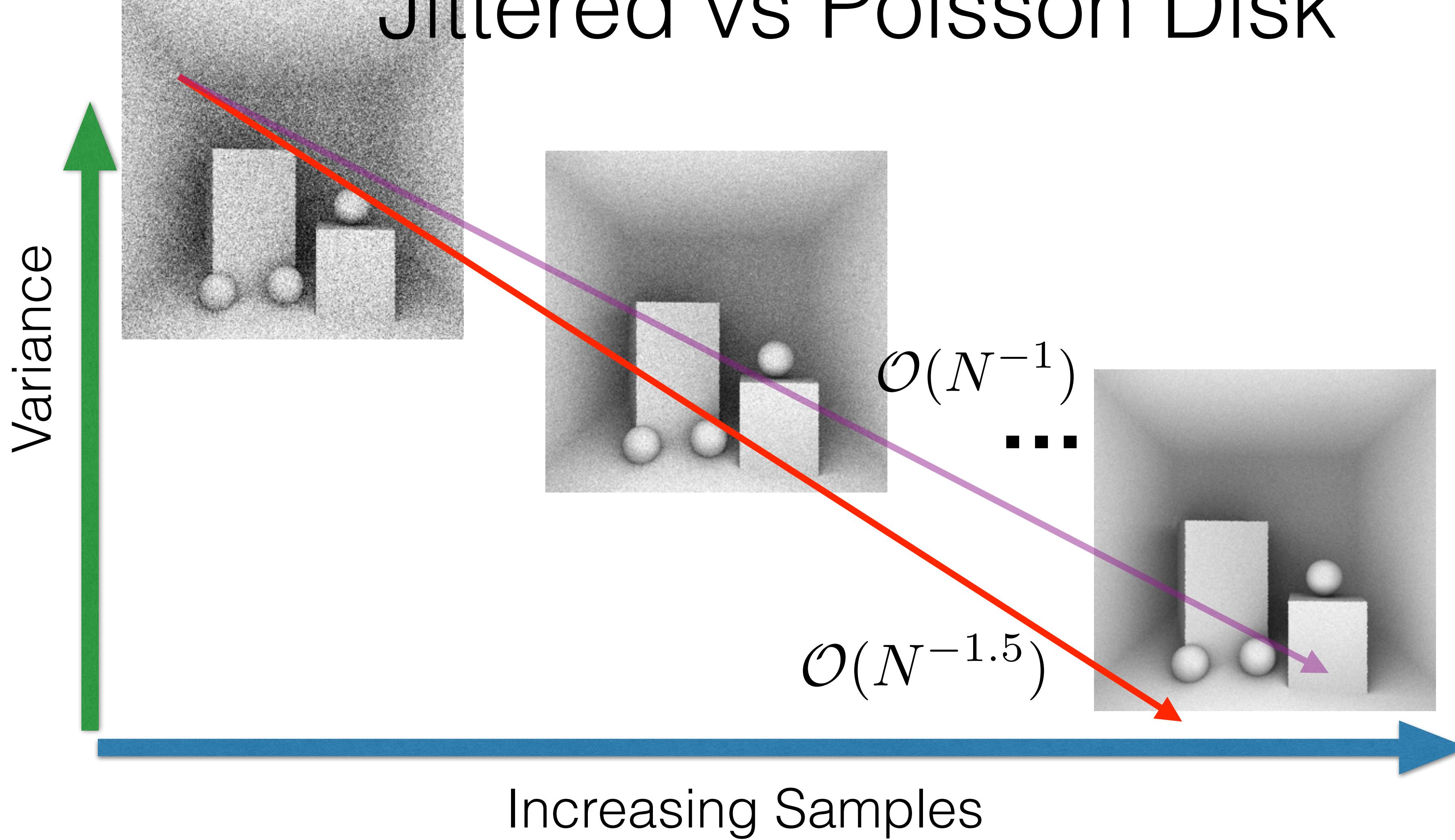
Convergence rate for Jittered Samples



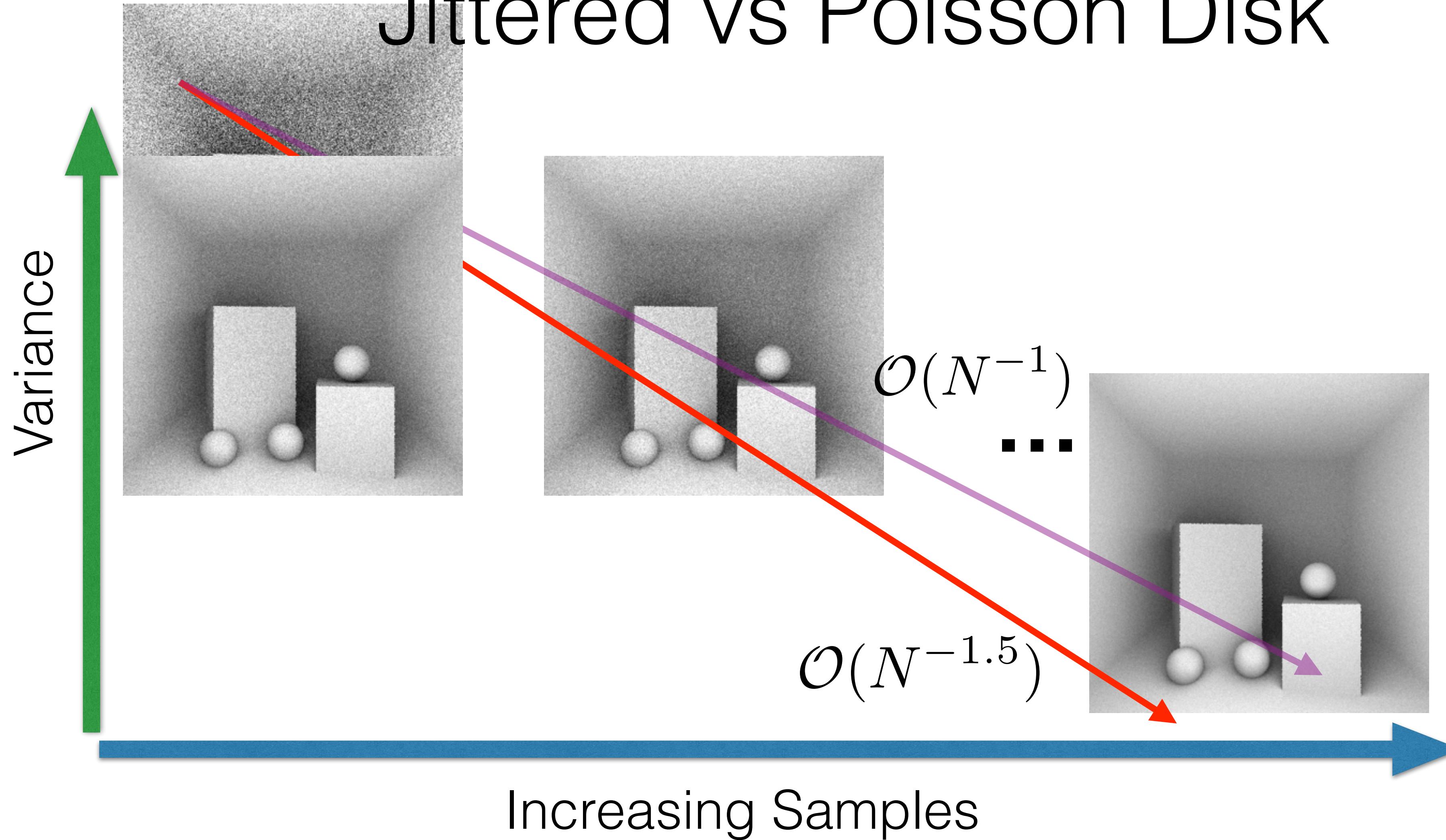
Convergence rate for Jittered Samples



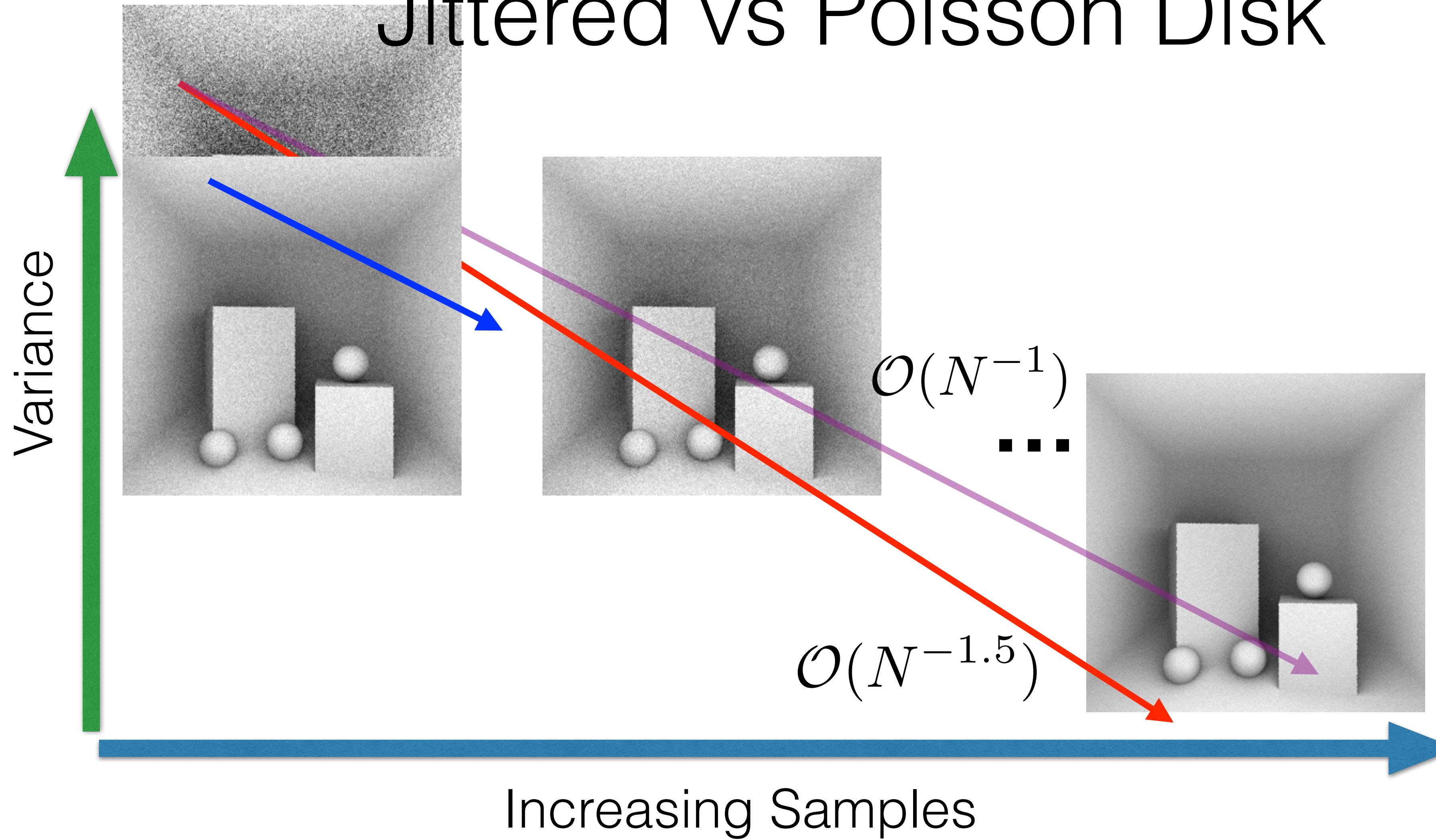
Convergence rate Jittered vs Poisson Disk



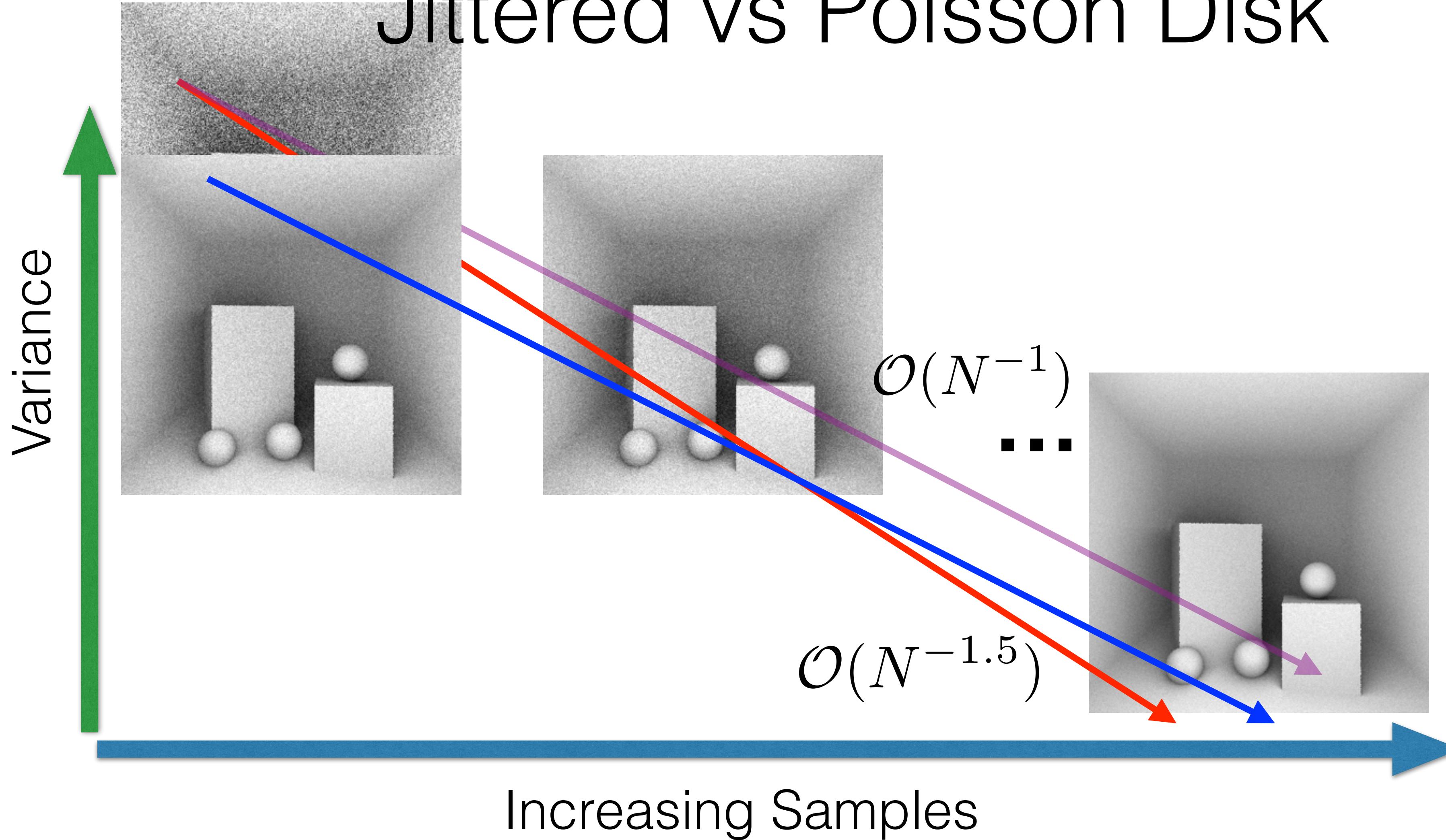
Convergence rate Jittered vs Poisson Disk



Convergence rate Jittered vs Poisson Disk

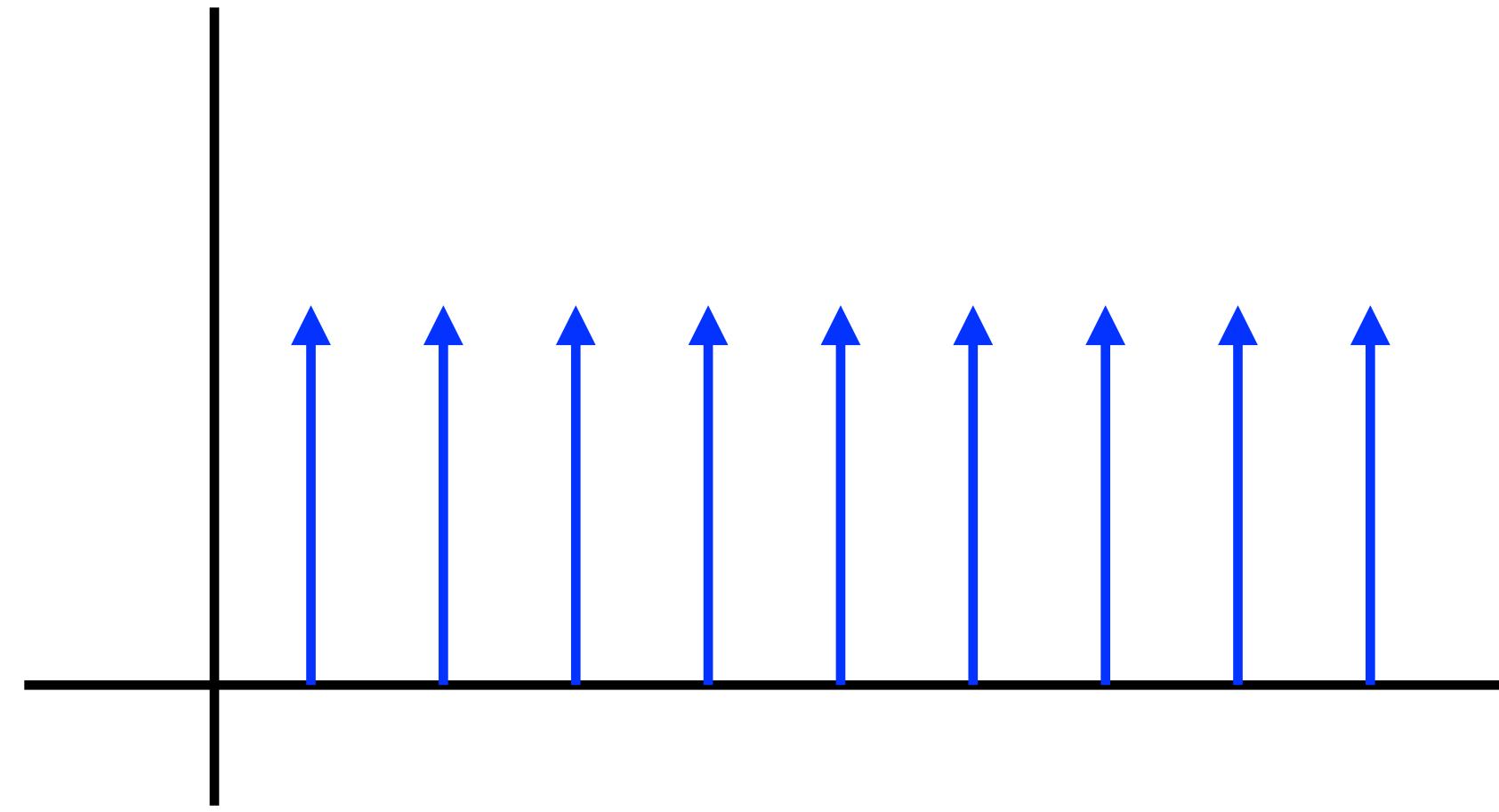


Convergence rate Jittered vs Poisson Disk



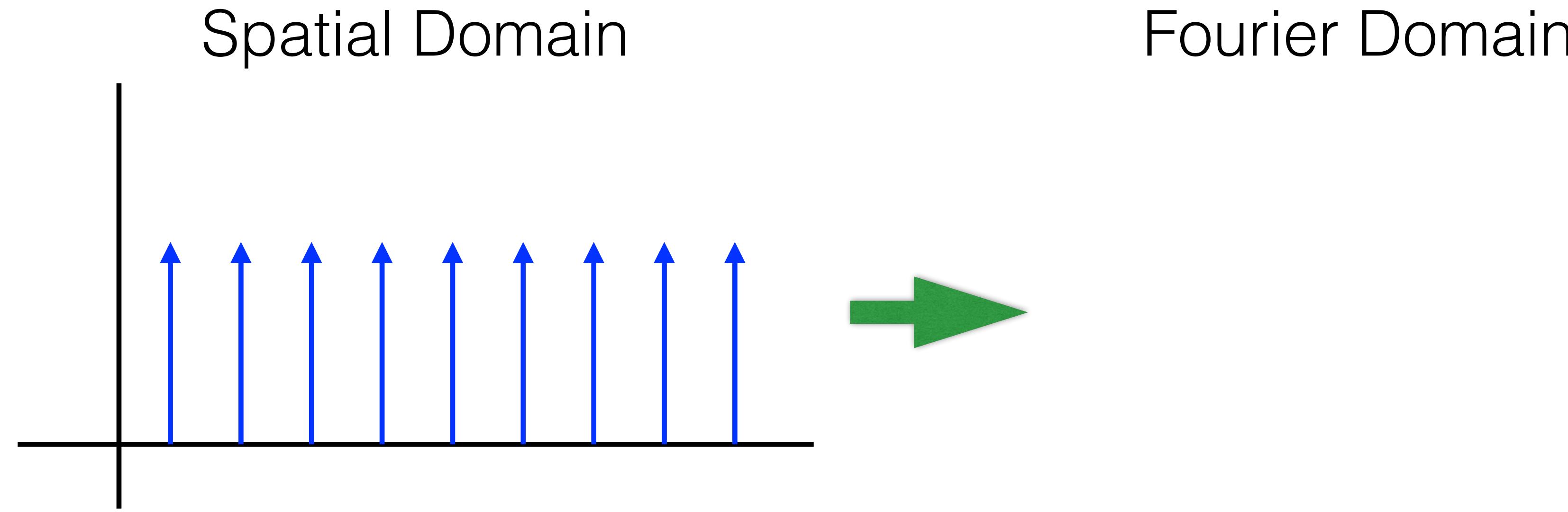
Samples and function in Fourier Domain

Spatial Domain

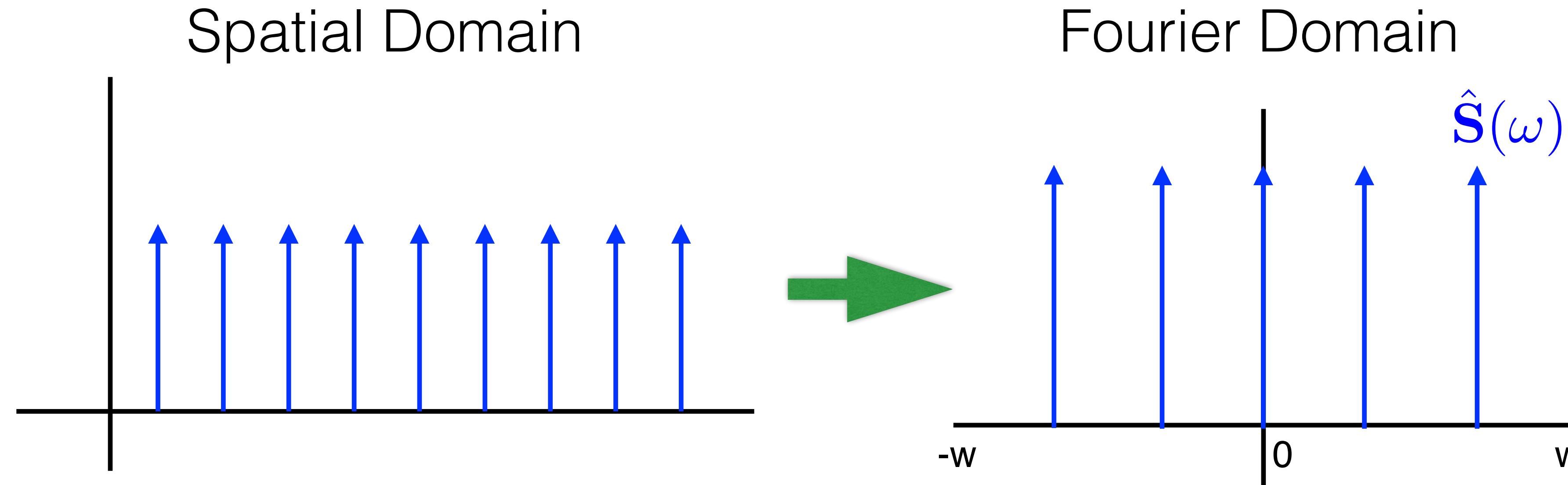


Fourier Domain

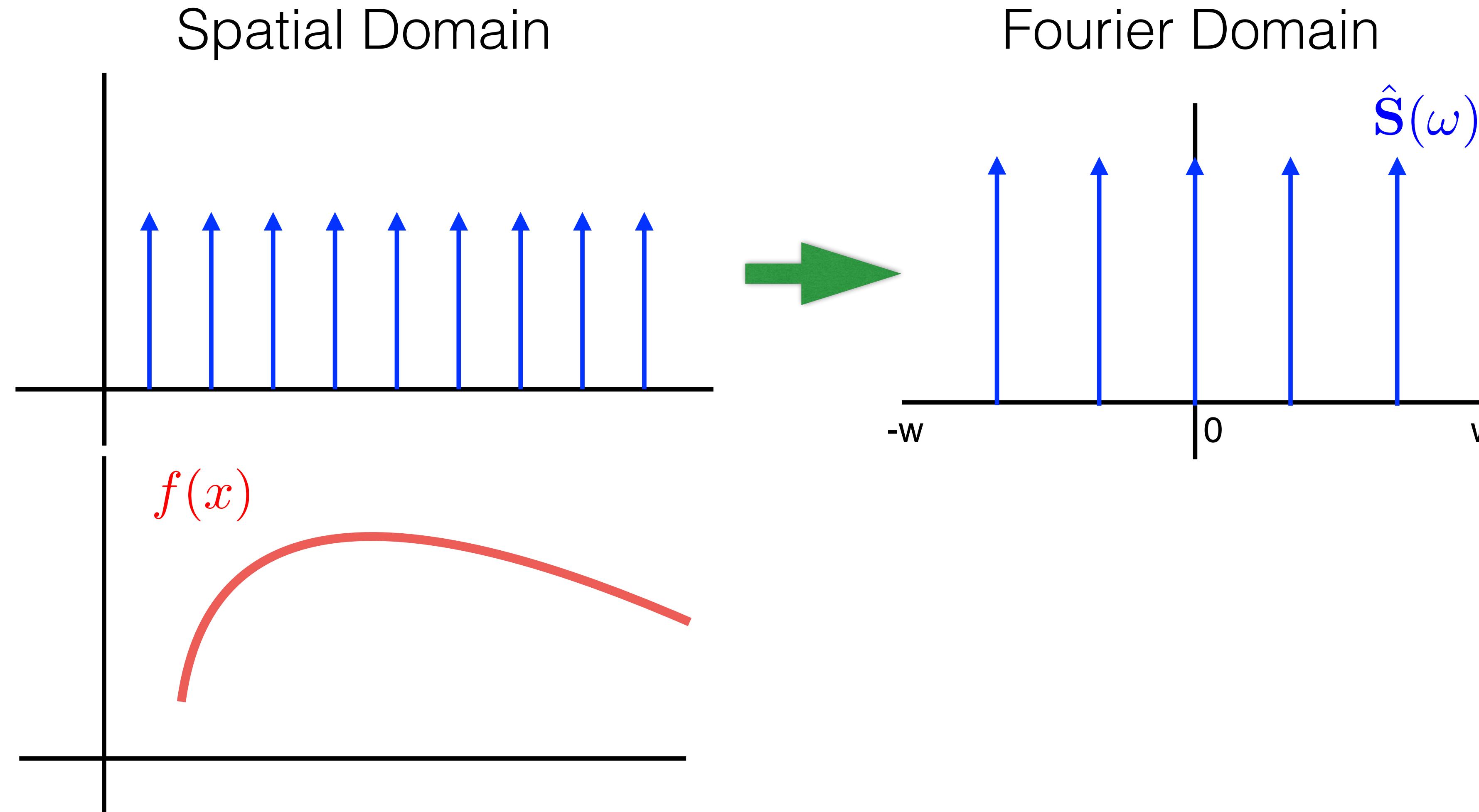
Samples and function in Fourier Domain



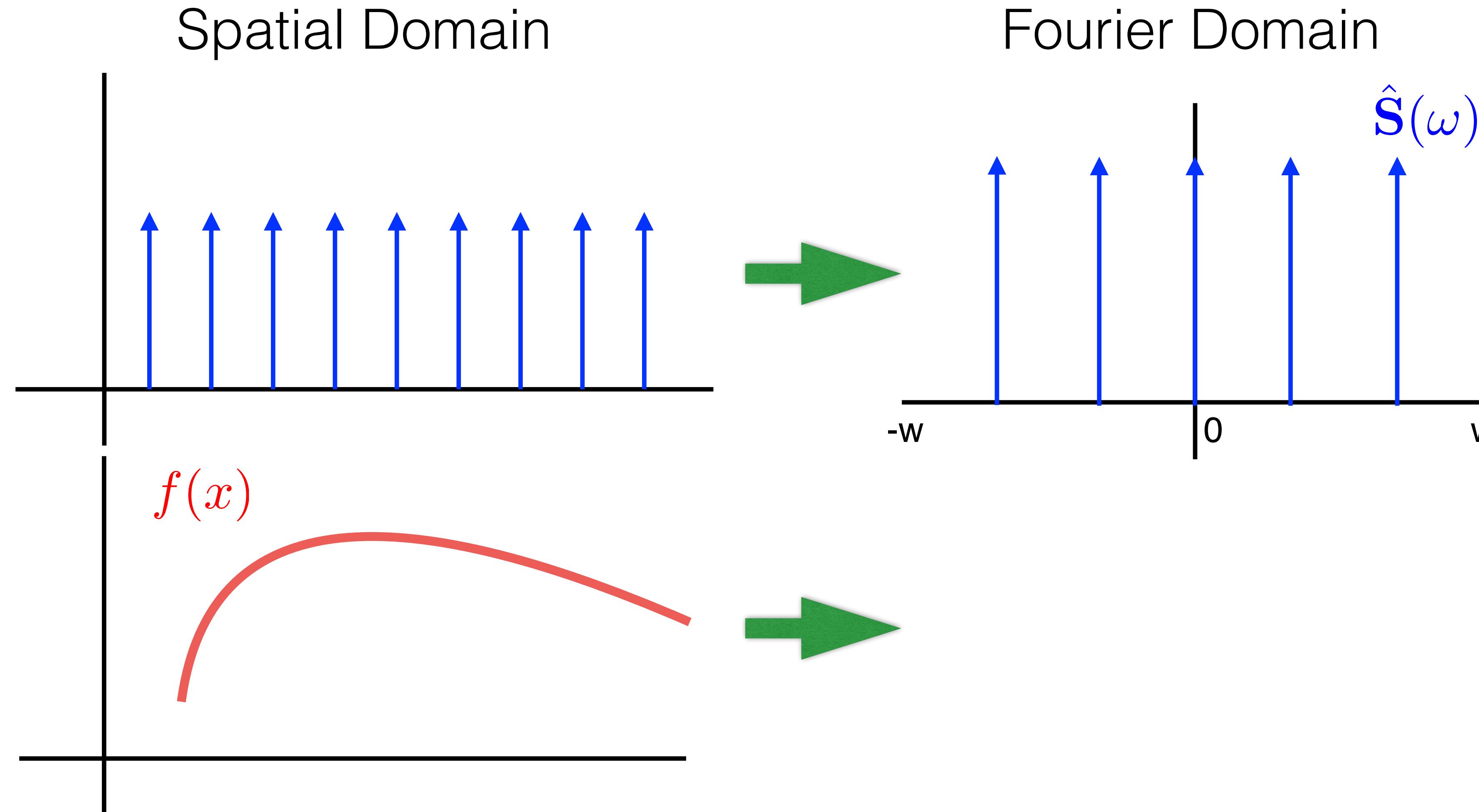
Samples and function in Fourier Domain



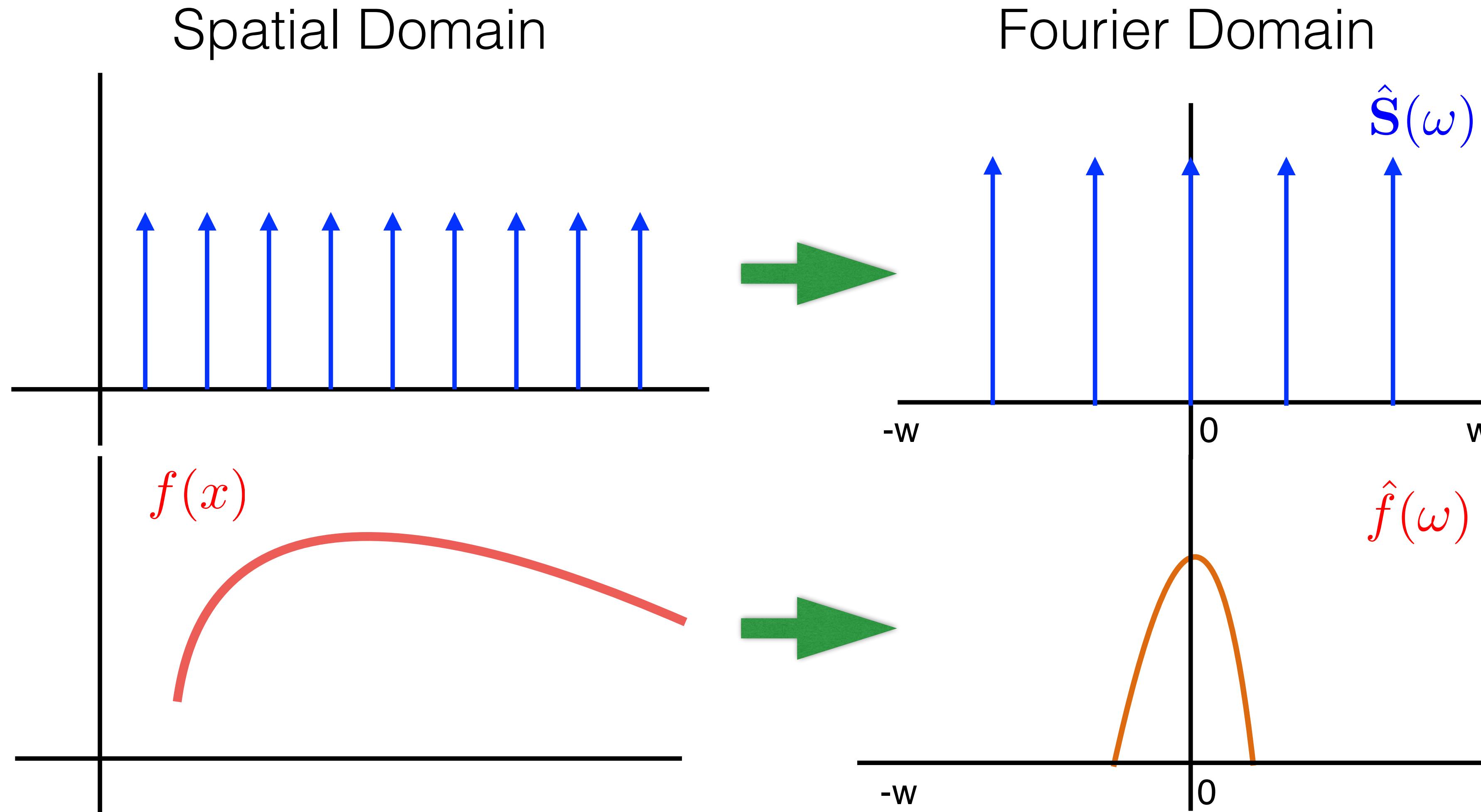
Samples and function in Fourier Domain



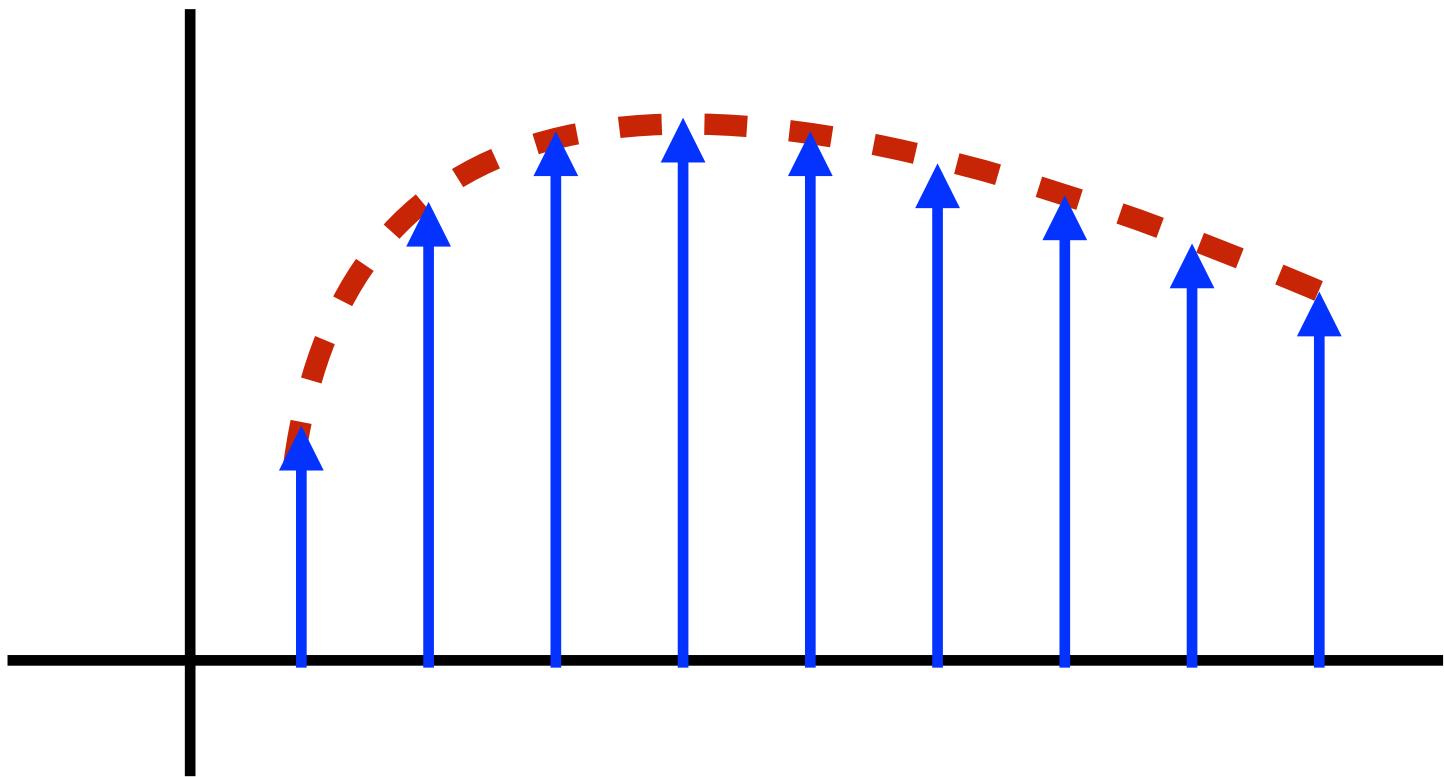
Samples and function in Fourier Domain



Samples and function in Fourier Domain

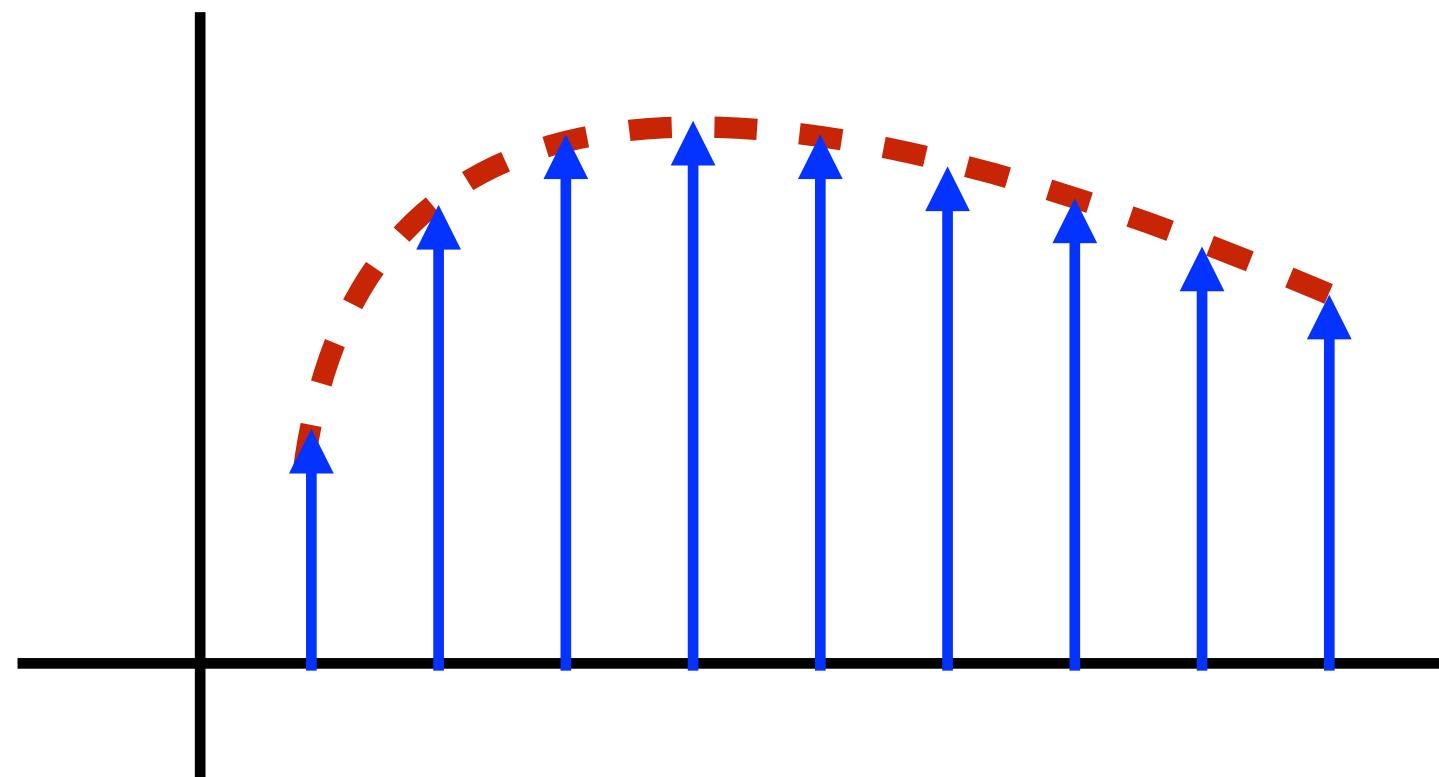


Sampling in Primal Domain is Convolution in Fourier Domain



$$f(x) \mathbf{S}(x)$$

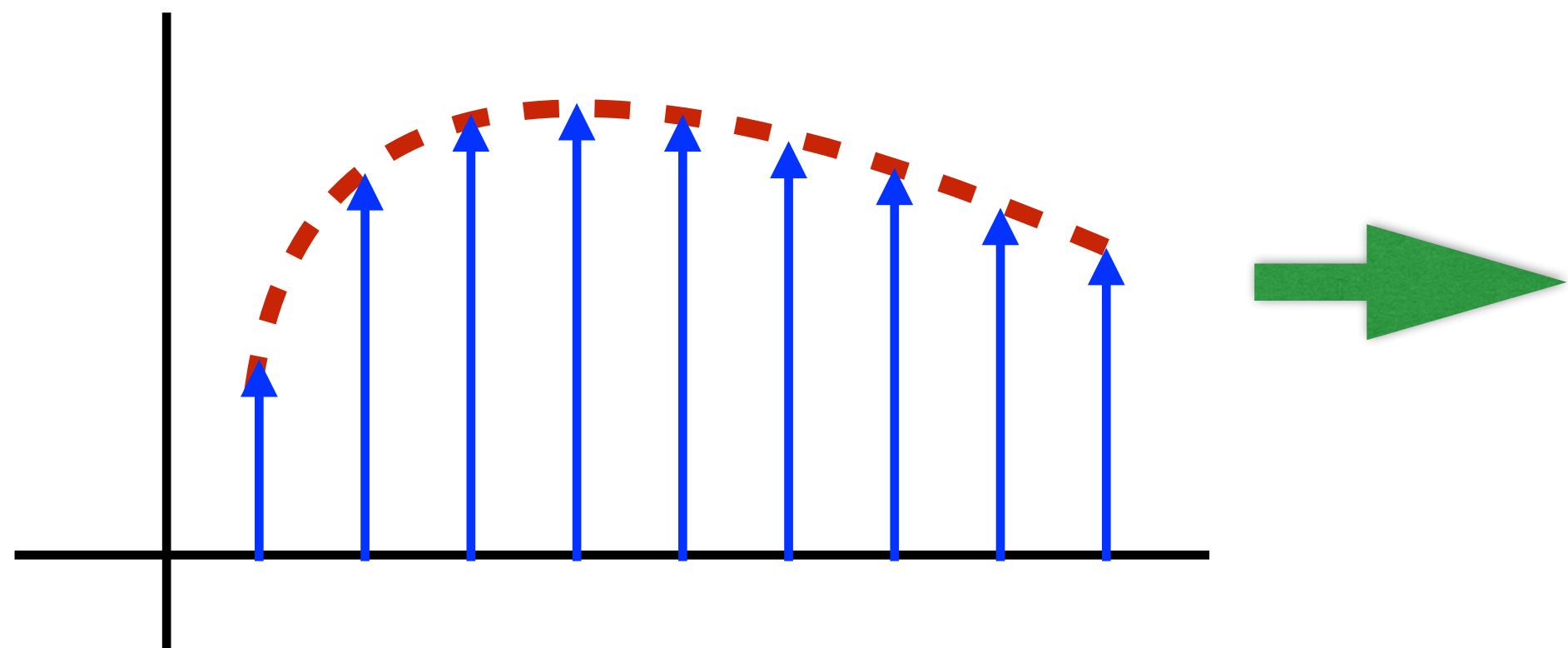
Sampling in Primal Domain is Convolution in Fourier Domain



$f(x) \mathbf{S}(x)$

Fredo Durand [2011]

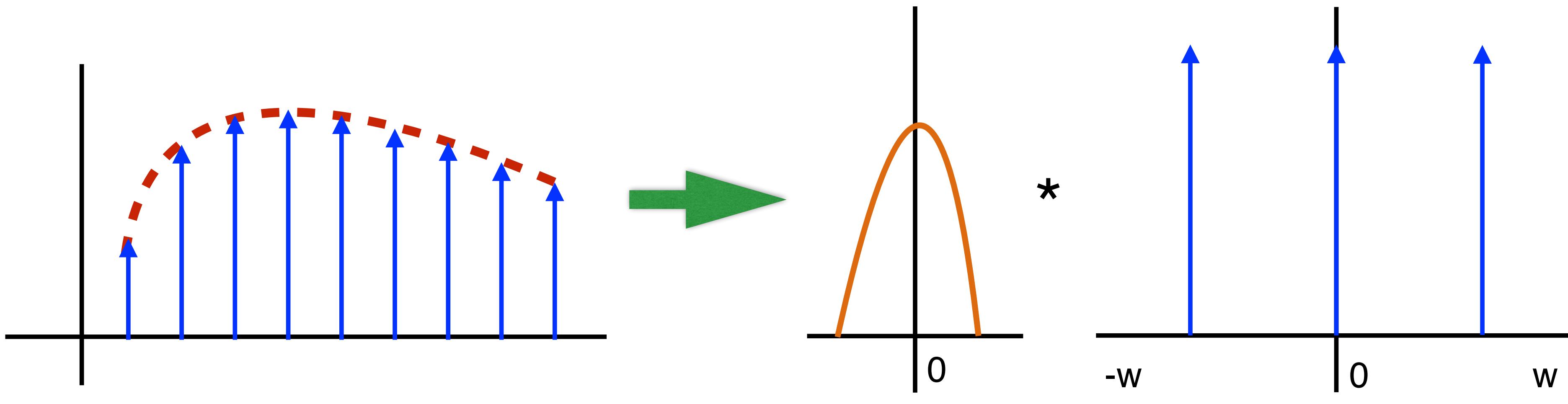
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Sampling in Primal Domain is Convolution in Fourier Domain



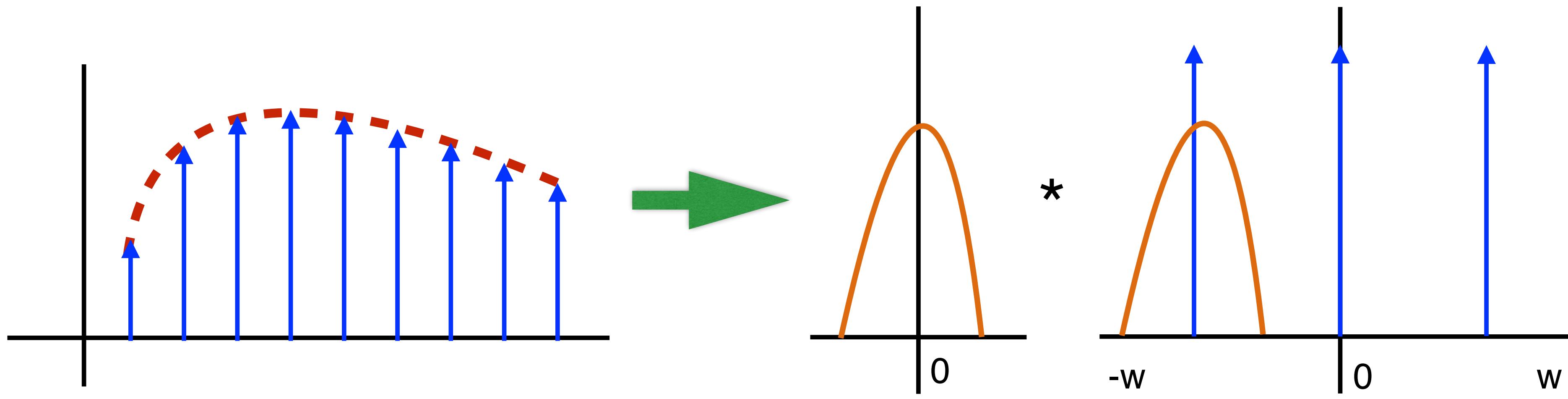
$$f(x) \mathbf{S}(x)$$

$$\hat{f}(\omega) \otimes \hat{\mathbf{S}}(\omega)$$

Fredo Durand [2011]



Sampling in Primal Domain is Convolution in Fourier Domain

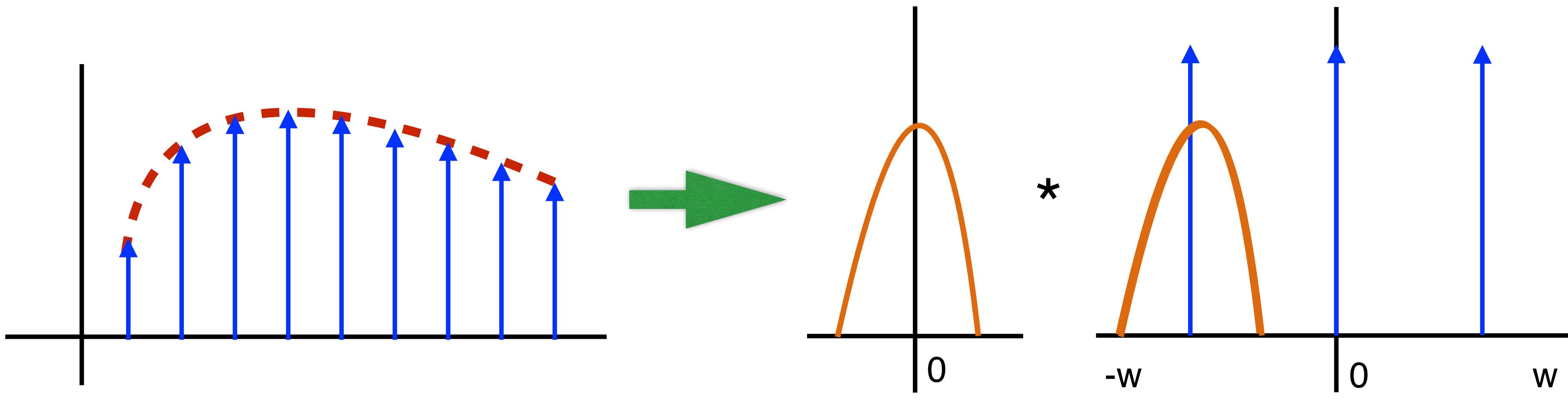


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Fredo Durand [2011]

Sampling in Primal Domain is Convolution in Fourier Domain

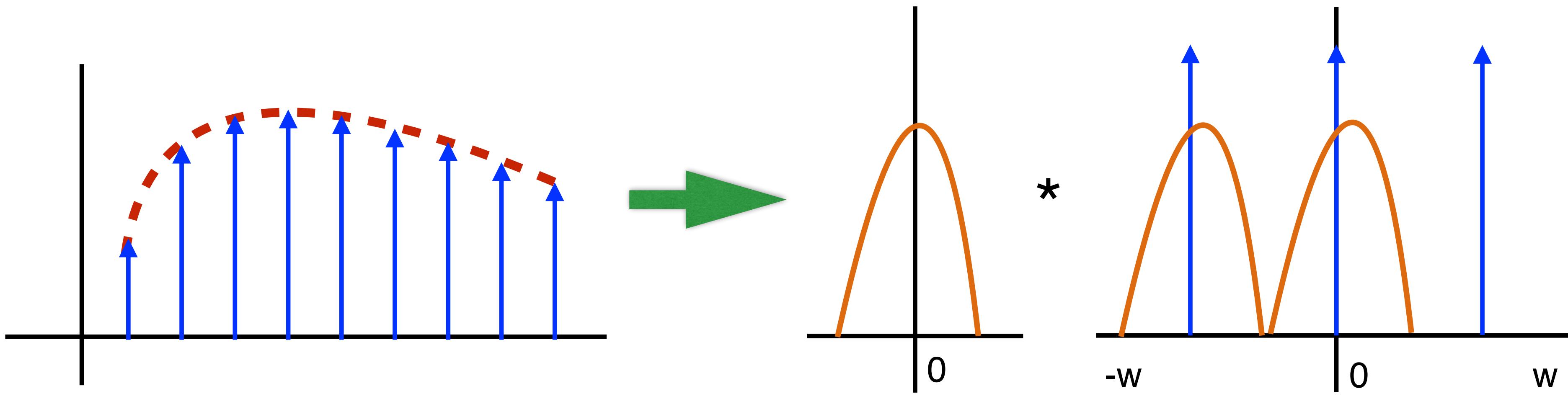


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Fredo Durand [2011]

Sampling in Primal Domain is Convolution in Fourier Domain



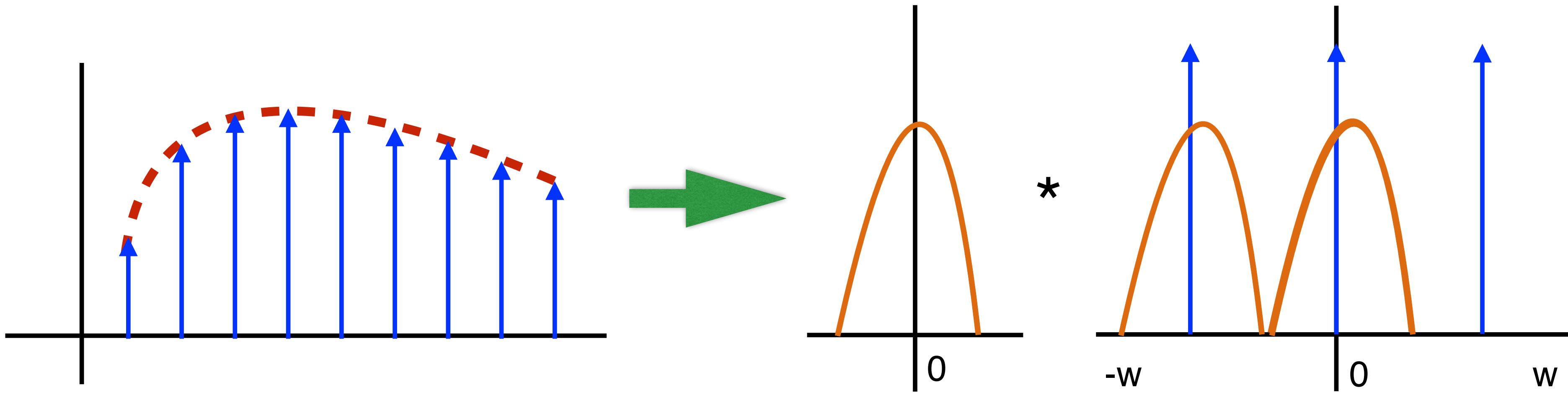
$$f(x) \mathbf{S}(x)$$

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Fredo Durand [2011]



Sampling in Primal Domain is Convolution in Fourier Domain



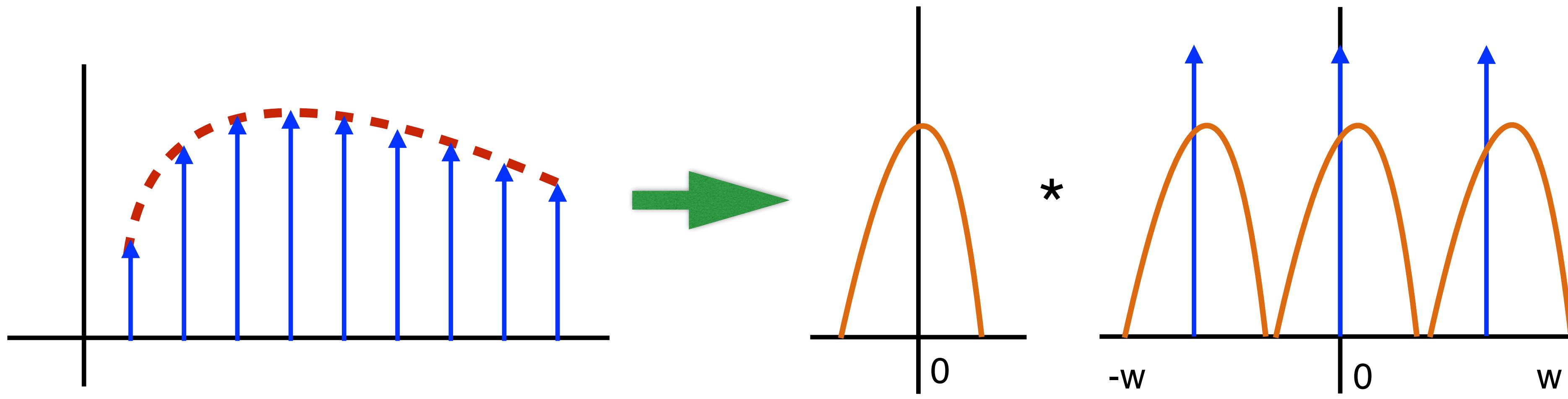
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Fredo Durand [2011]



Sampling in Primal Domain is Convolution in Fourier Domain

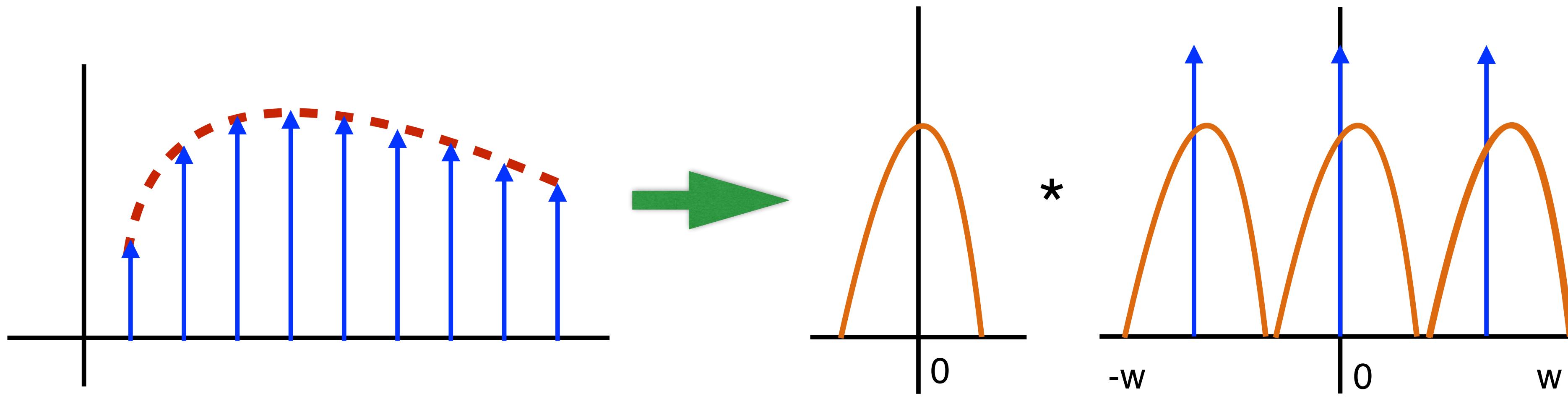


$$f(x) \mathbf{S}(x)$$

$$\hat{f}(\omega) \otimes \hat{\mathbf{S}}(\omega)$$

Fredo Durand [2011]

Sampling in Primal Domain is Convolution in Fourier Domain

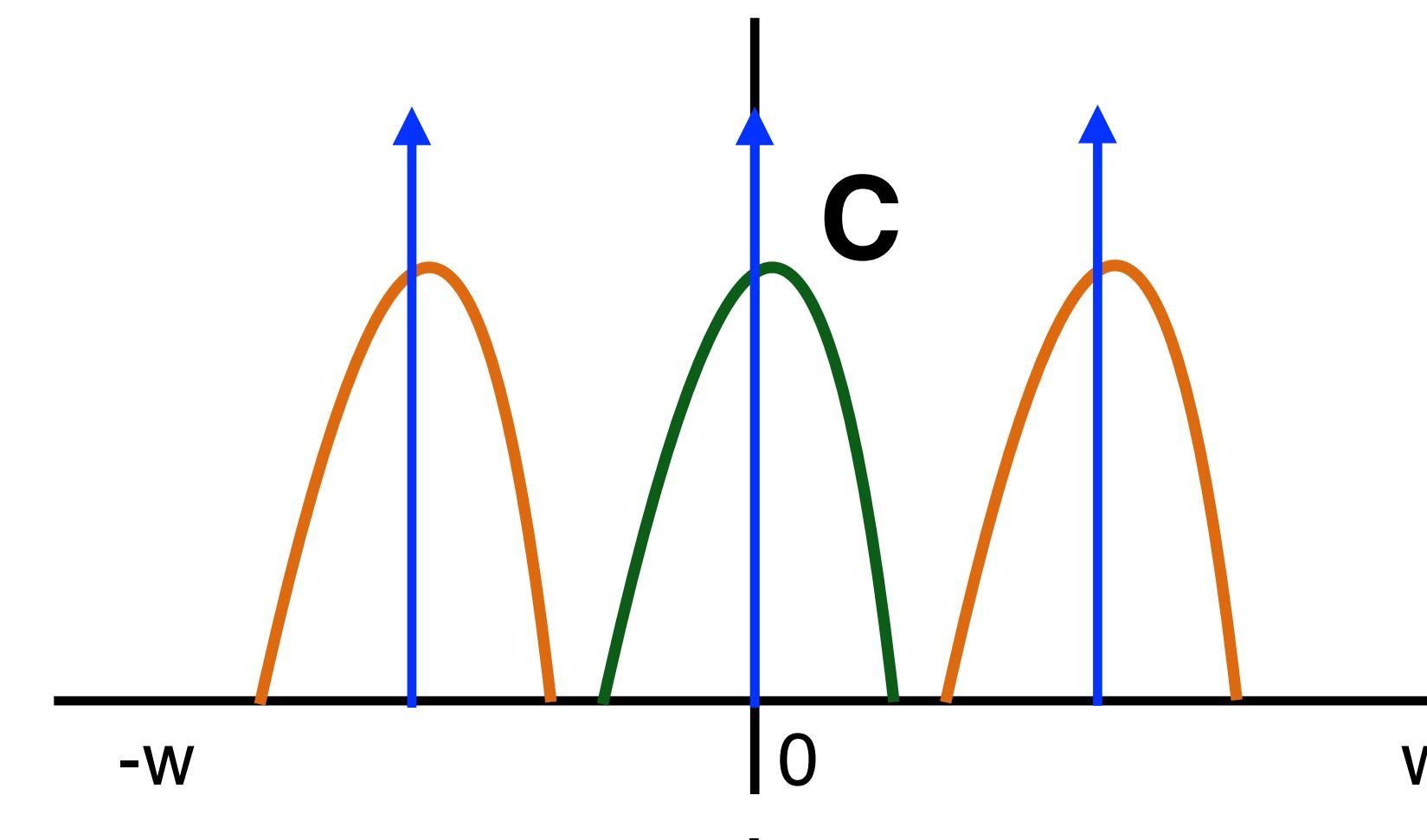
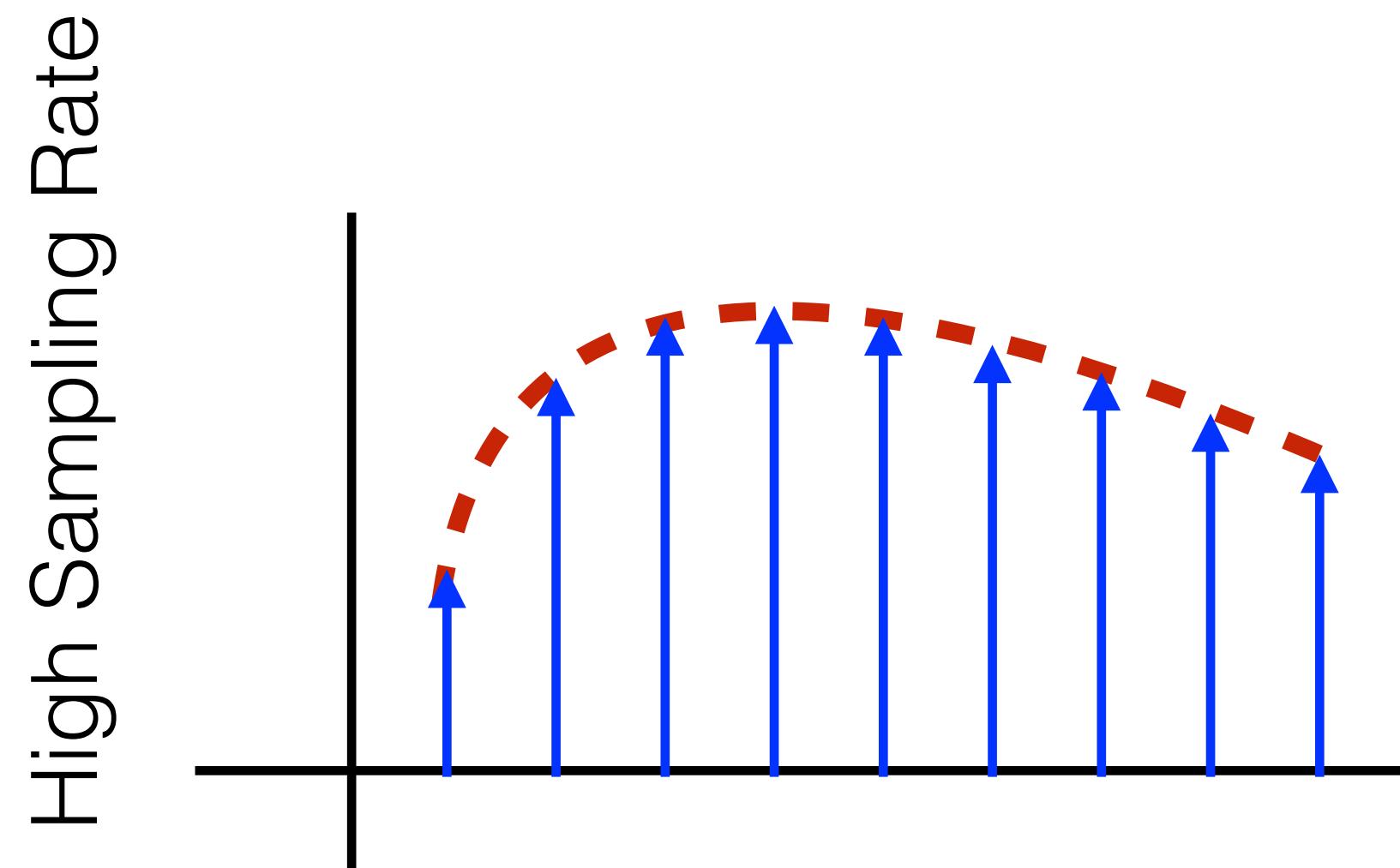


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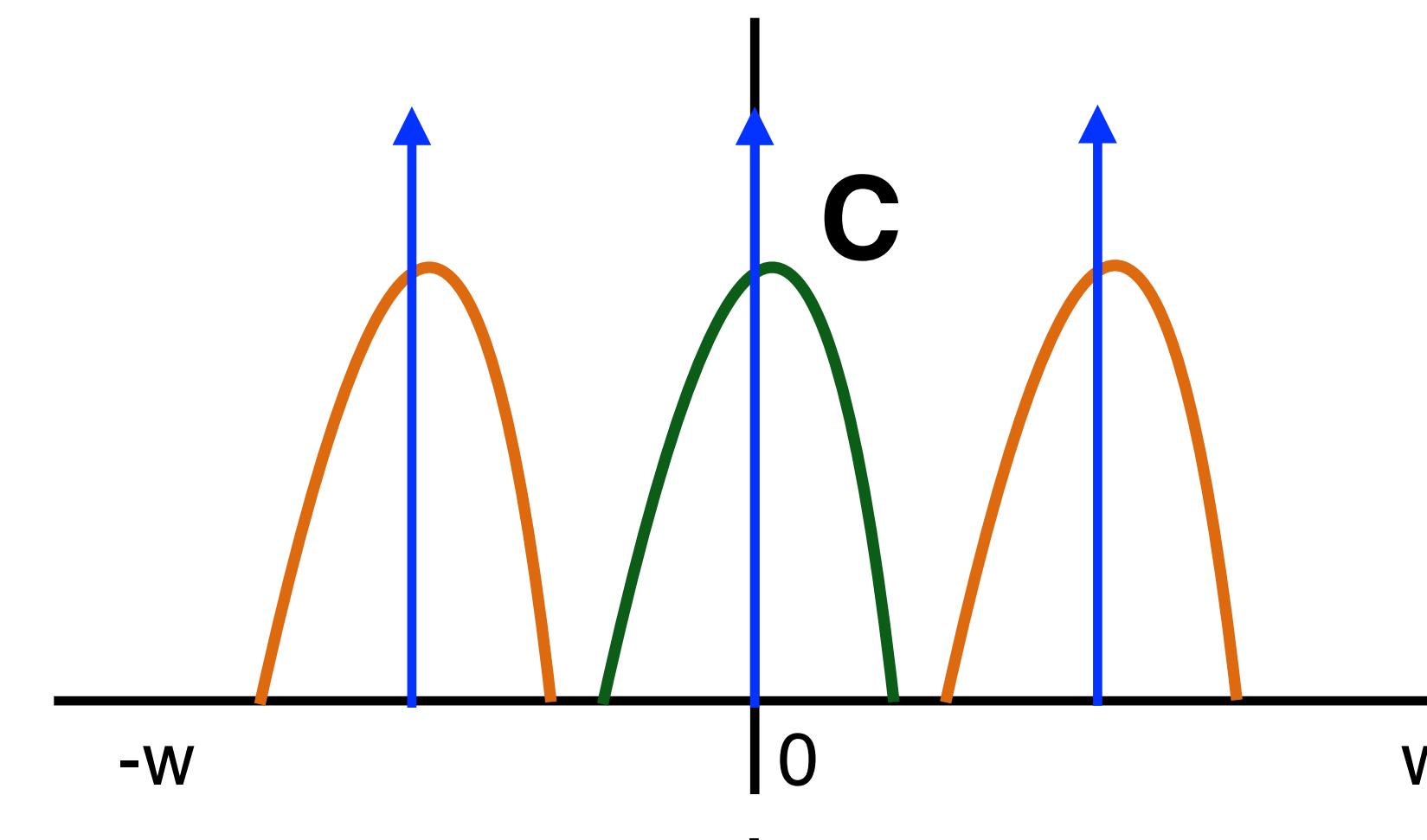
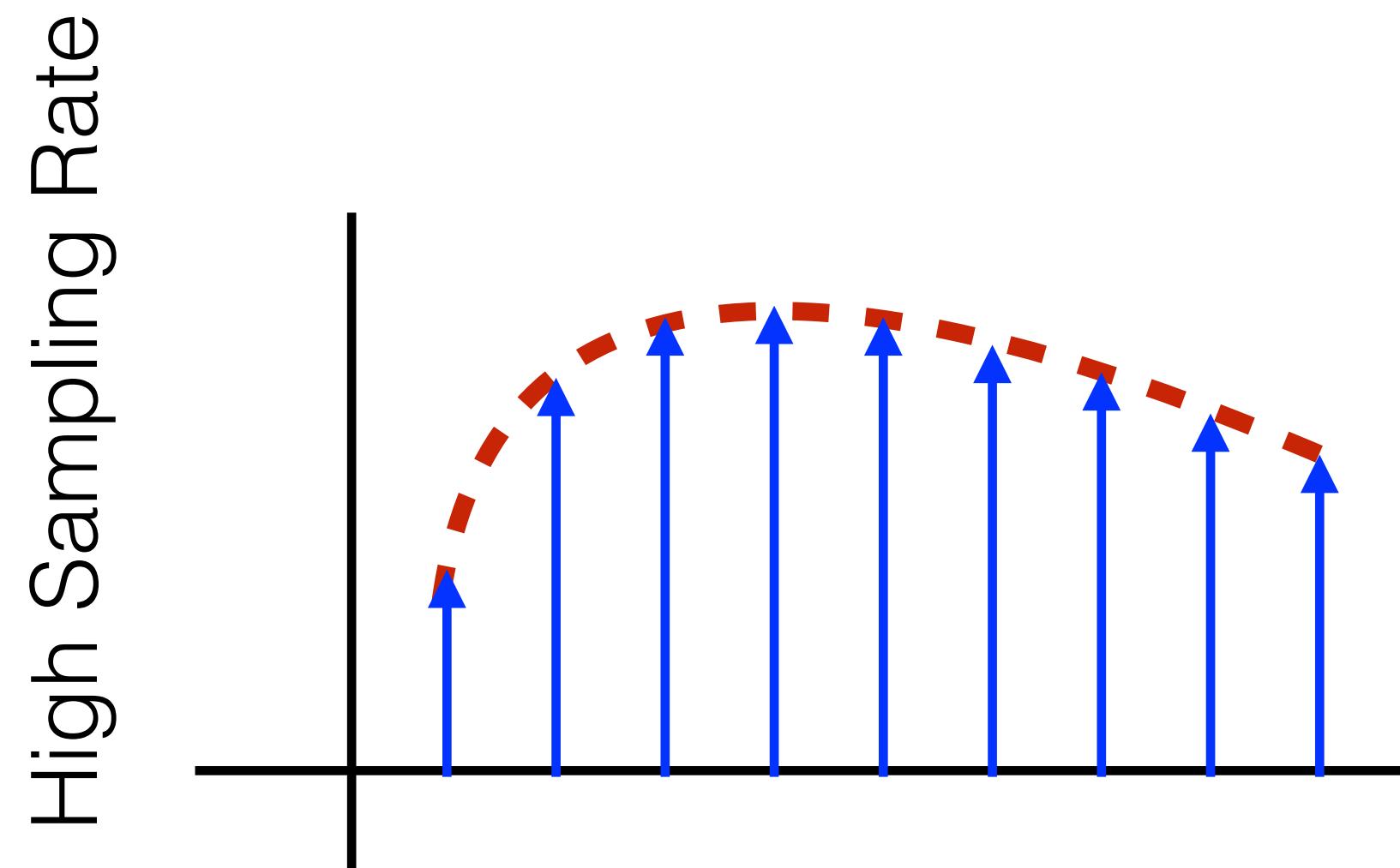
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Fredo Durand [2011]

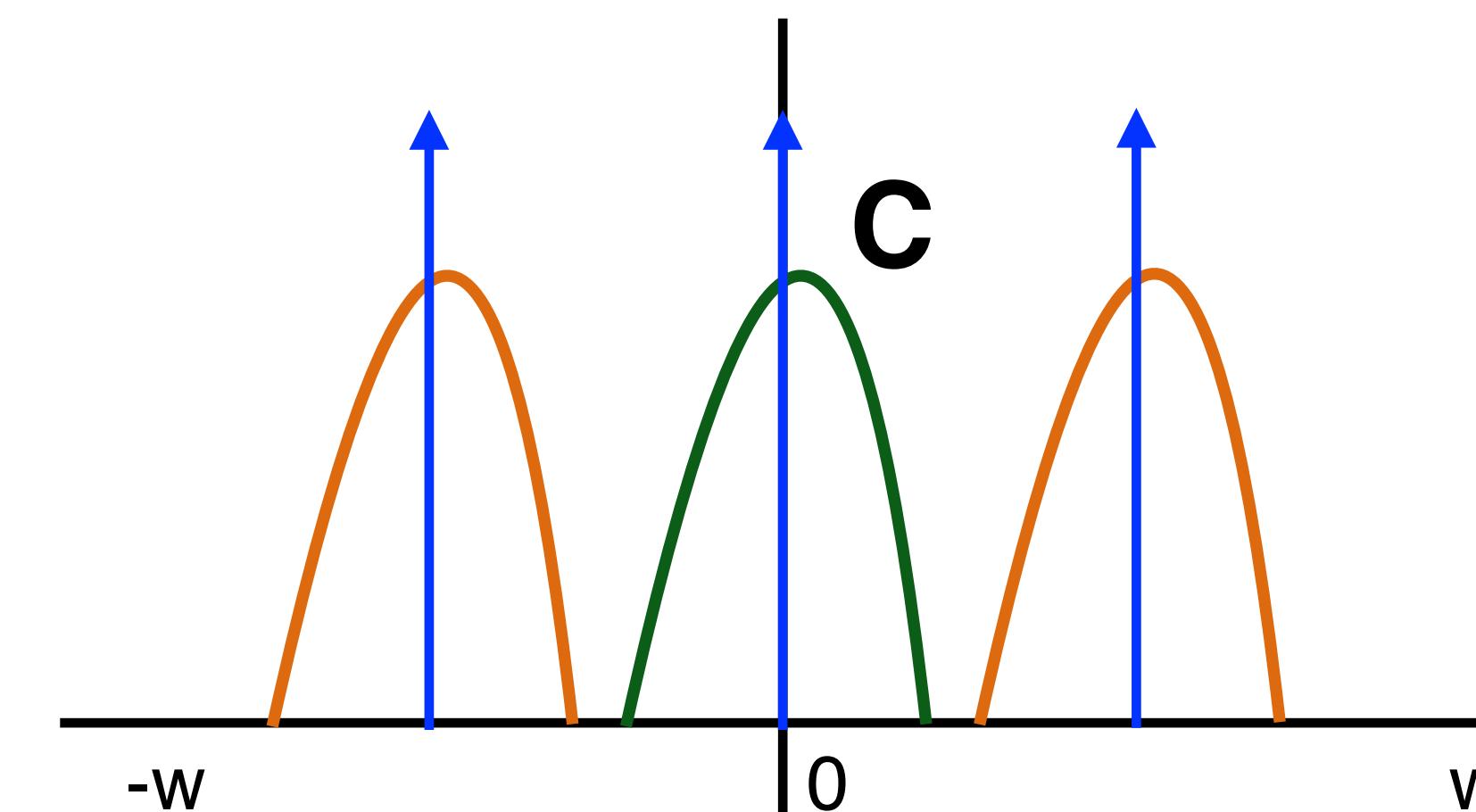
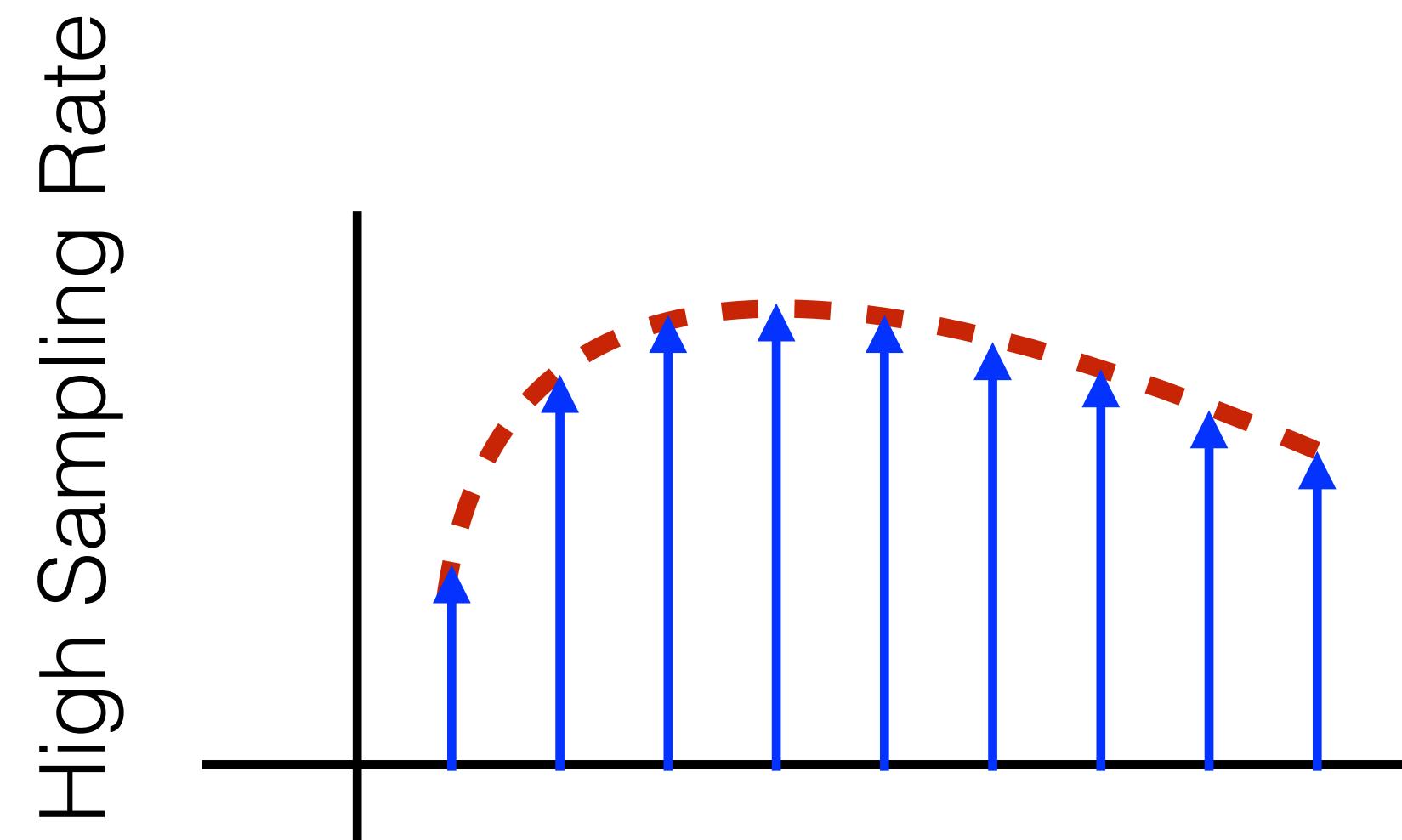
Aliasing in Reconstruction



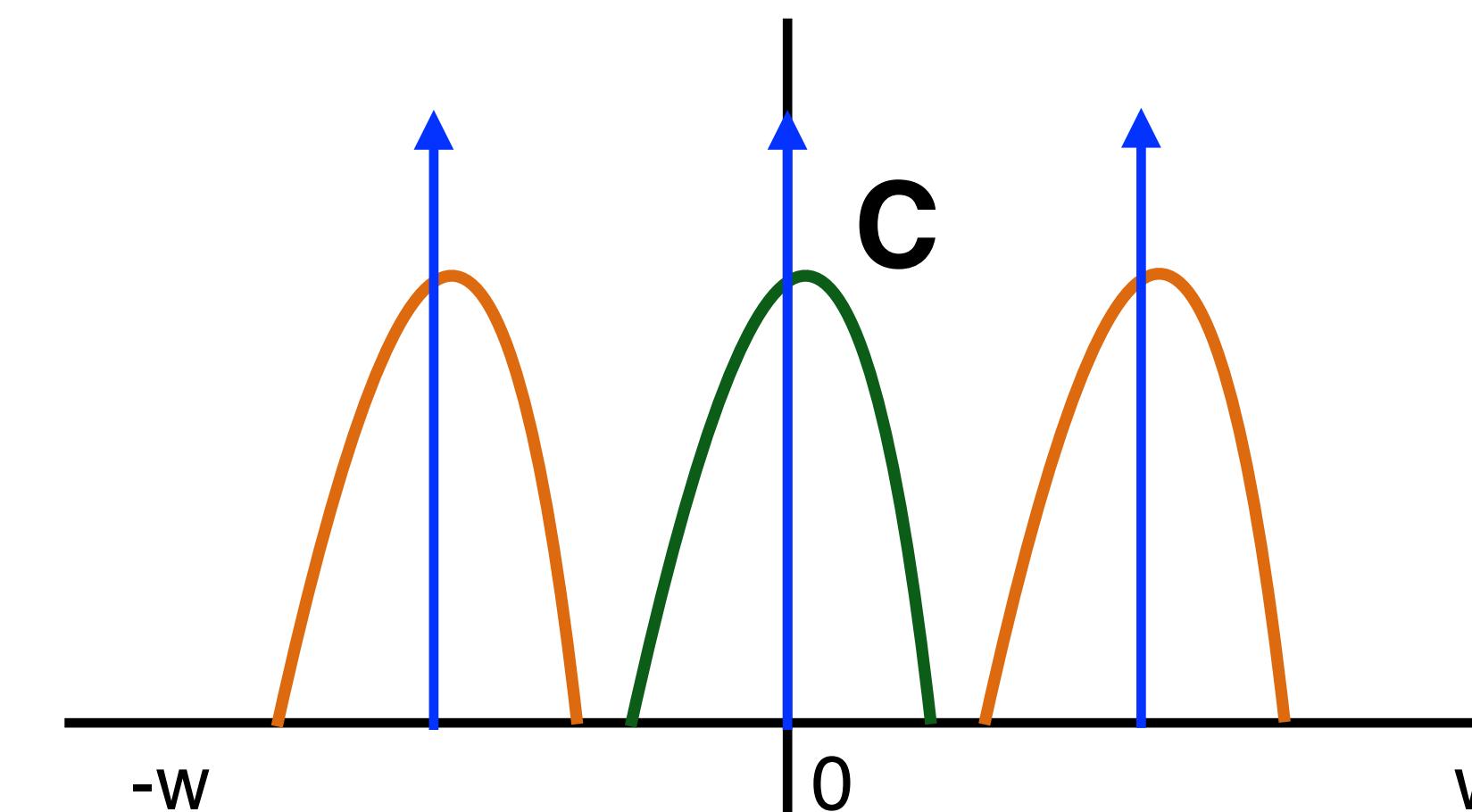
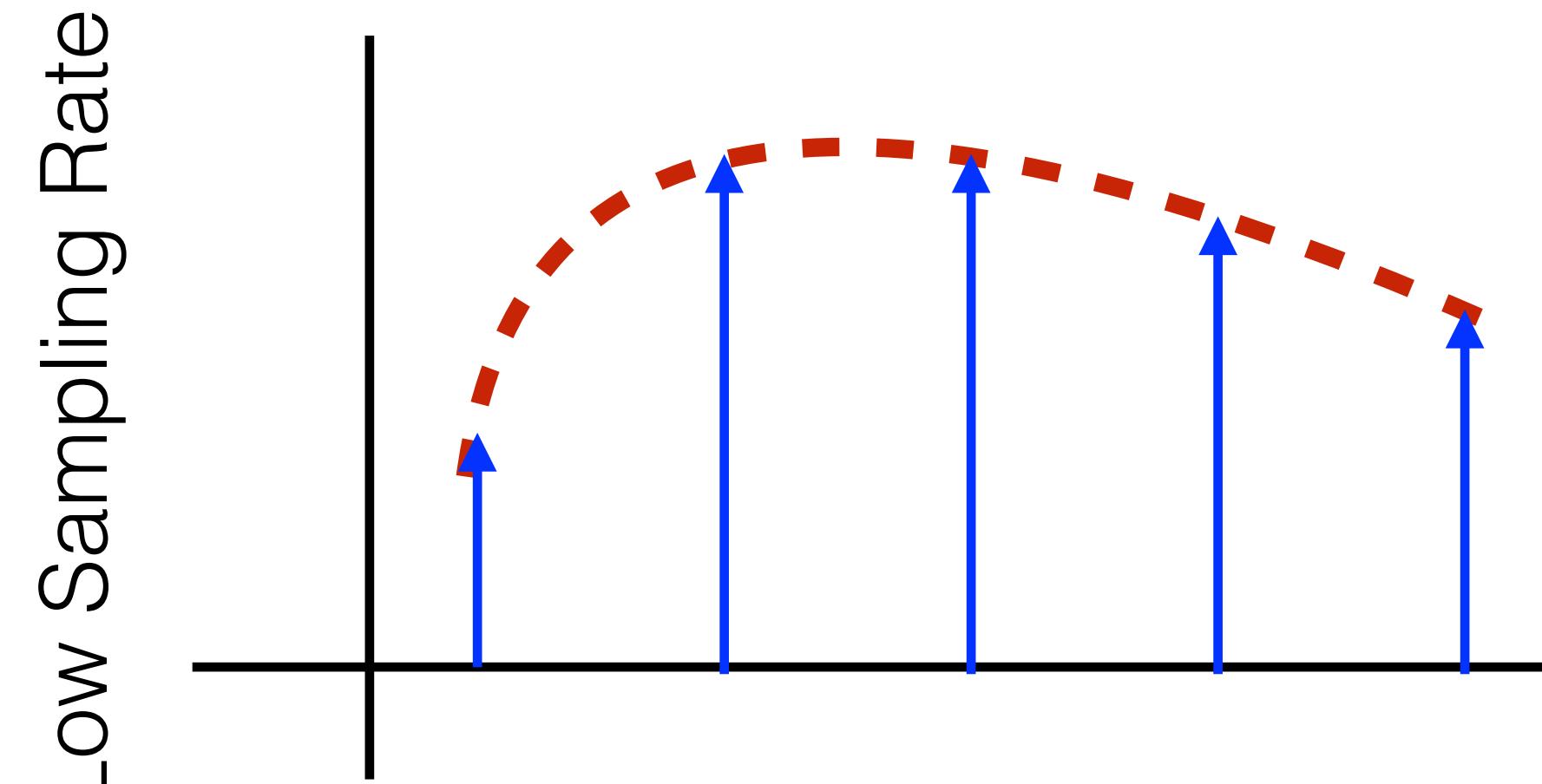
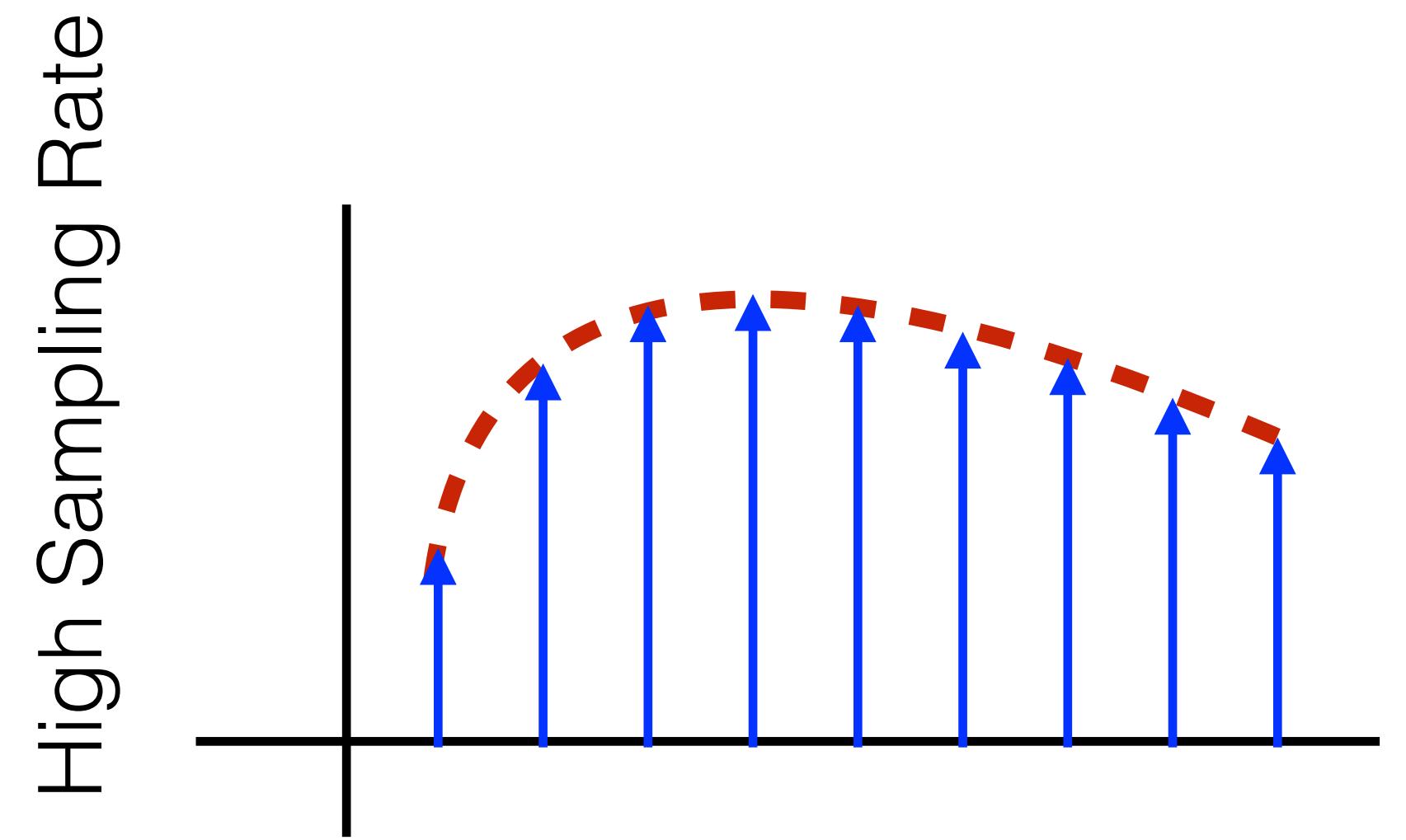
Aliasing in Reconstruction



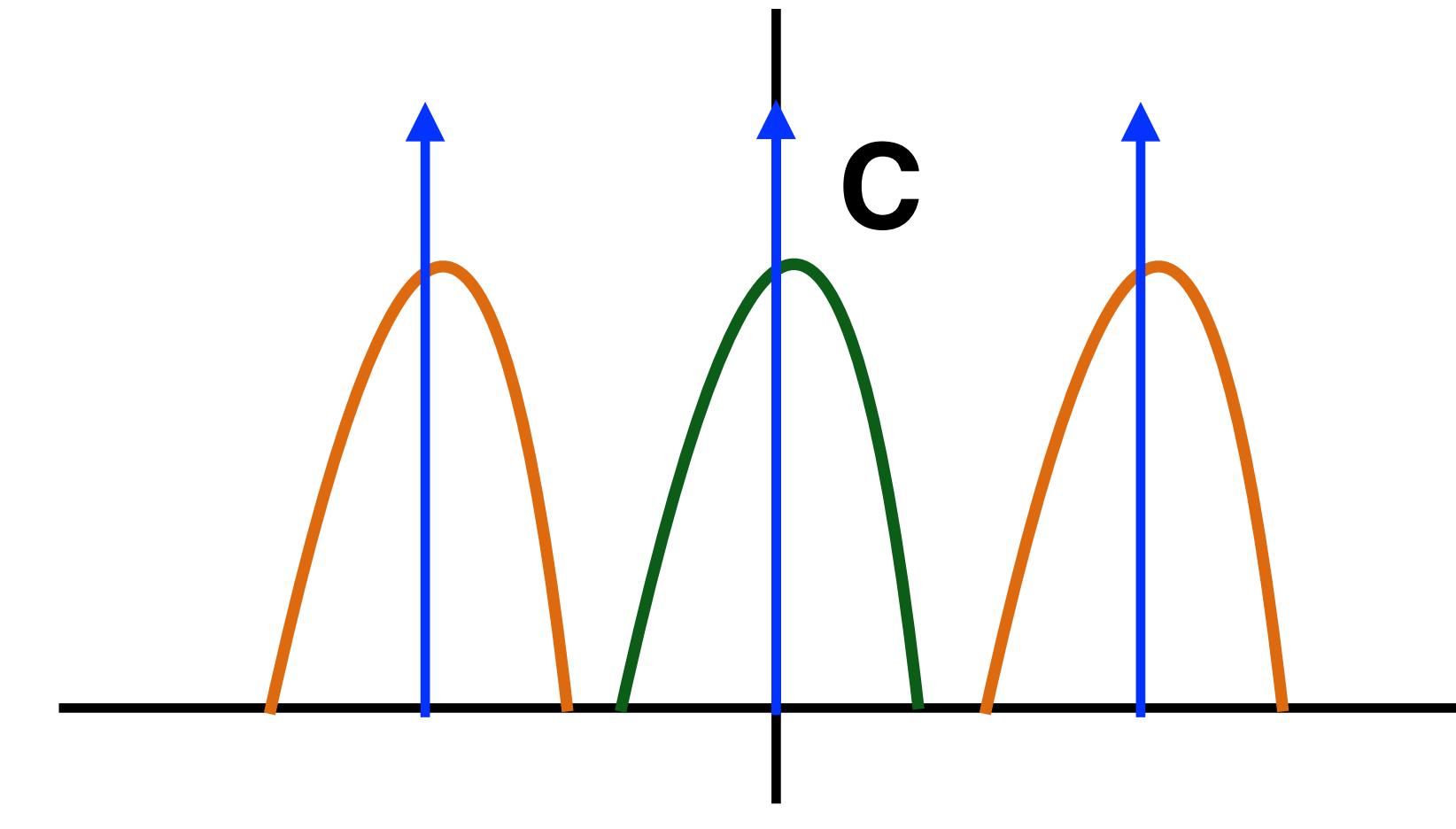
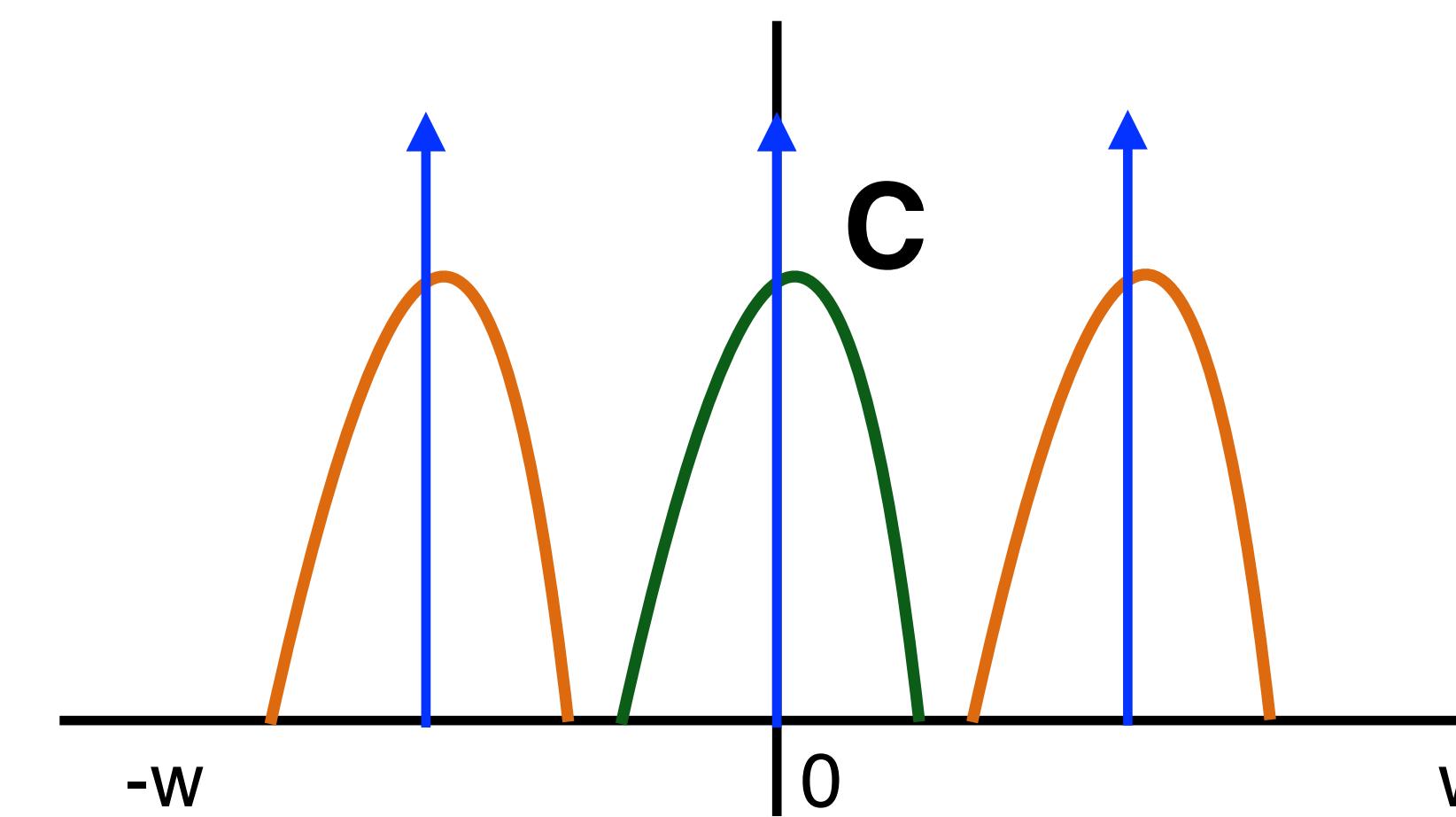
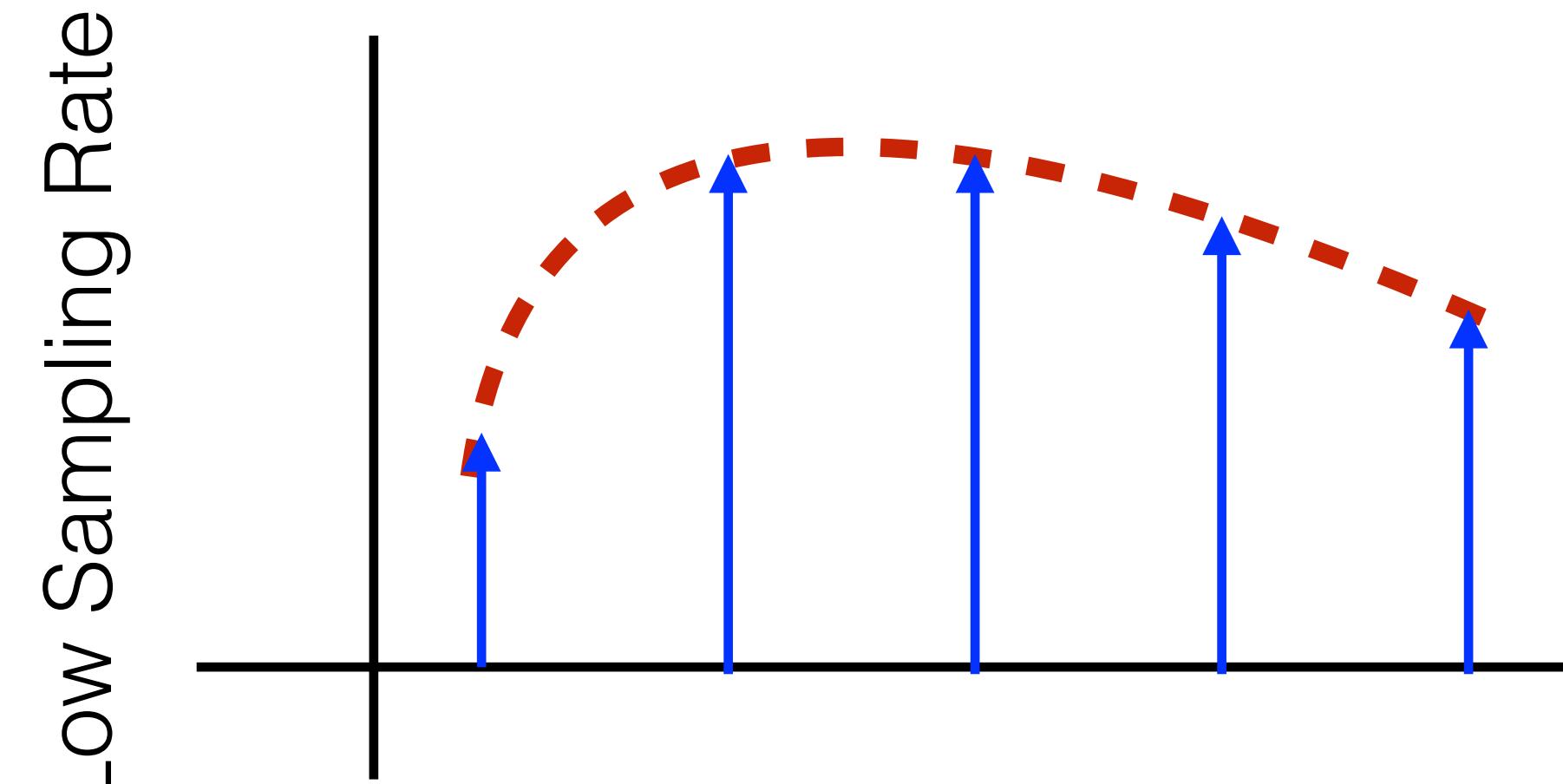
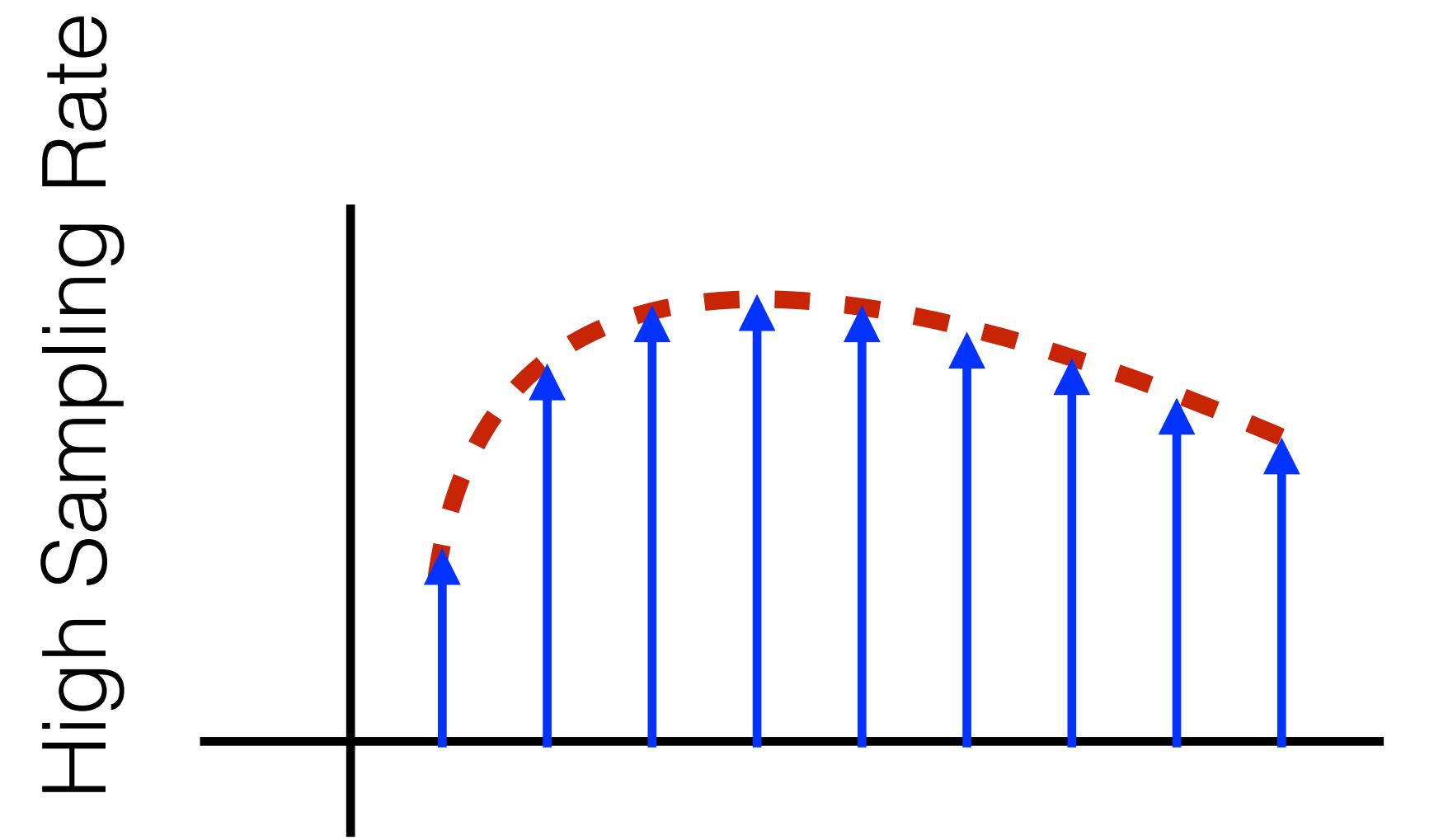
Aliasing in Reconstruction



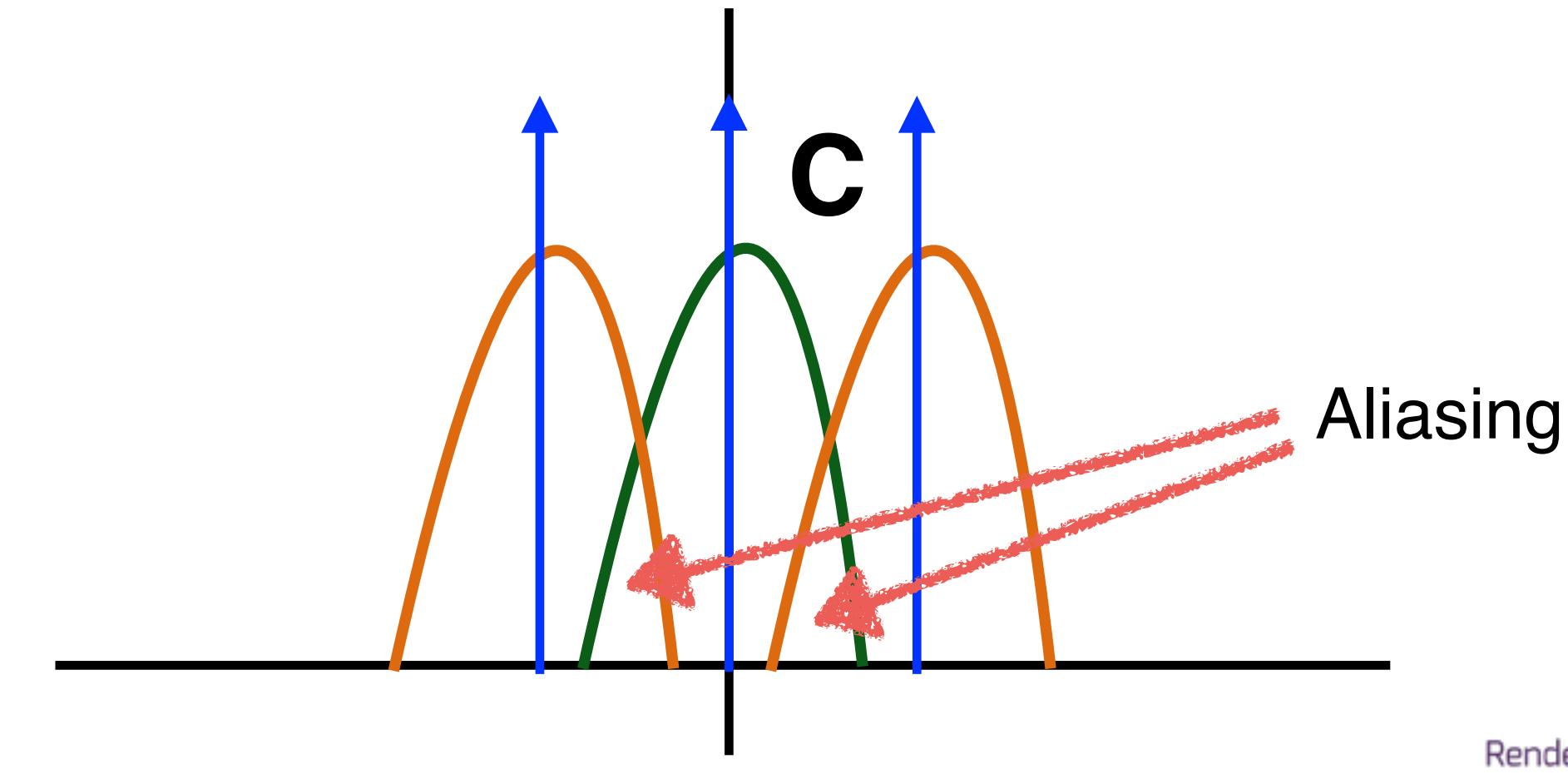
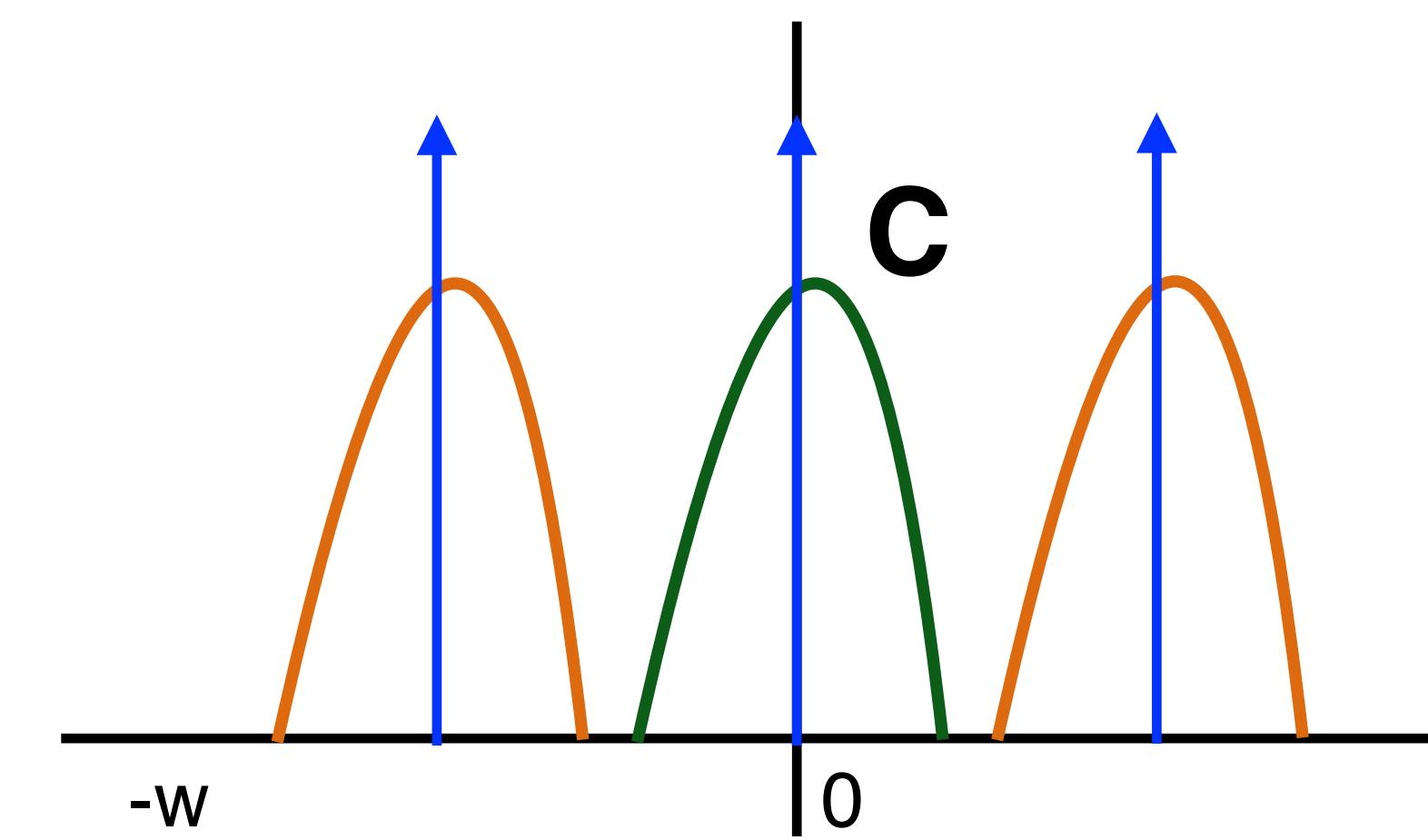
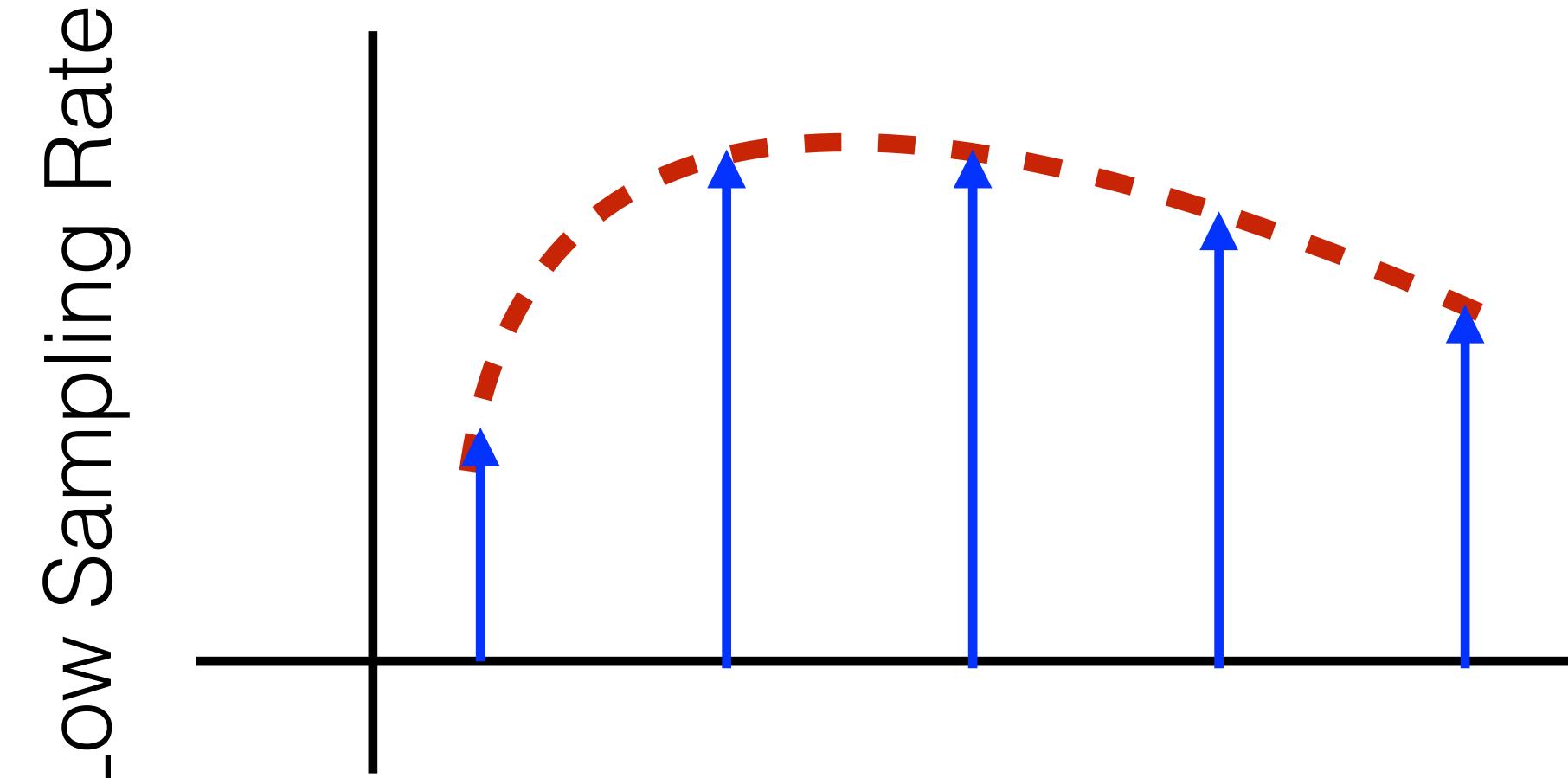
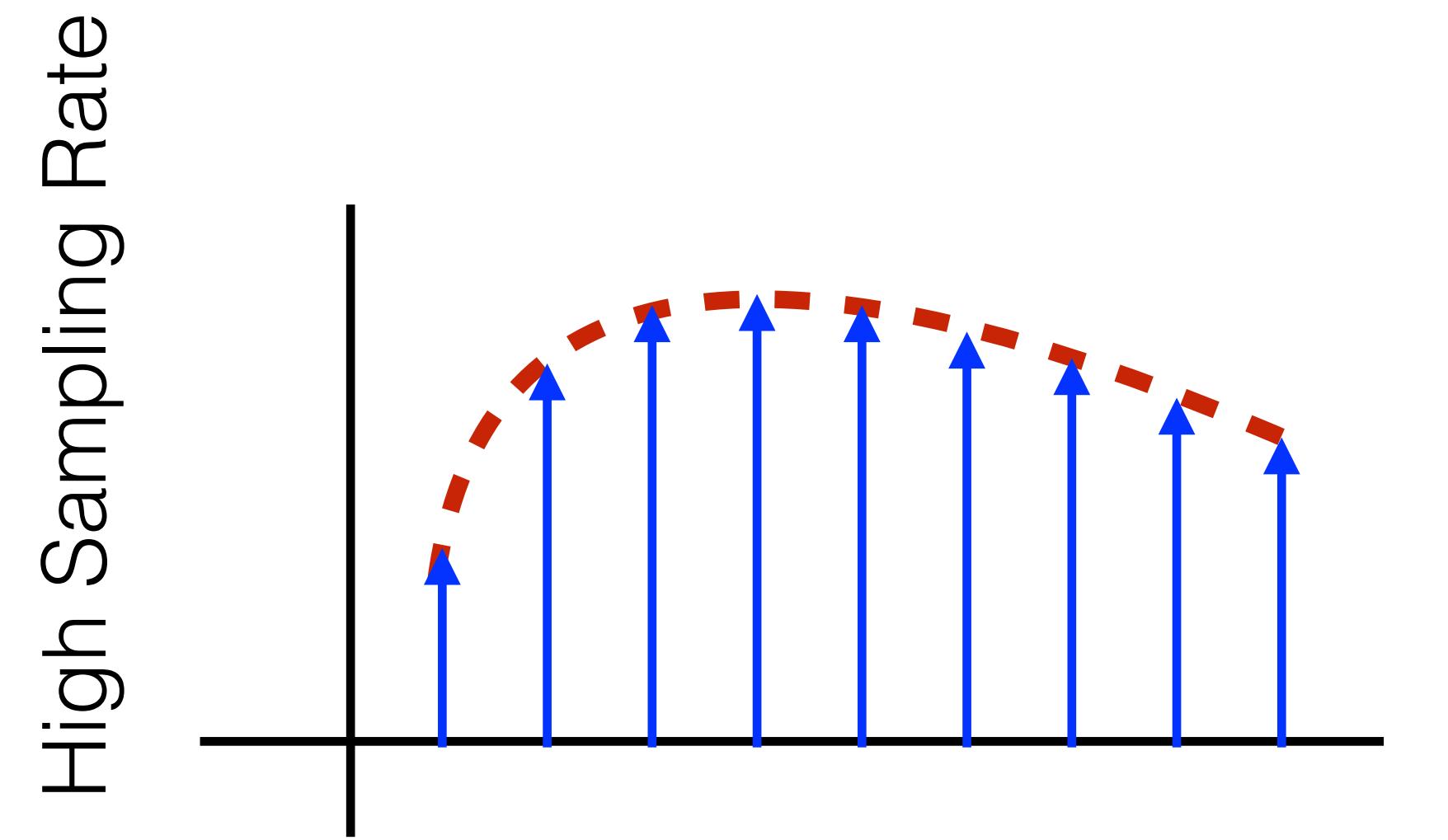
Aliasing in Reconstruction



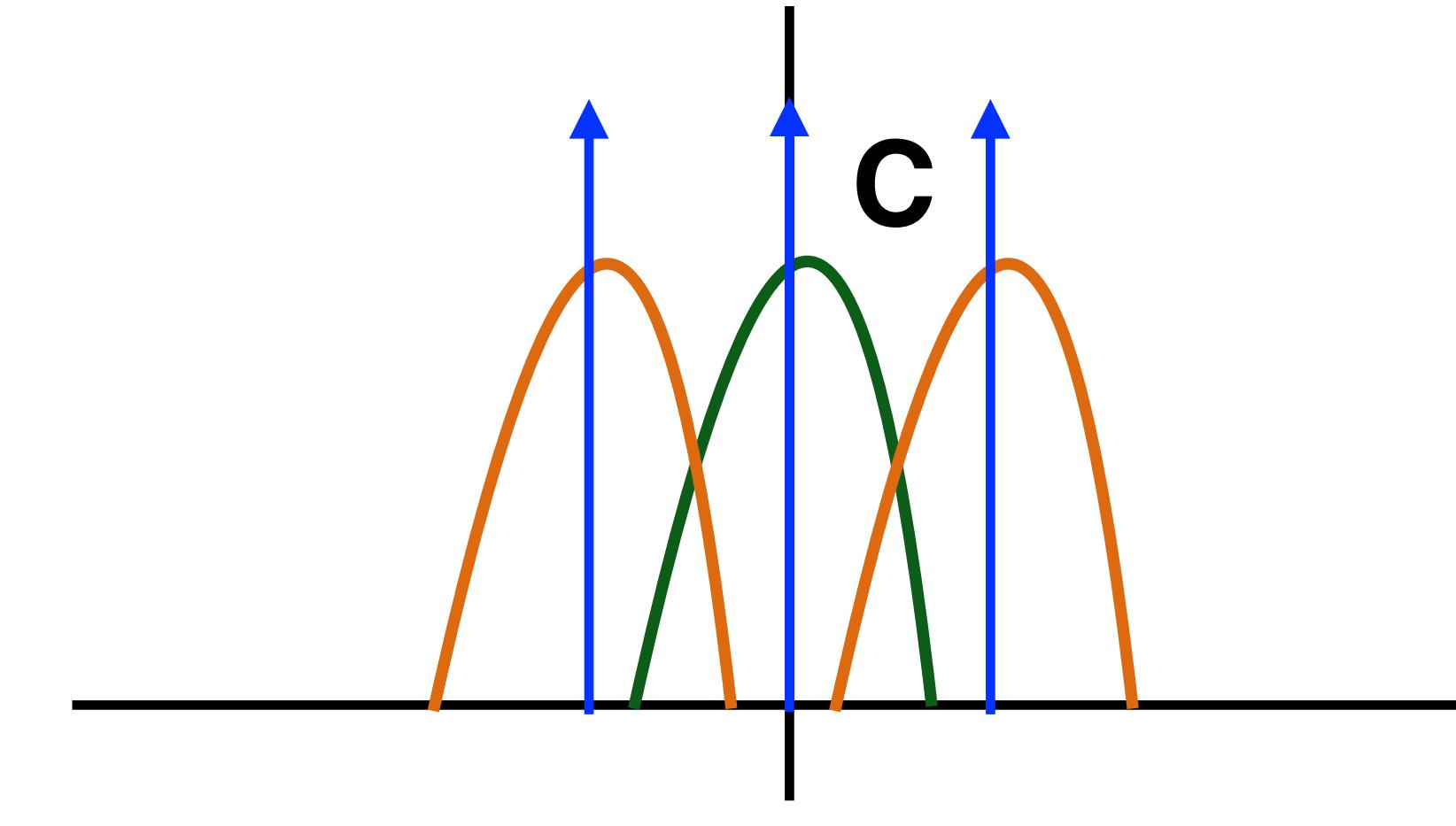
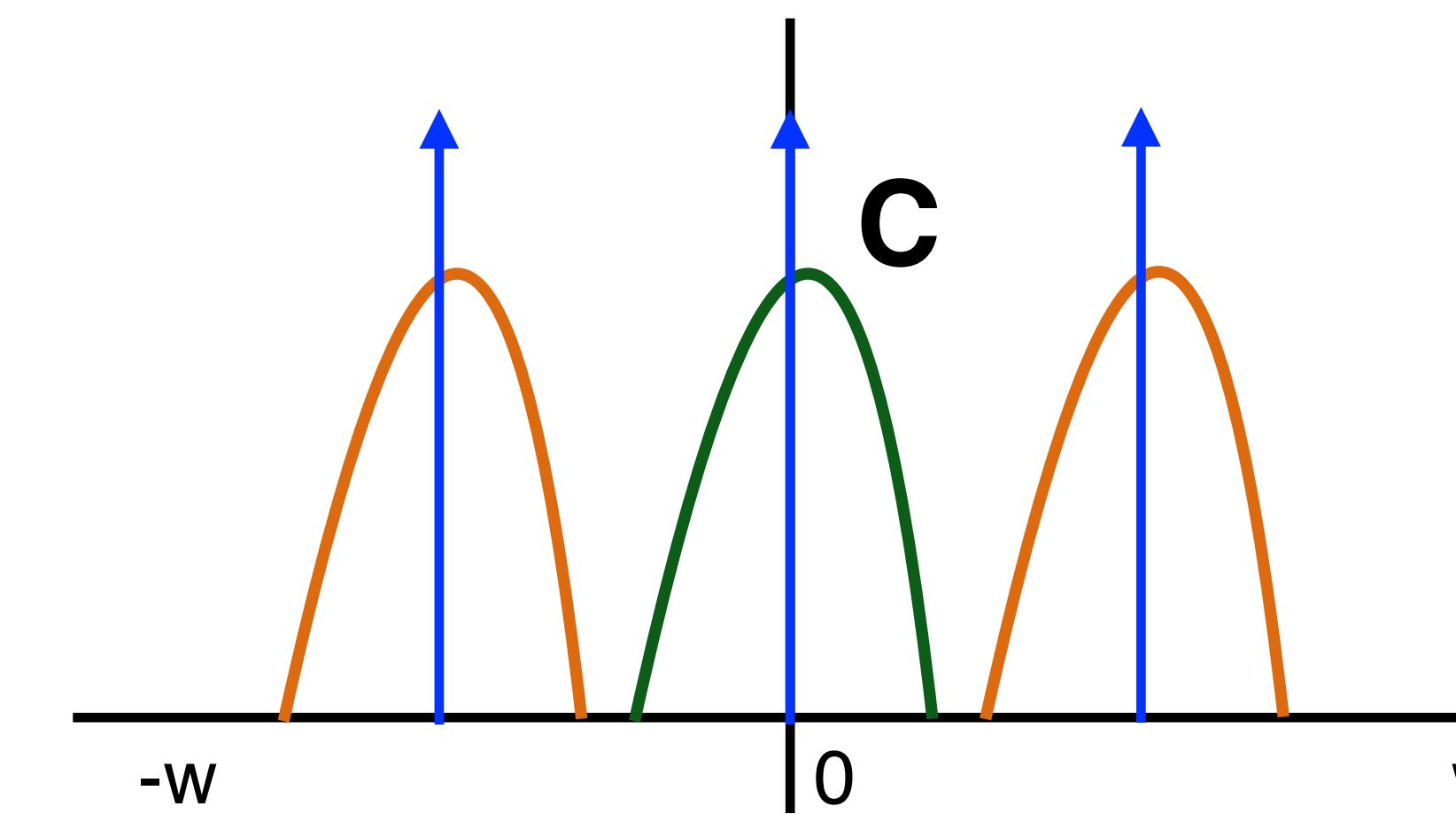
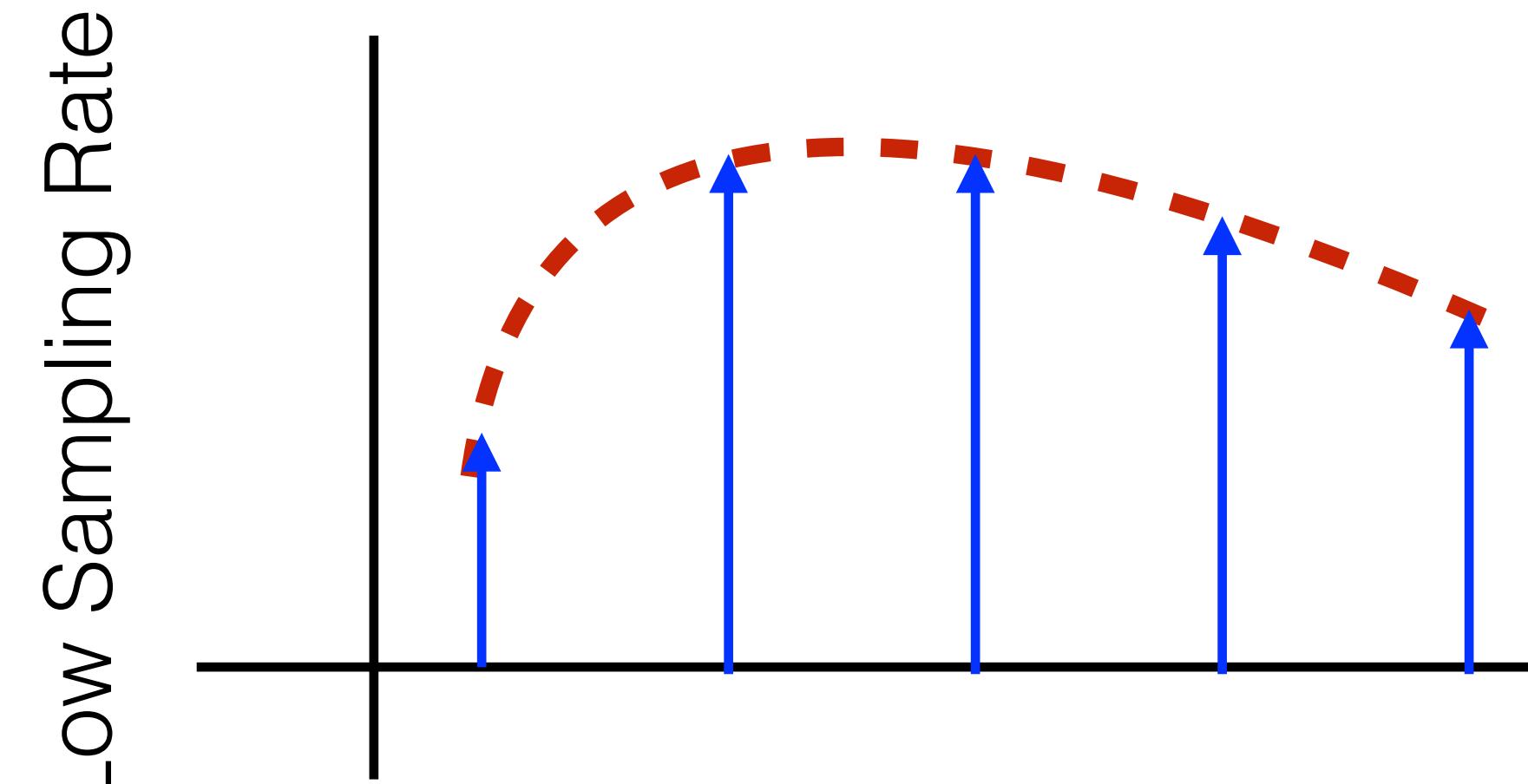
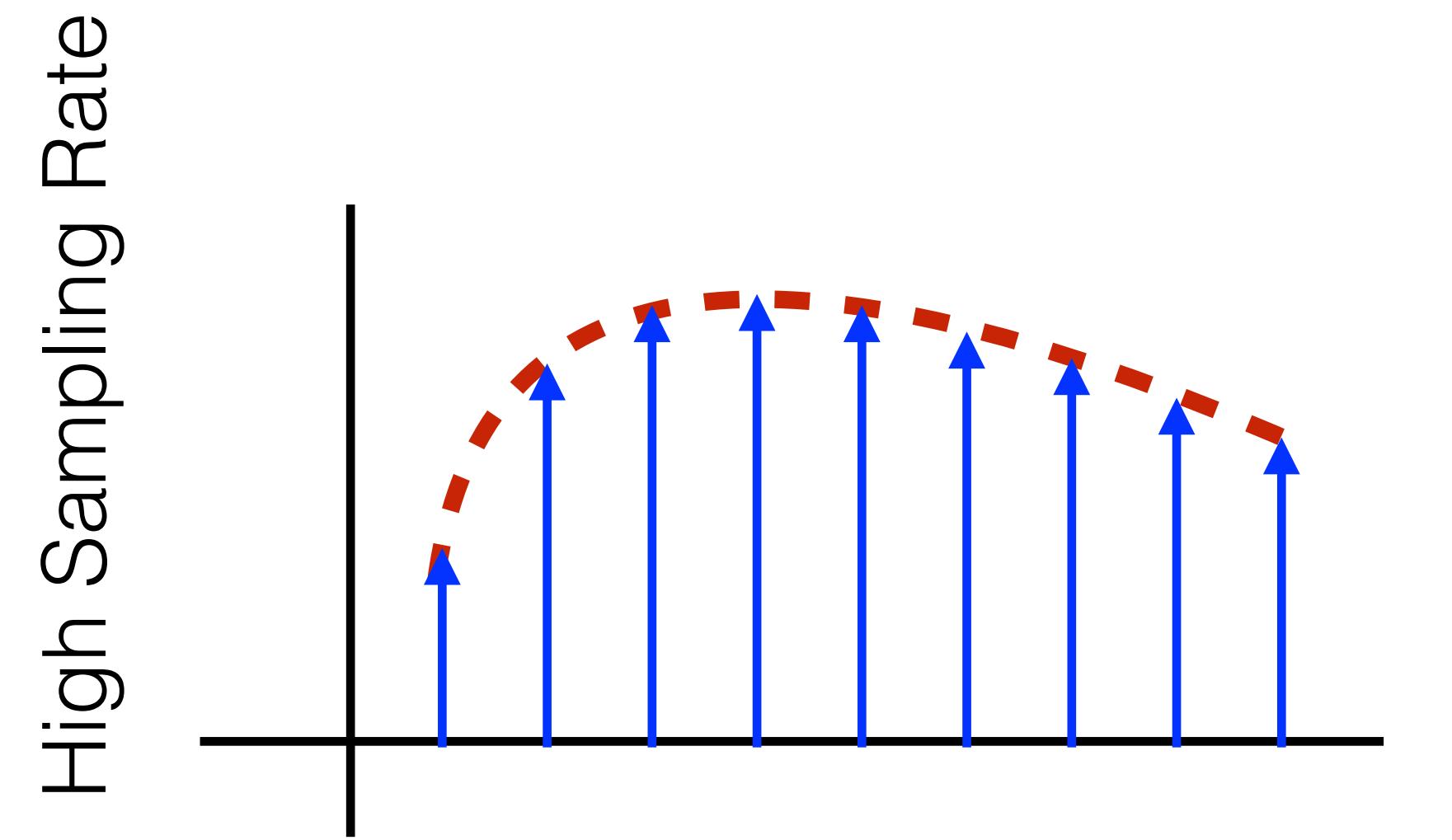
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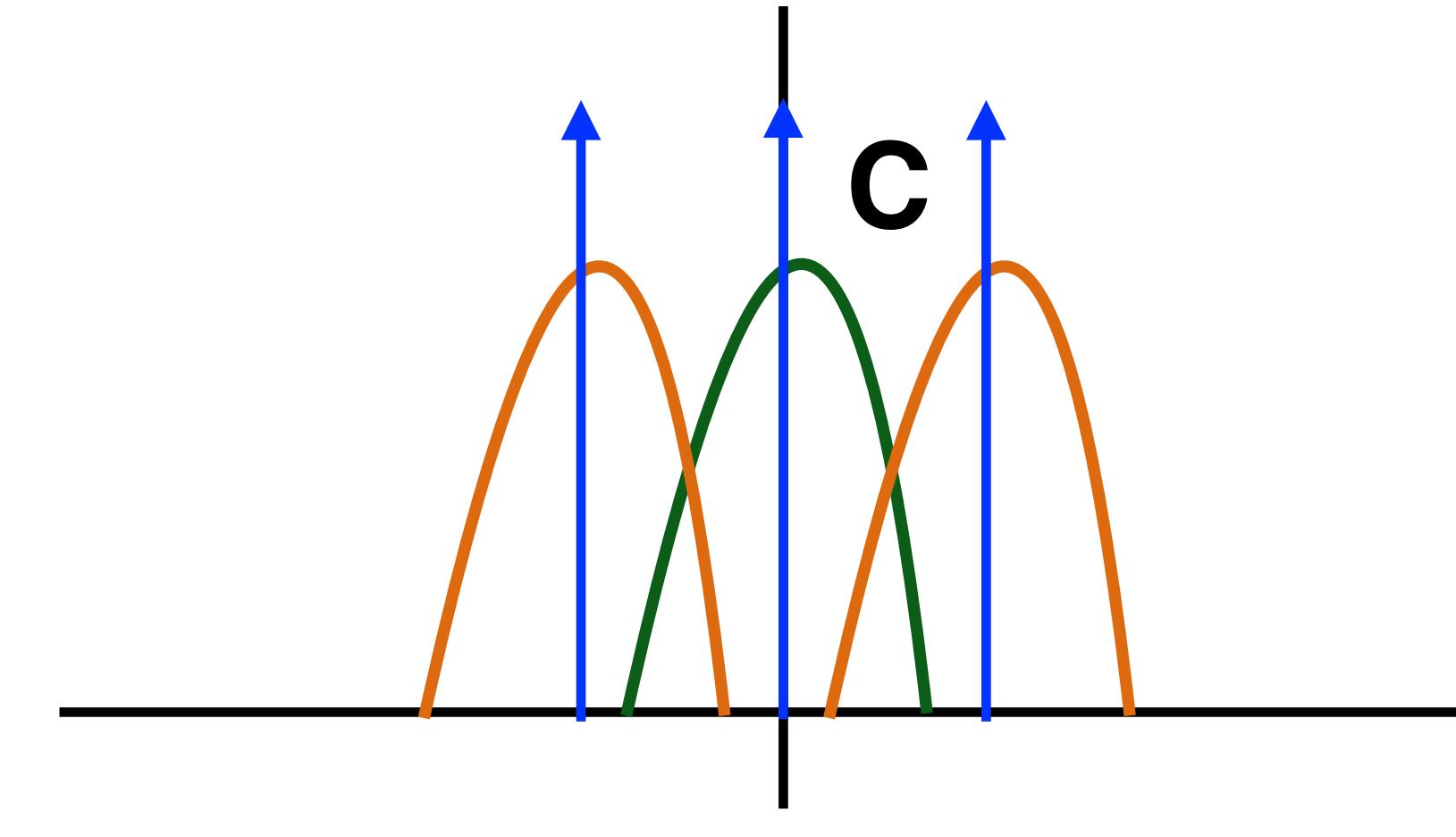
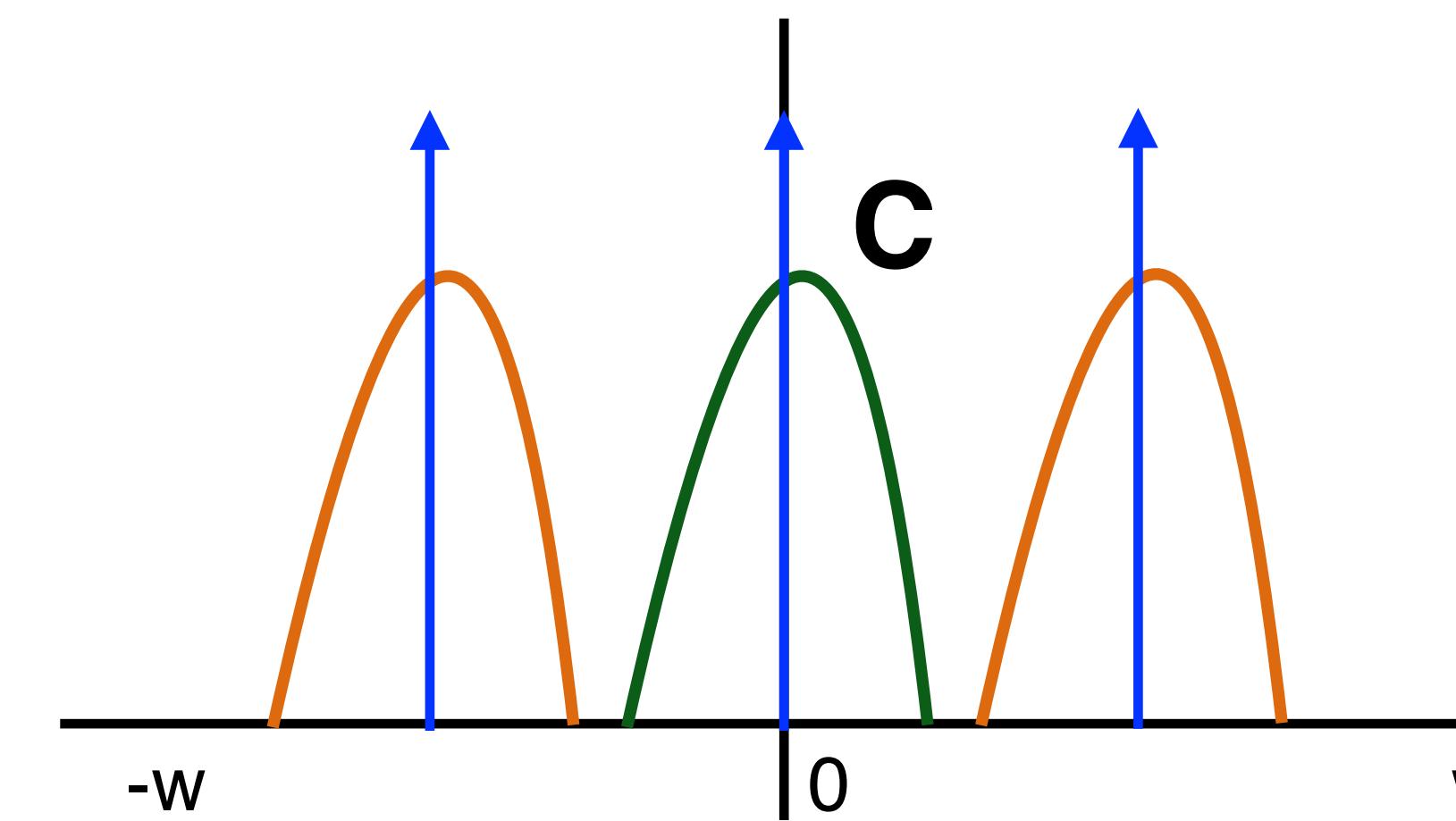
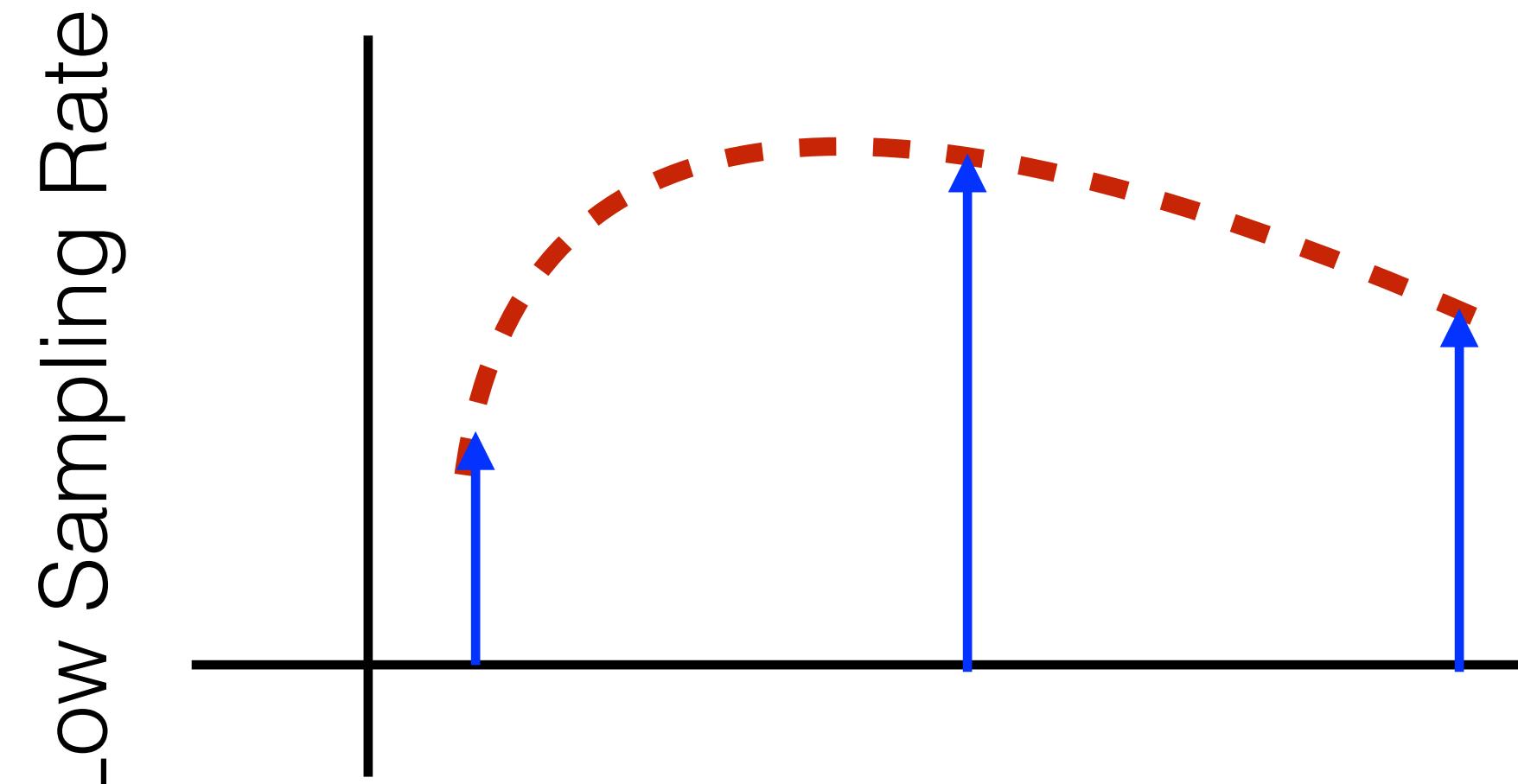
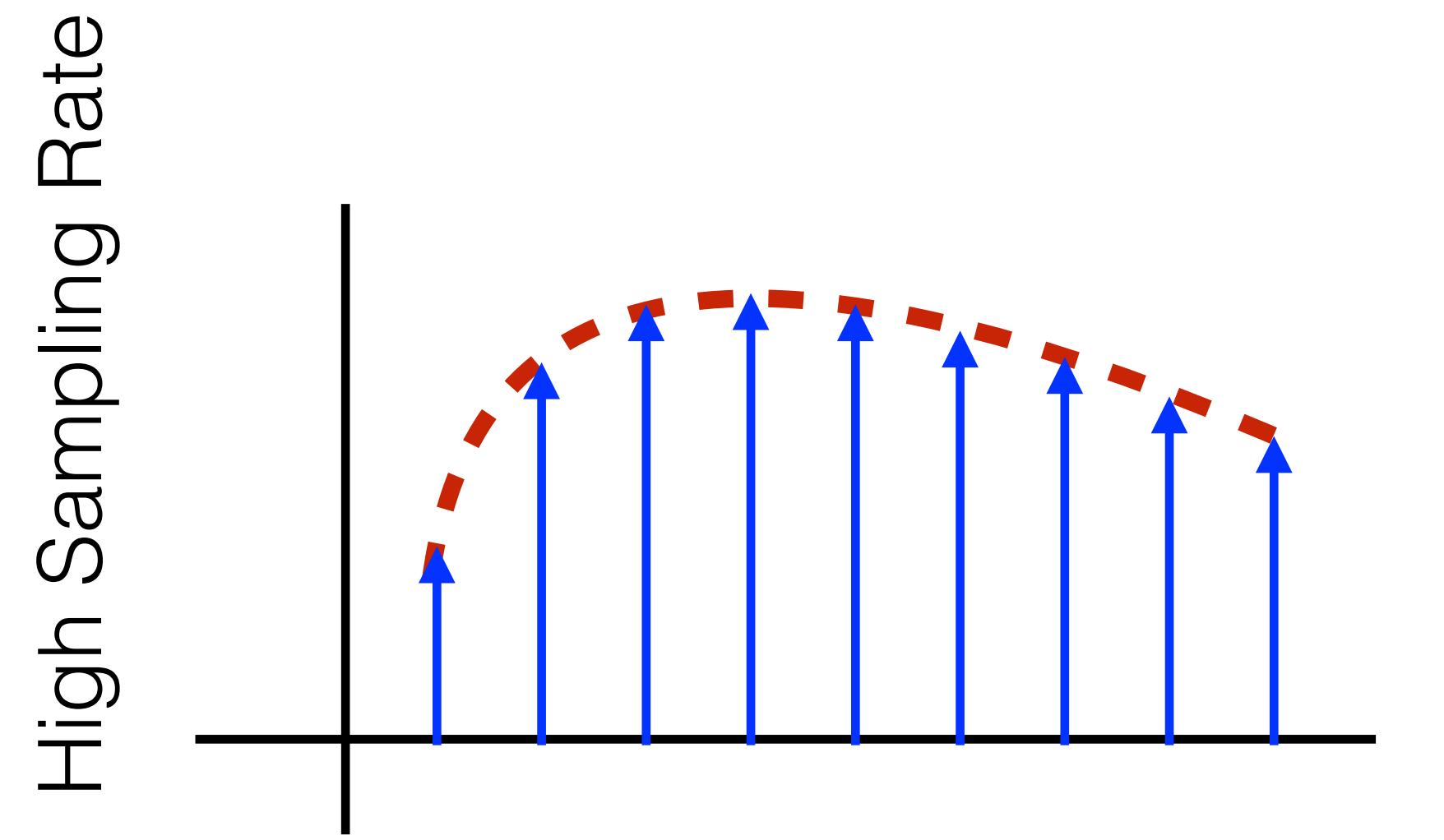
Aliasing in Reconstruction



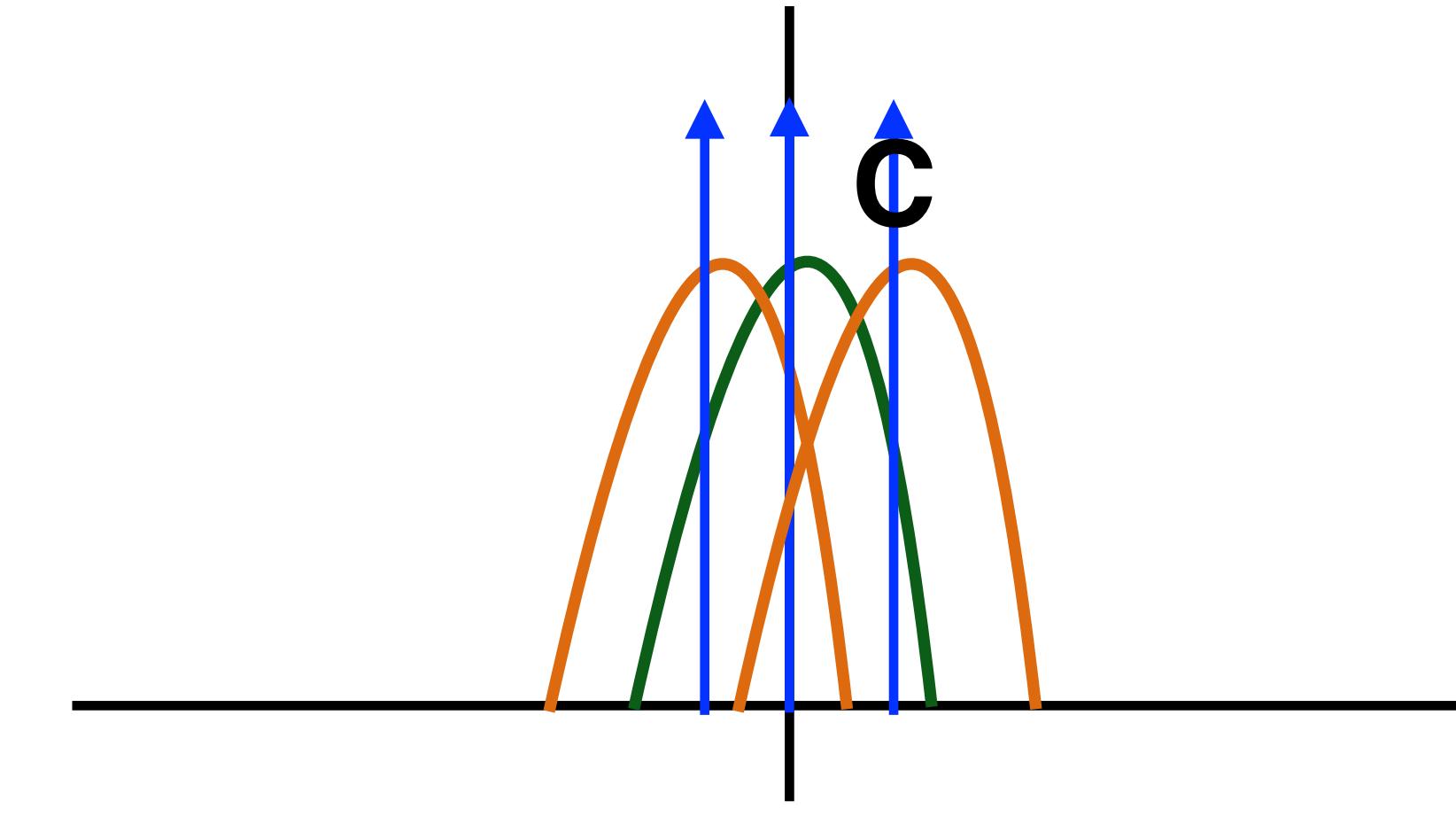
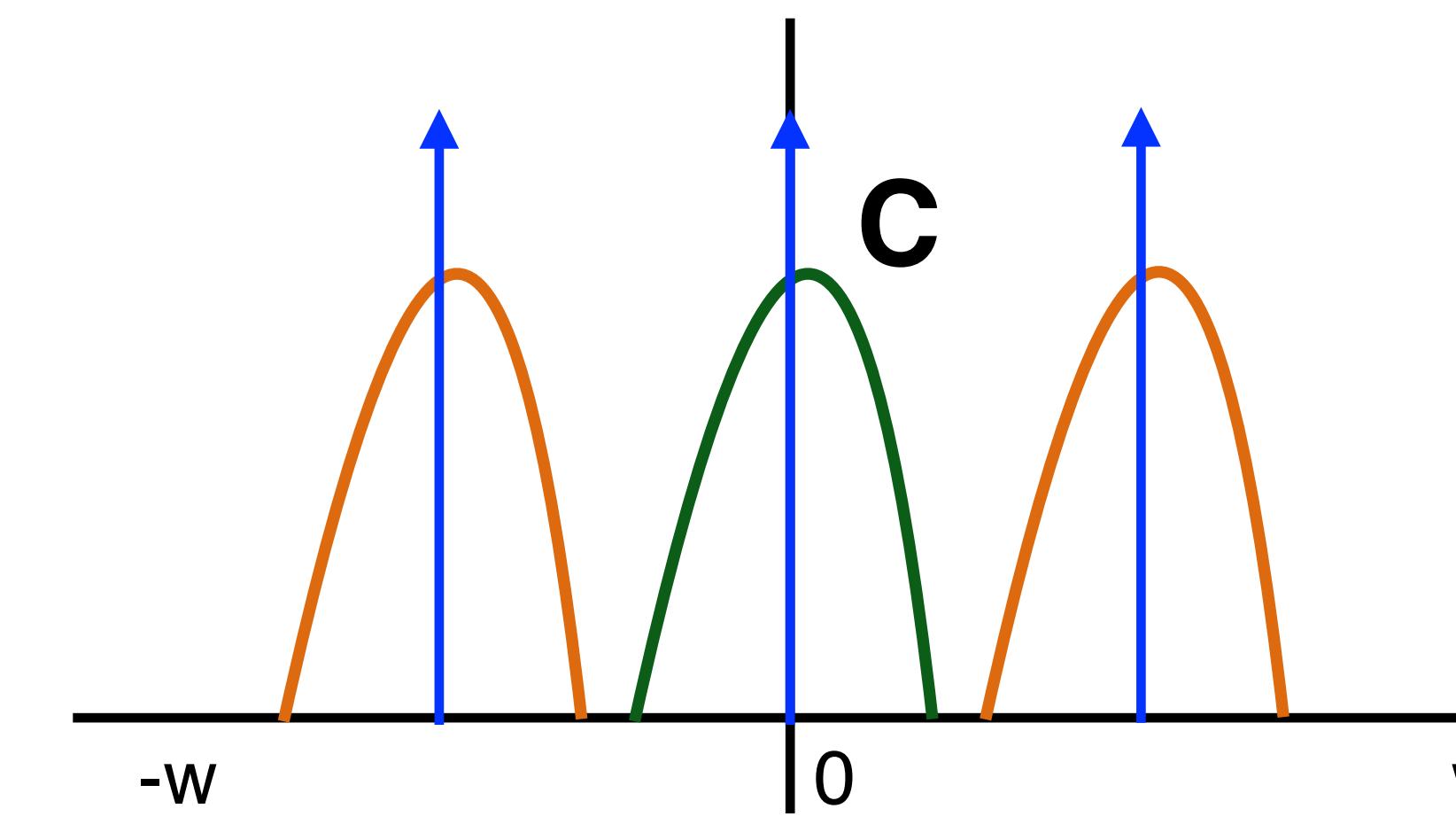
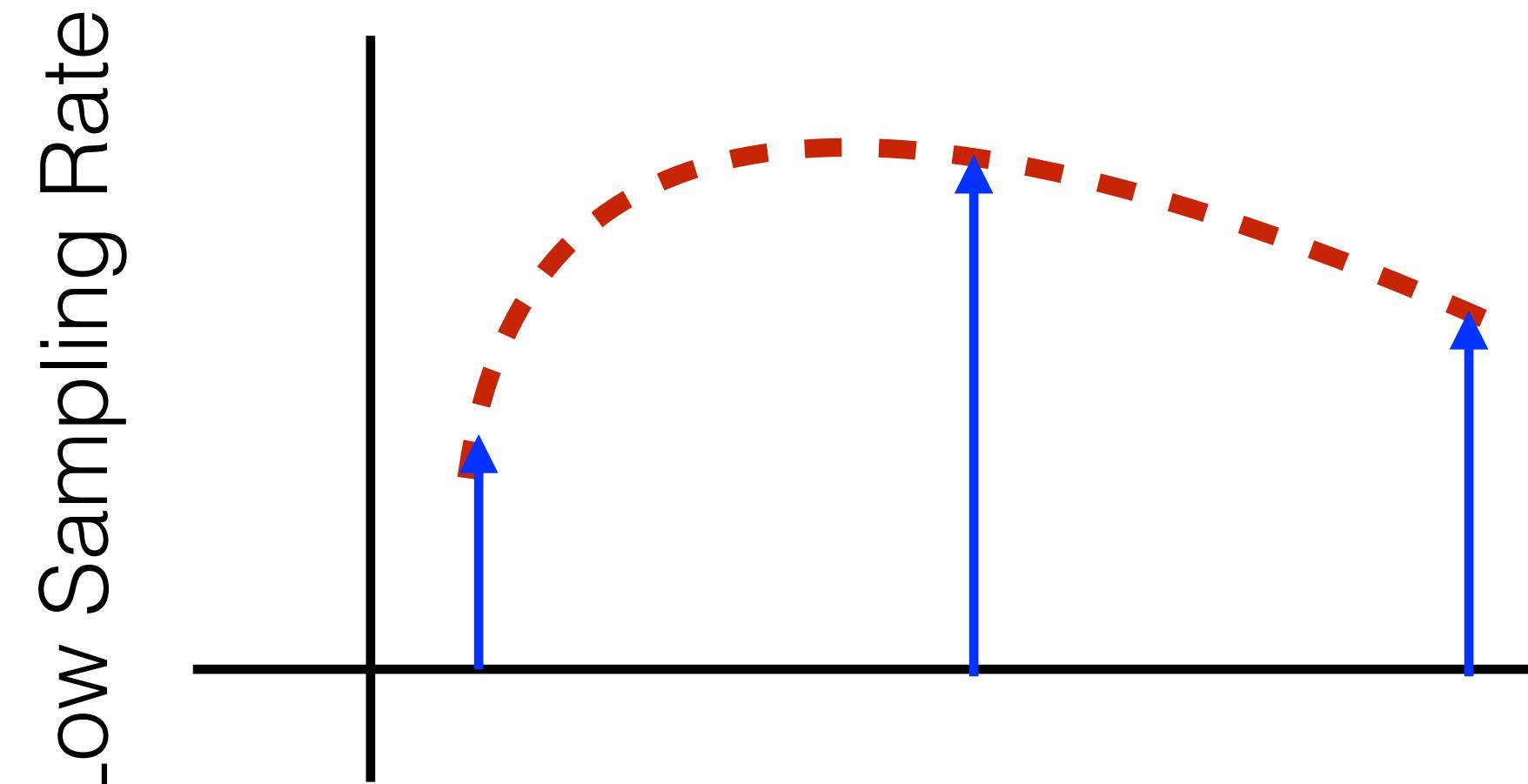
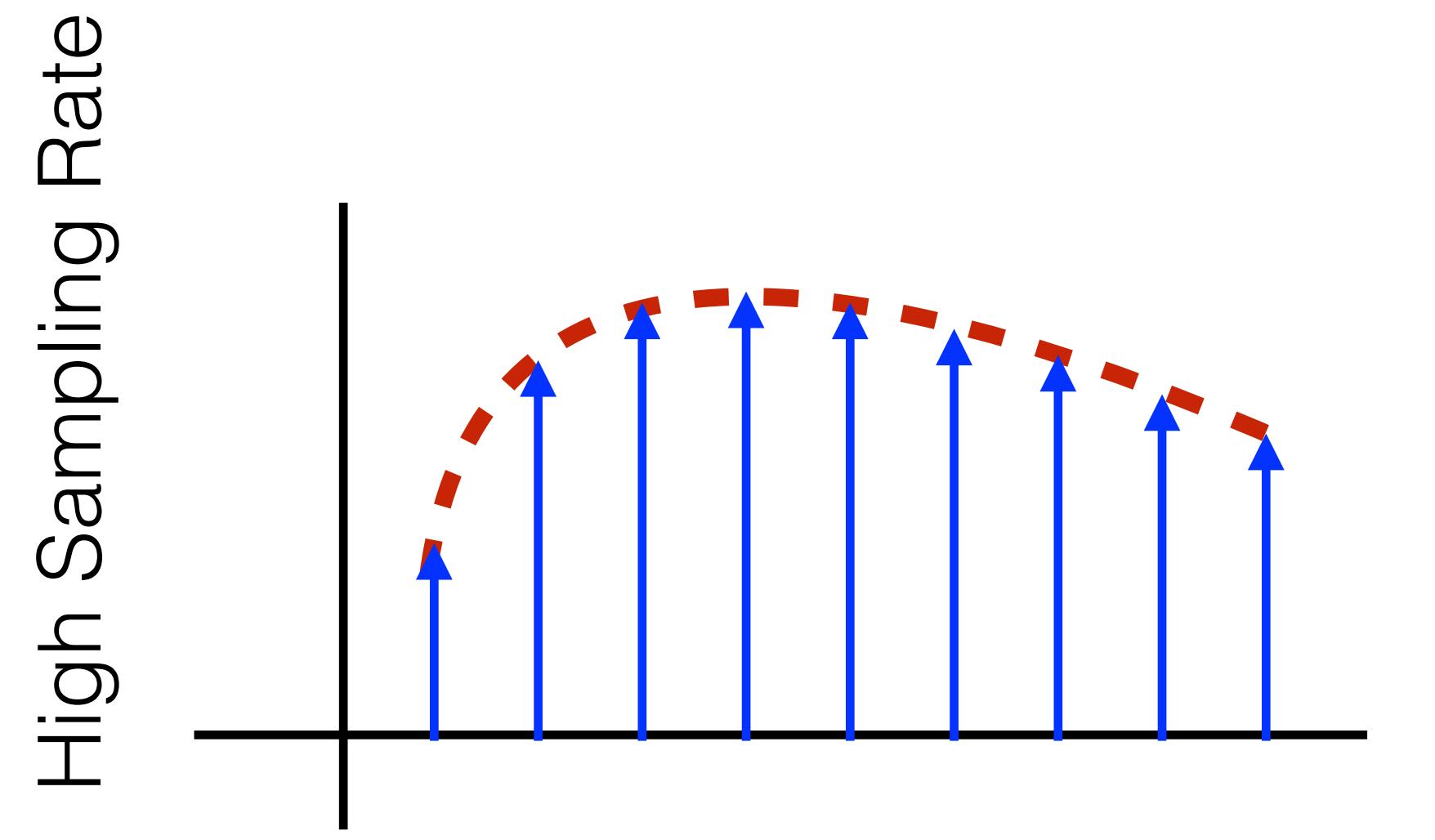
Aliasing in Reconstruction



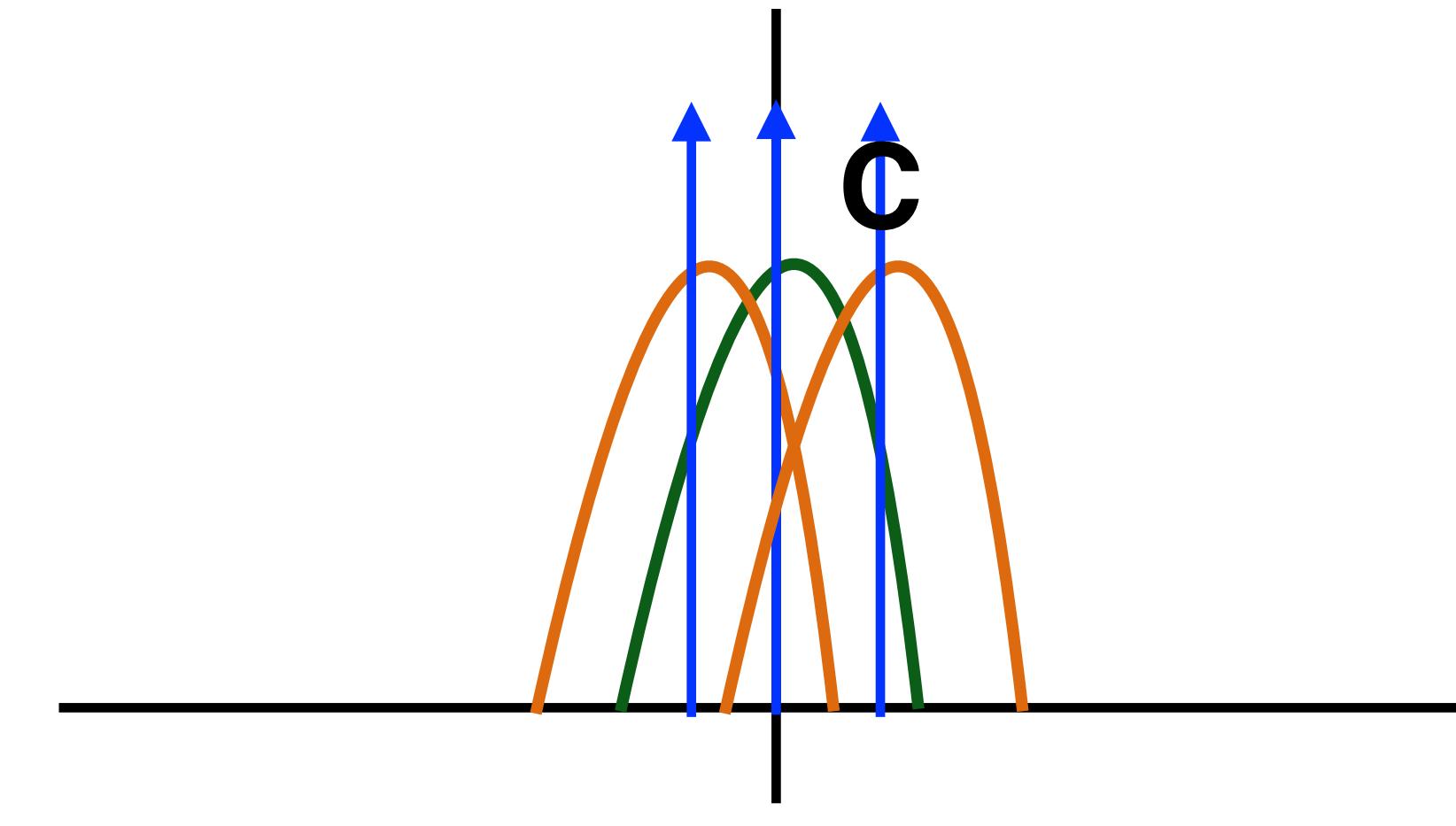
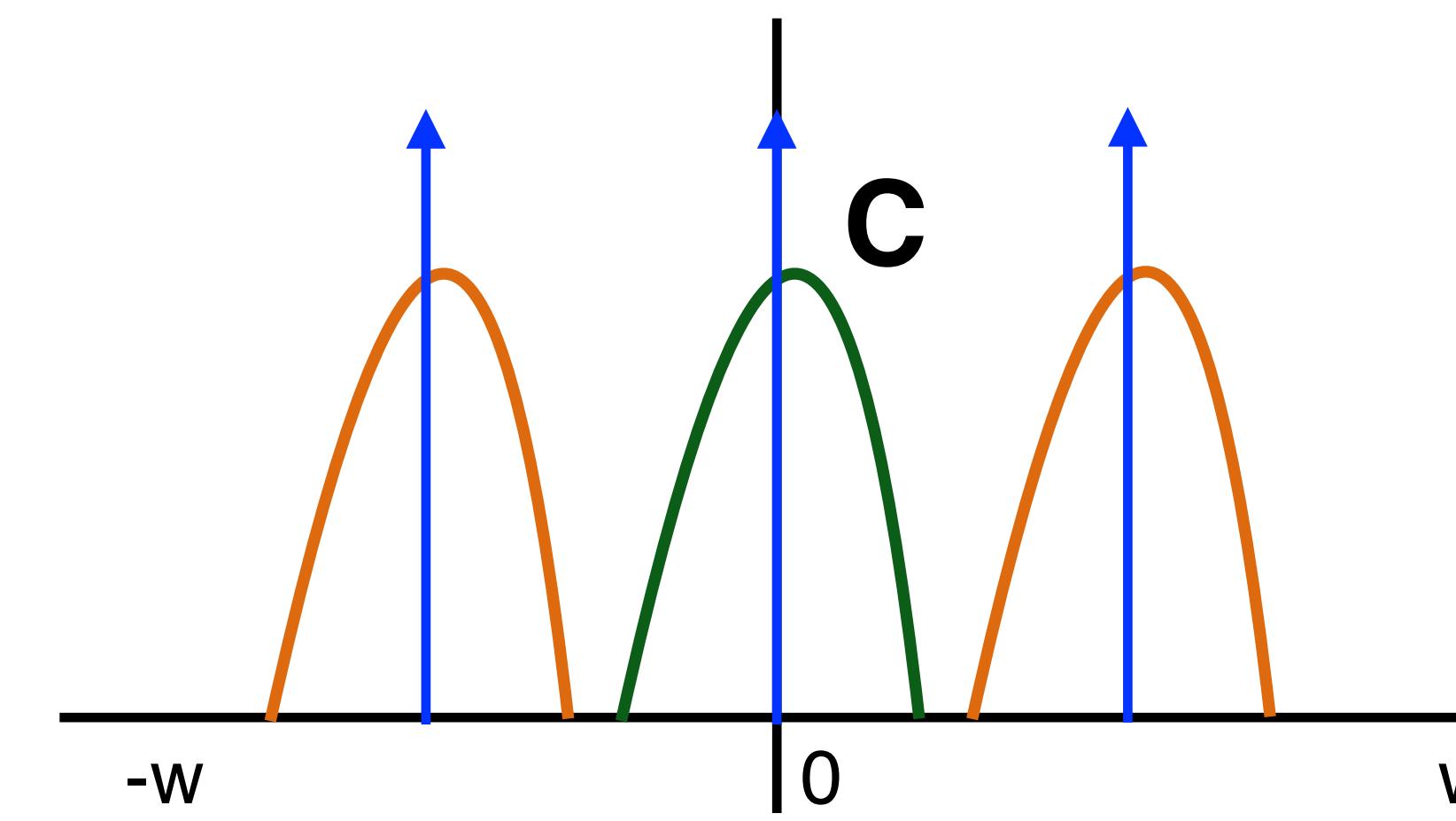
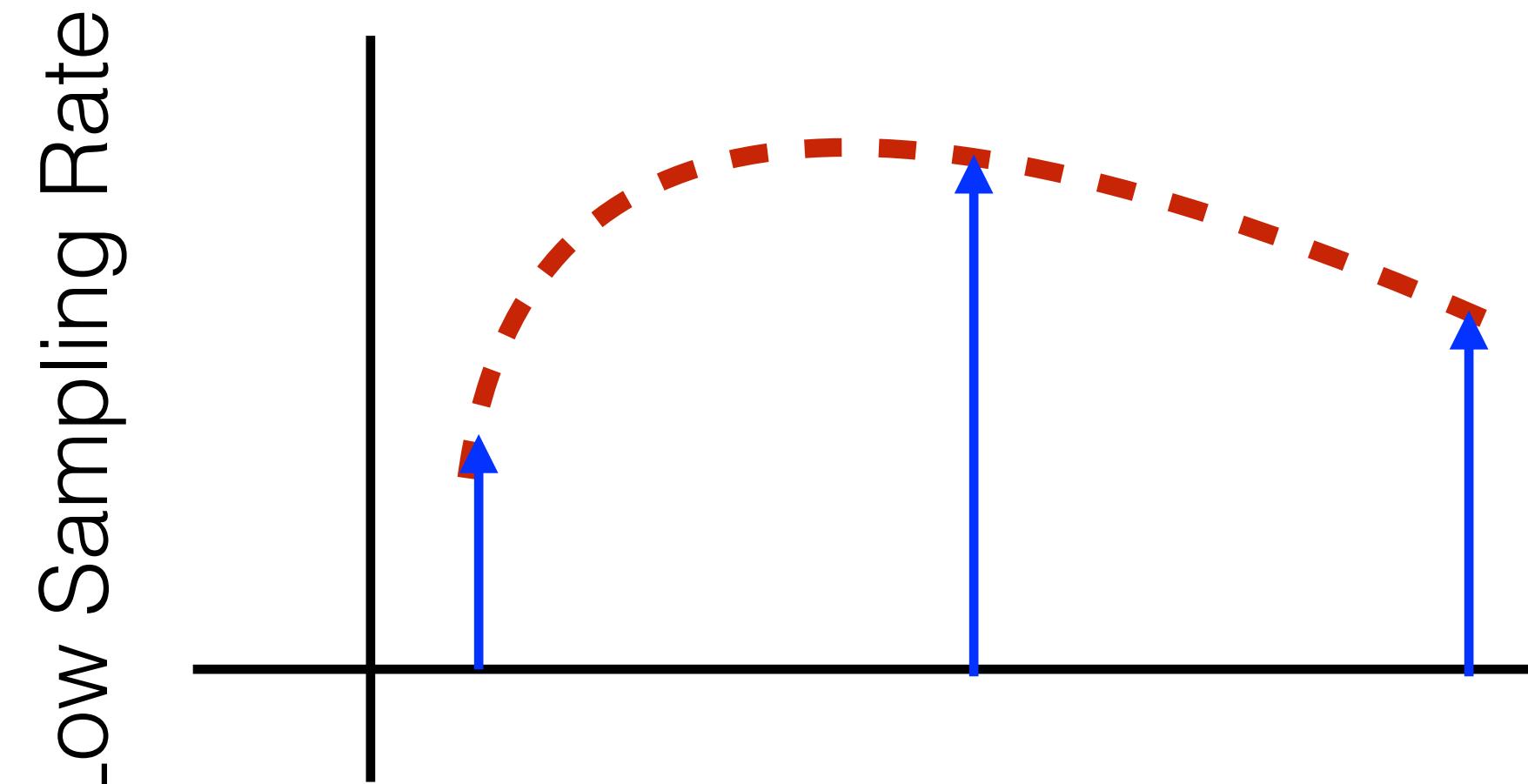
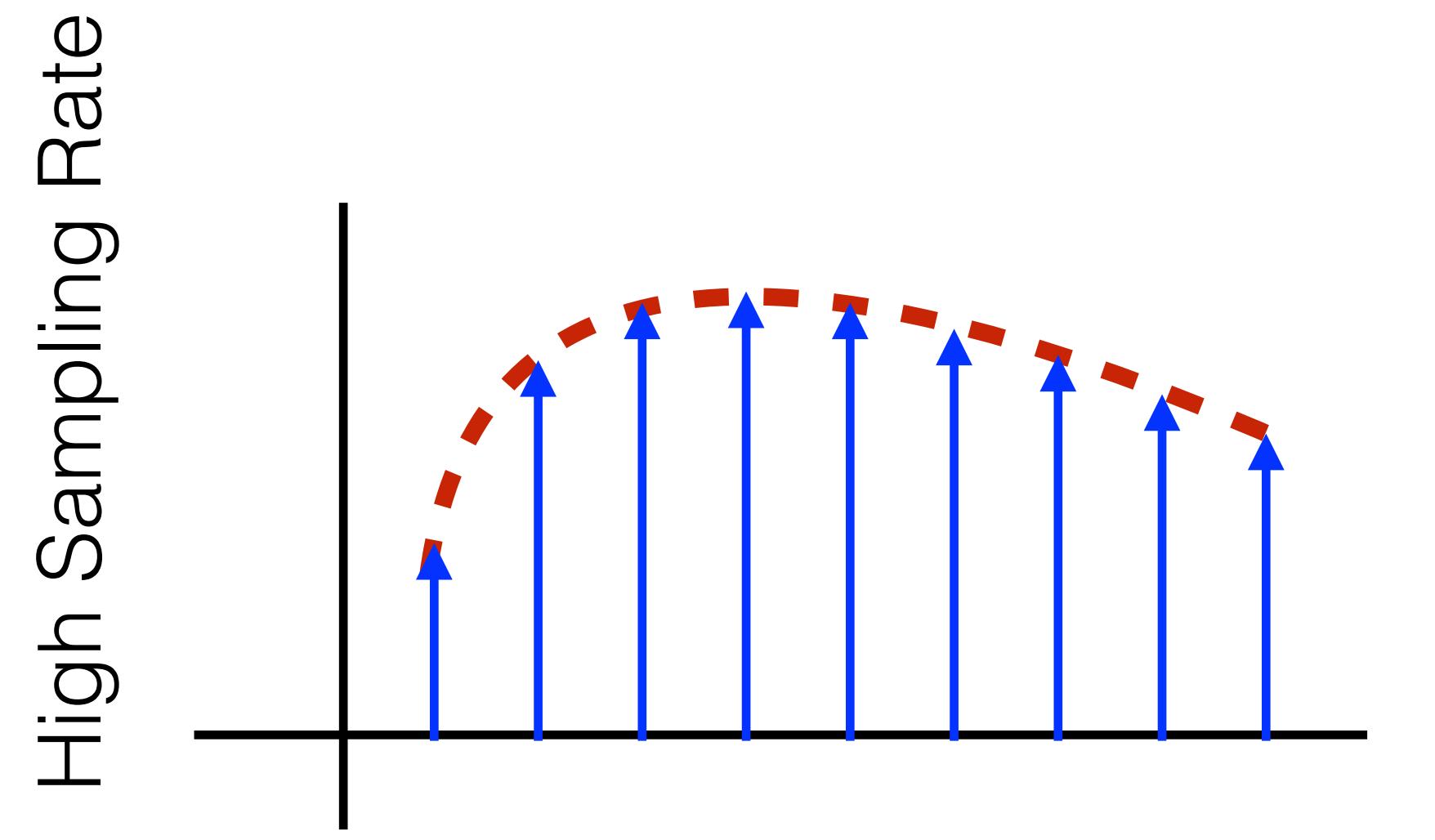
Error in Monte Carlo Integration



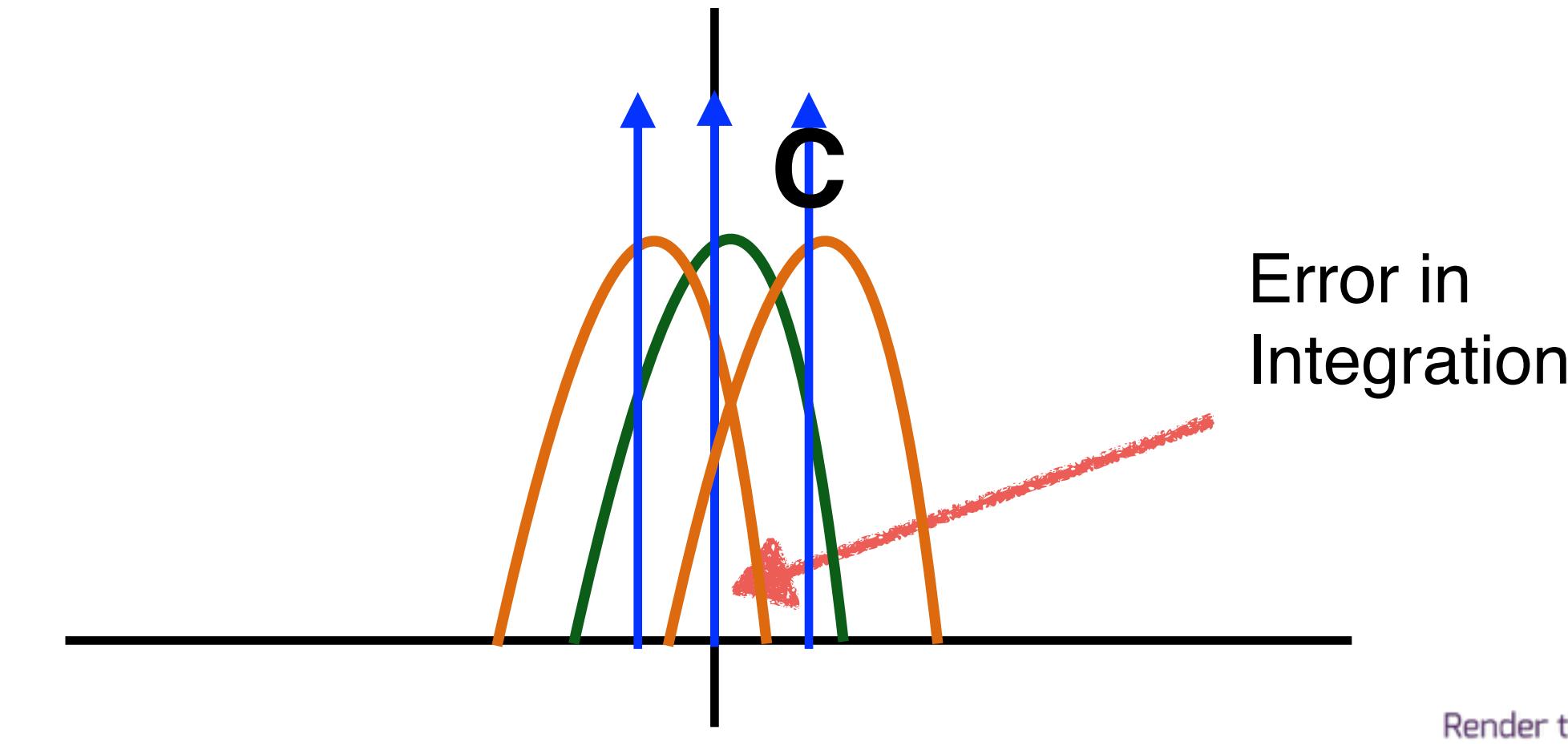
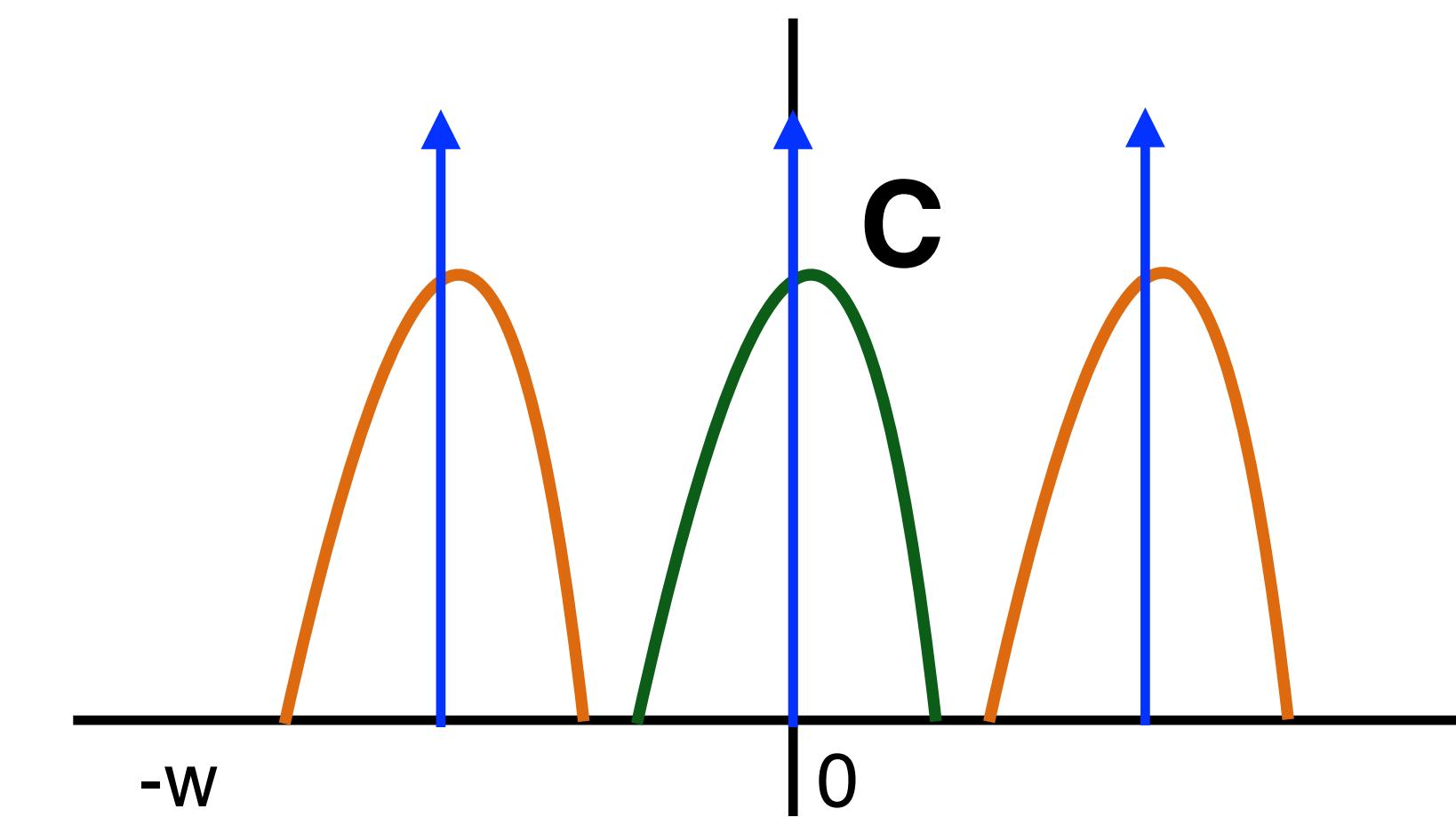
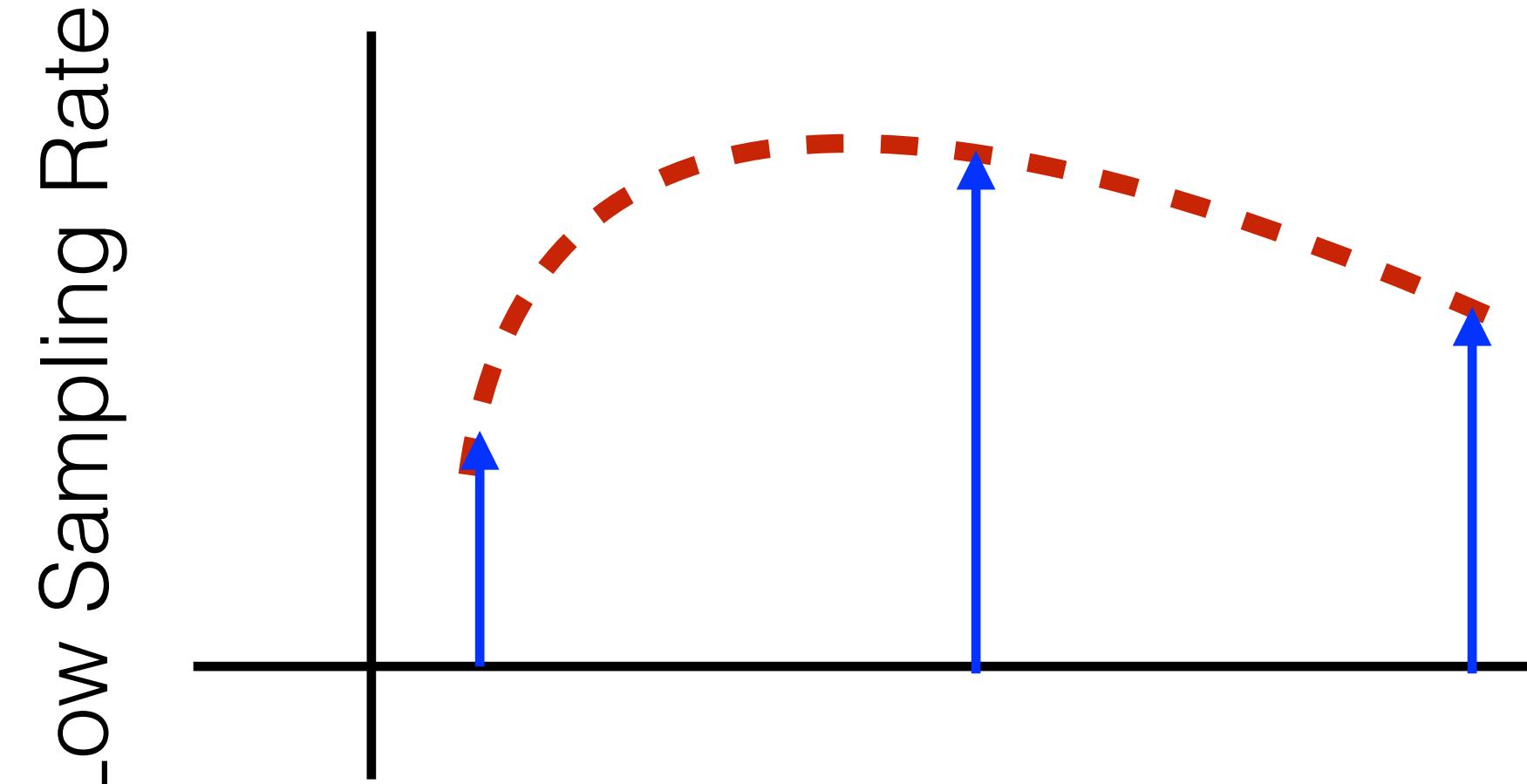
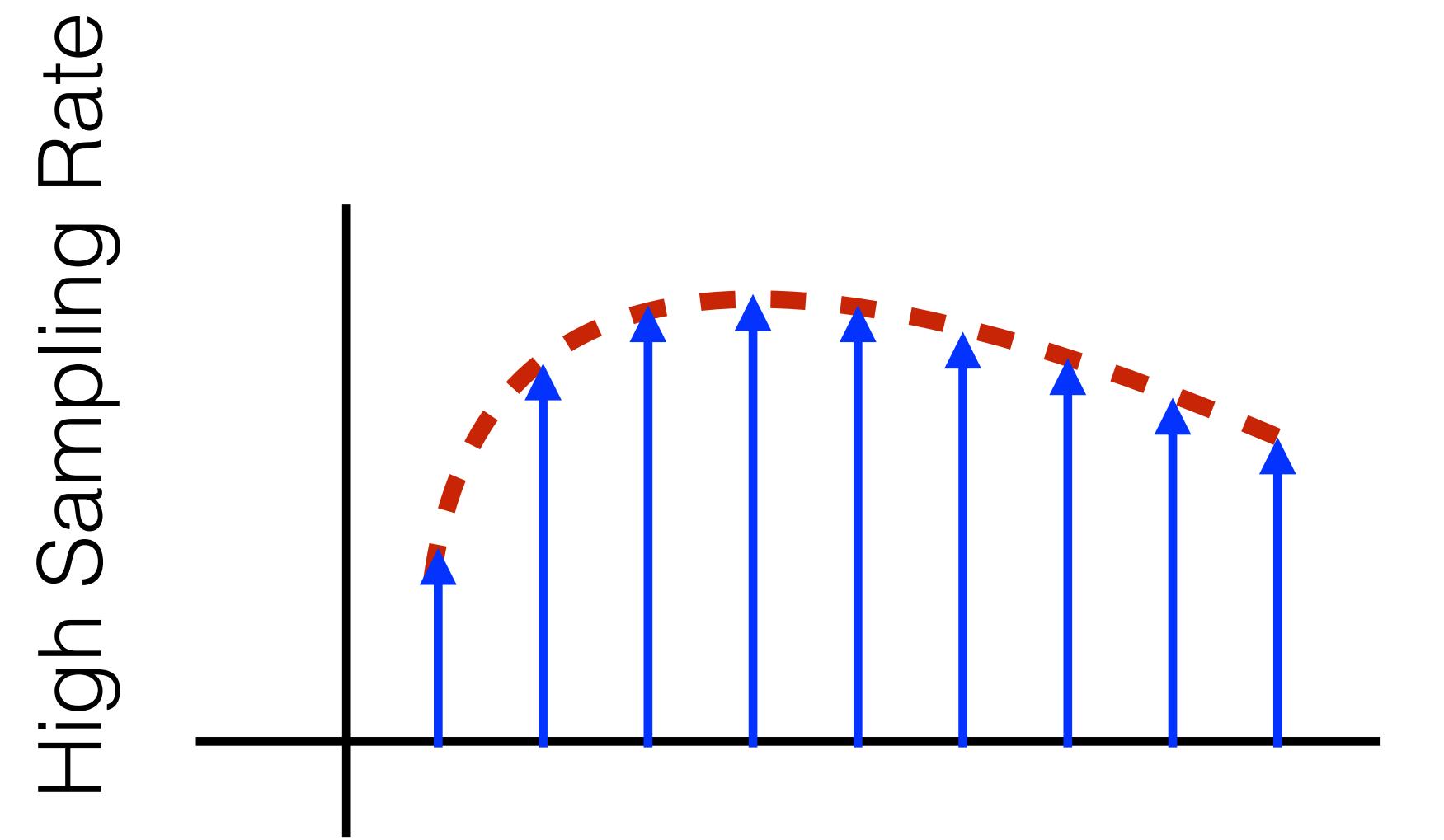
Error in Monte Carlo Integration



Error in Monte Carlo Integration



Error in Monte Carlo Integration

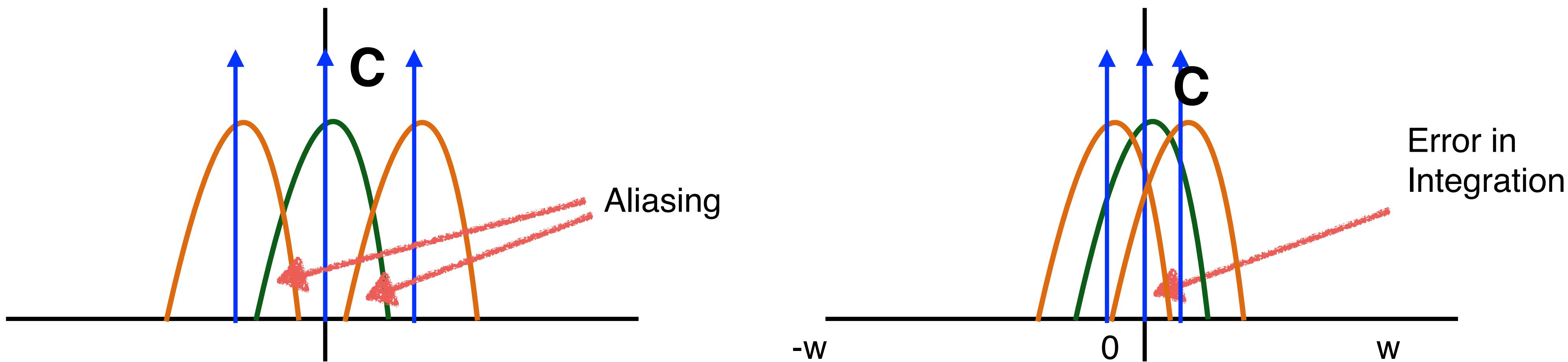


Aliasing (Reconstruction) vs. Error (Integration)



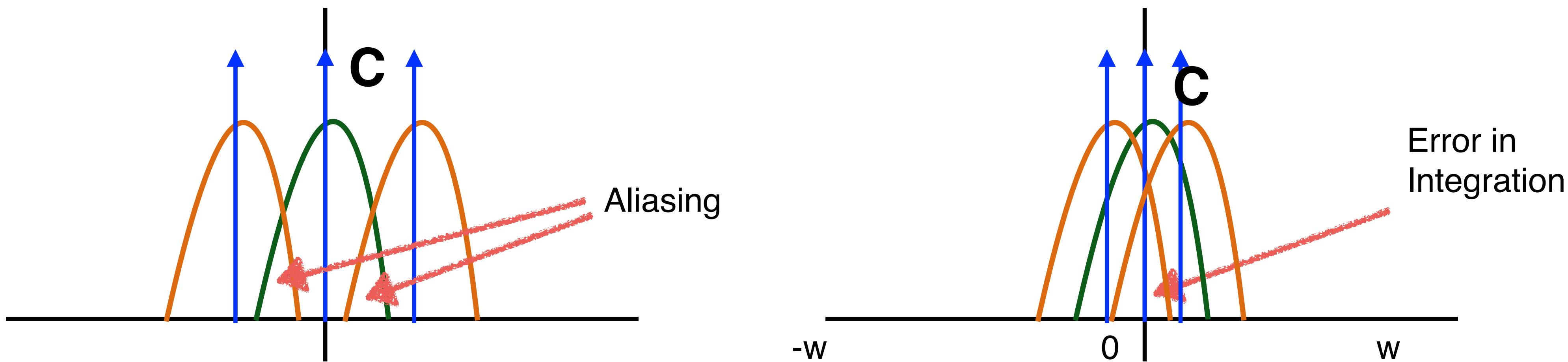
Aliasing (Reconstruction) vs. Error (Integration)

Fredo Durand [2011]
Belcour et al. [2013]



Aliasing (Reconstruction) vs. Error (Integration)

Fredo Durand [2011]
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Integration in the Fourier Domain

Integration is the DC term in the Fourier Domain

Spatial Domain:

$$I = \int_D f(x) dx$$

Integration is the DC term in the Fourier Domain

Spatial Domain:

$$I = \int_D f(x) dx$$

Fourier Domain:

Integration is the DC term in the Fourier Domain

Spatial Domain:

$$I = \int_D f(x) dx$$

Fourier Domain:

$$\hat{f}(0)$$

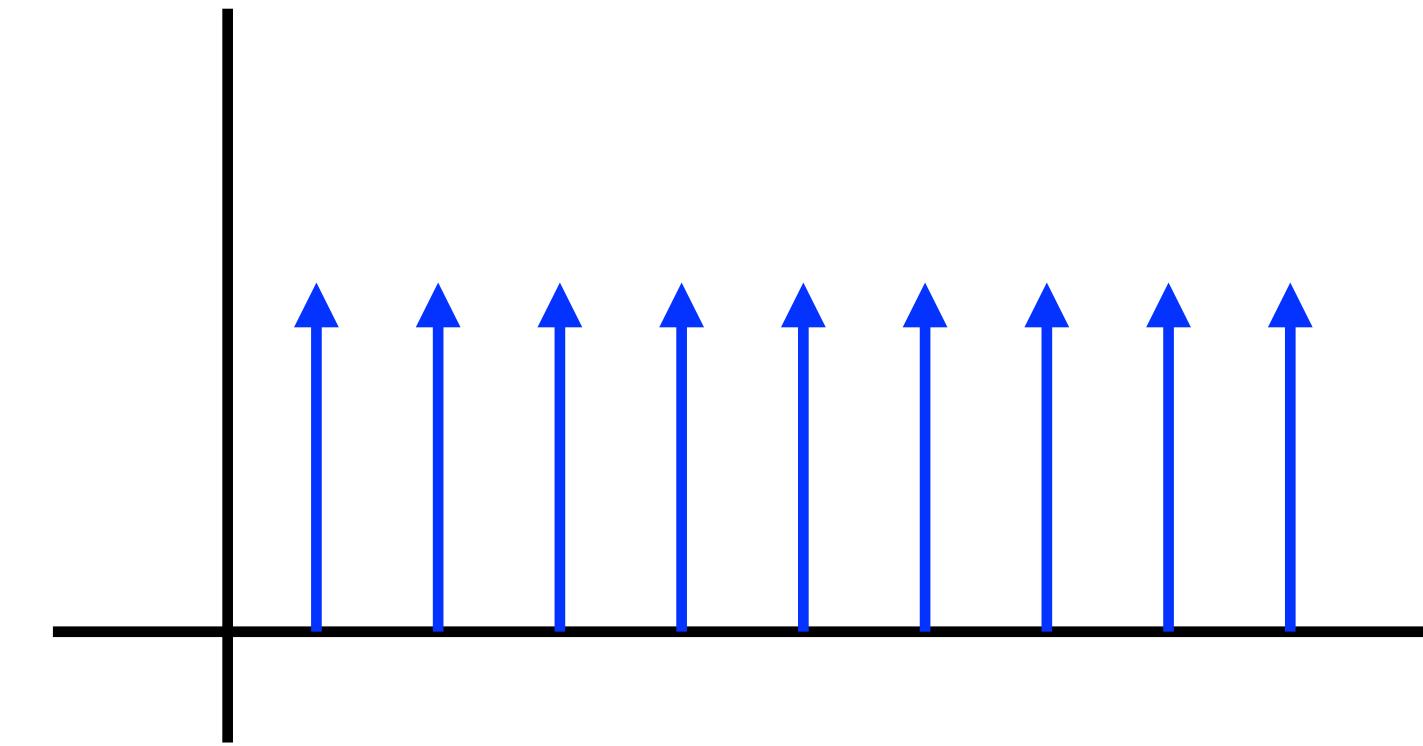
Monte Carlo Estimator in Spatial Domain

$$\tilde{\mu}_N = \int_D f(x) \mathbf{S}(x) dx$$

Monte Carlo Estimator in Spatial Domain

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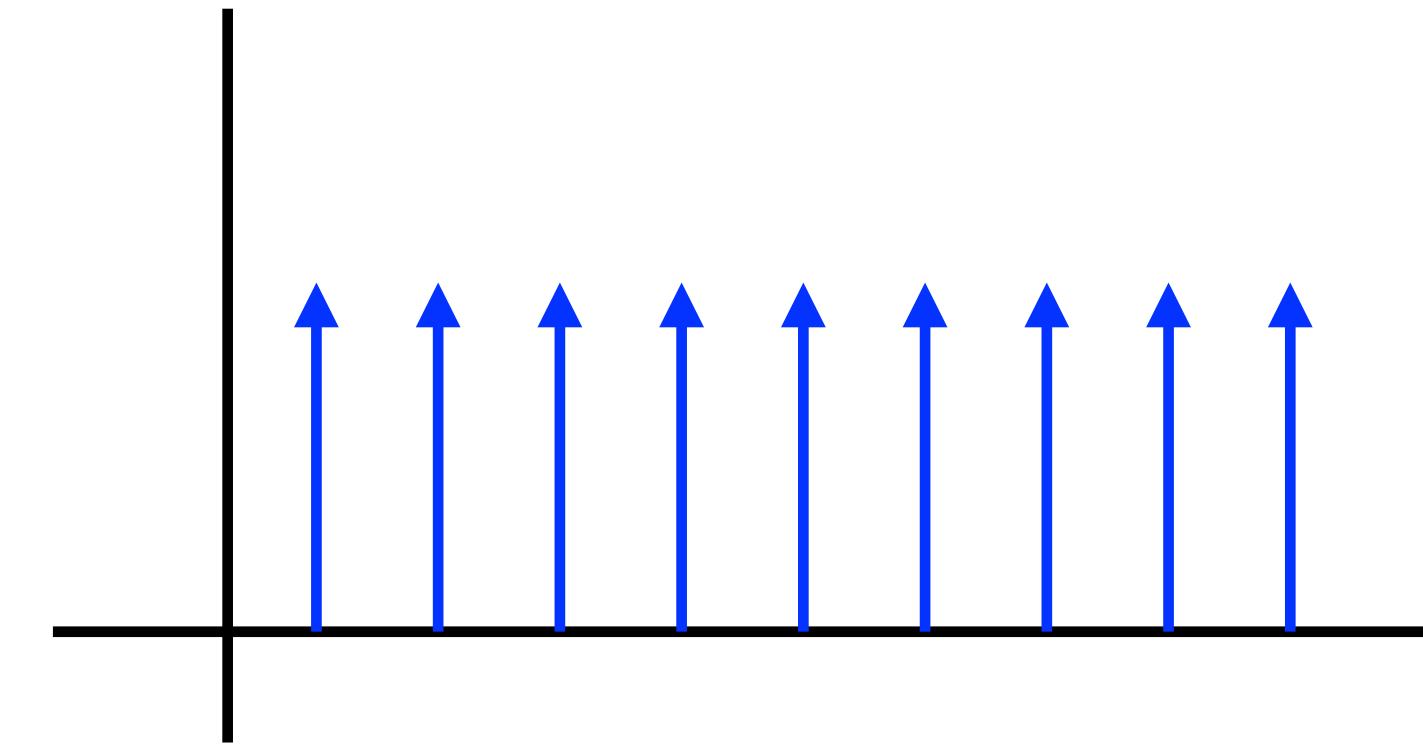
$$\mathbf{S}(x) = \frac{1}{N} \sum_{k=1}^N \delta(x - x_k)$$



Monte Carlo Estimator in Spatial Domain

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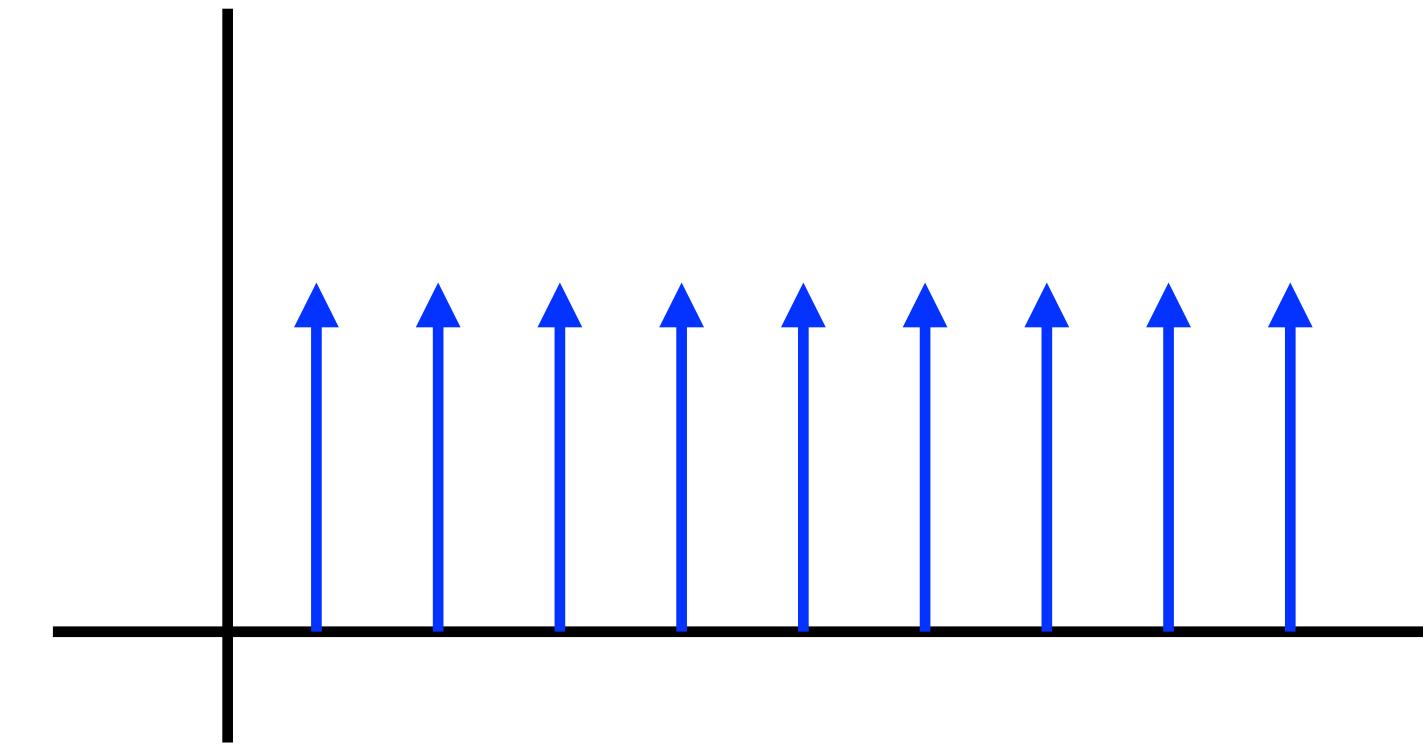
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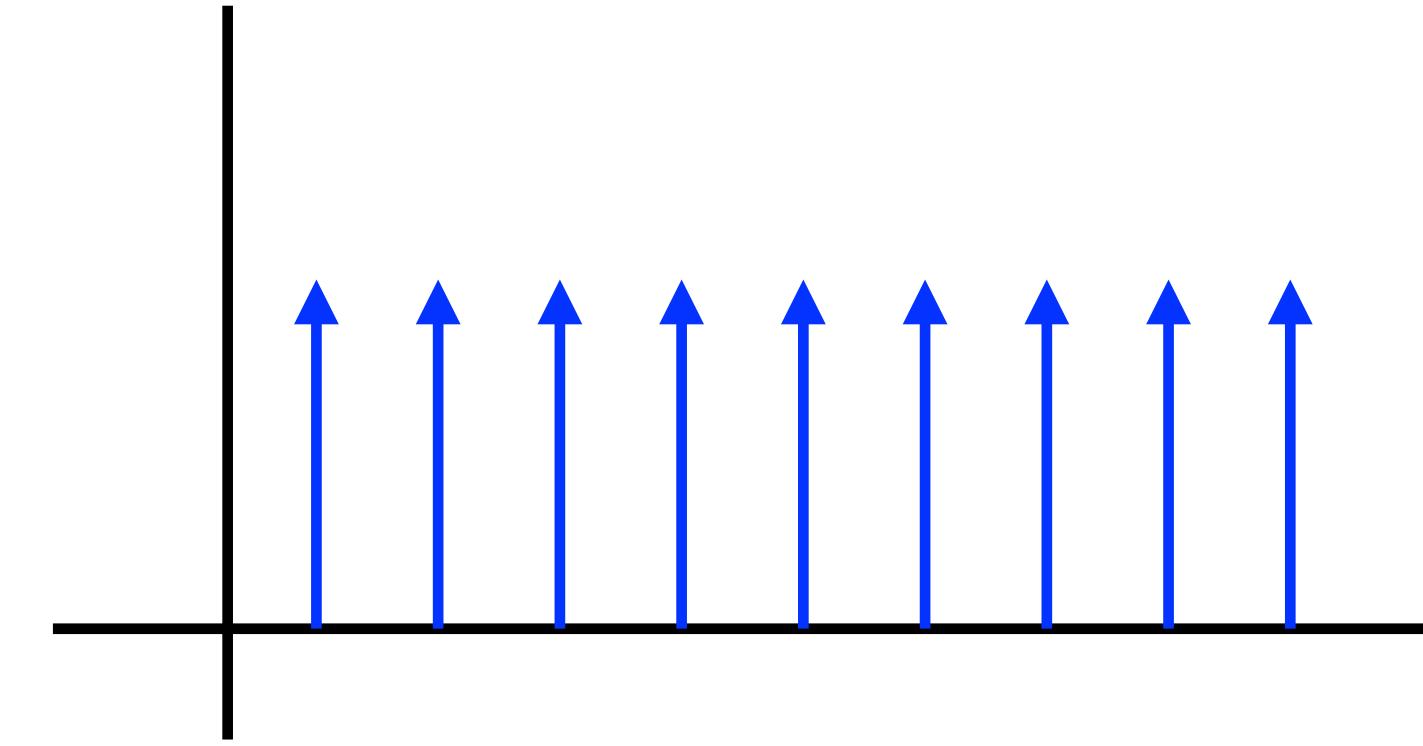
$$\boxed{\mathbf{S}(x)} = \frac{1}{N} \sum_{k=1}^N \delta(x - \boxed{x_k})$$



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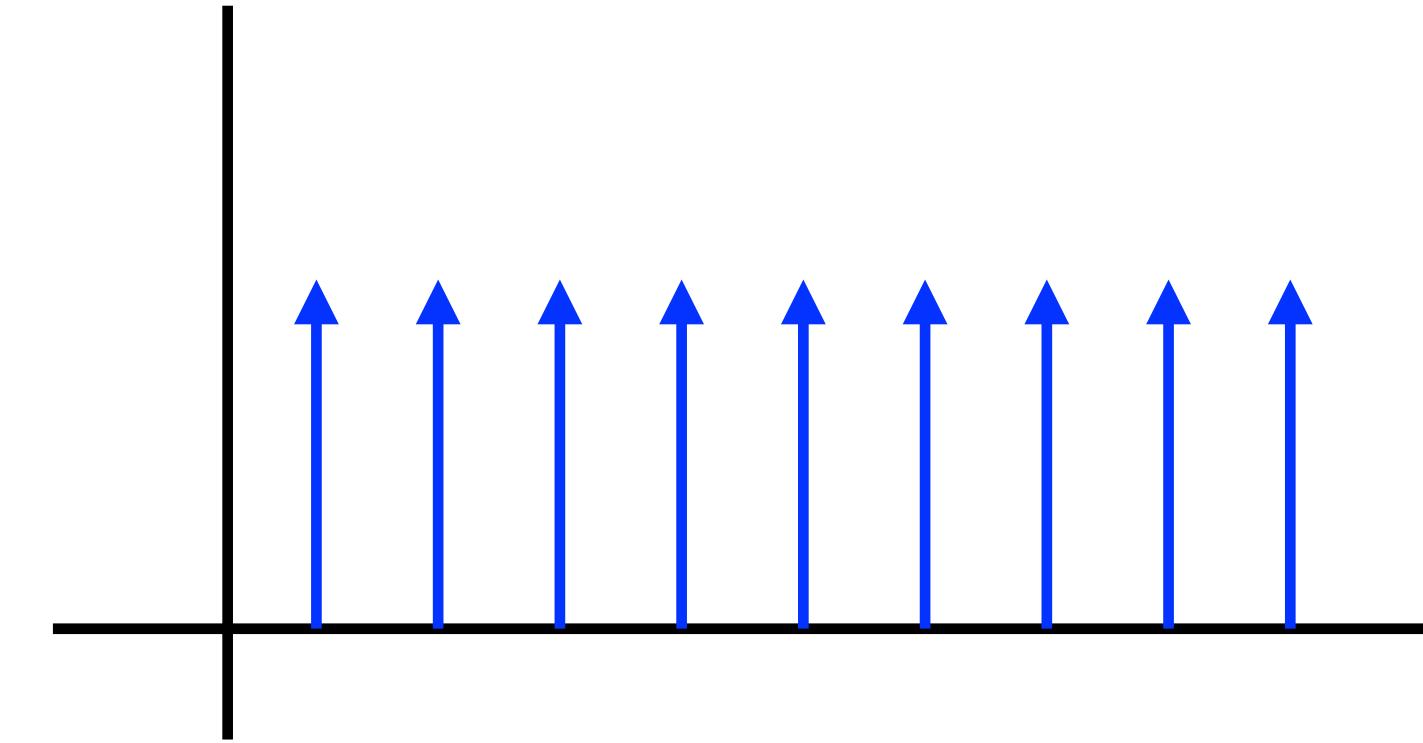
$$\mathbf{S}(x) = \frac{1}{N} \sum_{k=1}^N \delta(x - x_k)$$



Monte Carlo Estimator in Spatial Domain

$$\tilde{\mu}_N = \int_D f(x) \mathbf{S}(x) dx = \int_{\Omega} \hat{f}^*(\omega) \hat{\mathbf{S}}(\omega) d\omega$$

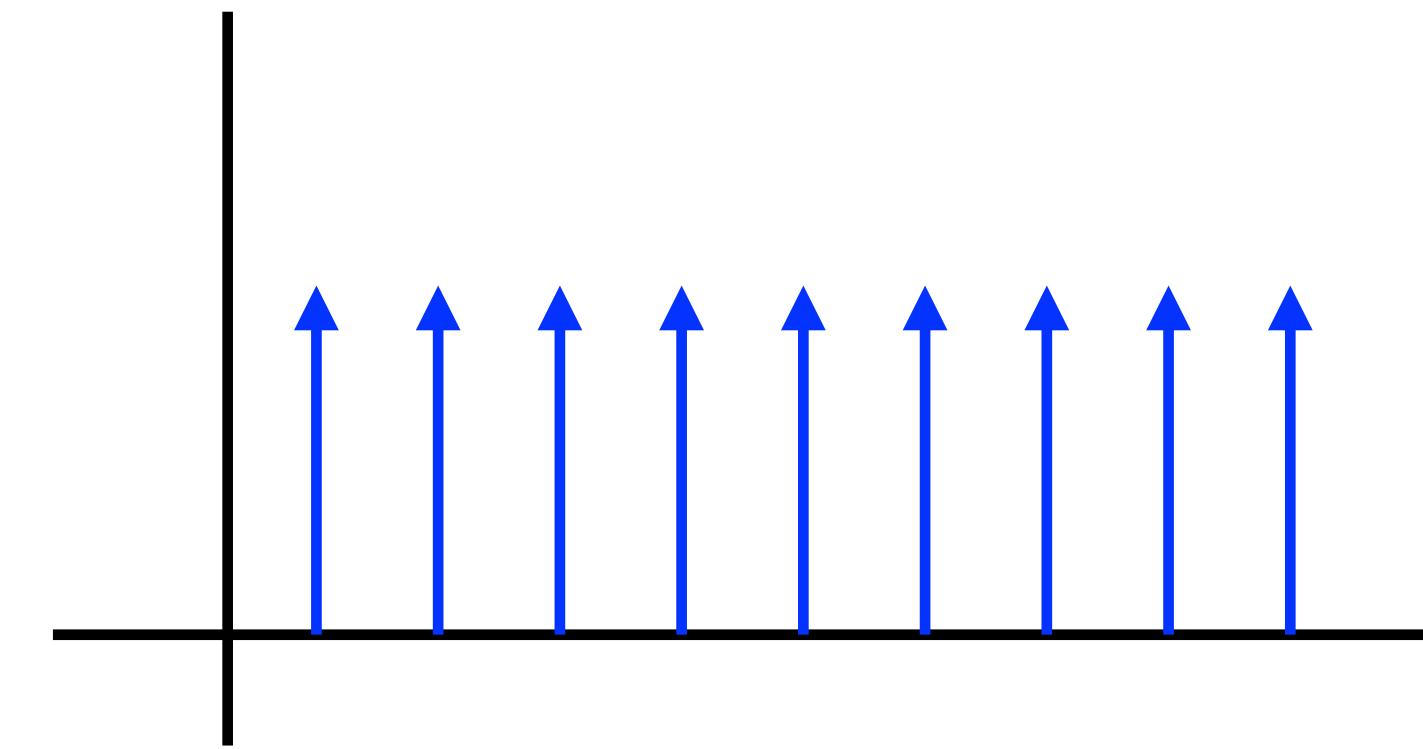
$$\mathbf{S}(x) = \frac{1}{N} \sum_{k=1}^N \delta(x - x_k)$$



Monte Carlo Estimator in Fourier Domain

$$\tilde{\mu}_N = \int_D f(x) \mathbf{S}(x) dx = \int_{\Omega} \hat{f}^*(\omega) \hat{\mathbf{S}}(\omega) d\omega$$

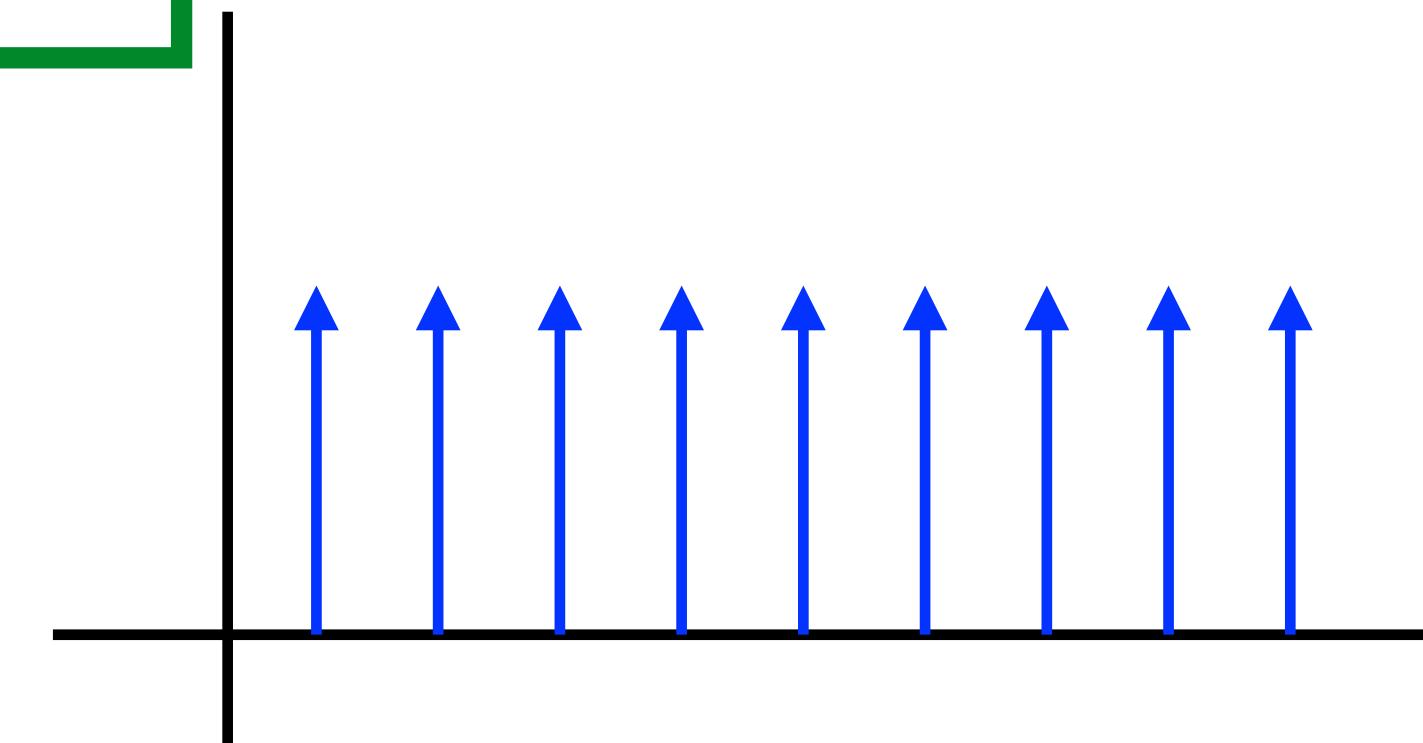
$$\mathbf{S}(x) = \frac{1}{N} \sum_{k=1}^N \delta(x - x_k)$$



Monte Carlo Estimator in Fourier Domain

$$\tilde{\mu}_N = \int_D f(x) \mathbf{S}(x) dx = \boxed{\int_{\Omega} \hat{f}^*(\omega) \hat{\mathbf{S}}(\omega) d\omega}$$

$$\mathbf{S}(x) = \frac{1}{N} \sum_{k=1}^N \delta(x - x_k)$$

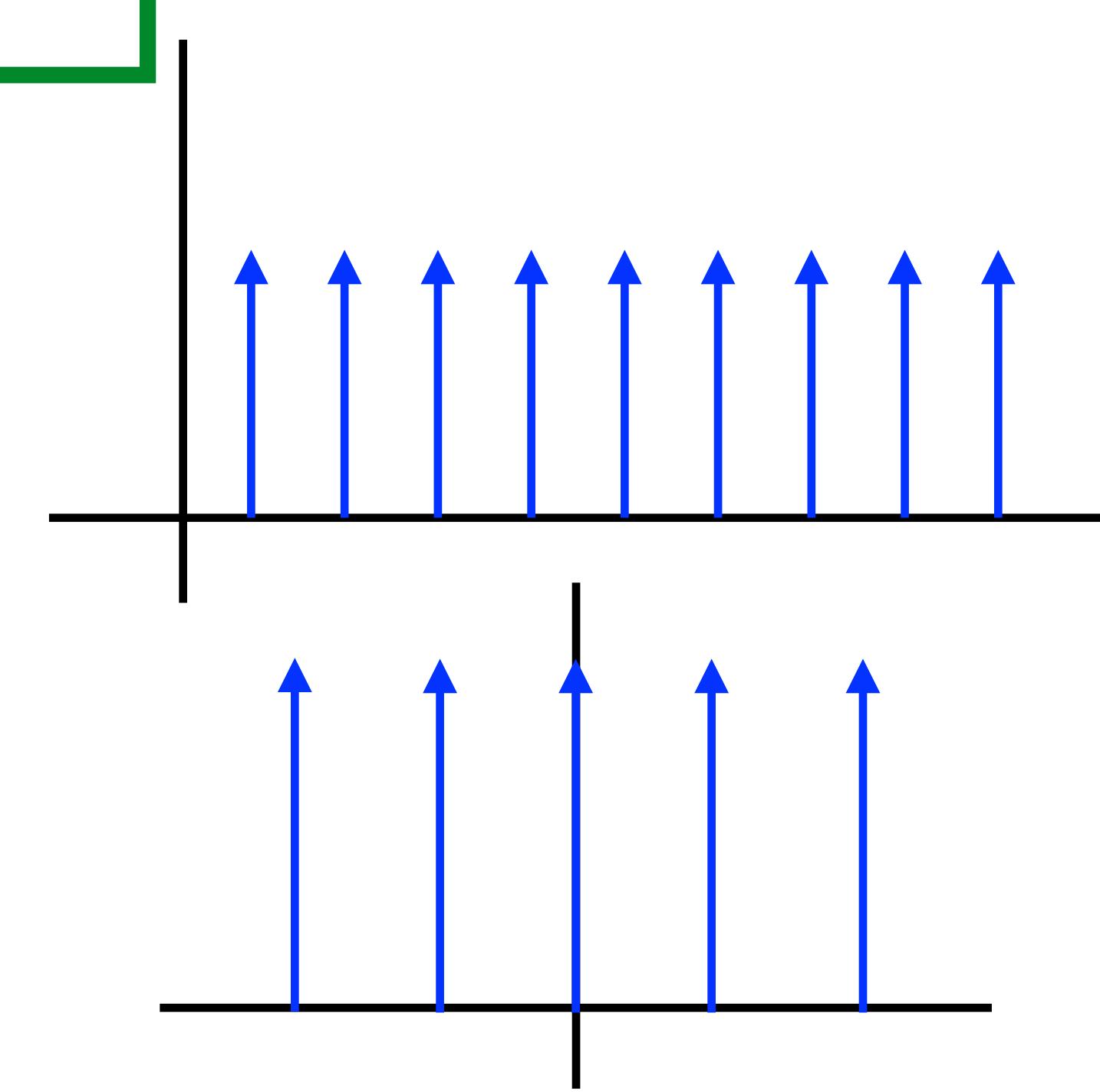


Monte Carlo Estimator in Fourier Domain

$$\tilde{\mu}_N = \int_D f(x) \mathbf{S}(x) dx = \boxed{\int_{\Omega} \hat{f}^*(\omega) \hat{\mathbf{S}}(\omega) d\omega}$$

$$\mathbf{S}(x) = \frac{1}{N} \sum_{k=1}^N \delta(x - x_k)$$

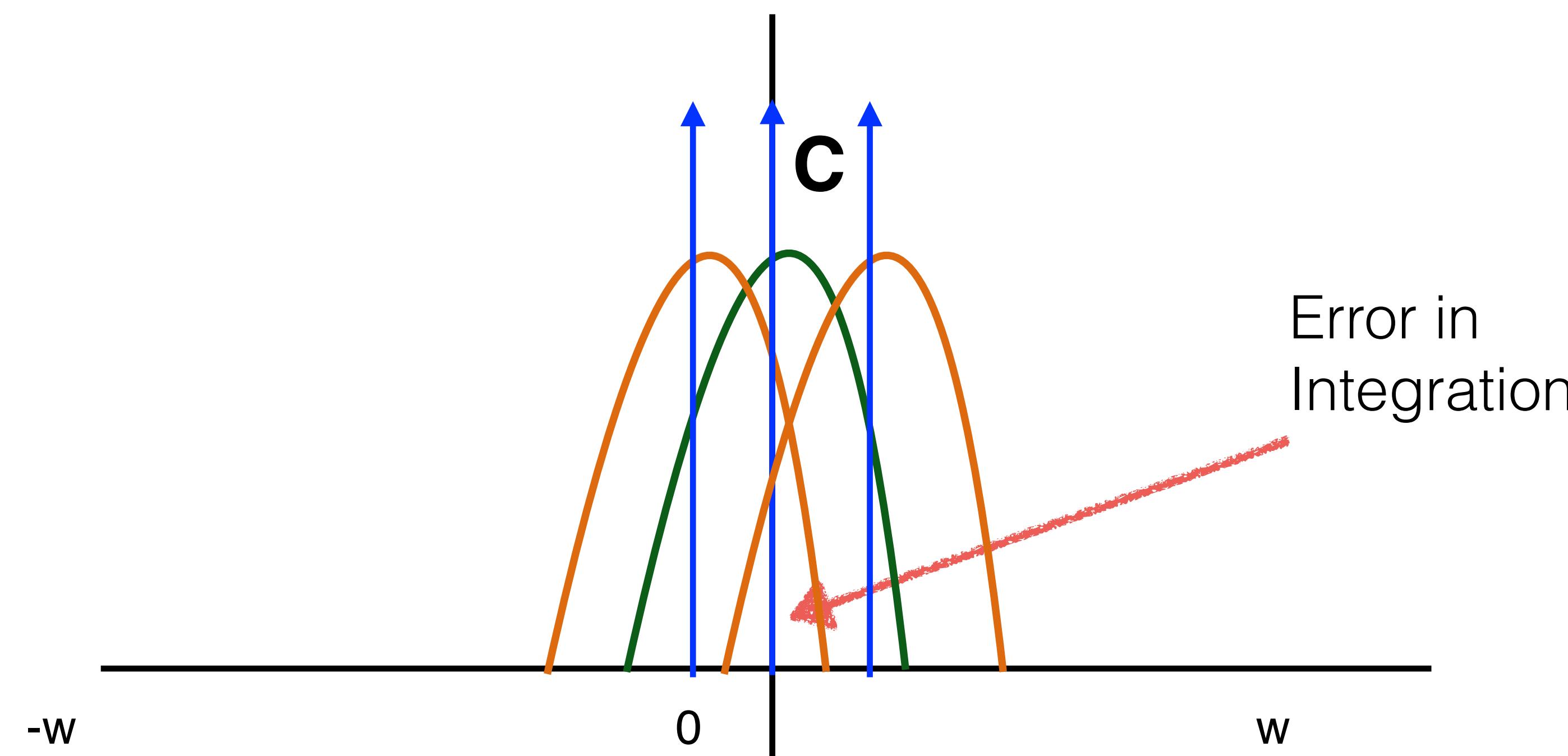
$$\hat{\mathbf{S}}(\omega) = \frac{1}{N} \sum_{k=1}^N e^{-i2\pi\omega x_k}$$



How to Formulate Error in Fourier Domain ?

$$I = \hat{f}(0)$$

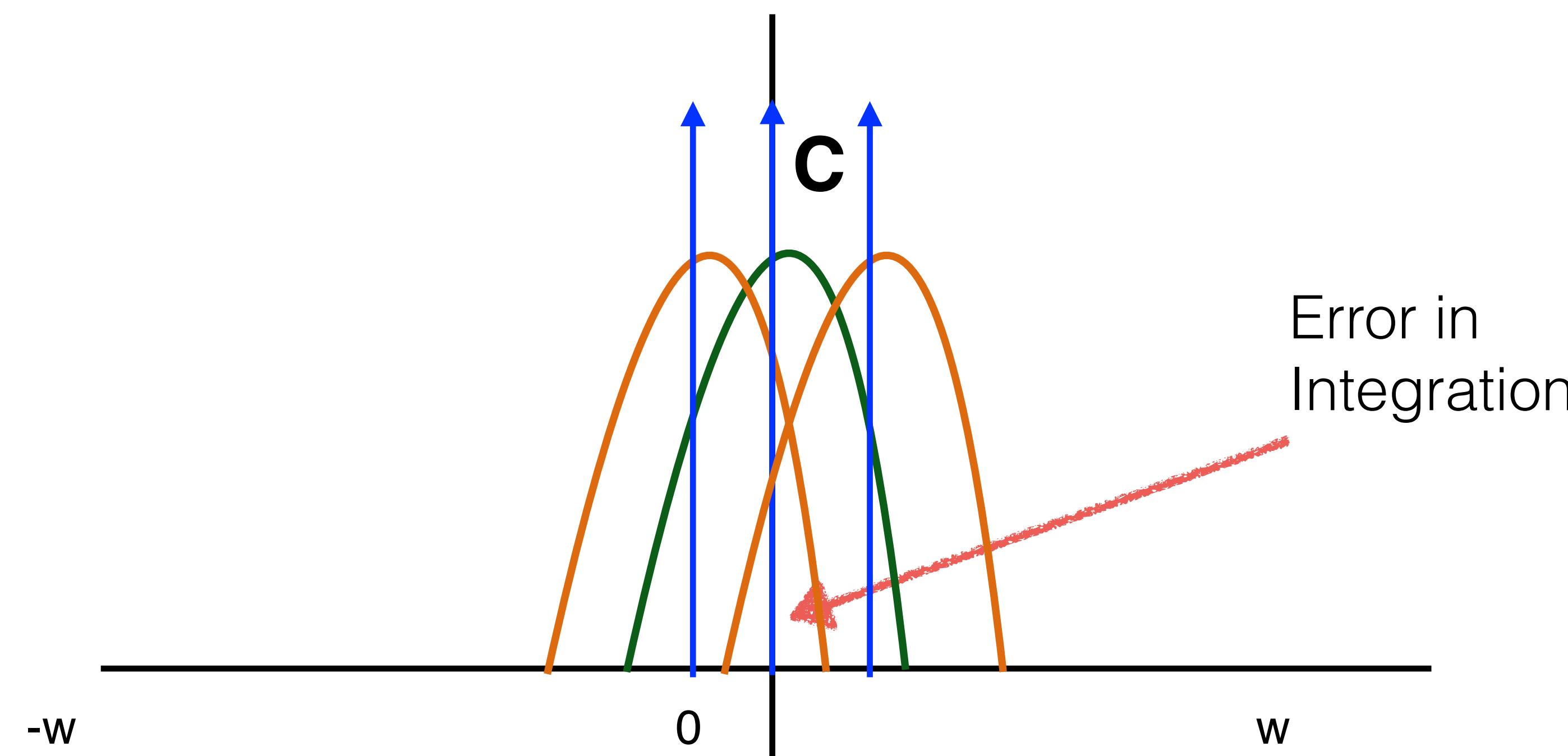
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Fredo Durand [2011]



Error in Spatial Domain

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Monte Carlo Estimator

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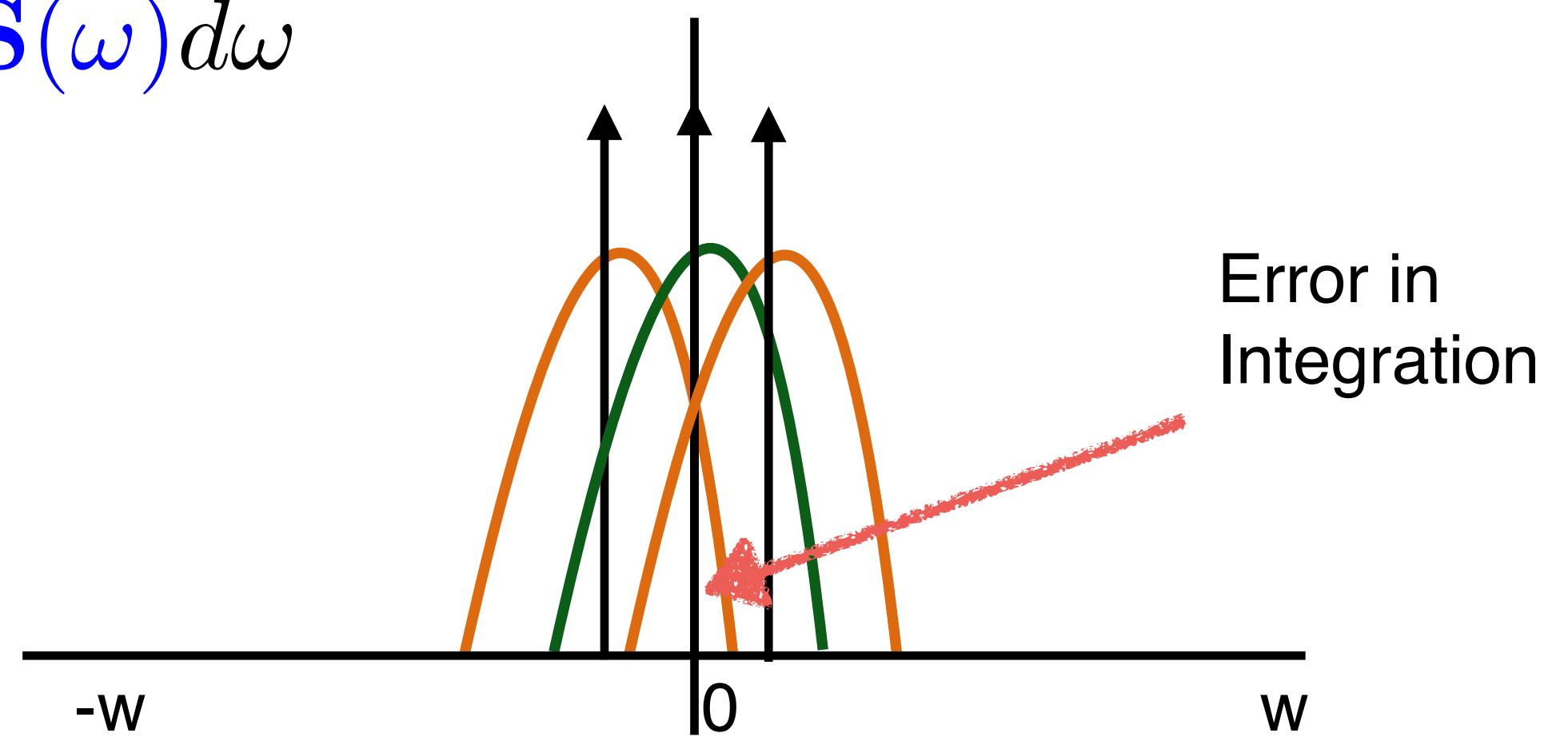
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Fredo Durand [2011]

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Fredo Durand [2011]

$$\text{Error} = \text{Bias}^2 + \text{Variance}$$

Properties of Error

- Bias
- Variance

Properties of Error

- Bias: Expected value of the Error
- Variance

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- Bias: Expected value of the Error $\langle I - \tilde{\mu}_N \rangle$
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Subr and Kautz [2013]

Bias in the Monte Carlo Estimator

Bias in Fourier Domain

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Bias:

$$\langle I - \tilde{\mu}_N \rangle$$

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Subr and Kautz [2013]

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To obtain an unbiased estimator:

Subr and Kautz [2013]

$$\langle \hat{\mathbf{S}}(\omega) \rangle = 0$$

for frequencies other than zero

How to obtain $\langle \hat{\mathbf{S}}(\omega) \rangle = 0$?

Complex form in Amplitude and Phase

$$\langle \hat{\mathbf{S}}(\omega) \rangle = |\langle \hat{\mathbf{S}}(\omega) \rangle| e^{-\Phi(\langle \hat{\mathbf{S}}(\omega) \rangle)}$$

Complex form in Amplitude and Phase

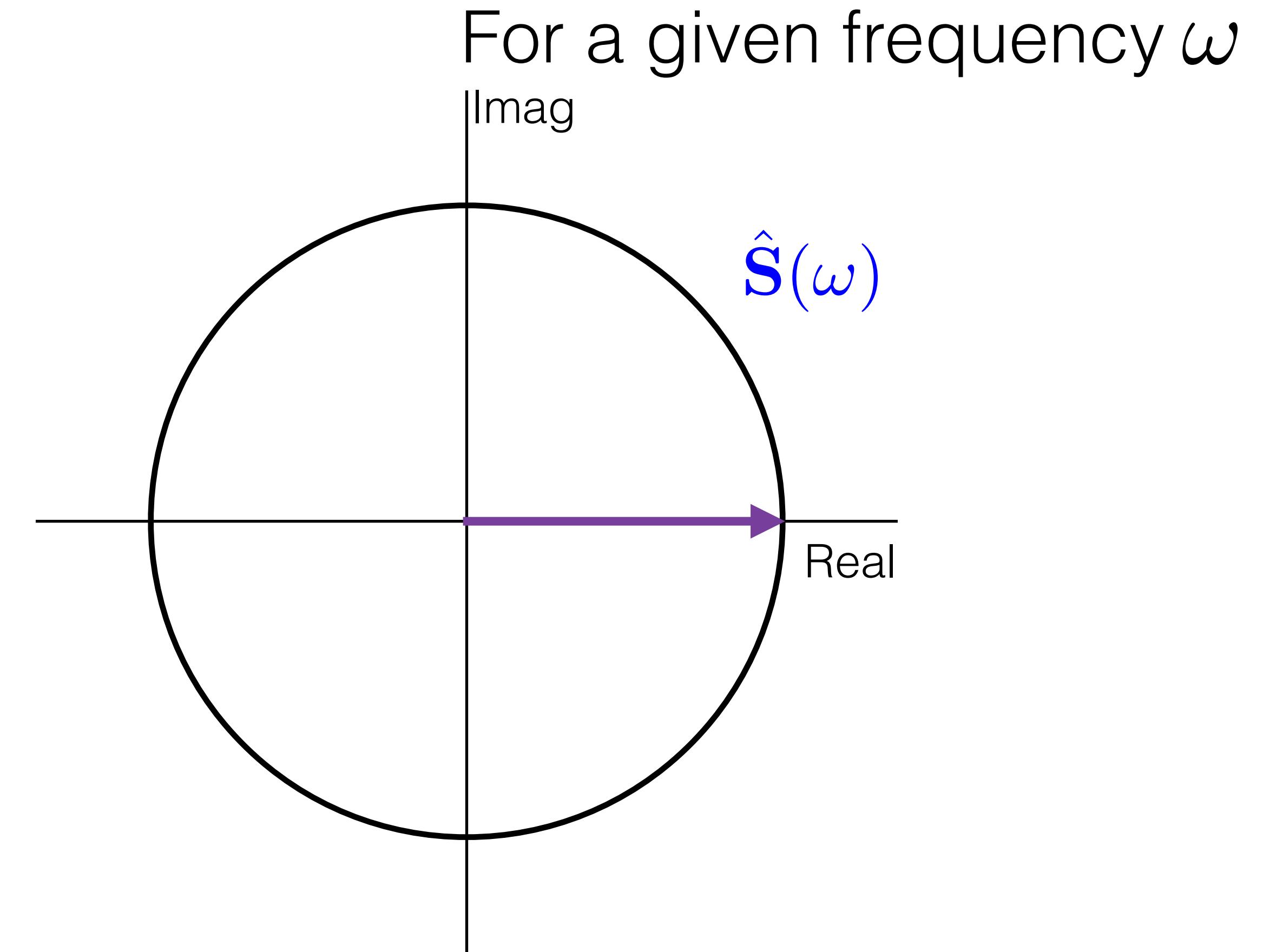
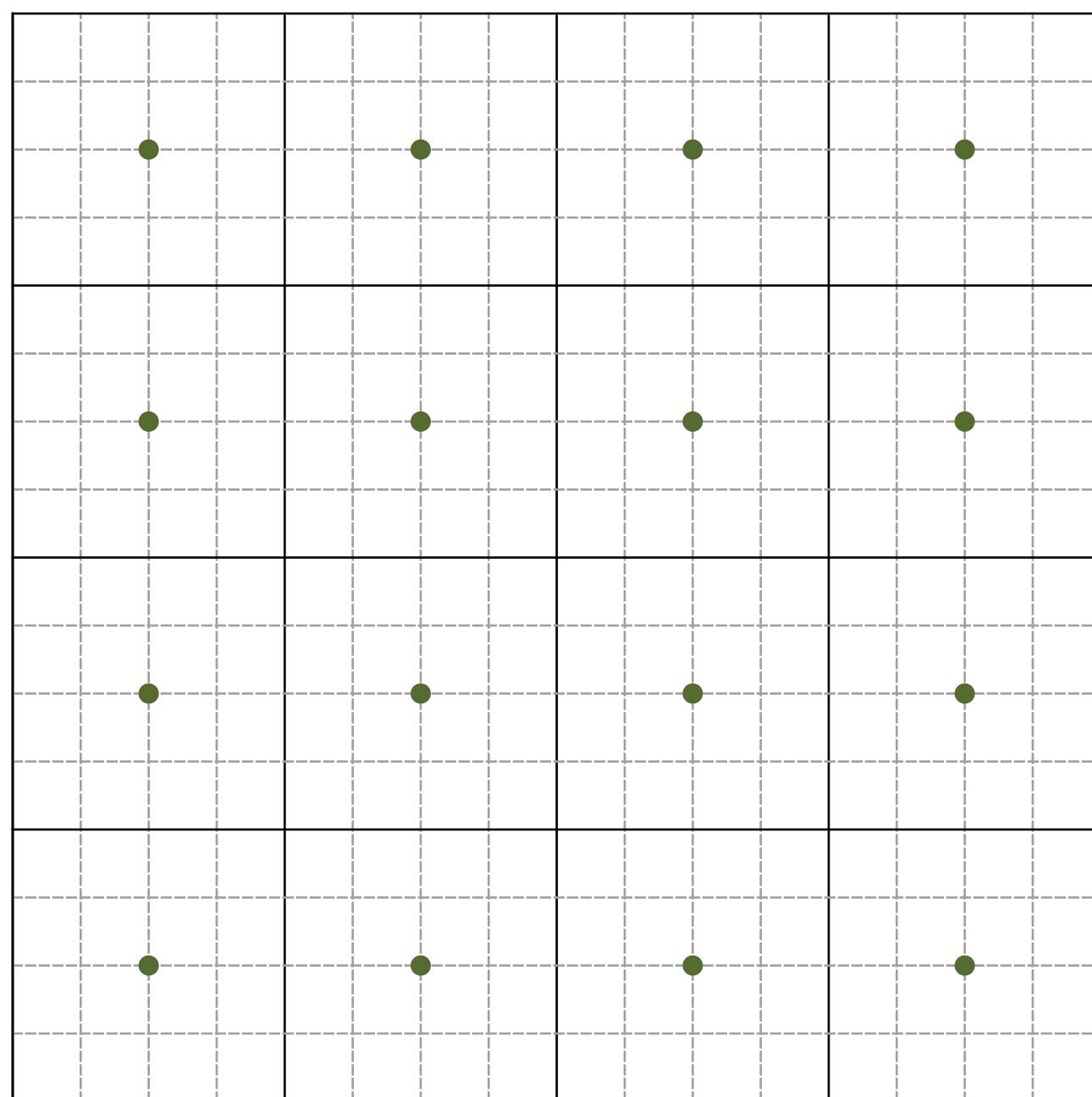
Amplitude

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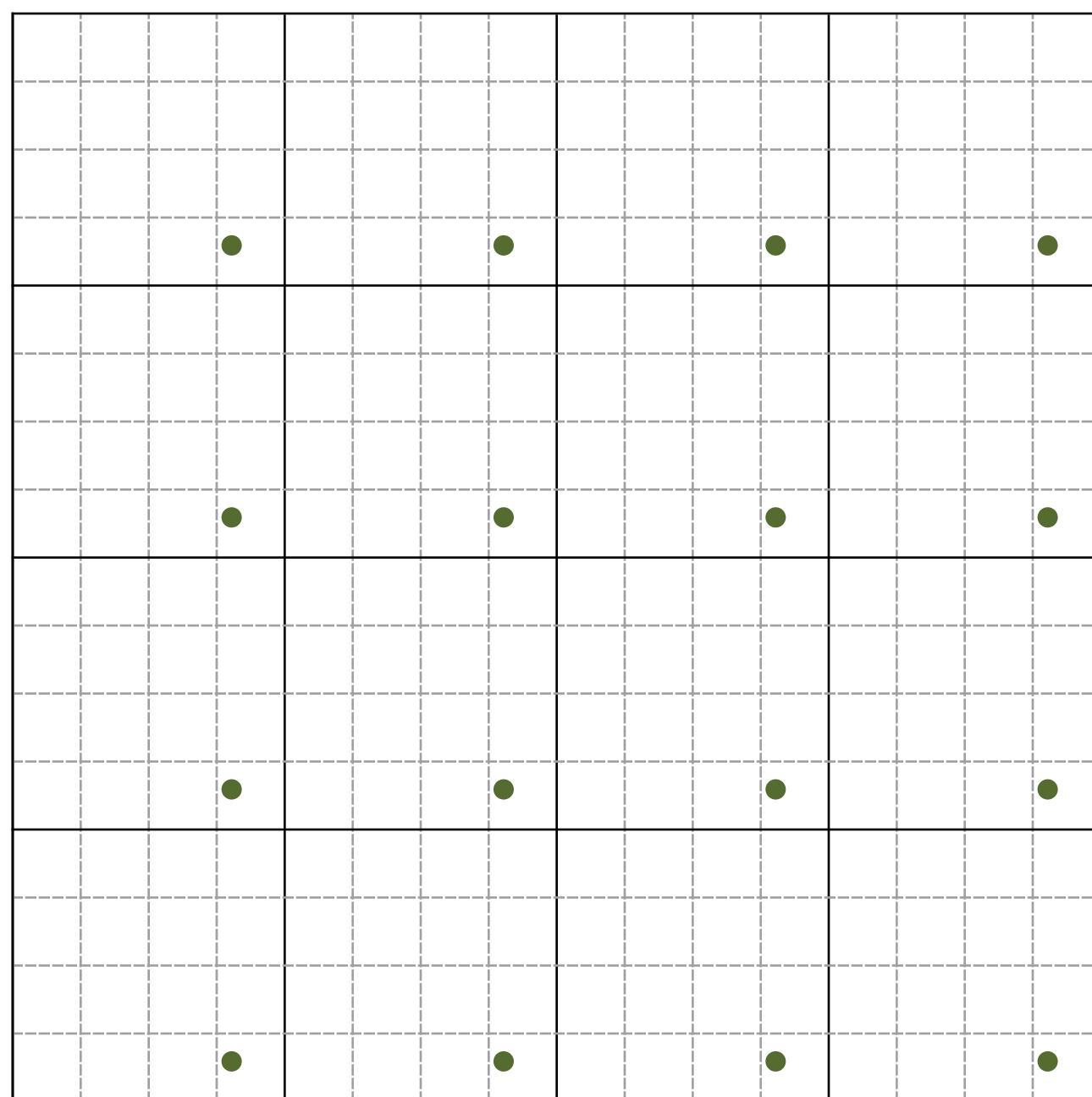
Complex form in Amplitude and Phase

$$\langle \hat{\mathbf{S}}(\omega) \rangle = \boxed{\text{Amplitude}} |\langle \hat{\mathbf{S}}(\omega) \rangle| e^{-j\boxed{\text{Phase}} \Phi(\langle \hat{\mathbf{S}}(\omega) \rangle)}$$

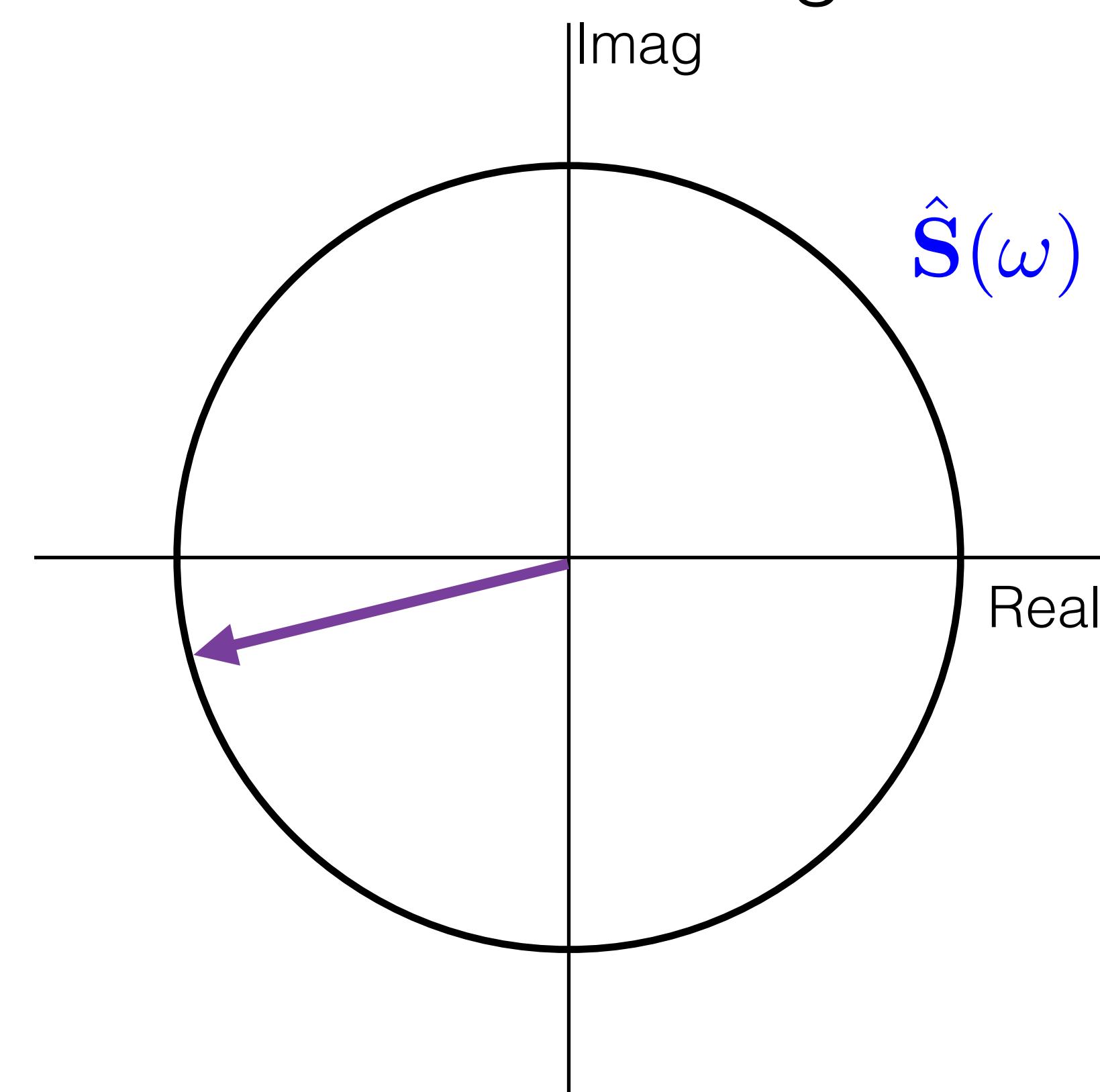
Phase change due to Random Shift



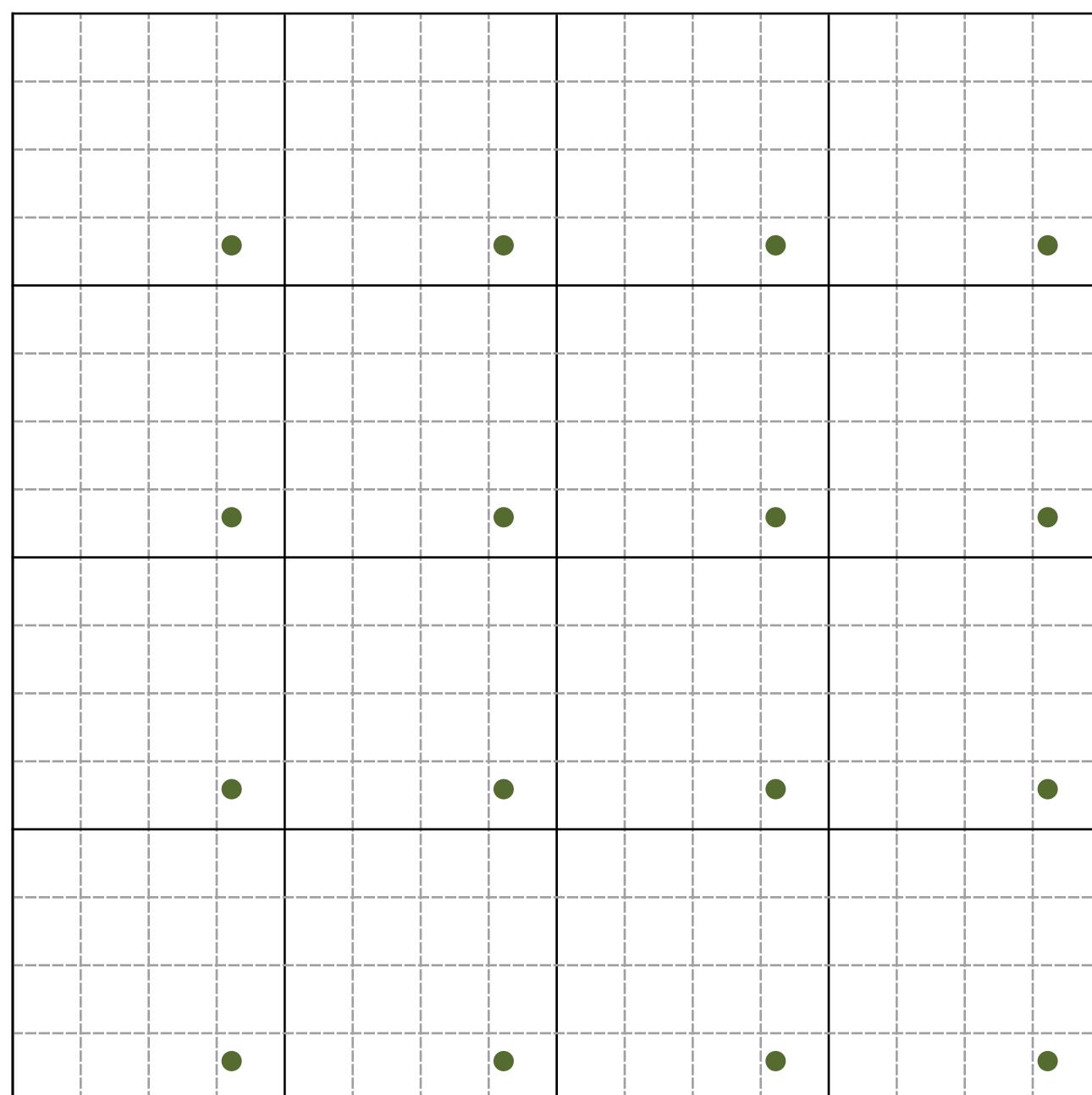
Phase change due to Random Shift



For a given frequency ω

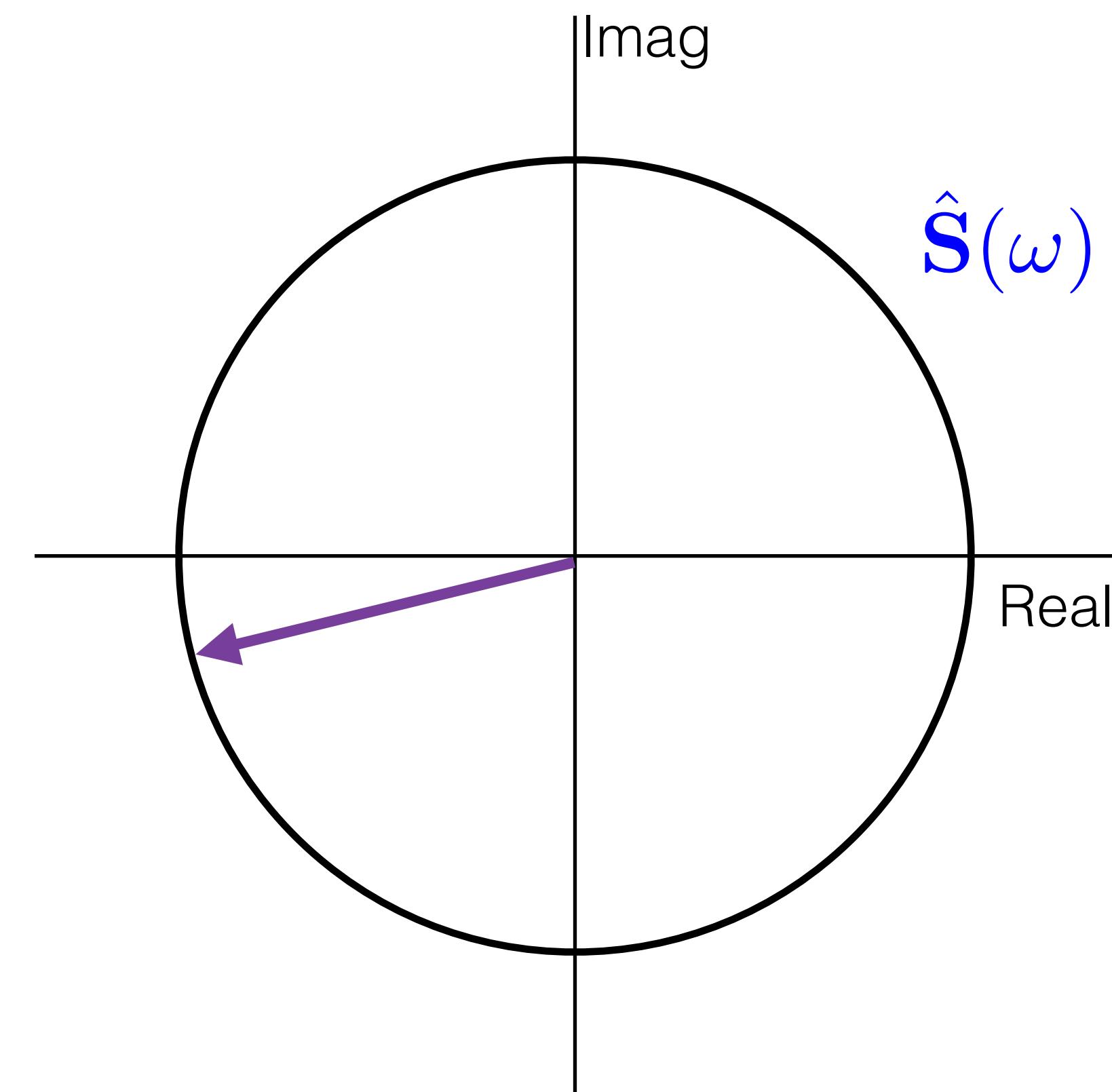


Phase change due to Random Shift

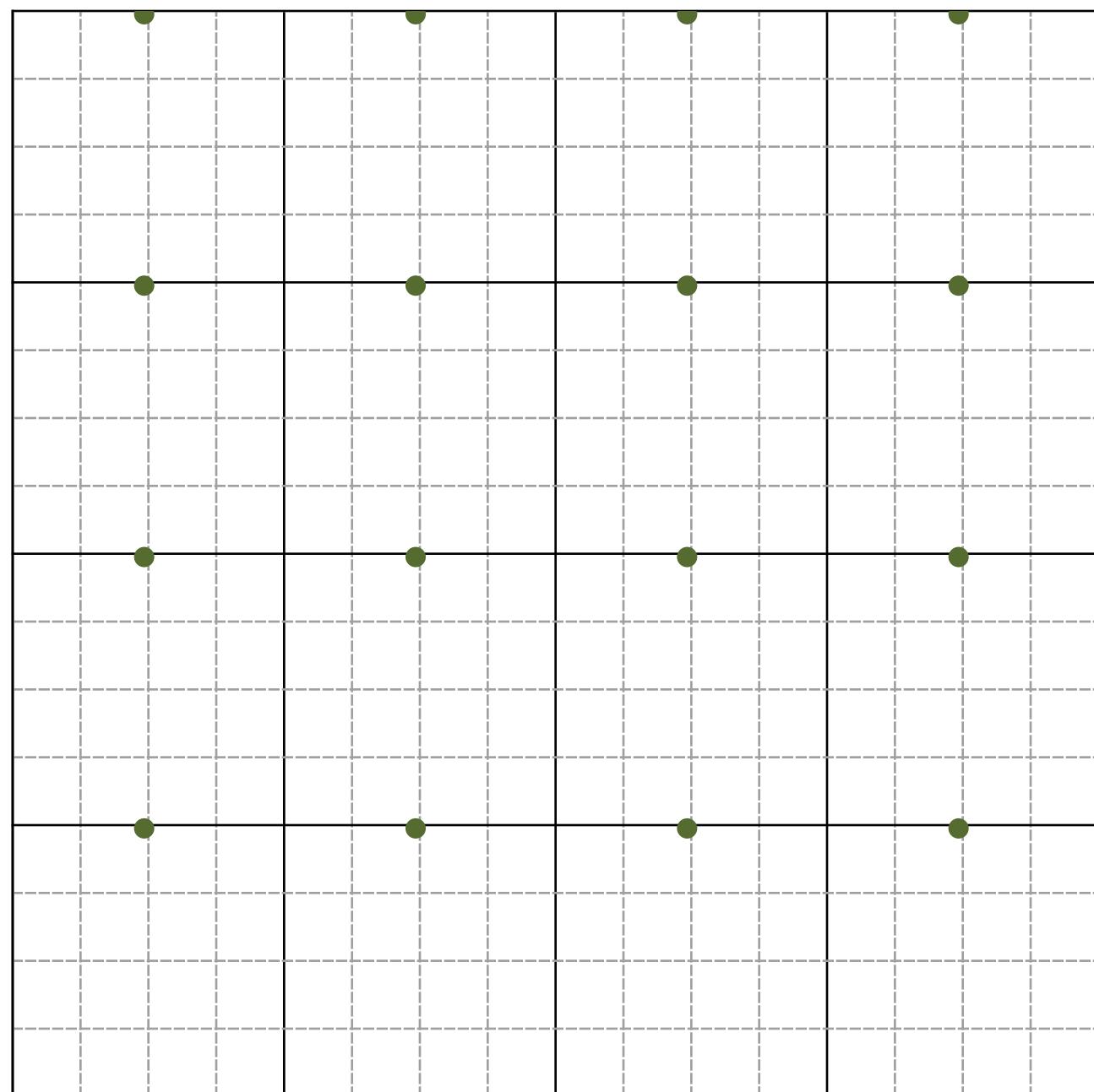


Pauly et al. [2000]
Ramamoorthi et al. [2012]

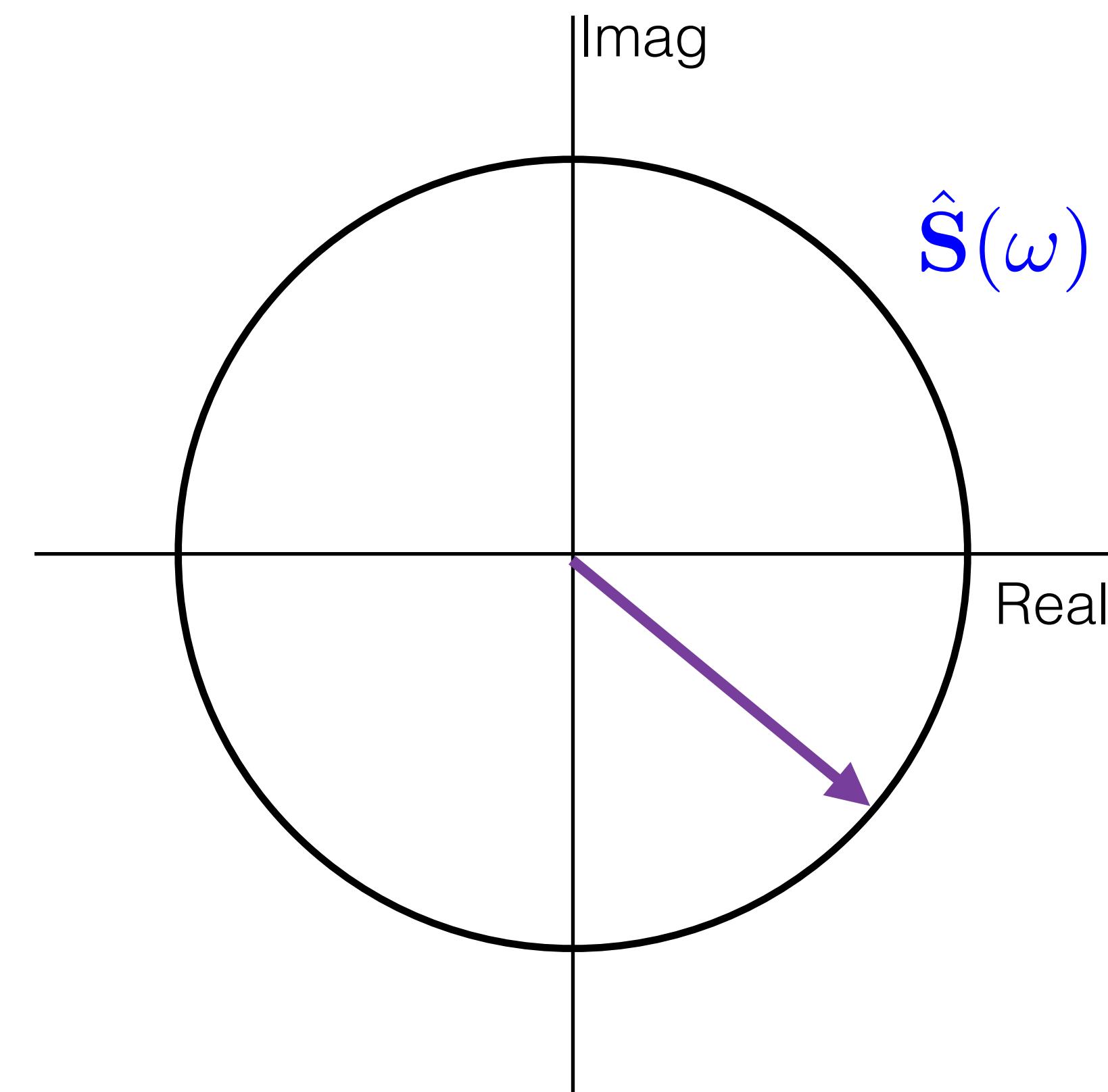
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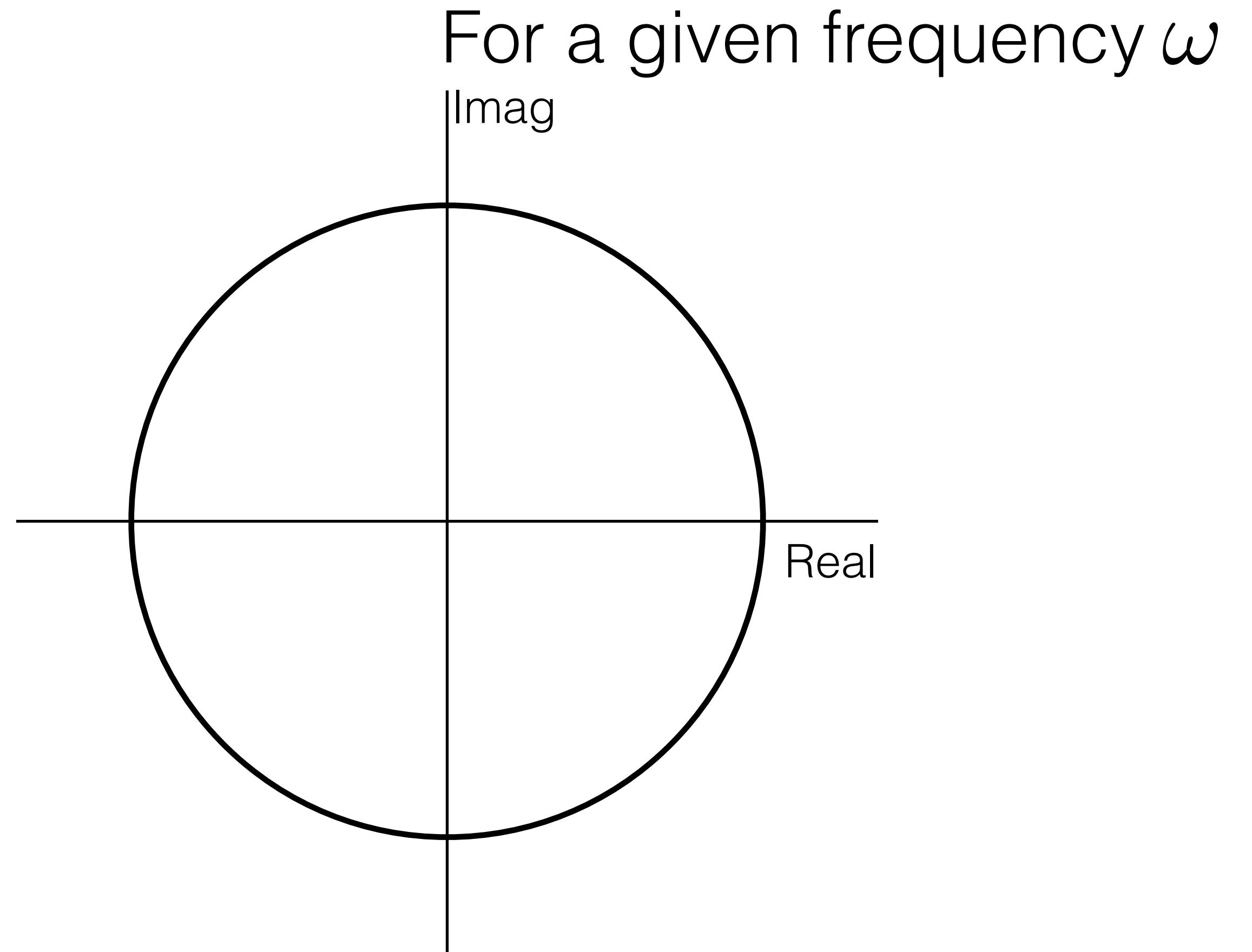


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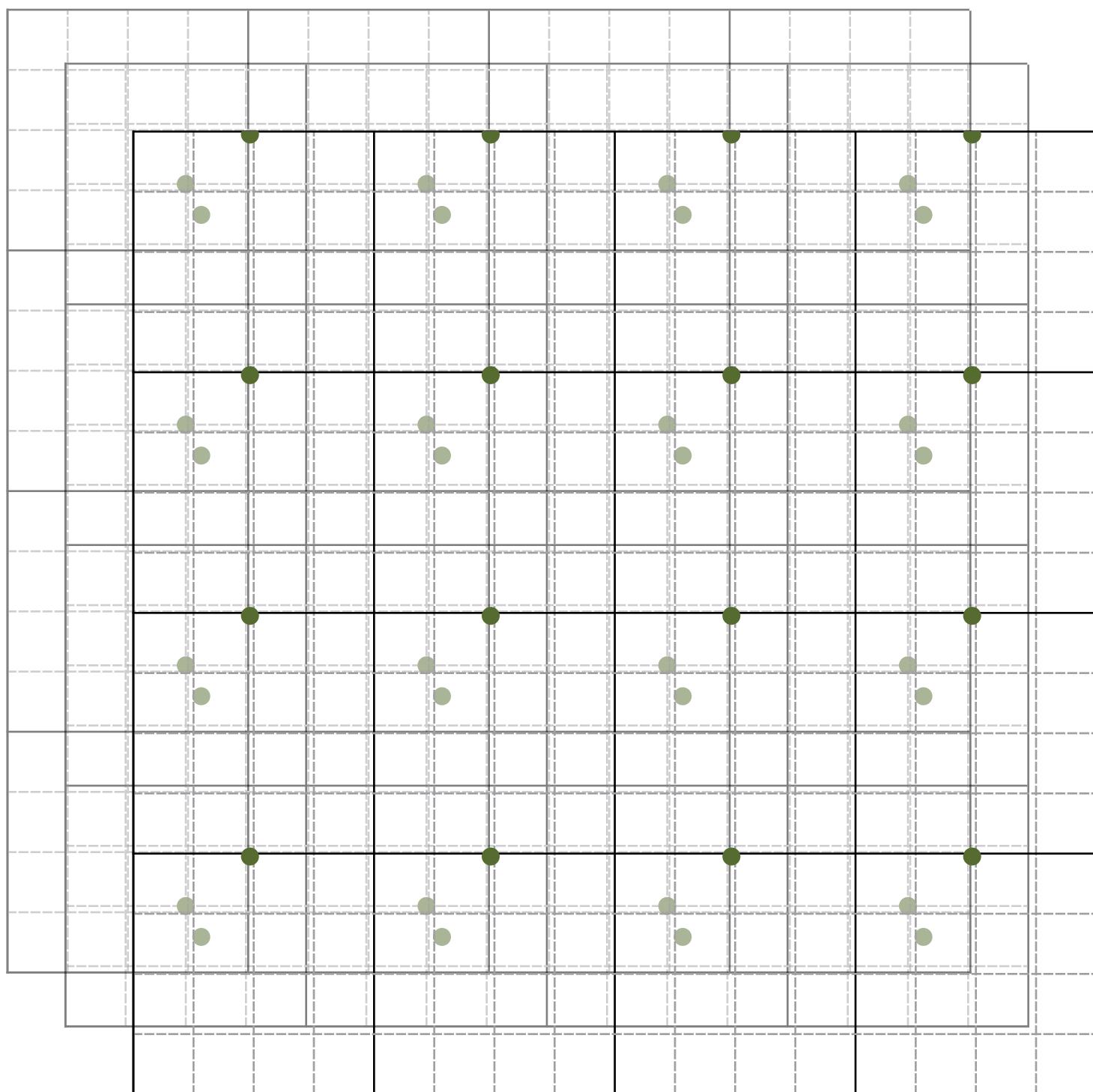
Phase change due to Random Shift

Multiple realizations

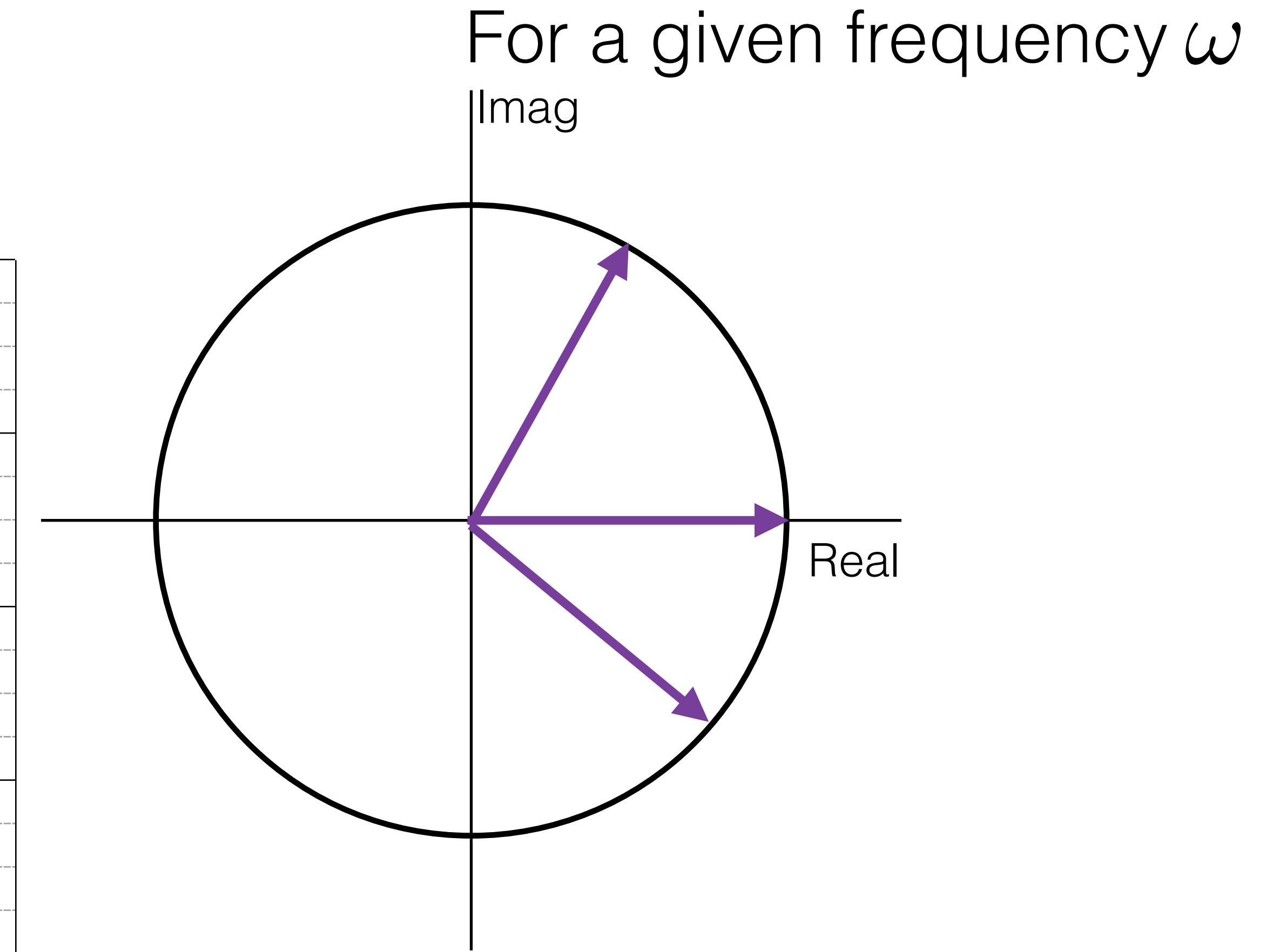


Phase change due to Random Shift

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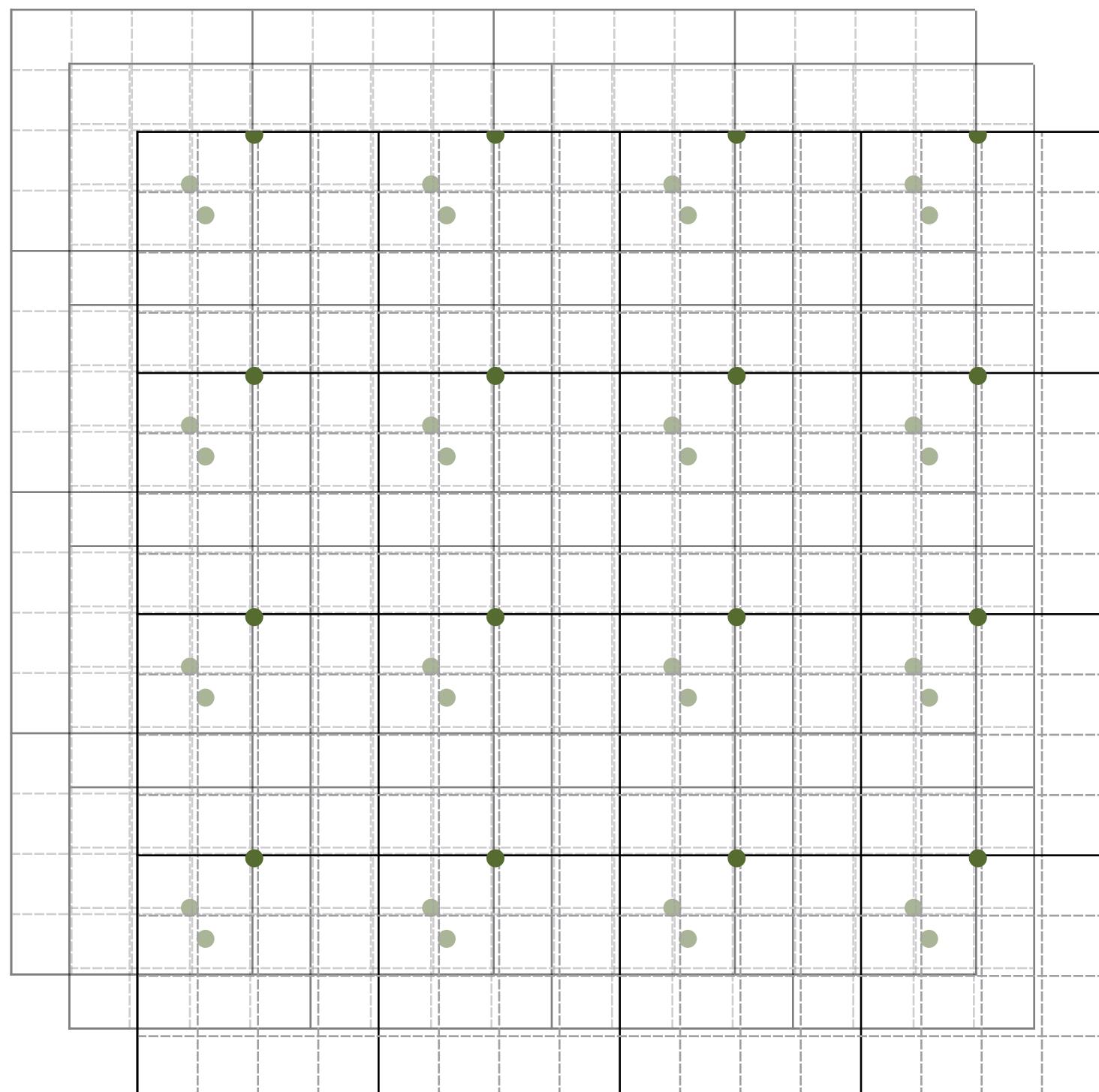


For a given frequency ω

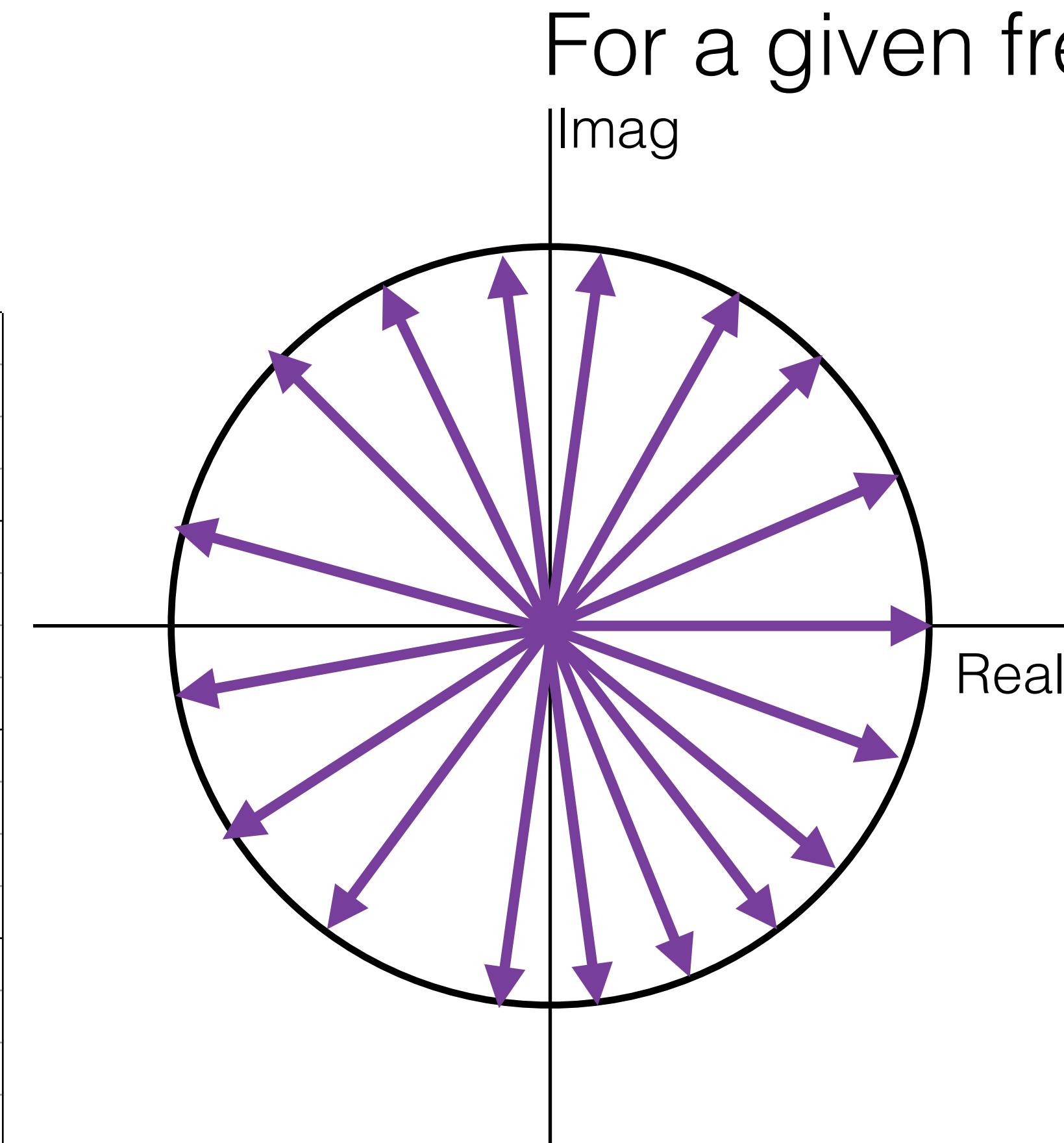


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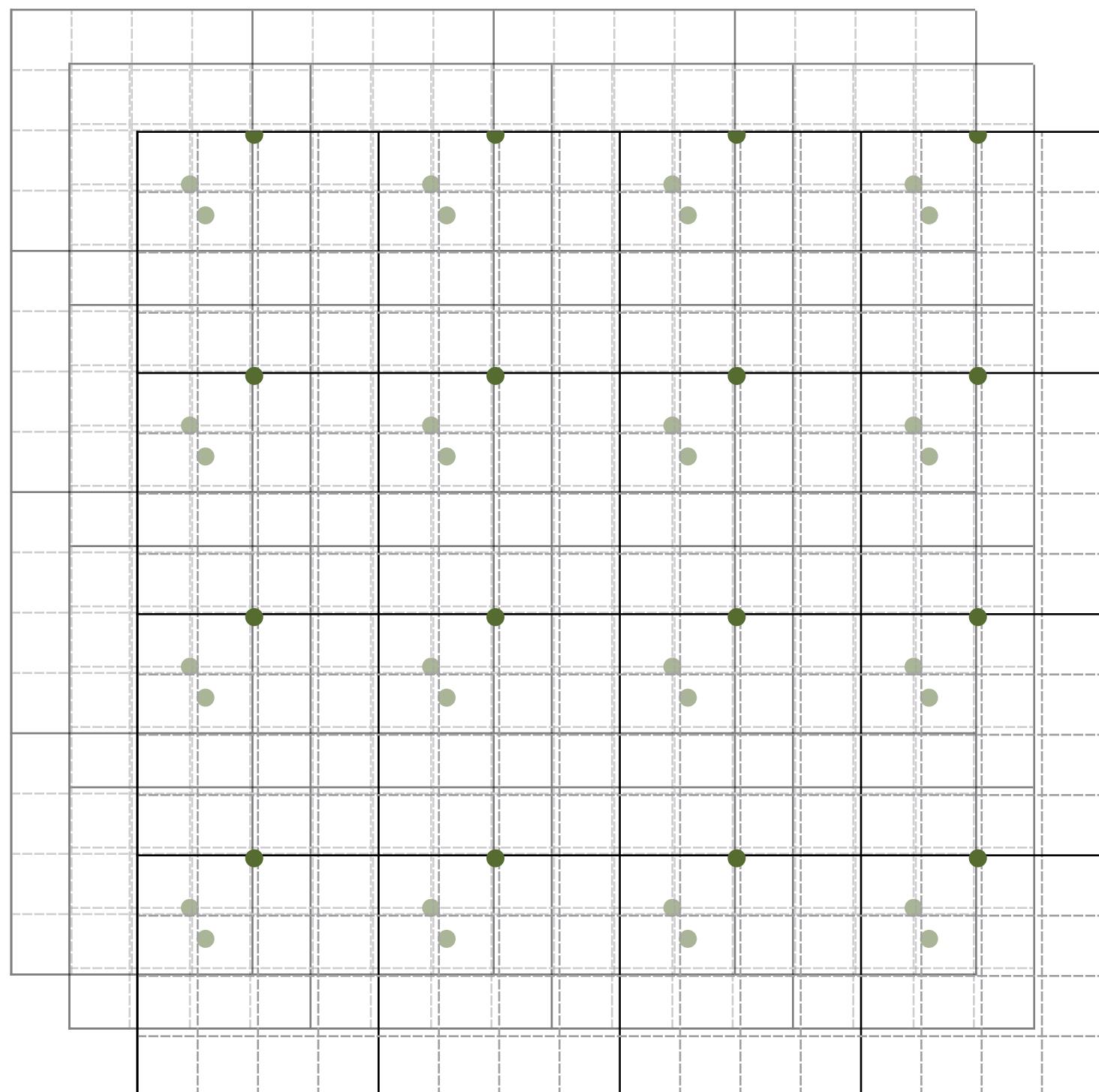


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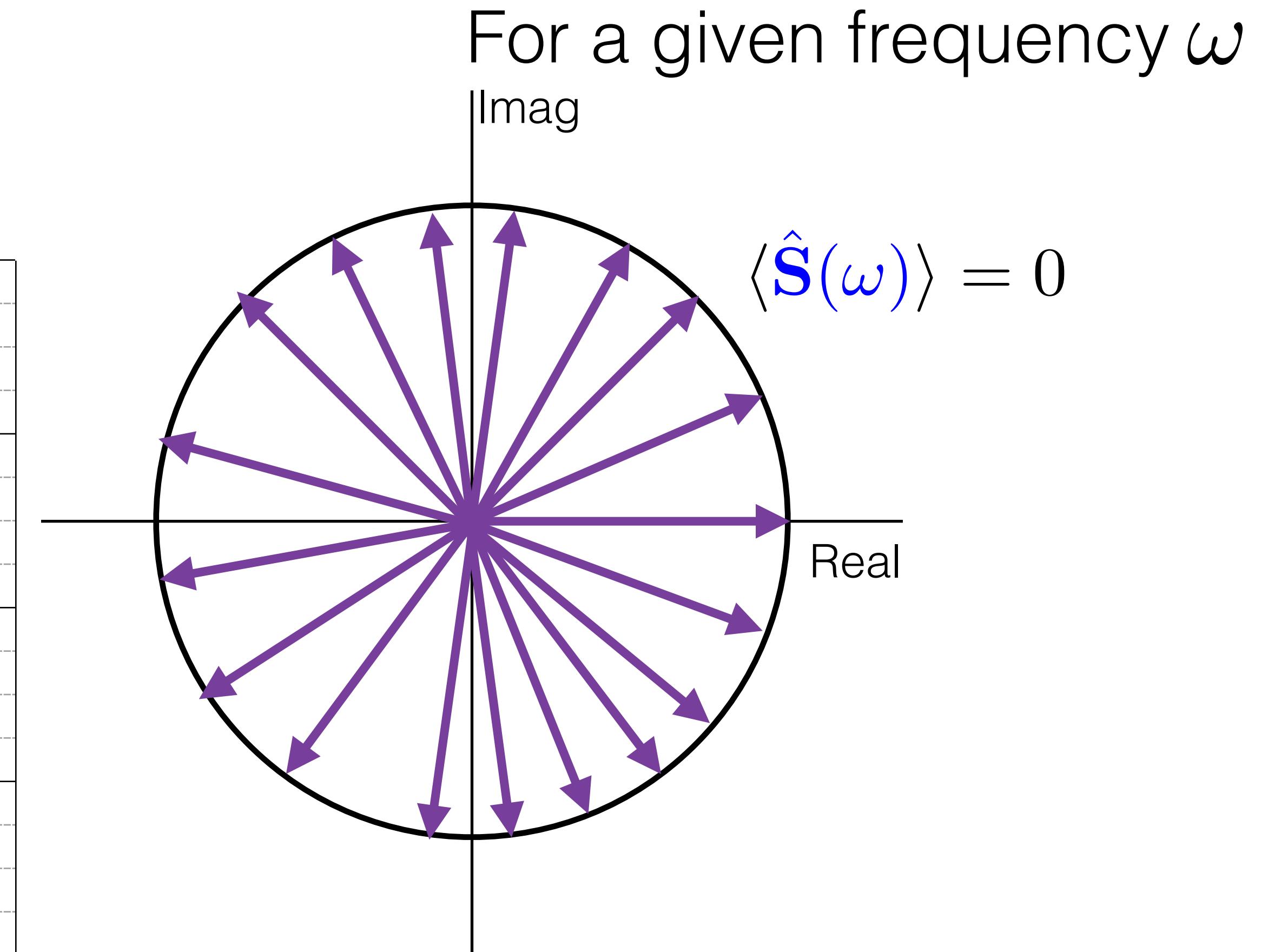


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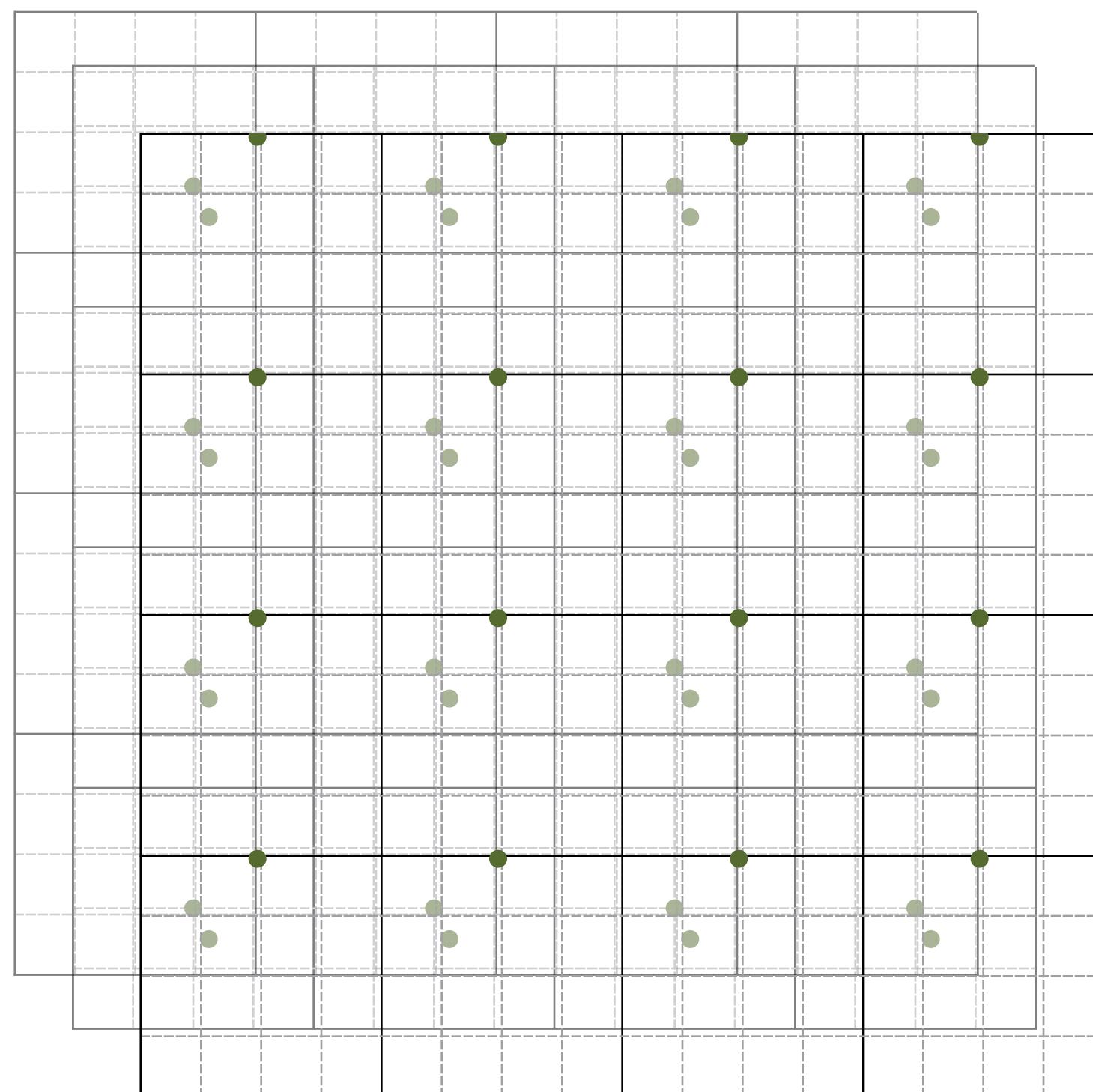


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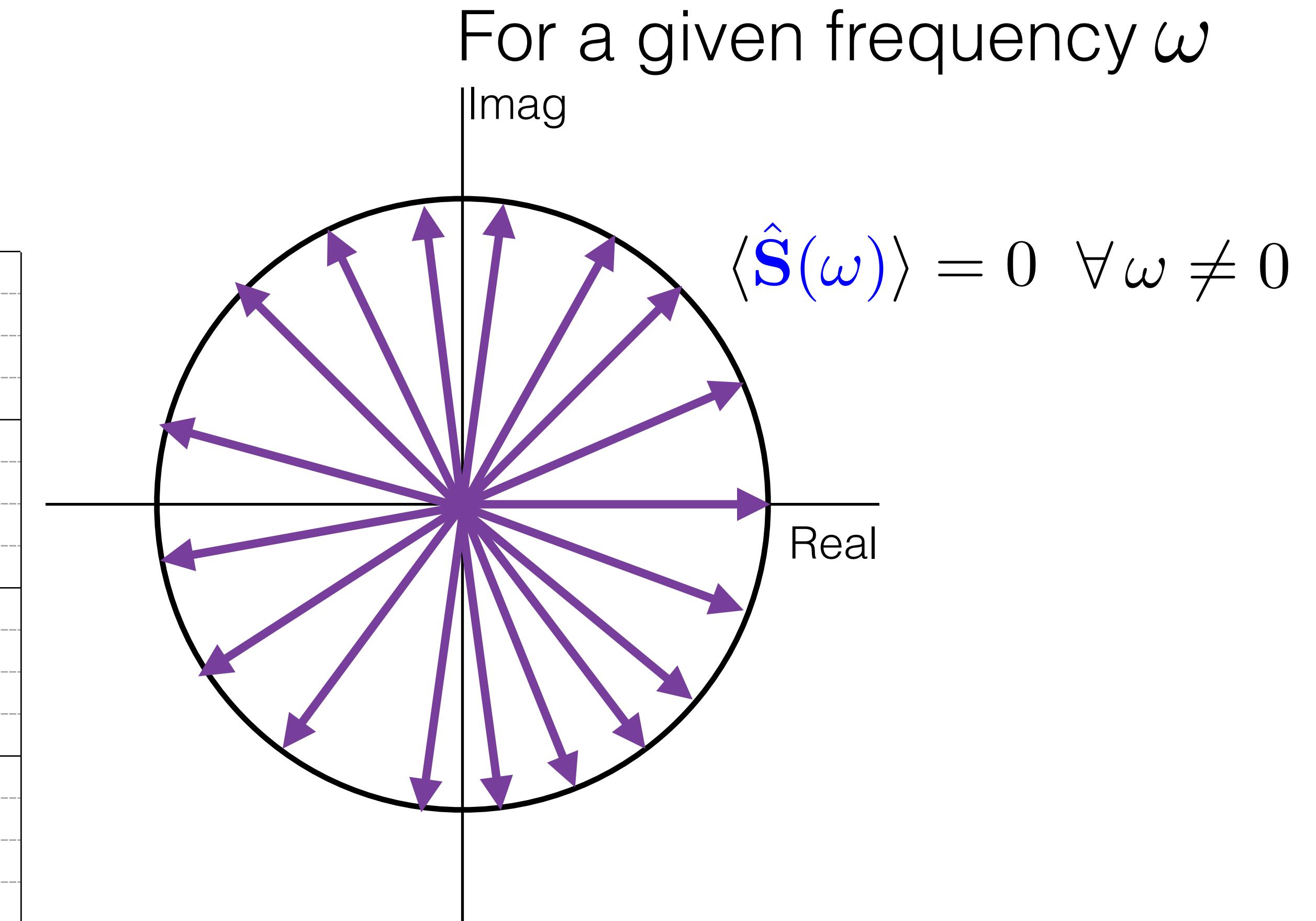


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For a given frequency ω



$$\text{Error} = \text{Bias}^2 + \text{Variance}$$

Error = ~~Bias²~~ + Variance

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- Homogenization allows representation of error only in terms of variance
- We can take any sampling pattern and homogenize it to make the Monte Carlo estimator unbiased.

Variance in the Fourier domain

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Error:

$$I - \tilde{\mu}_N = \hat{f}(0) - \int_{\Omega} \hat{f}^*(\omega) \hat{\mathbf{S}}(\omega) d\omega$$

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where,

$$P_f(\omega) = |\hat{f}^*(\omega)|^2 \quad \text{Power Spectrum}$$

Variance in the Fourier domain

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Subr and Kautz [2013]

Variance in the Fourier domain

$$\text{Var}(\tilde{\mu}_N) = \int_{\Omega} P_f(\omega) \text{Var}(\hat{\mathbf{S}}(\omega)) d\omega$$

Subr and Kautz [2013]

This is a general form, both for homogenised as well as non-homogenised sampling patterns

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Fredo Durand [2011]

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Fredo Durand [2011]

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Pilleboue et al. [2015]

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Variance using Homogenized Samples

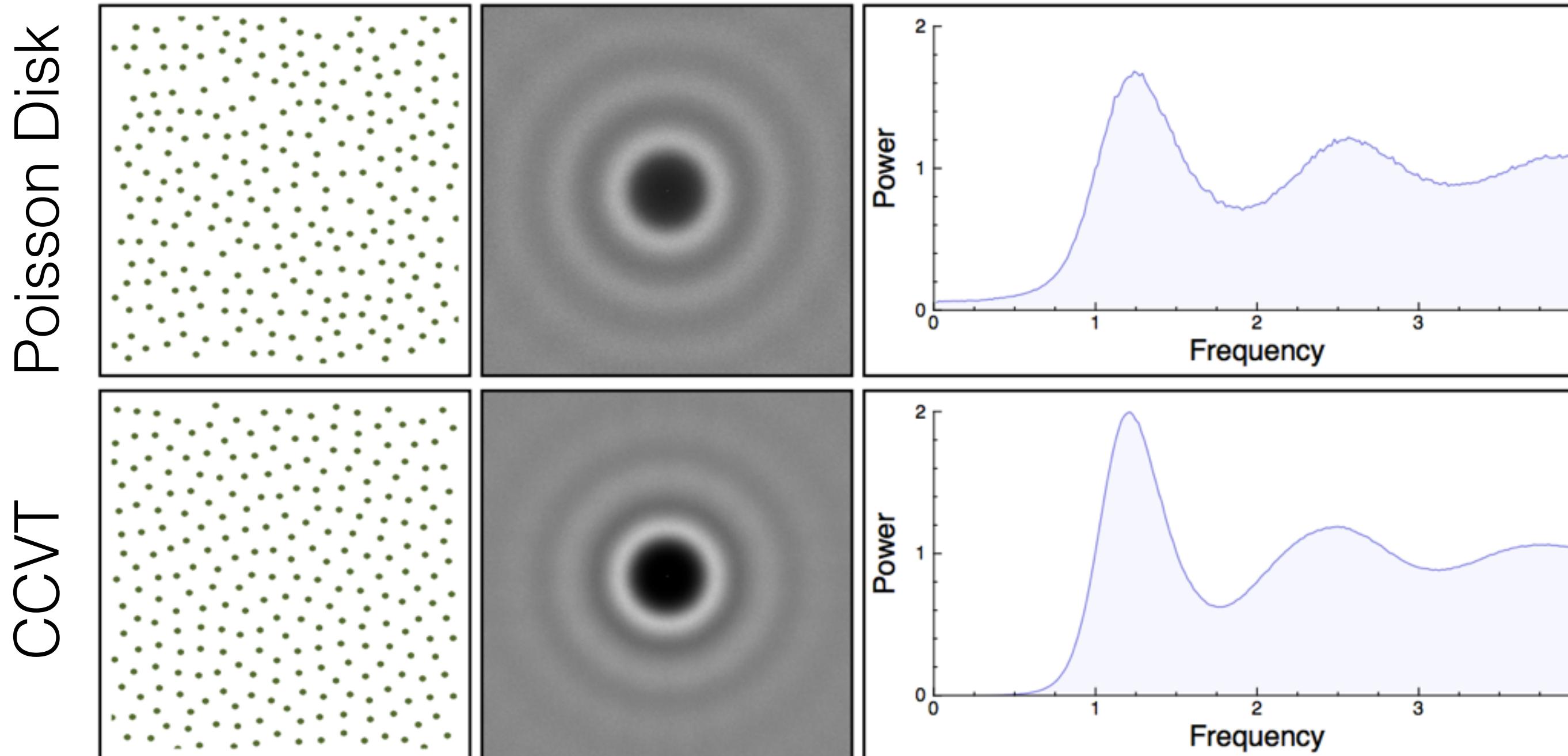
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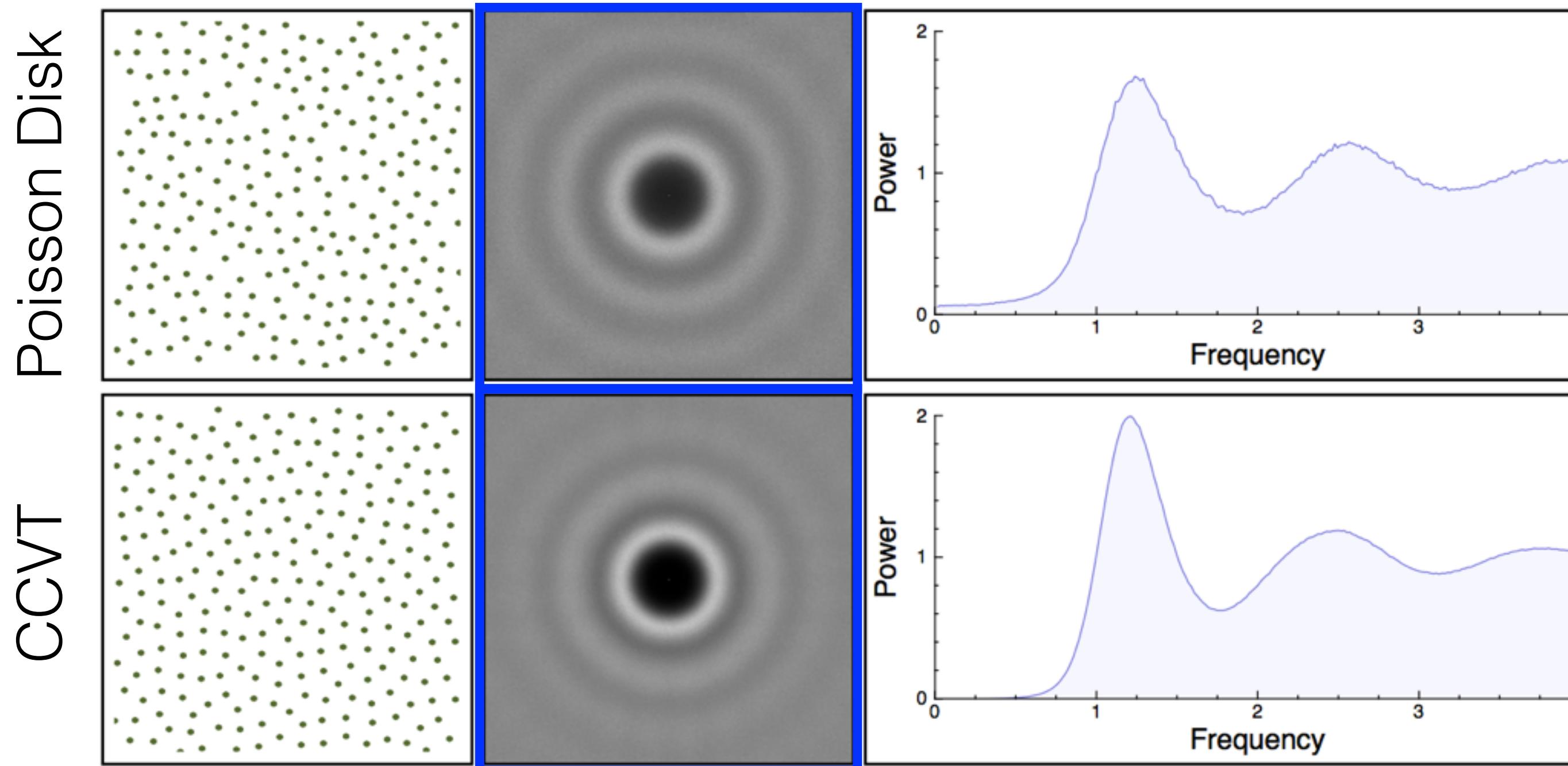
Variance in terms of n-dimensional Power Spectra

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Variance in the Polar Coordinates

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Variance for Isotropic Power Spectra

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Variance for Isotropic Power Spectra

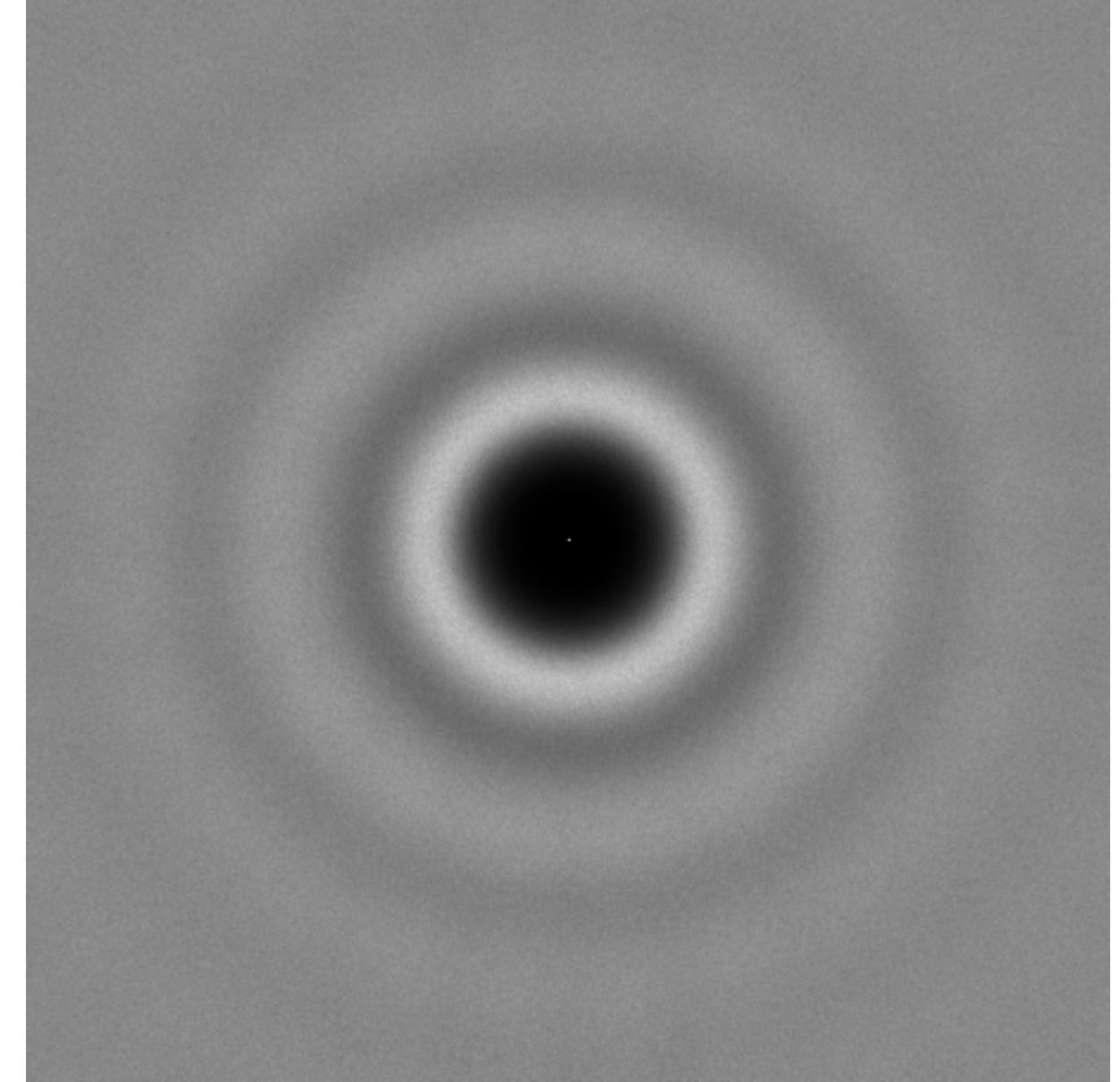
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For isotropic power spectra:

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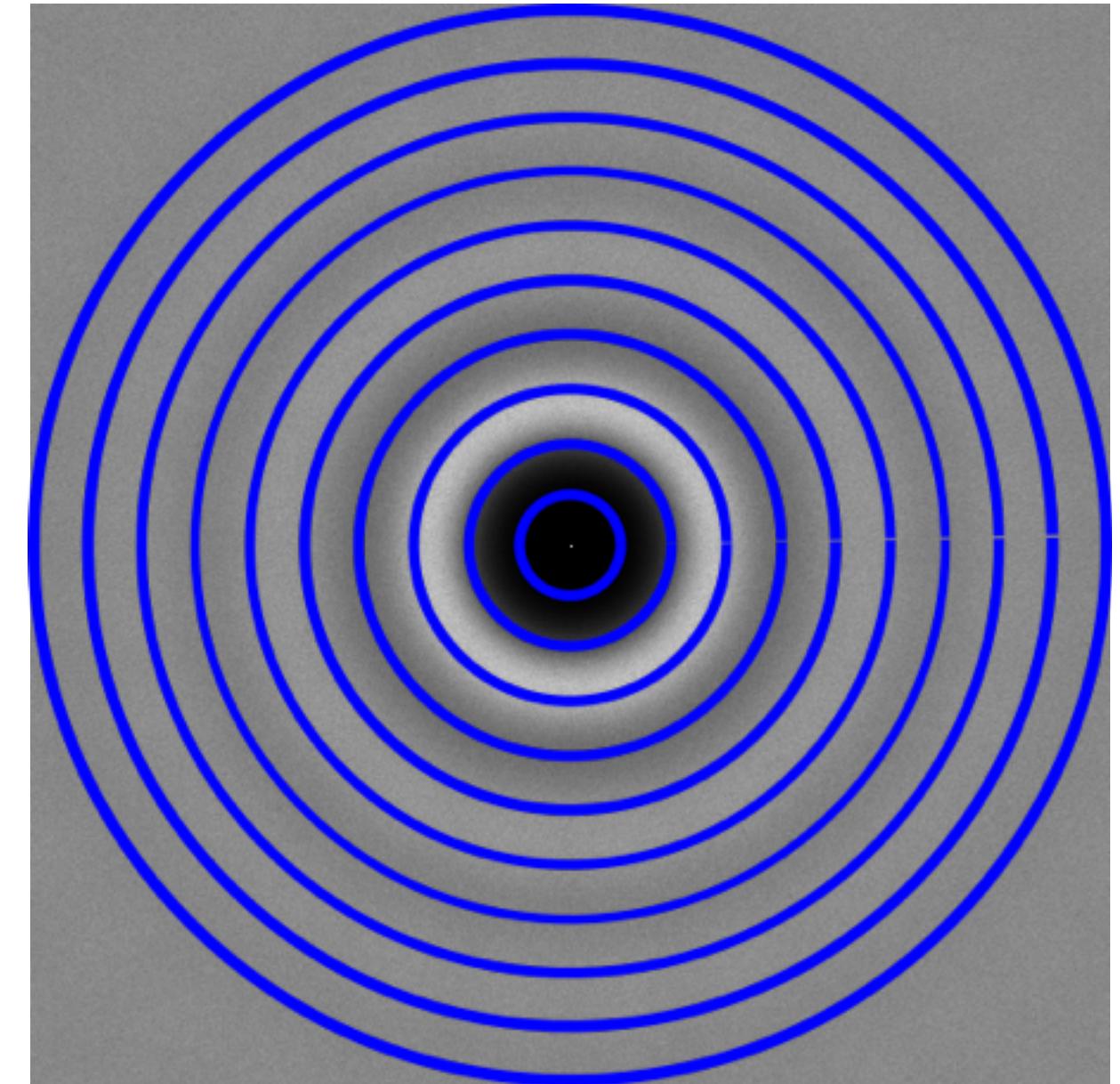
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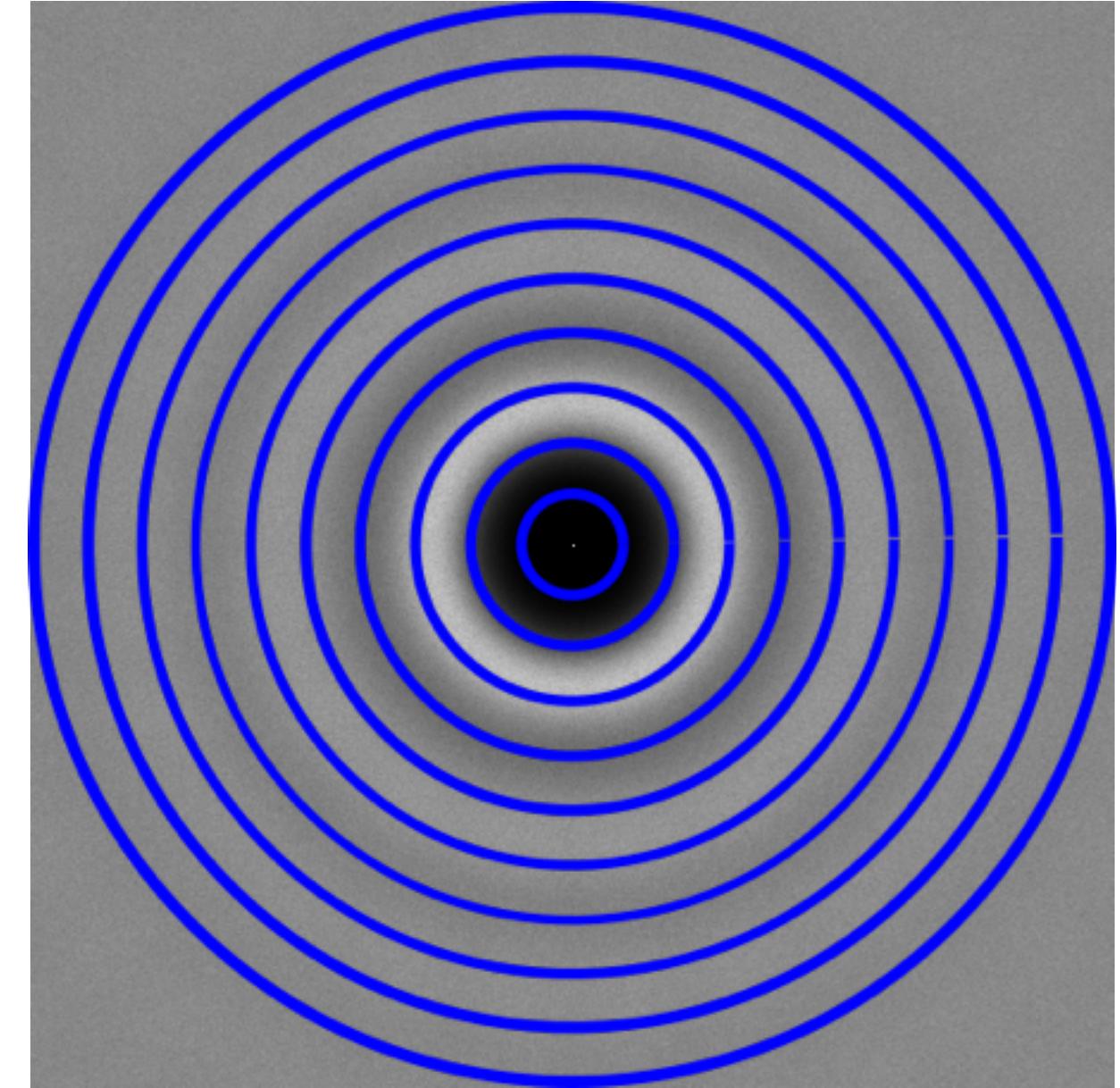


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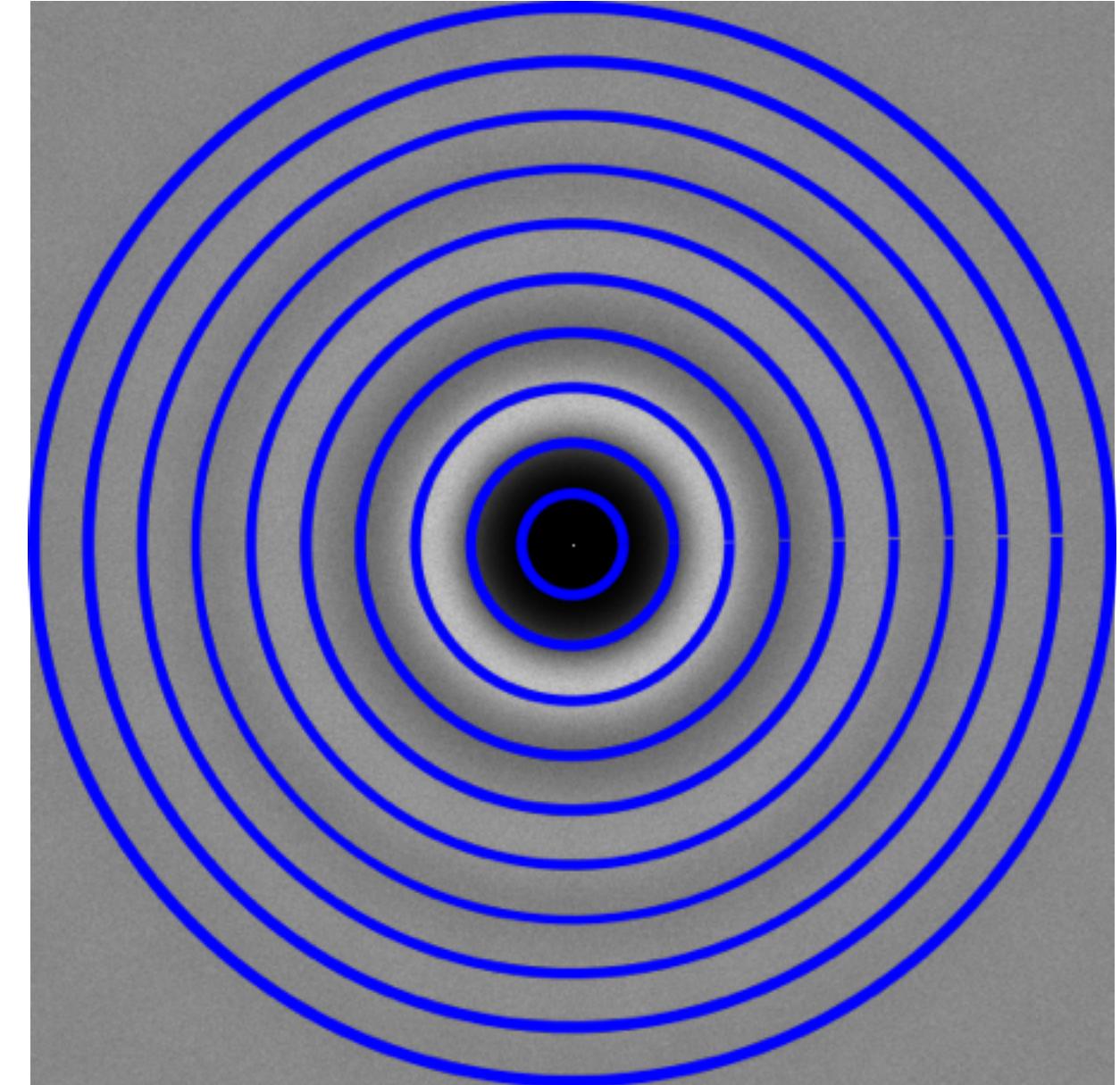


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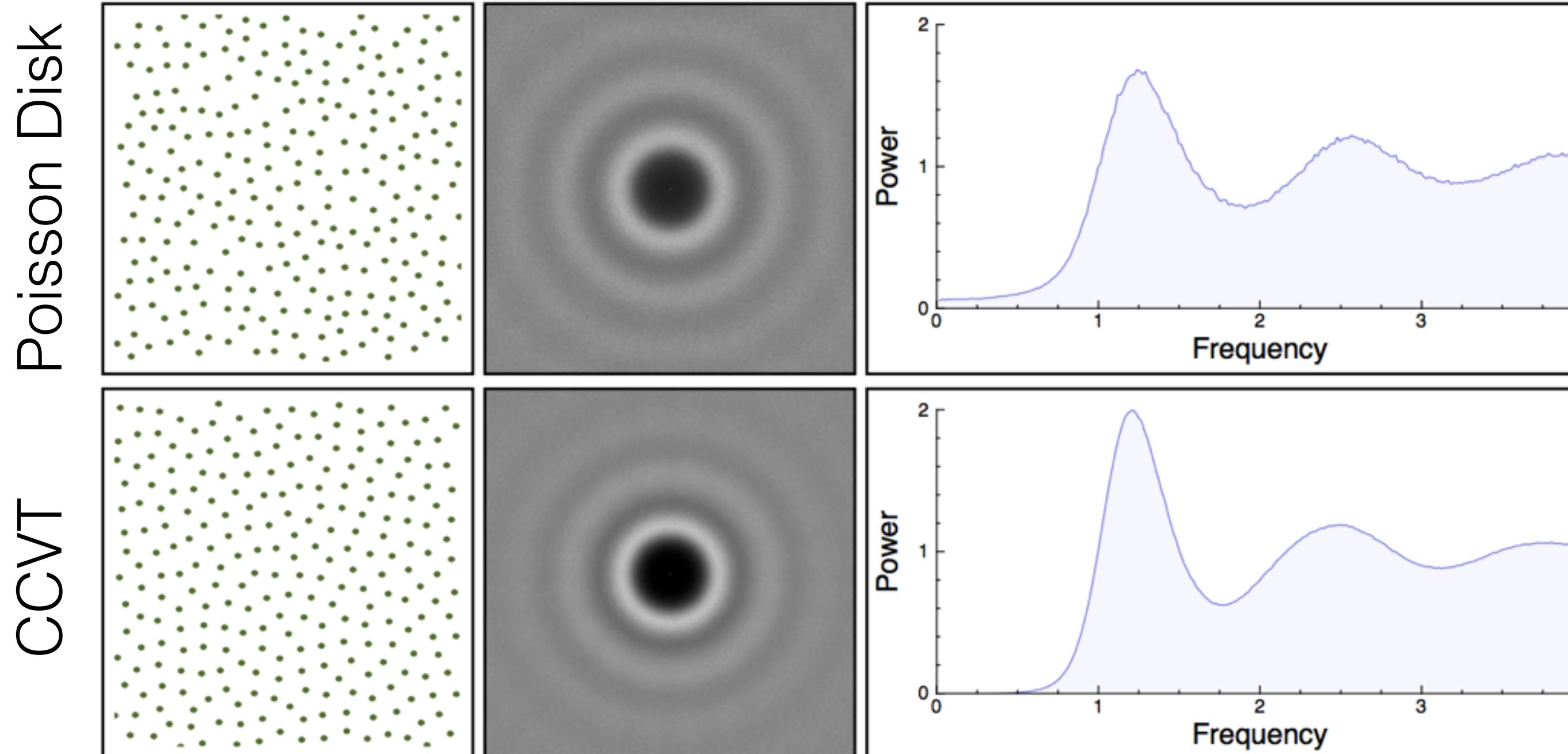
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For isotropic power spectra:

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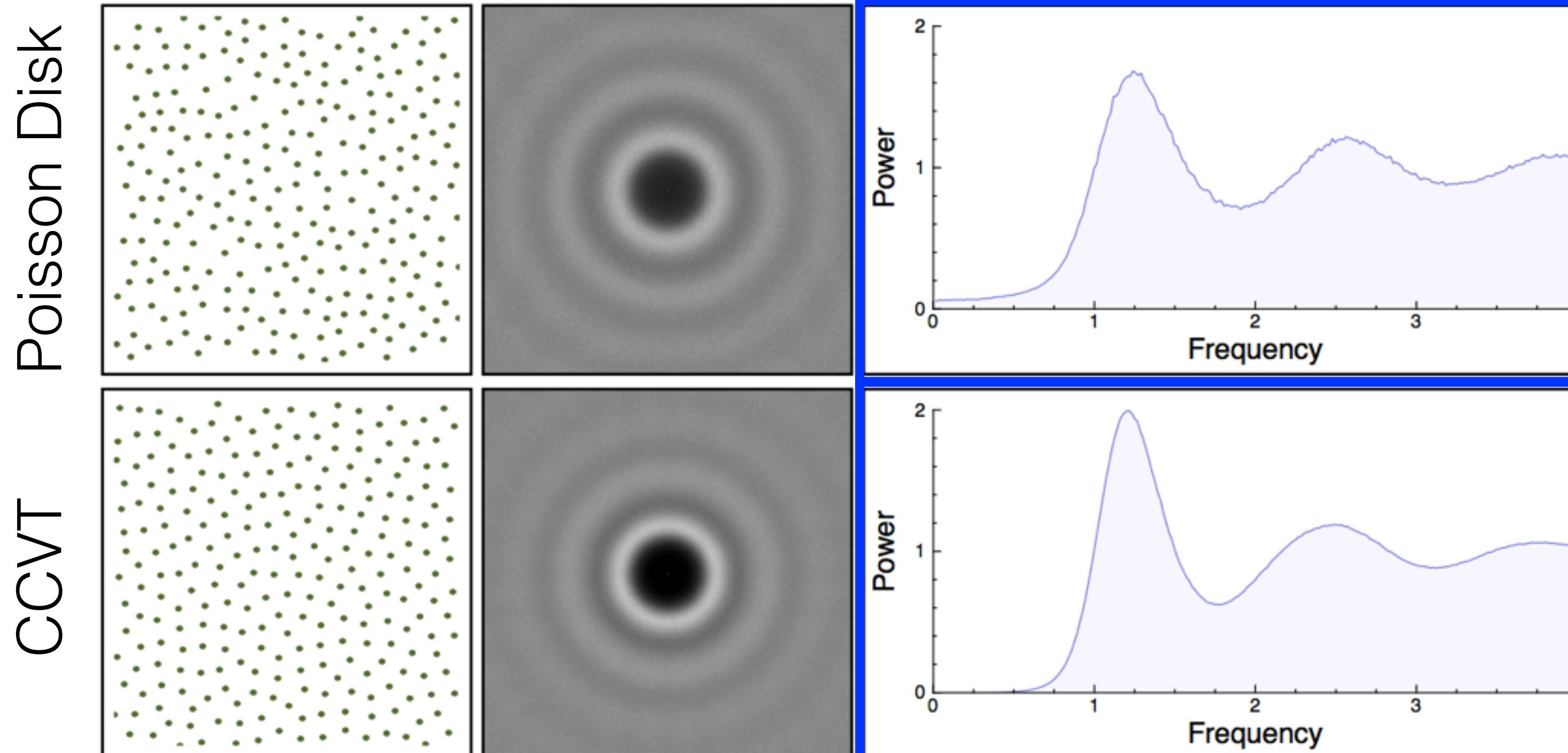
Variance in terms of 1-dimensional Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \langle \tilde{P}_{\mathbf{S}}(\rho) \rangle d\rho$$



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Variance: Integral over Product of Power Spectra

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Integrand Radial Power Spectrum

Sampling Radial Power Spectrum

For given number of Samples

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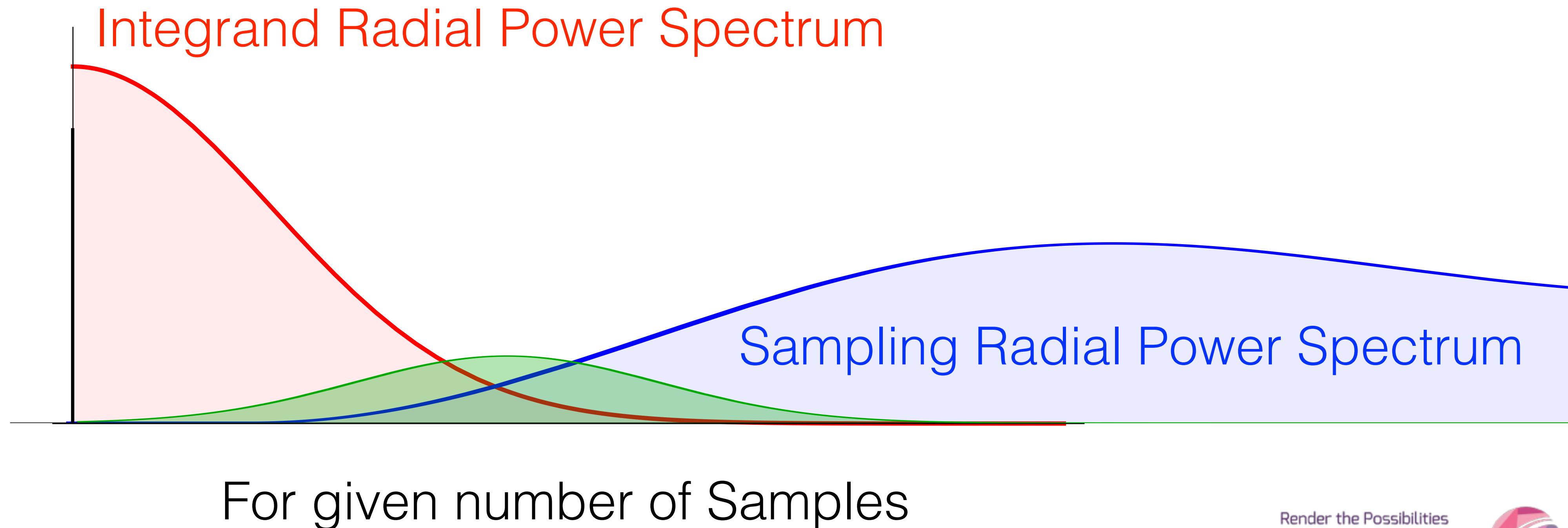
Integrand Radial Power Spectrum

Sampling Radial Power Spectrum

For given number of Samples

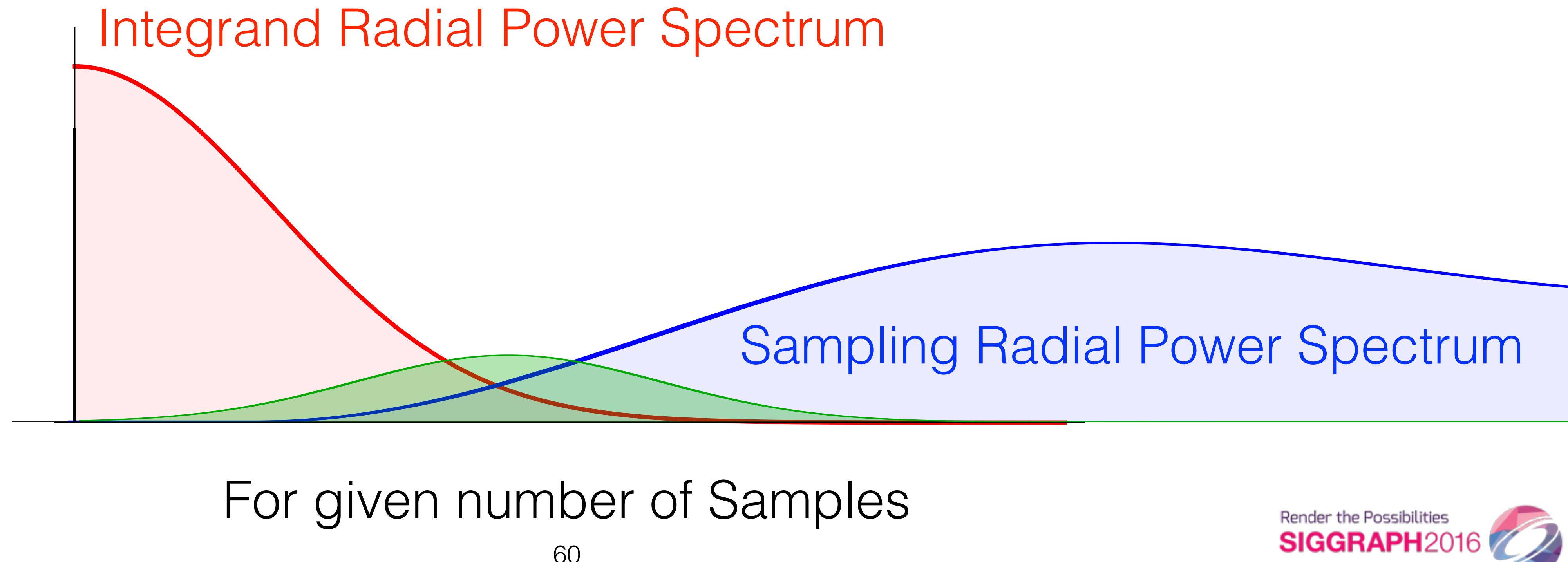
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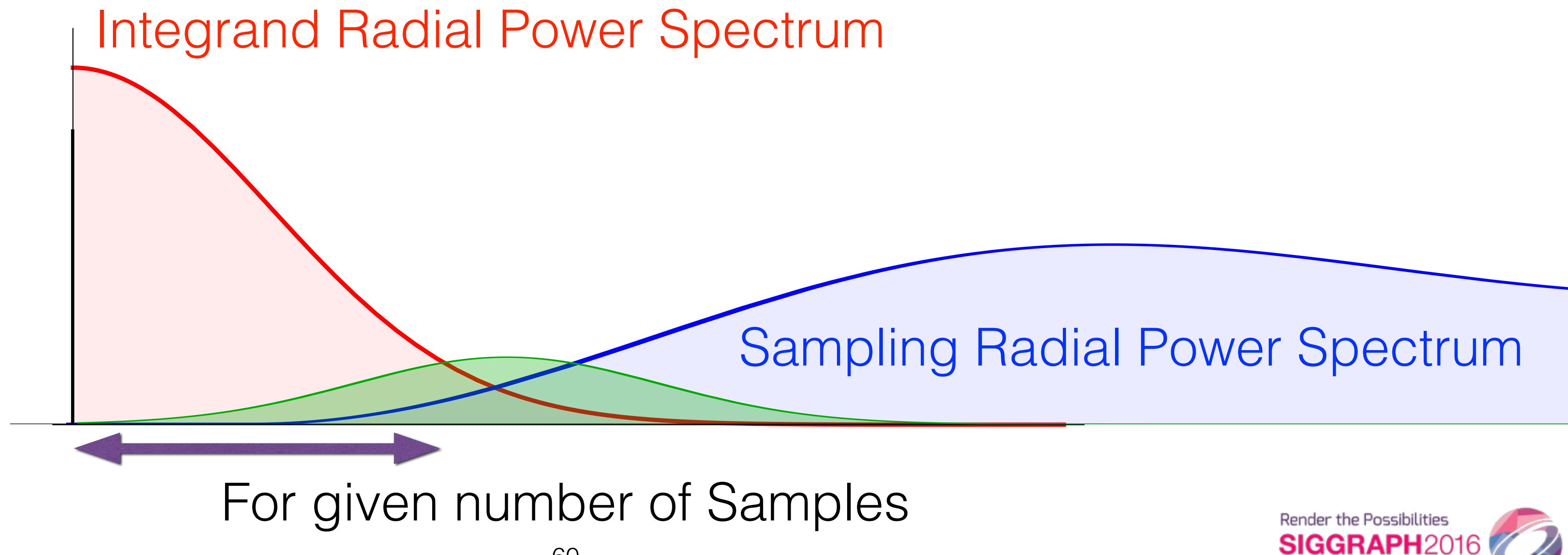
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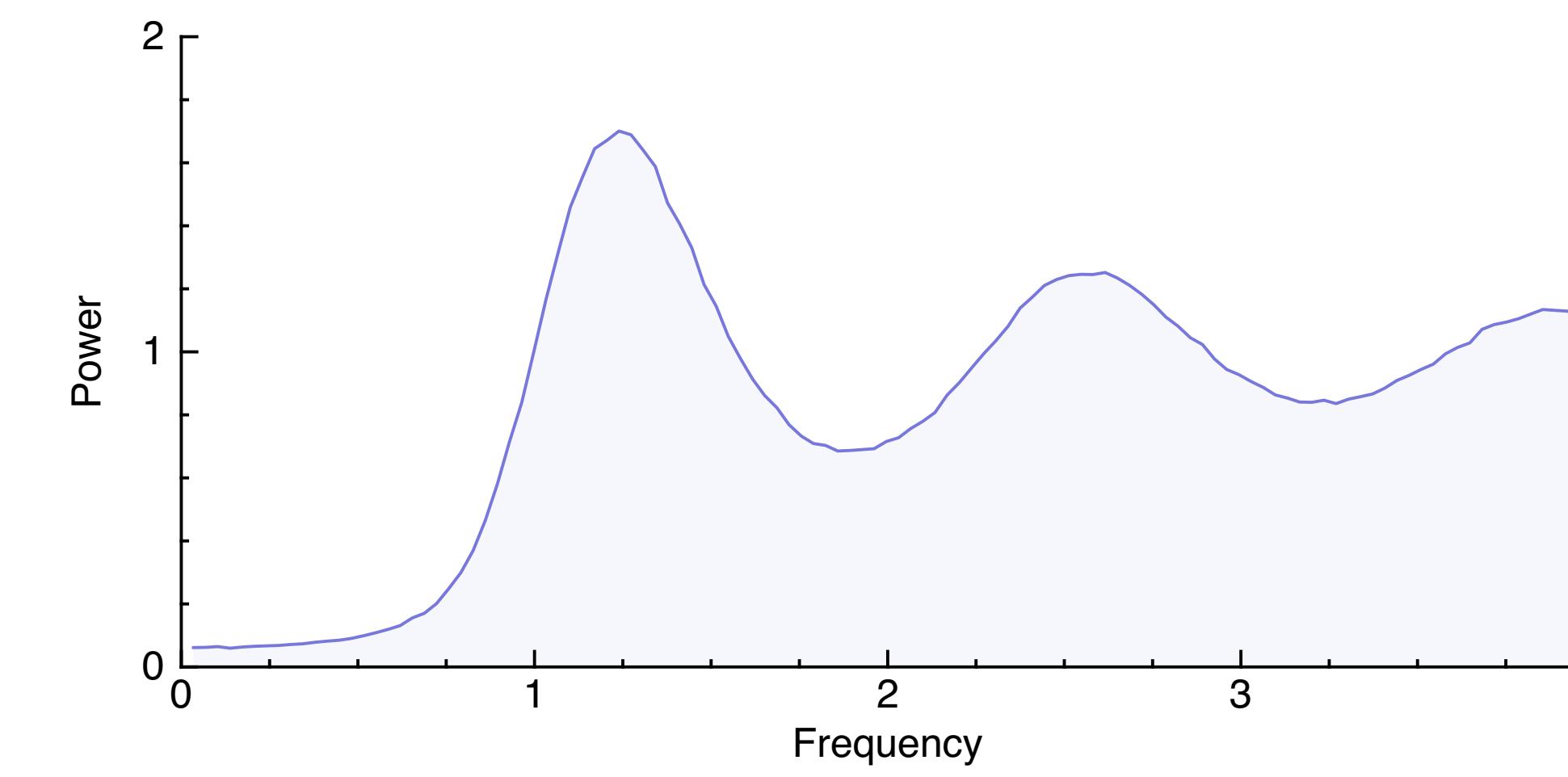
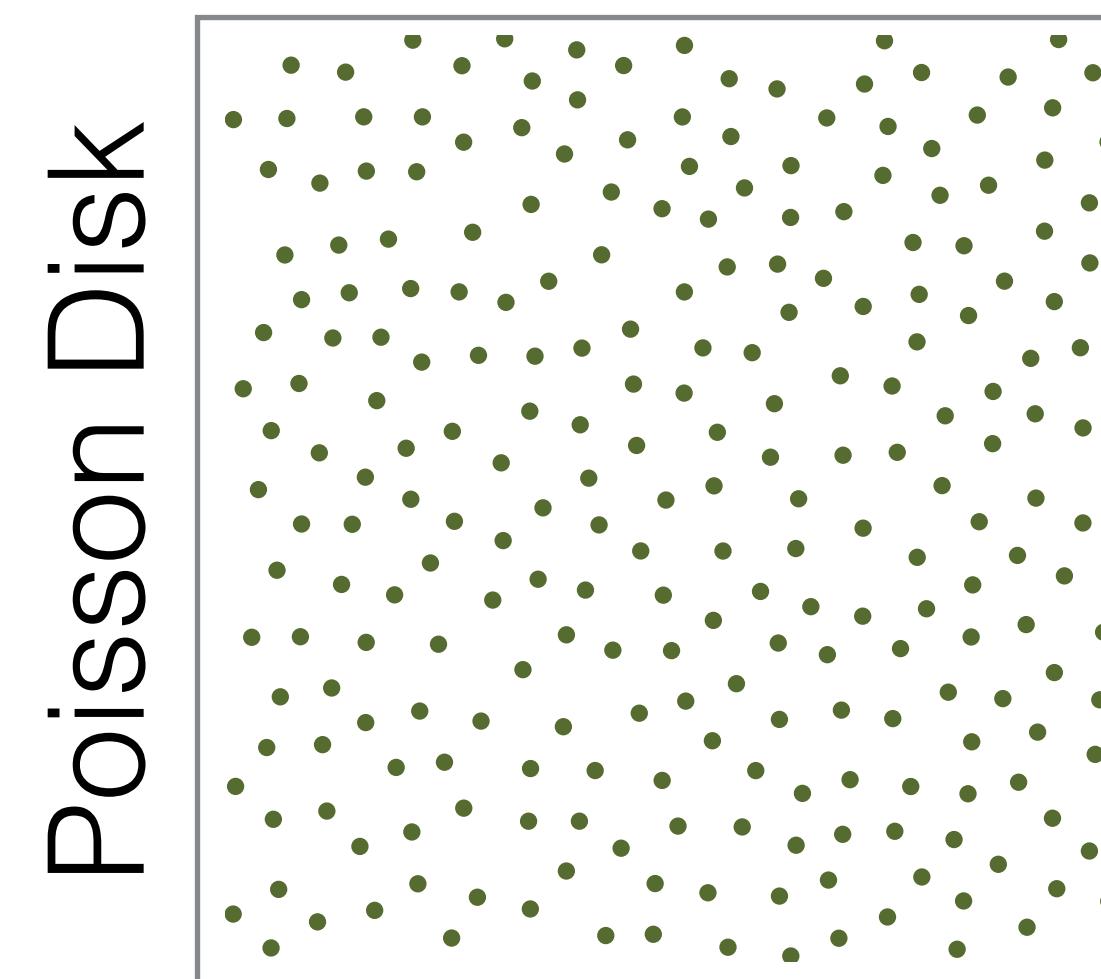
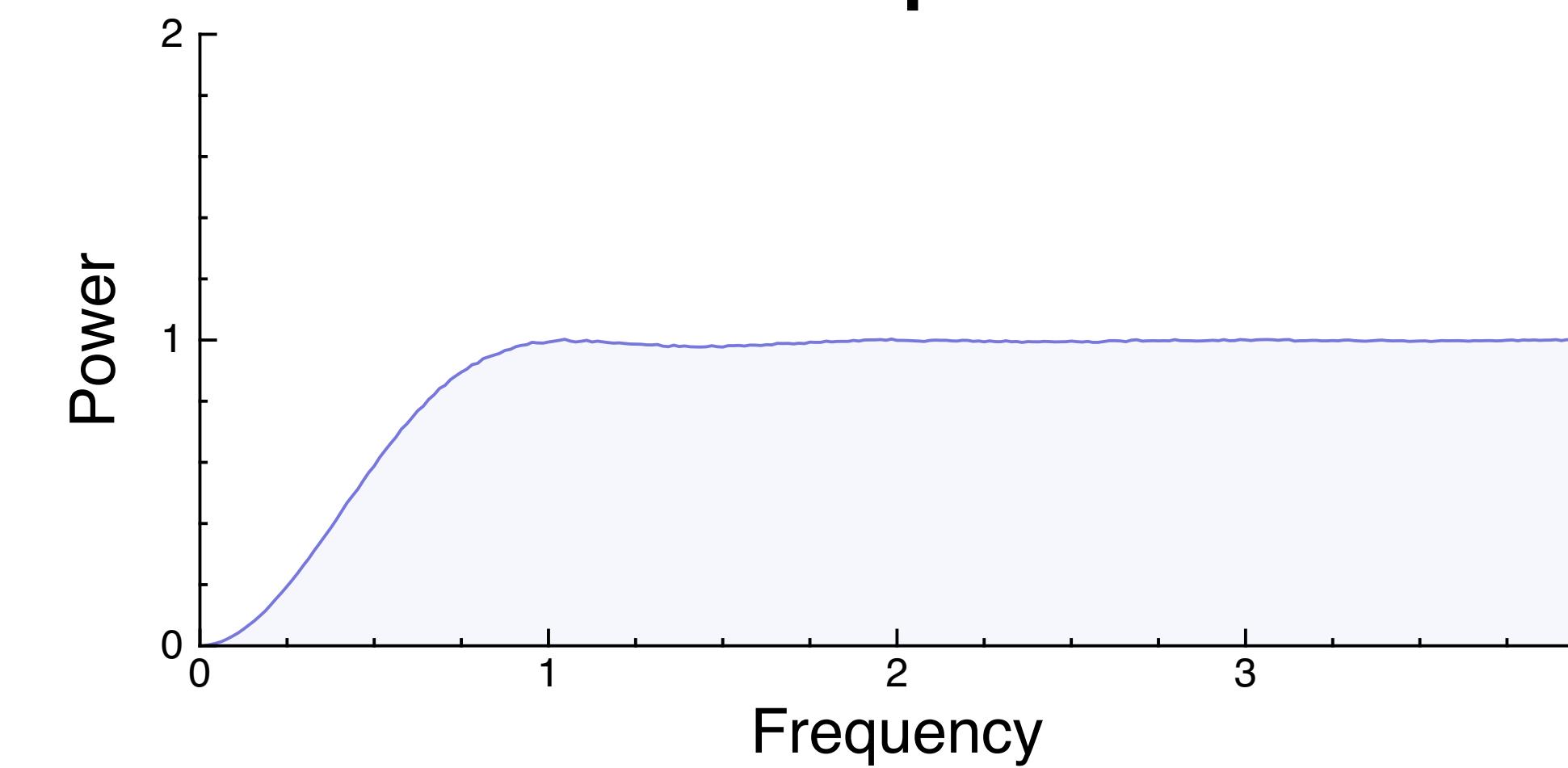
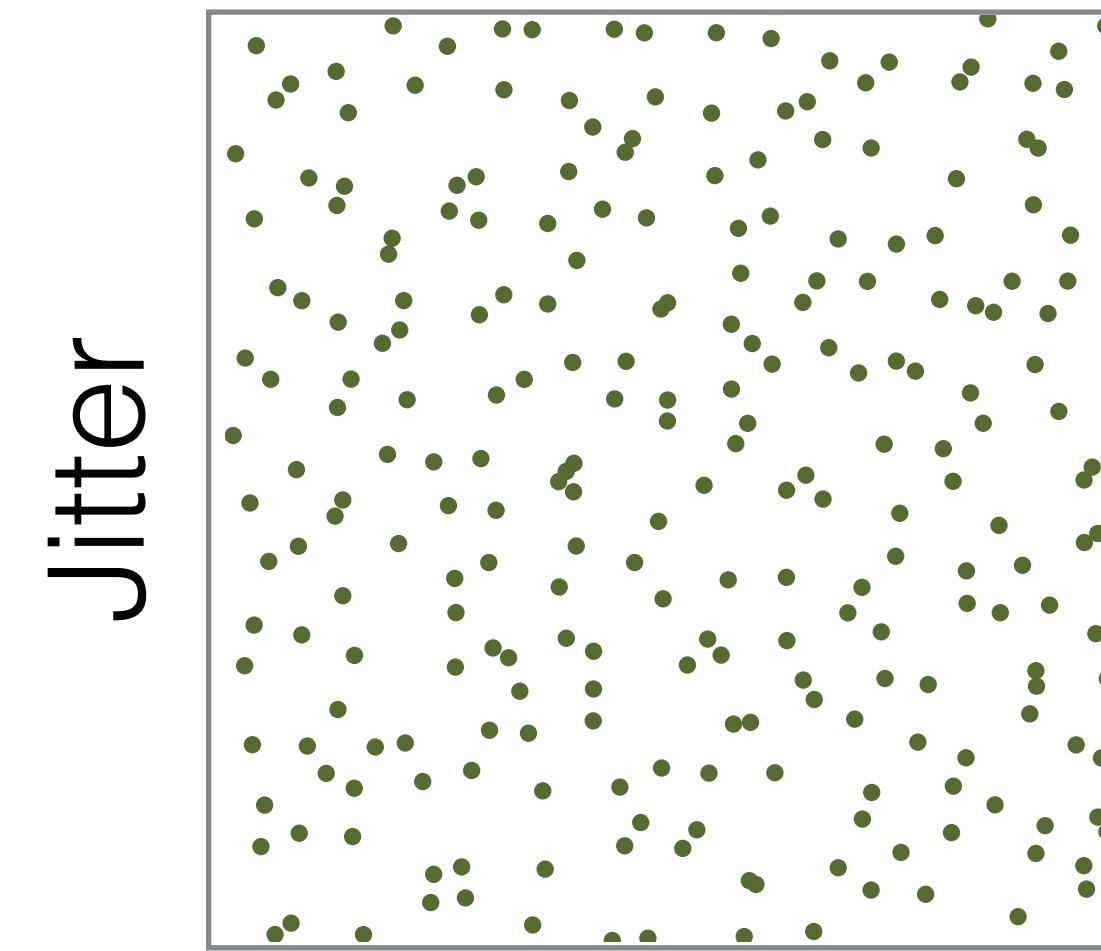


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Spatial Distribution vs Radial Mean Power Spectra



For 2-dimensions

Samplers	Worst Case	Best Case
Random		
Jitter		
Poisson Disk		
CCVT		

Pilleboue et al. [2015]

For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	
Jitter		
Poisson Disk		
CCVT		

Pilleboue et al. [2015]

For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter		
Poisson Disk		
CCVT		

Pilleboue et al. [2015]

For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	
Poisson Disk		
CCVT		

Pilleboue et al. [2015]

For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk		
CCVT		

Pilleboue et al. [2015]

For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk	$\mathcal{O}(N^{-1})$	
CCVT		

Pilleboue et al. [2015]



For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
CCVT		

Pilleboue et al. [2015]

For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
CCVT	$\mathcal{O}(N^{-1.5})$	

Pilleboue et al. [2015]

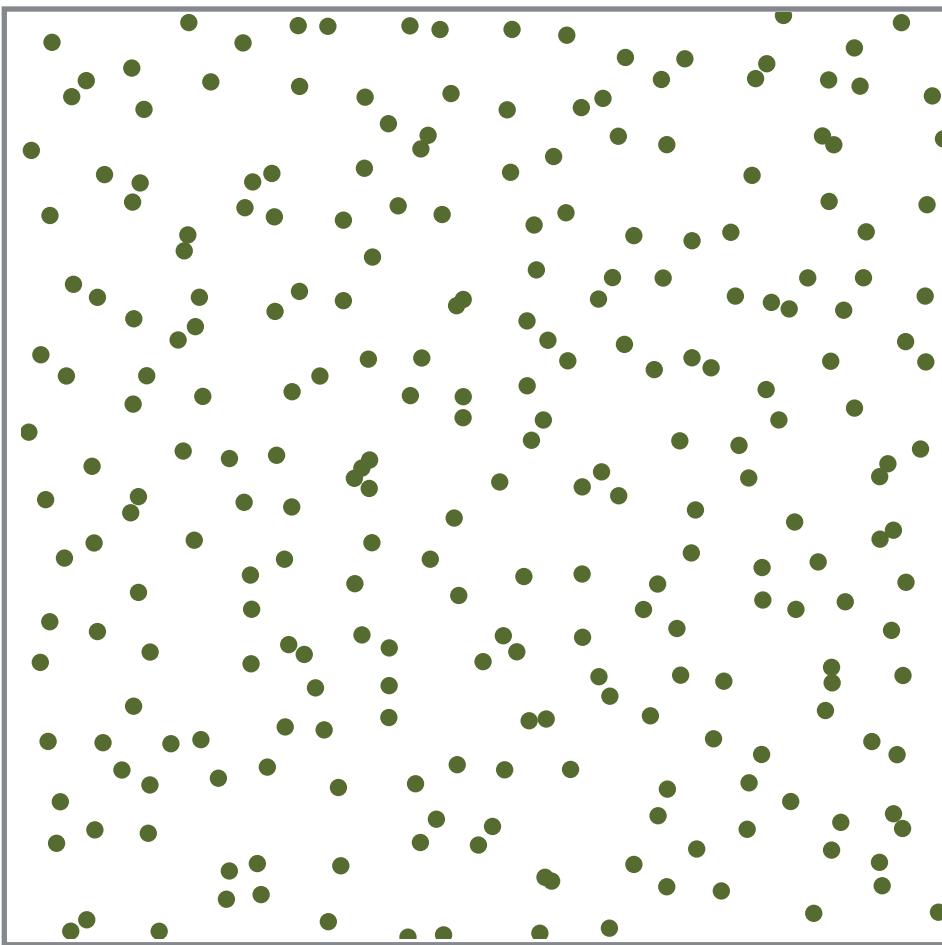
For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
CCVT	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-3})$

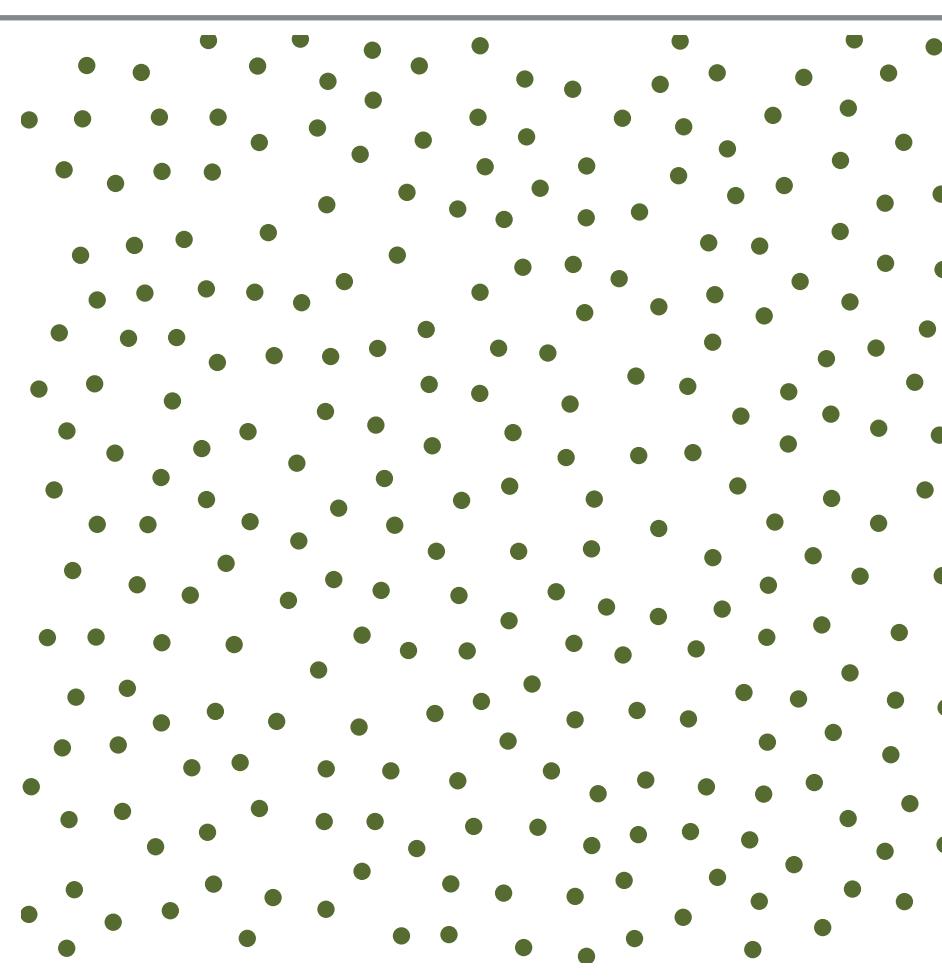
Pilleboue et al. [2015]

For 2-dimensions

Jitter



Poisson Disk

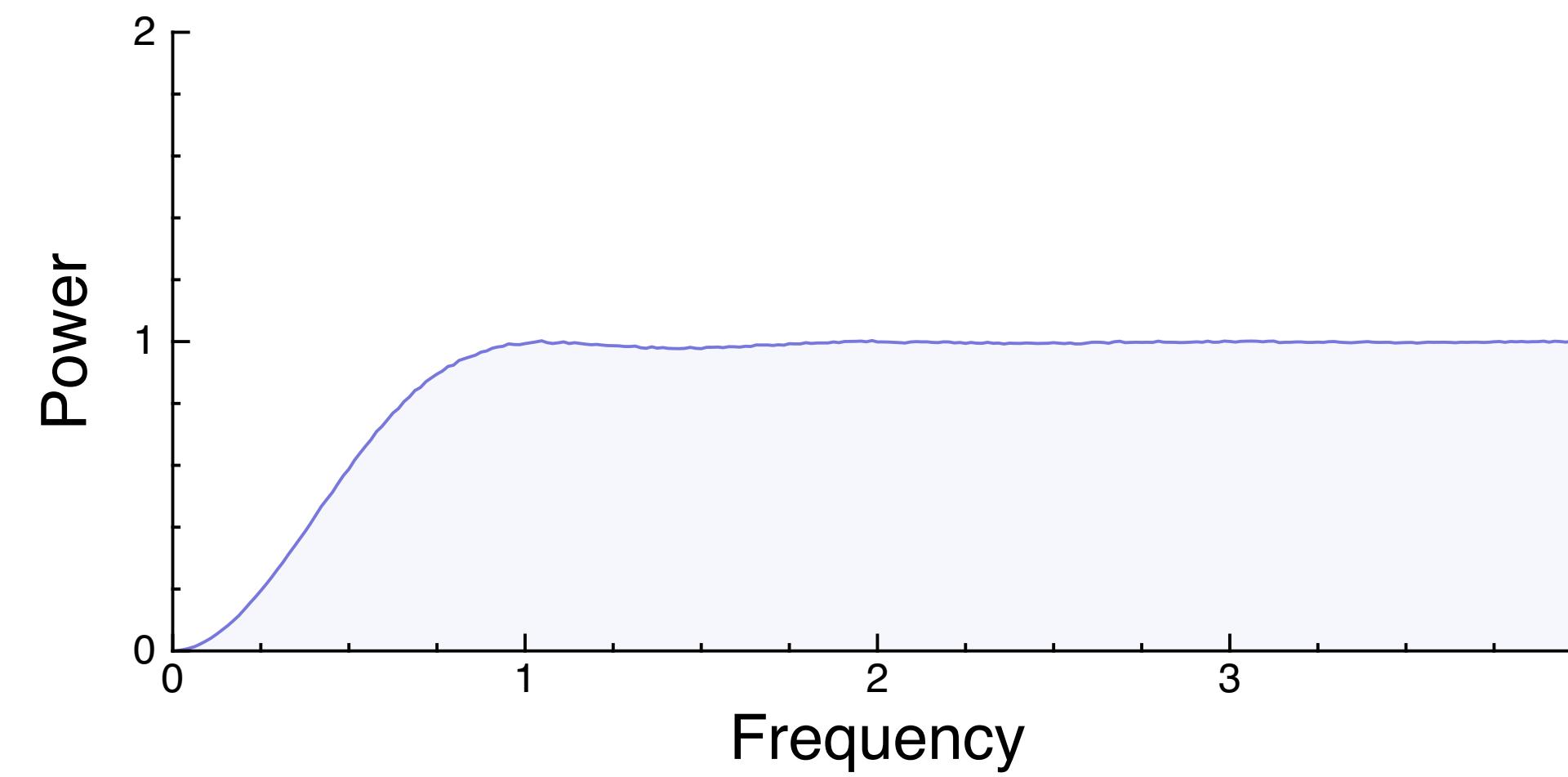


Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
CCVT	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-3})$

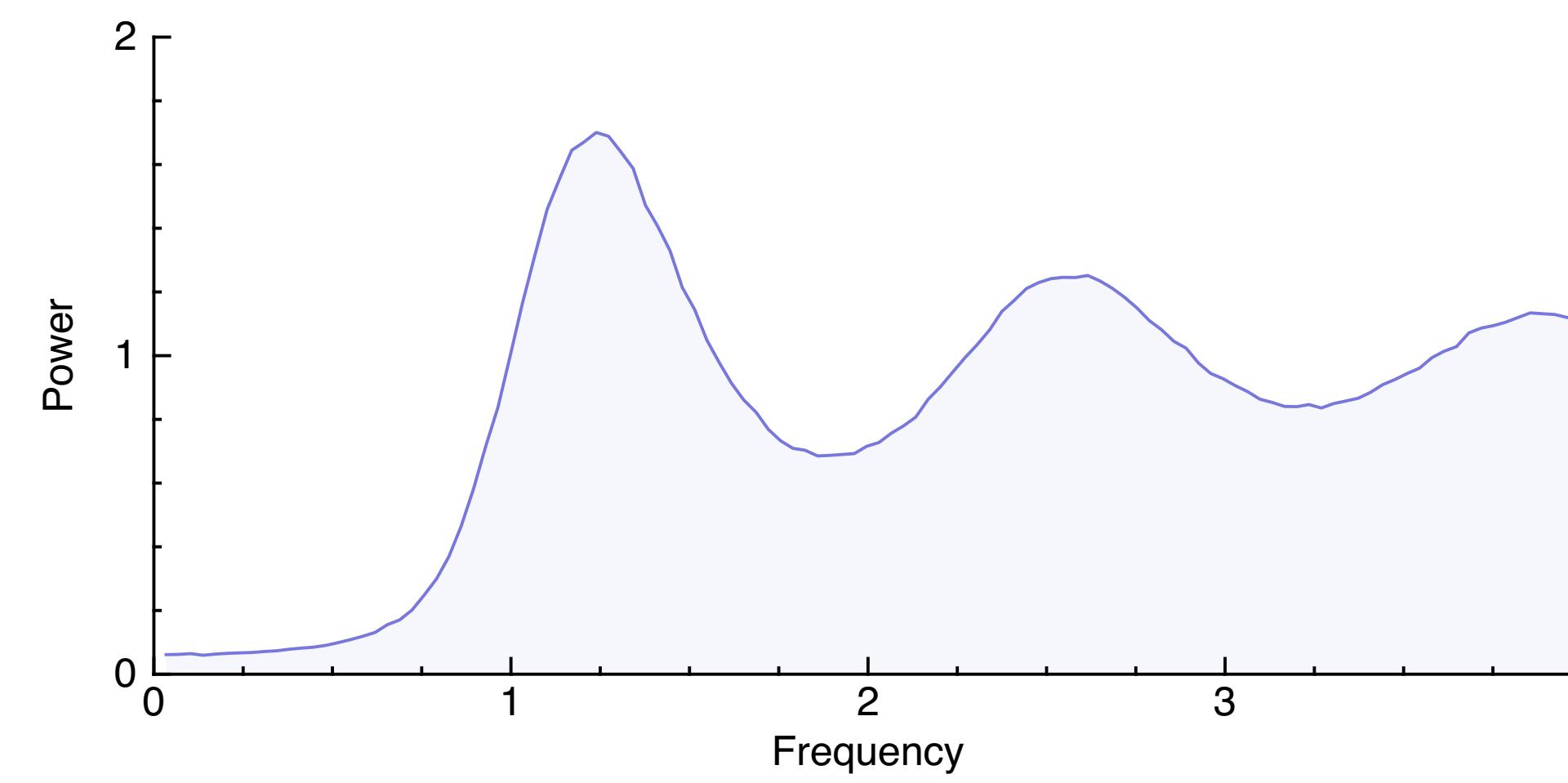
Pilleboue et al. [2015]

Low Frequency Region

Jitter

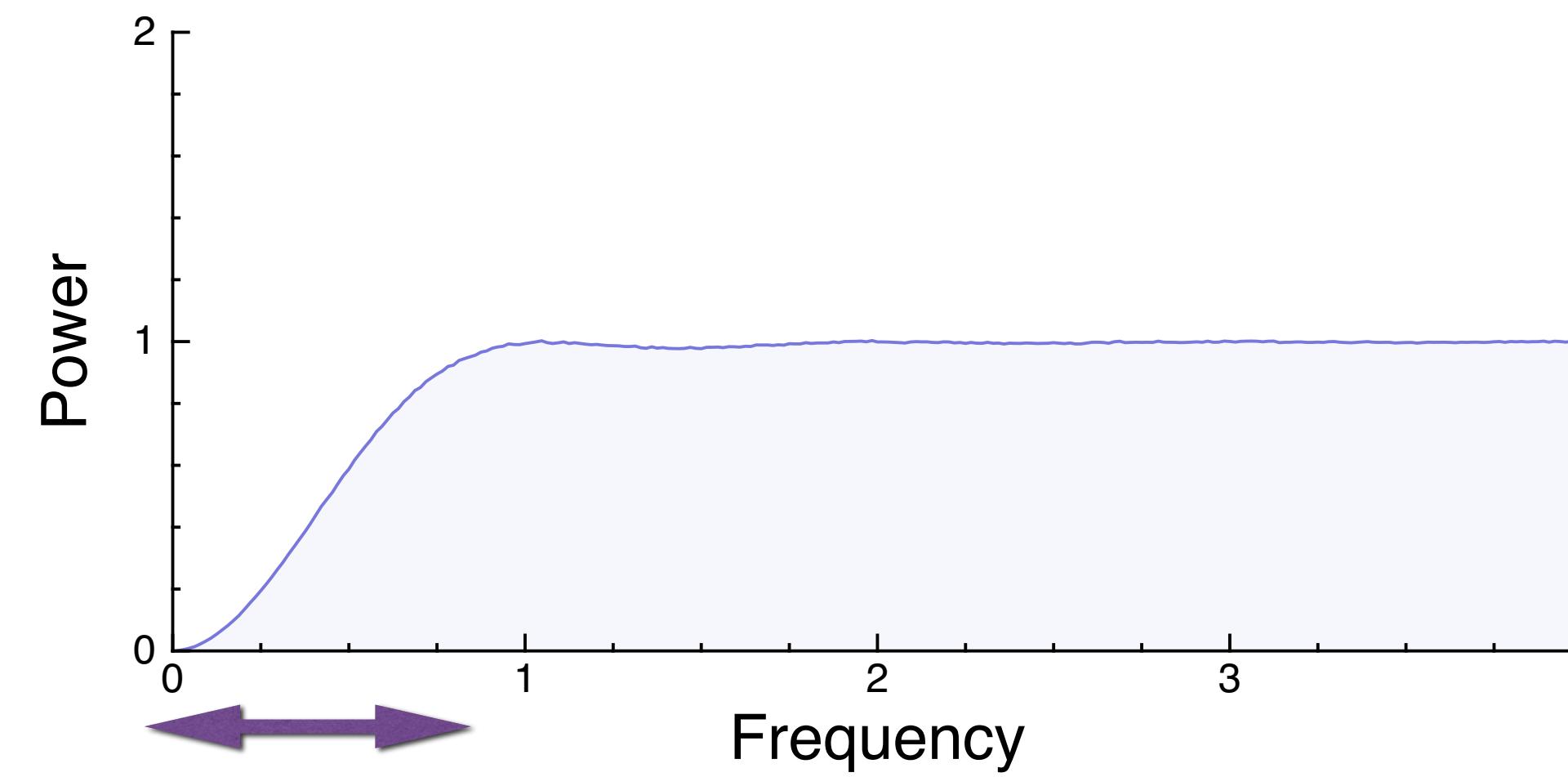


Poisson Disk

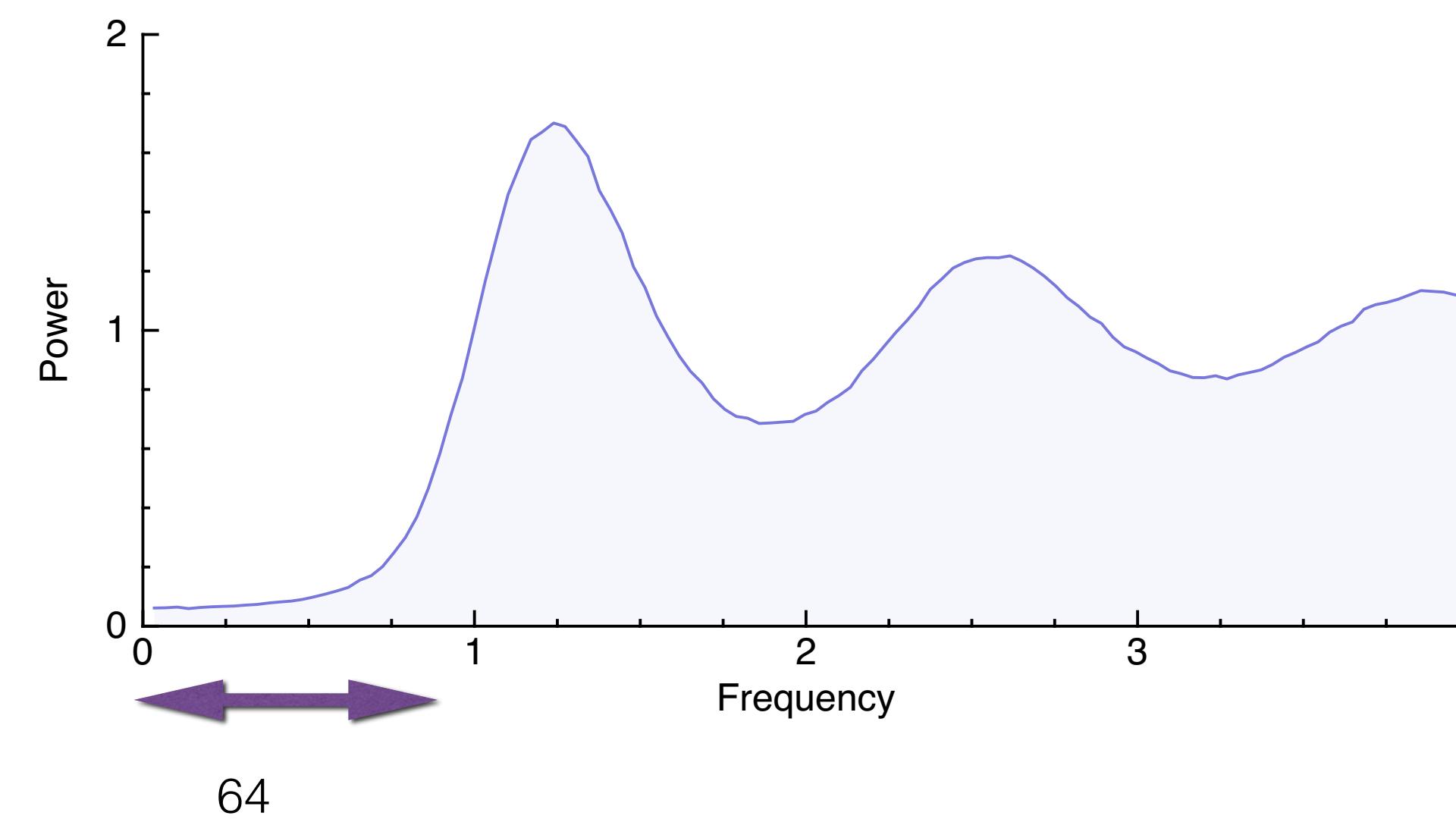


Low Frequency Region

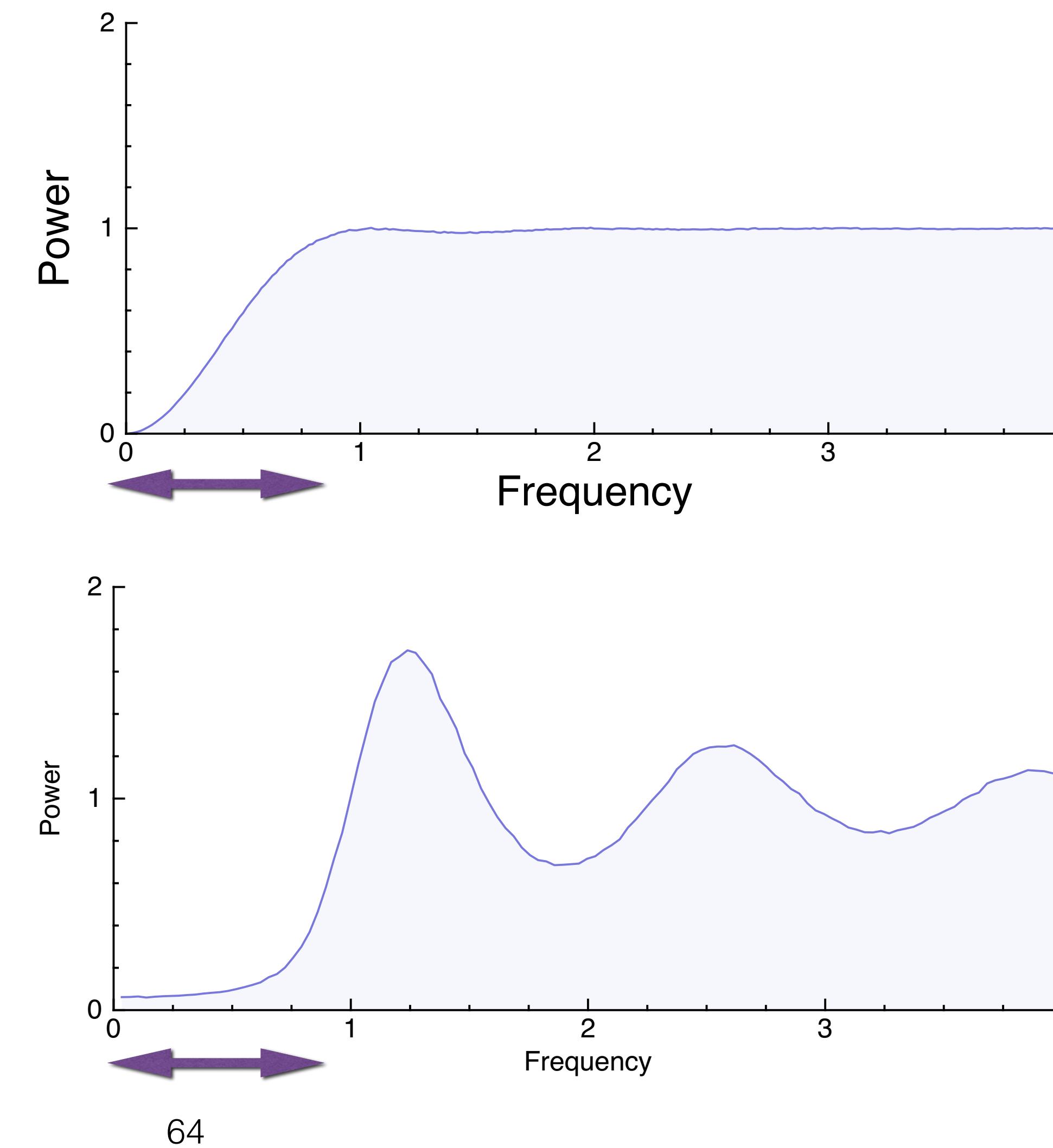
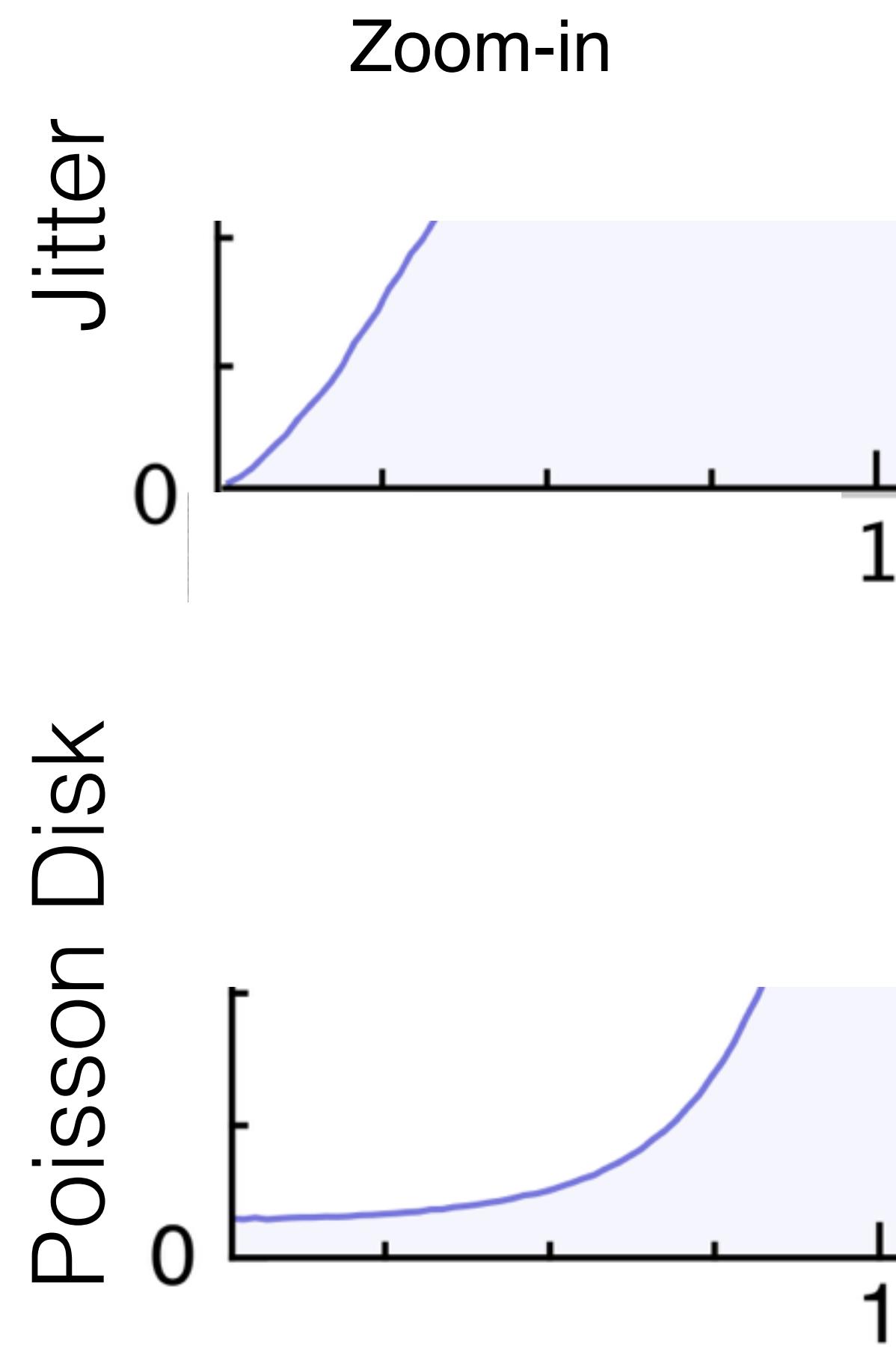
Jitter



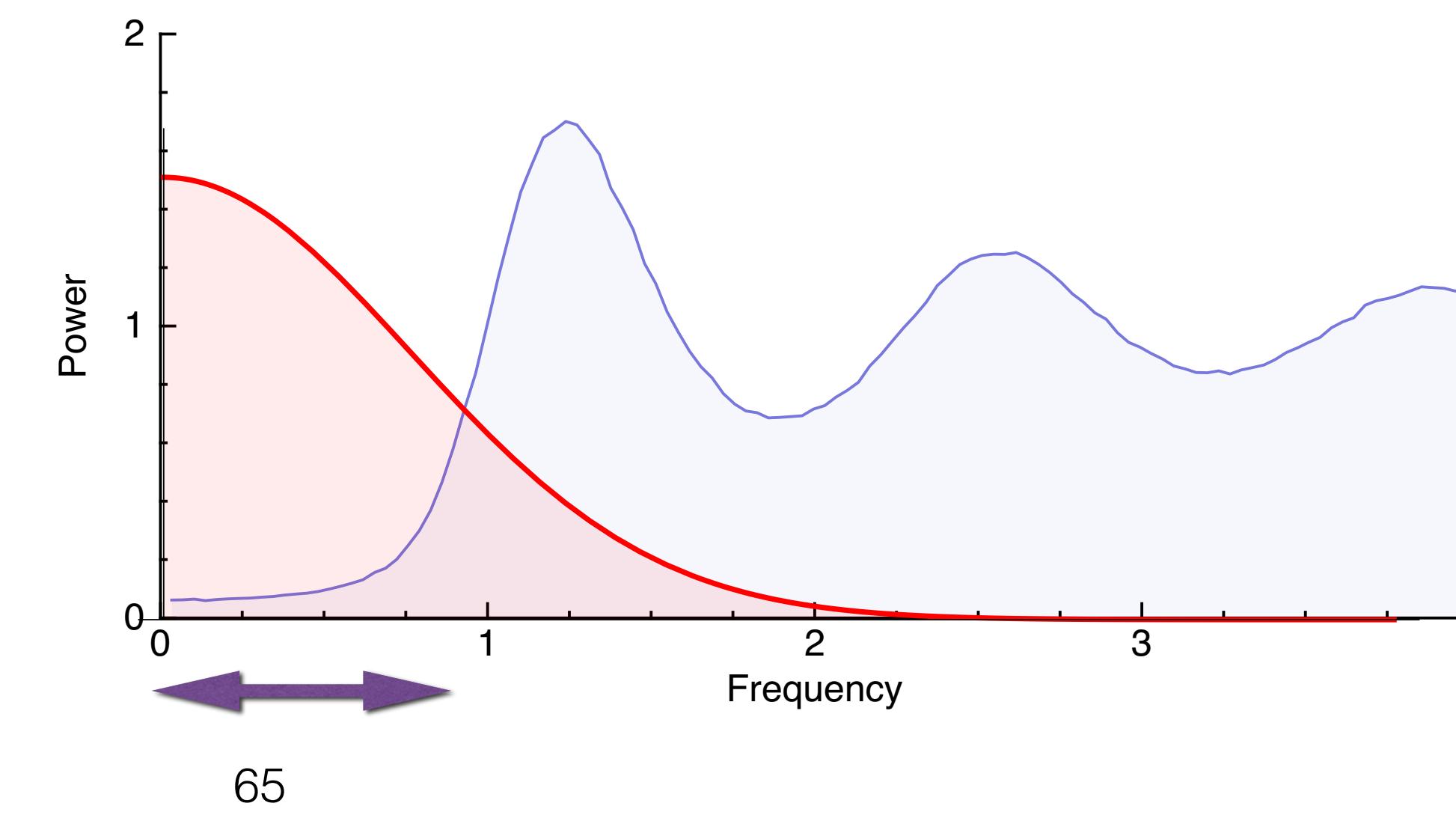
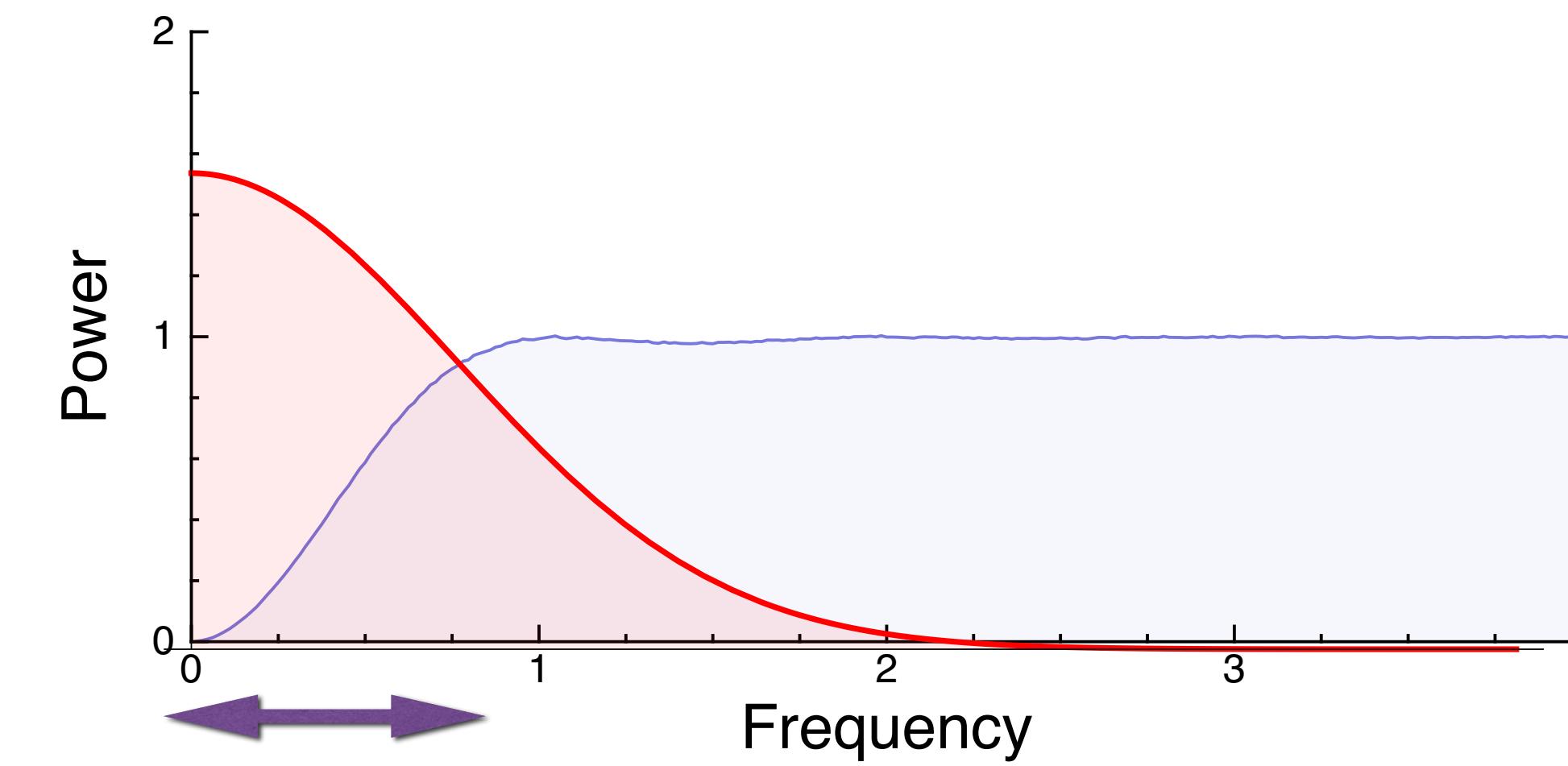
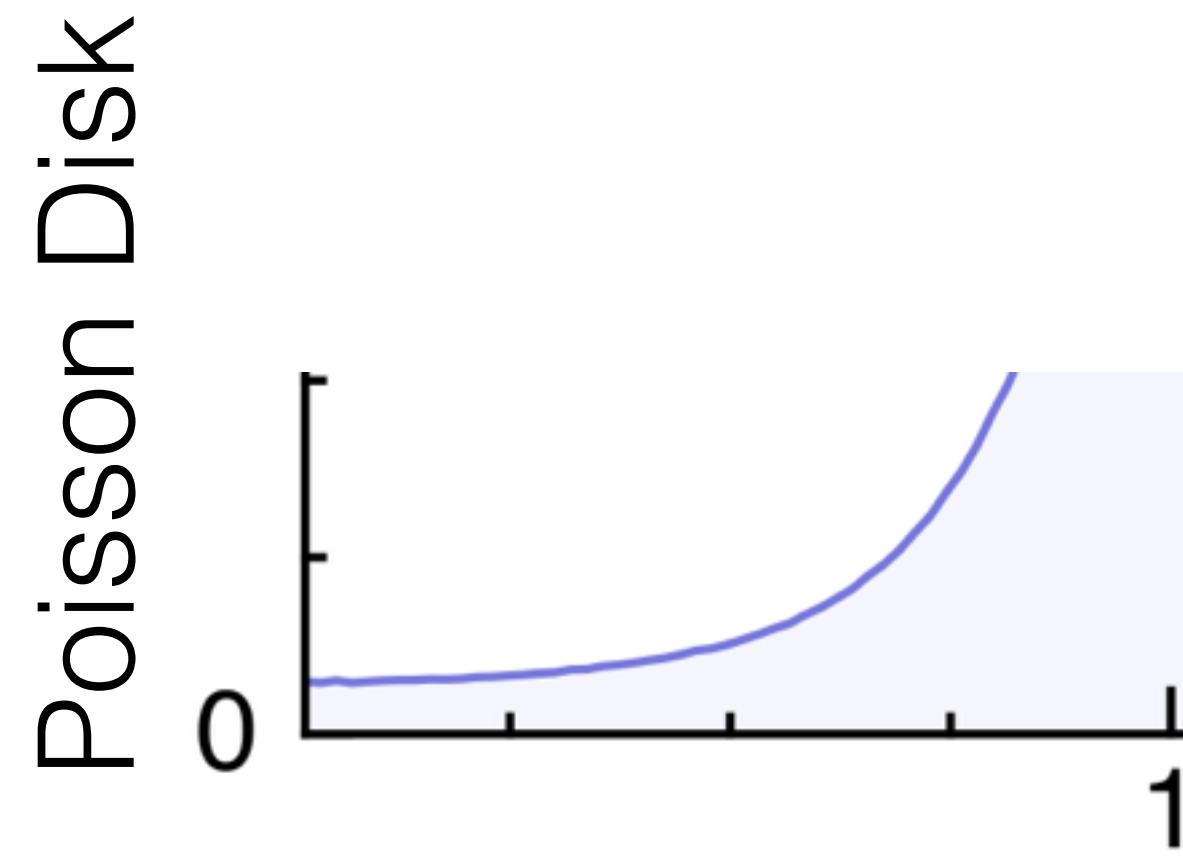
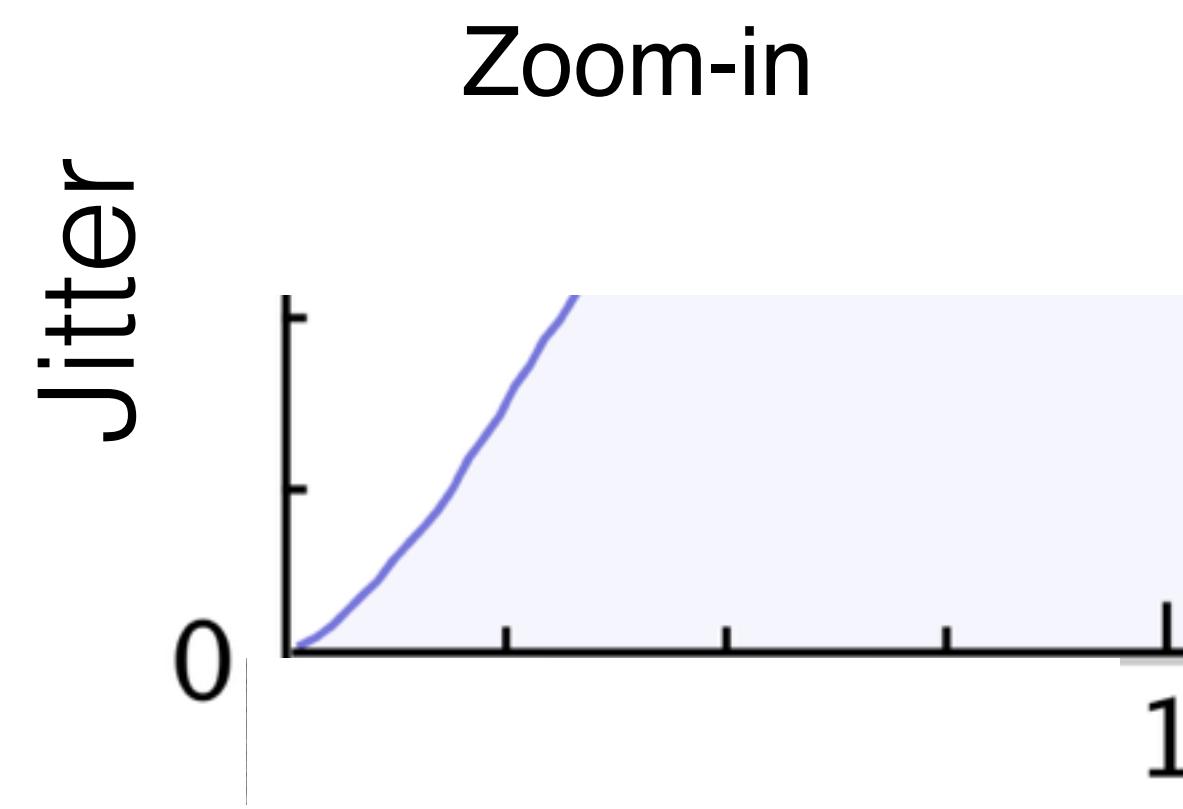
Poisson Disk



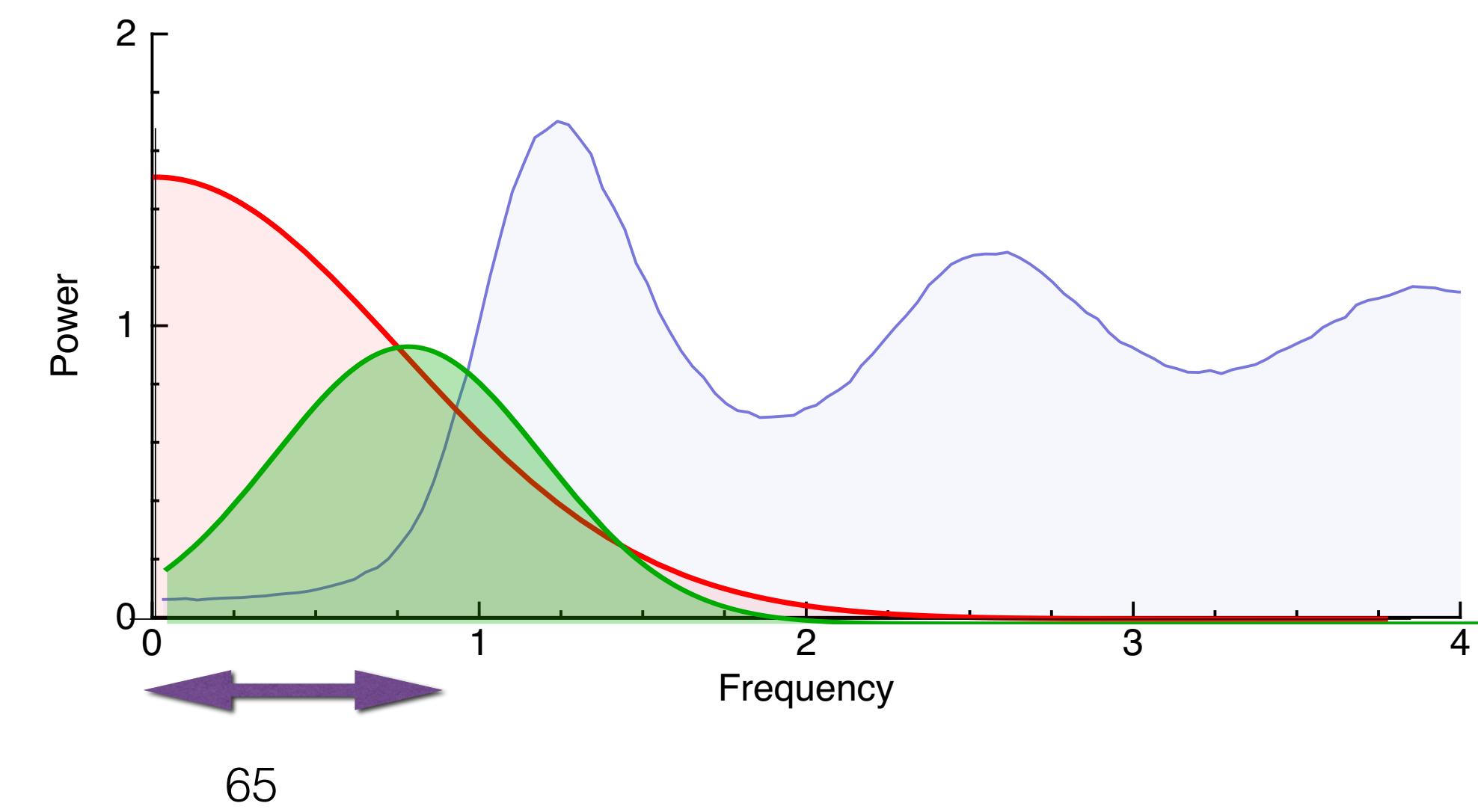
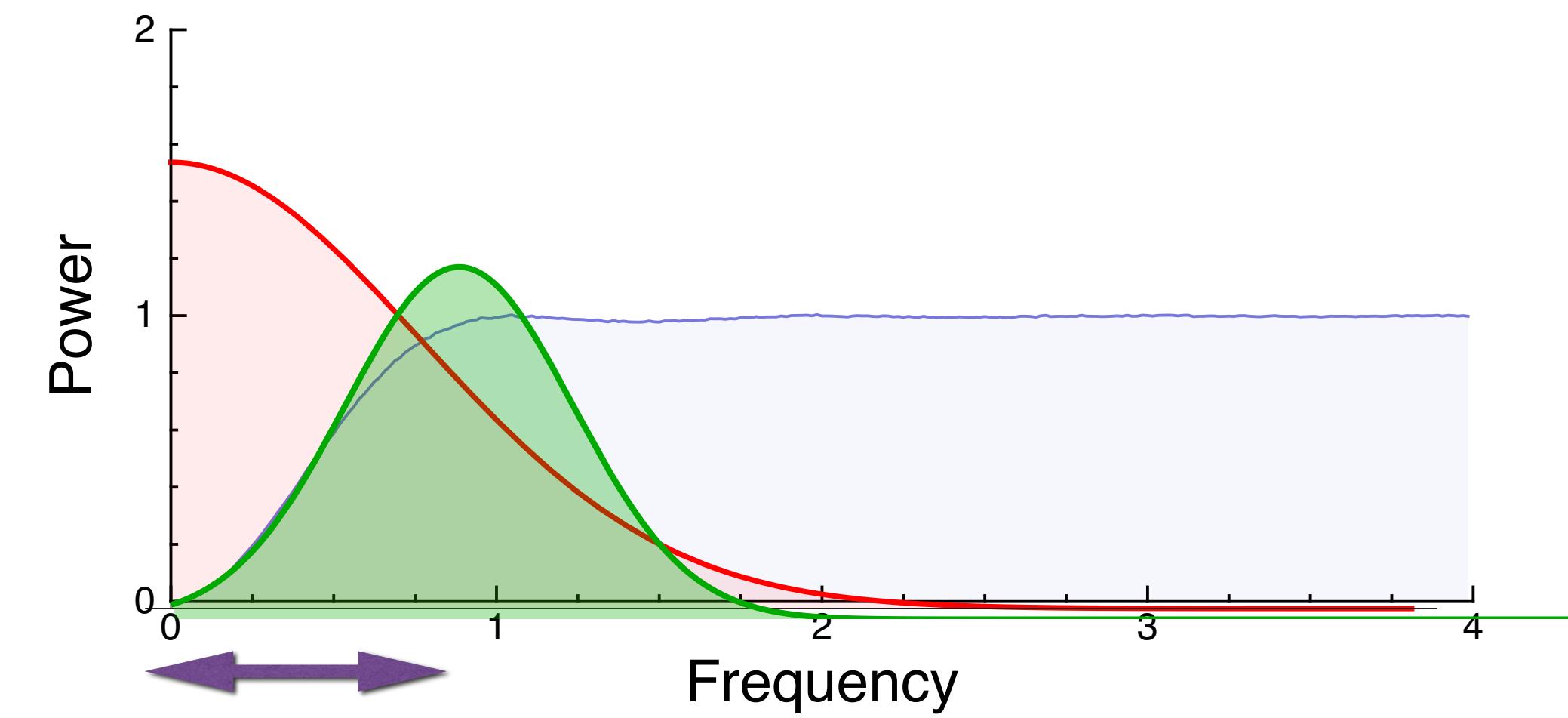
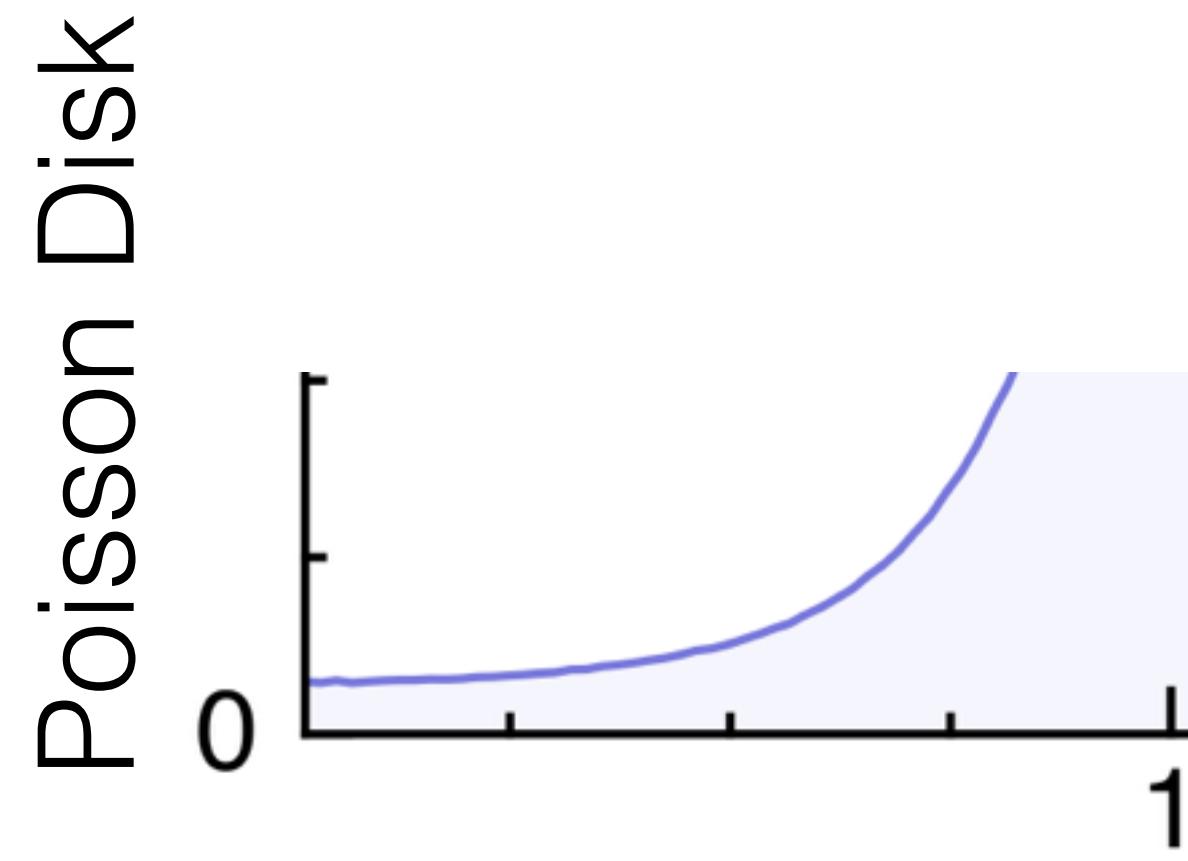
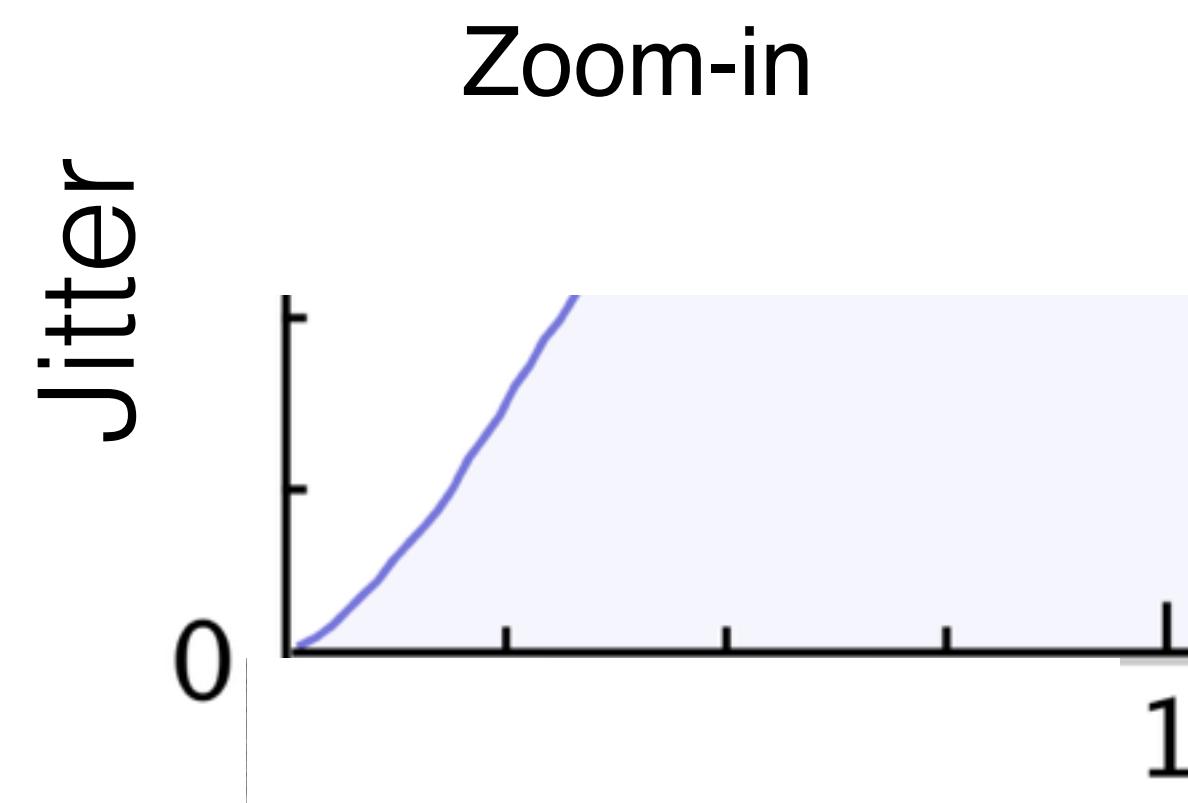
Low Frequency Region



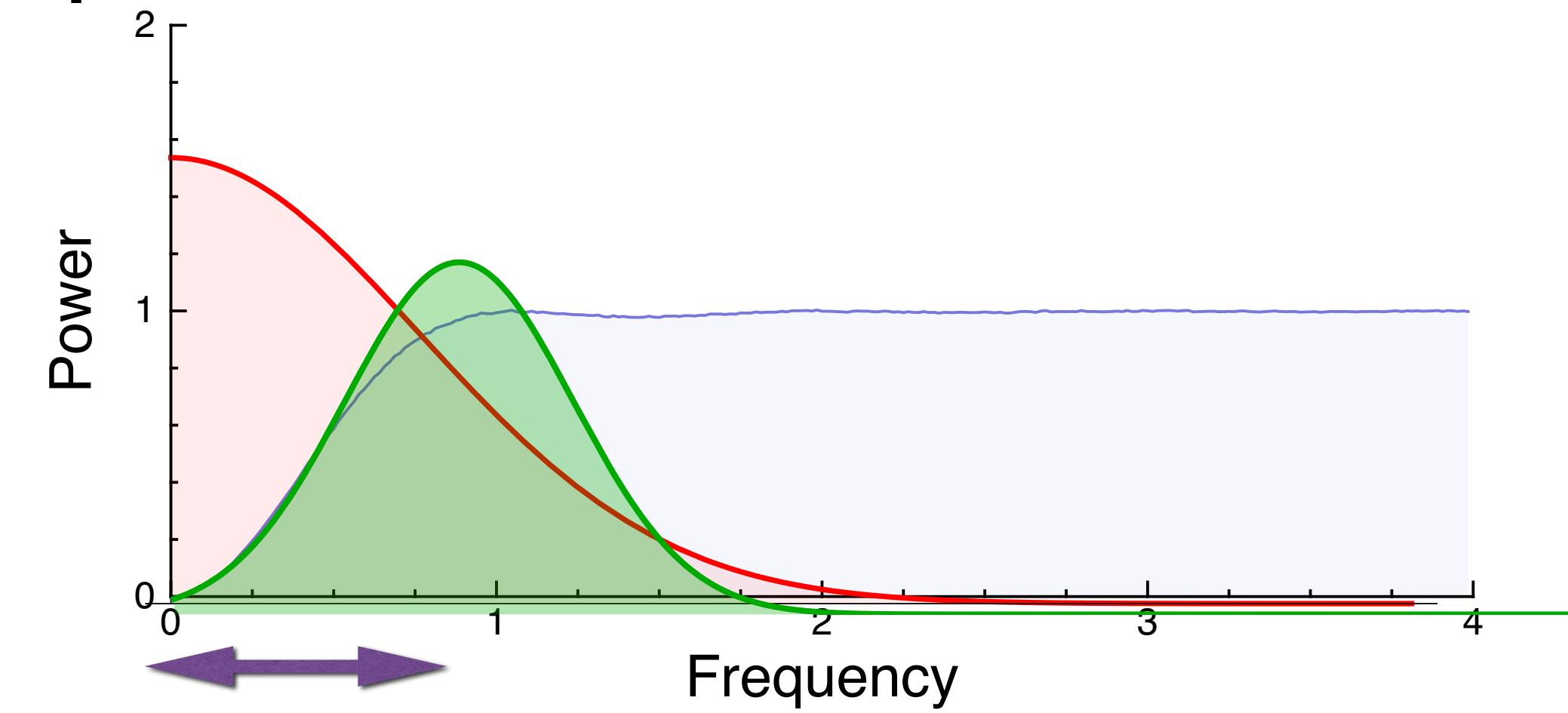
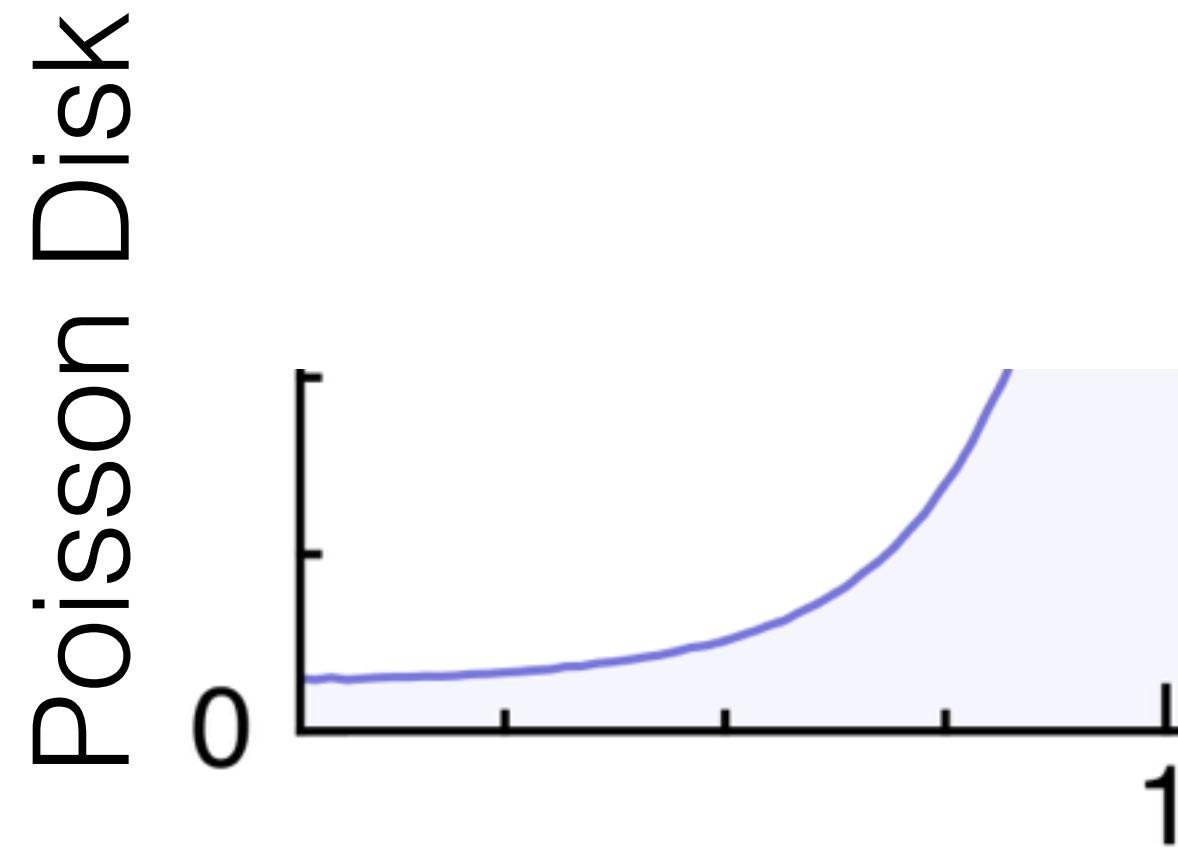
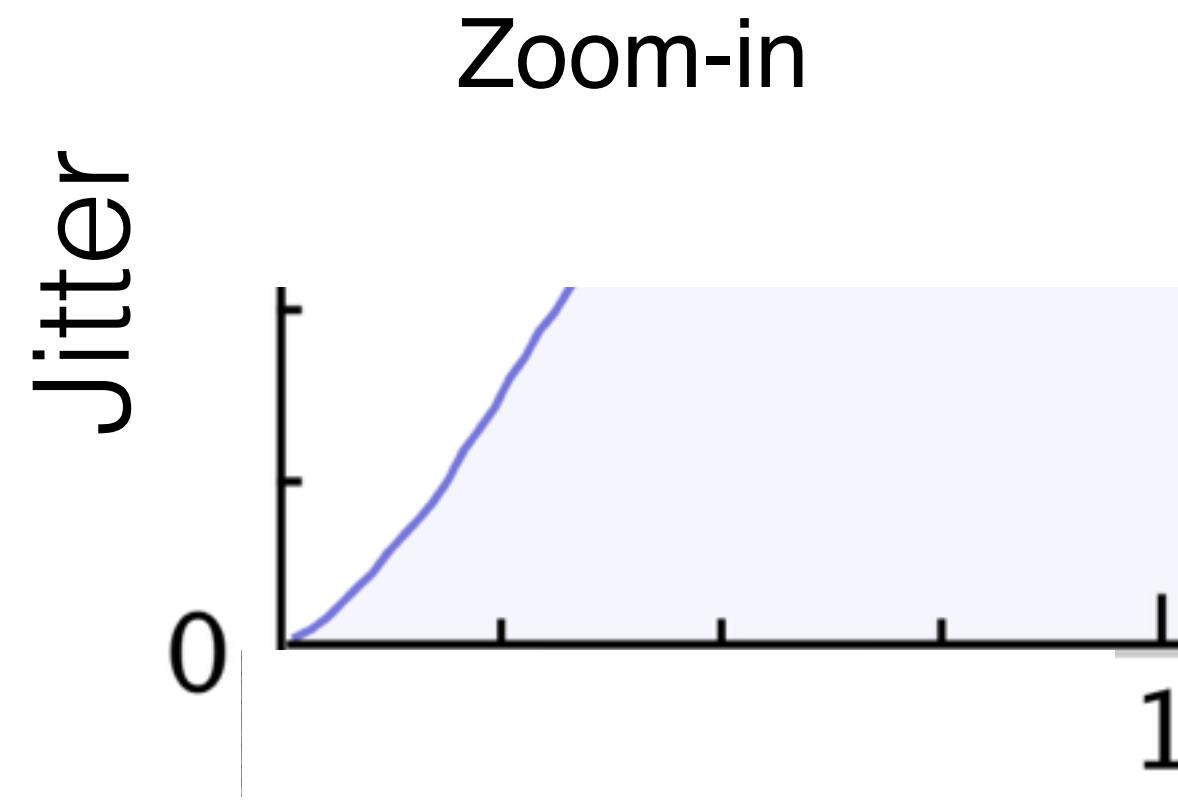
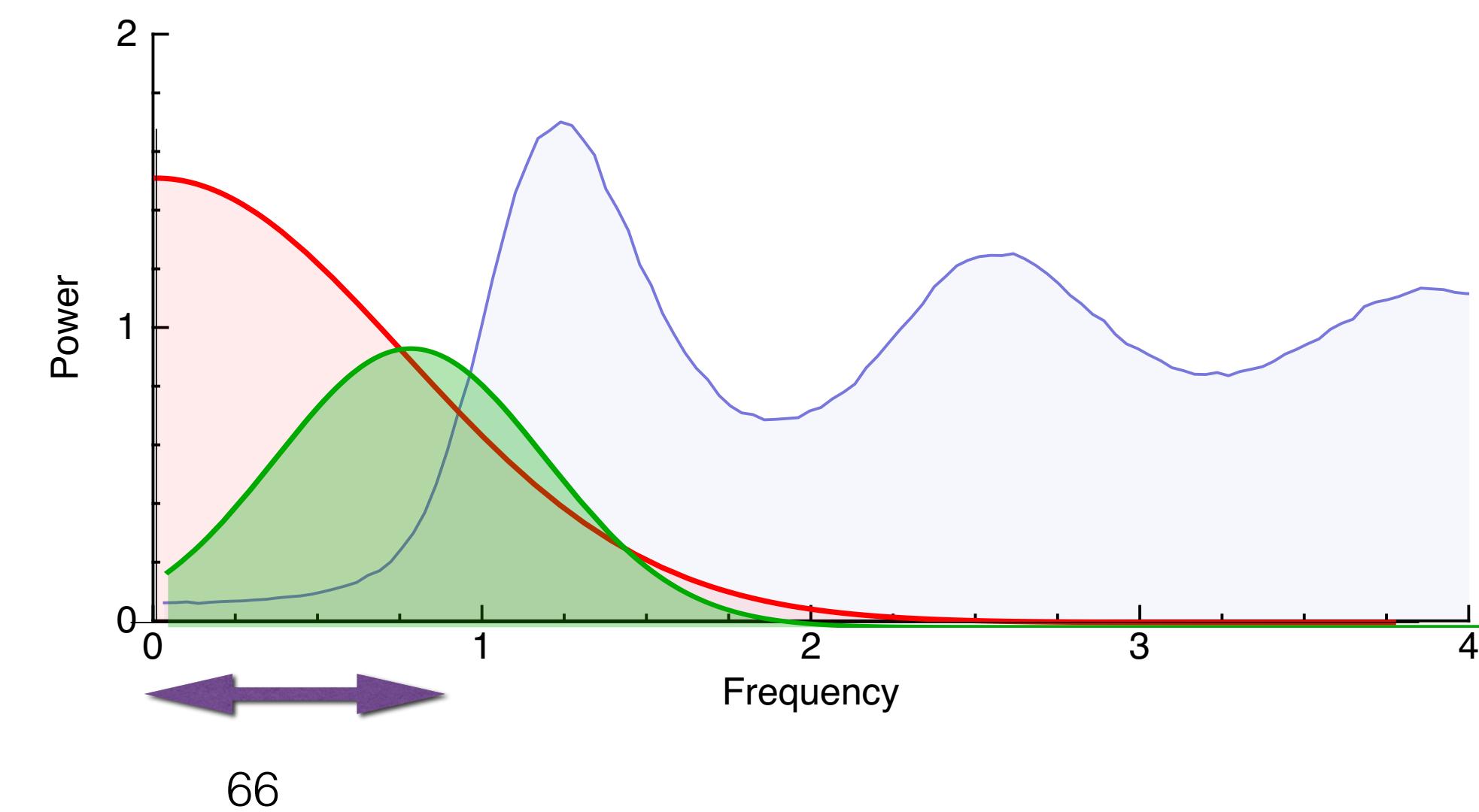
Variance for Low Sample Count



Variance for Low Sample Count

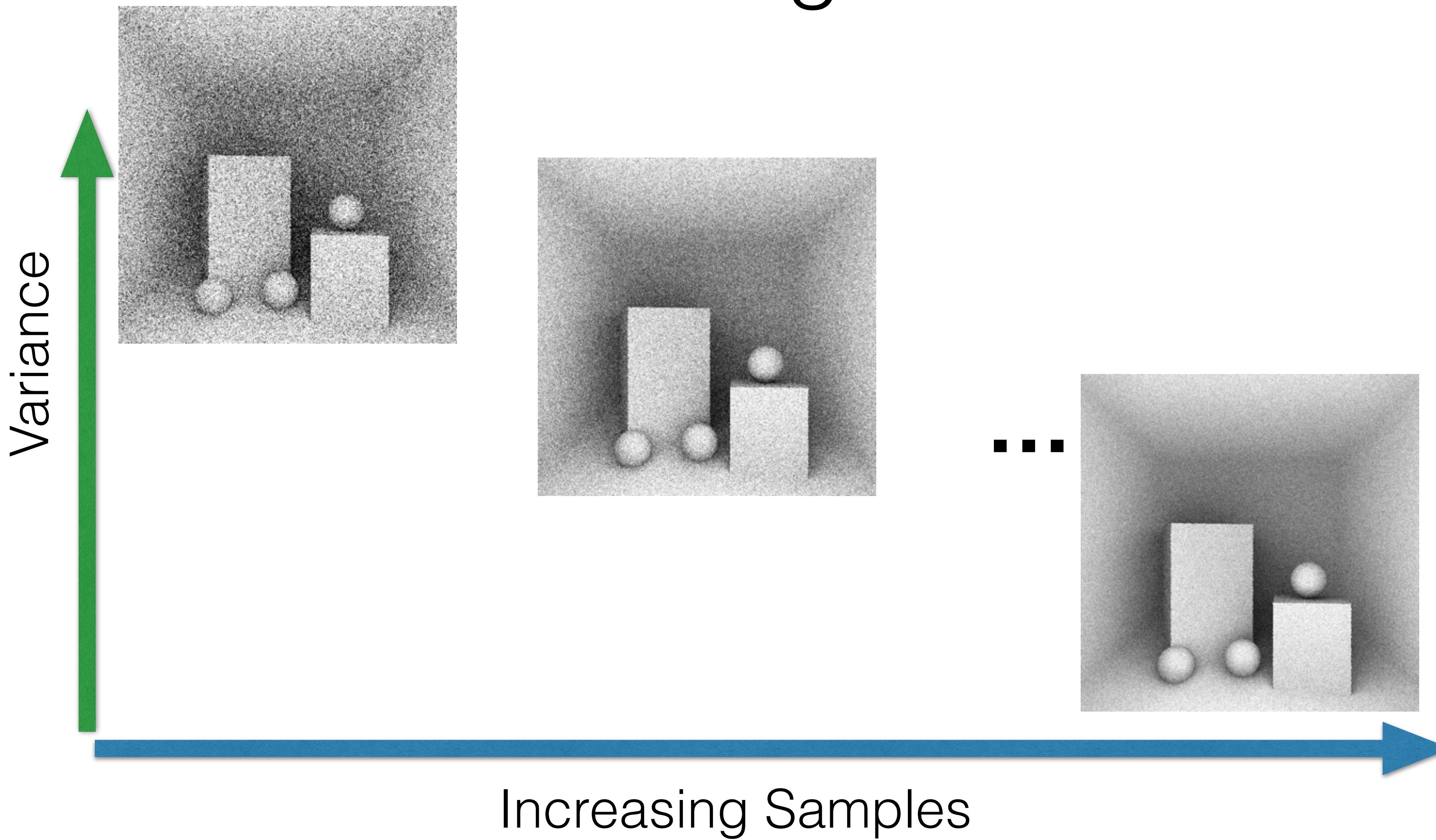


Variance for Increasing Sample Count

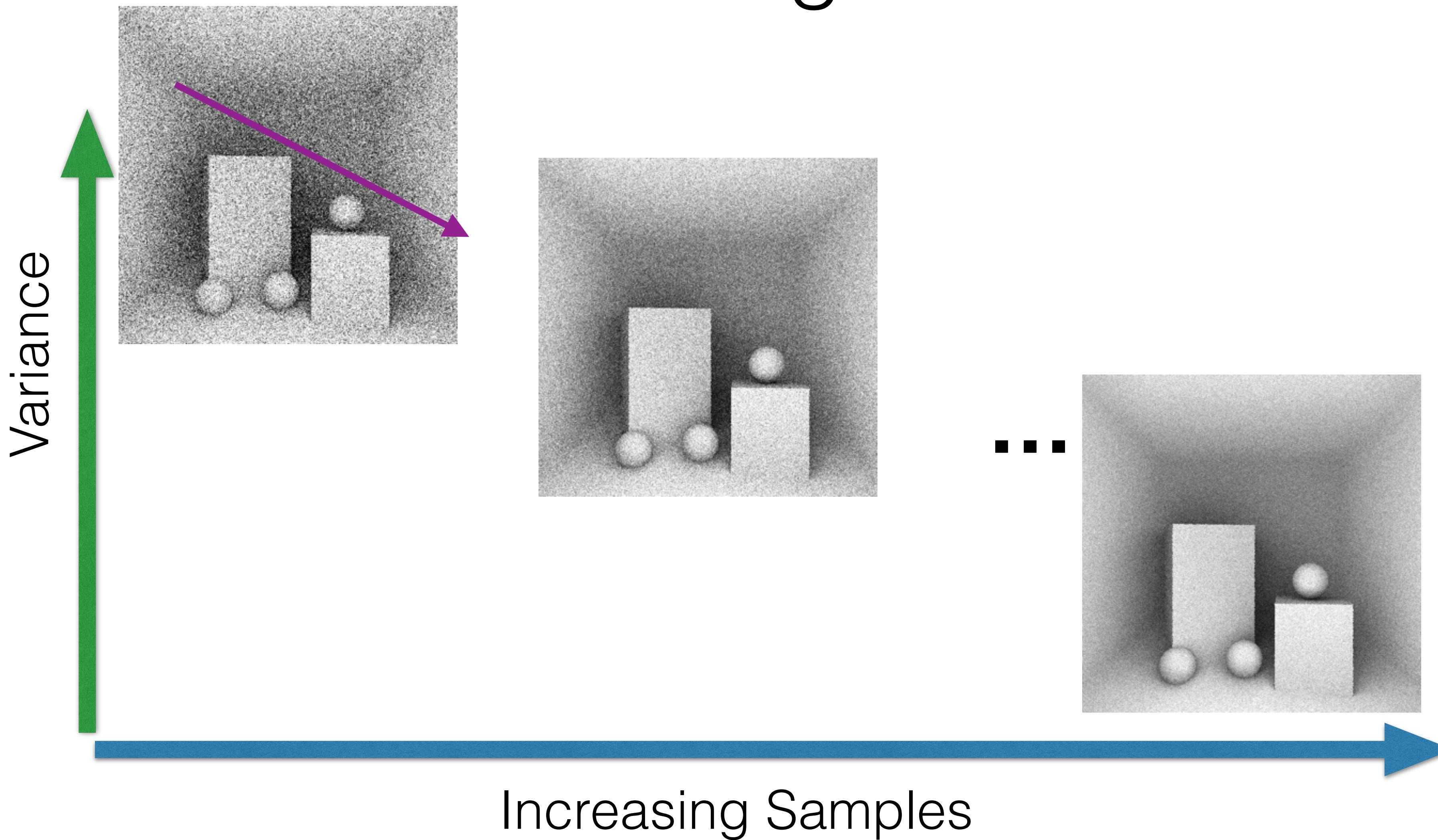
 $\mathcal{O}(N^{-2})$  $\mathcal{O}(N^{-1})$

Experimental Verification

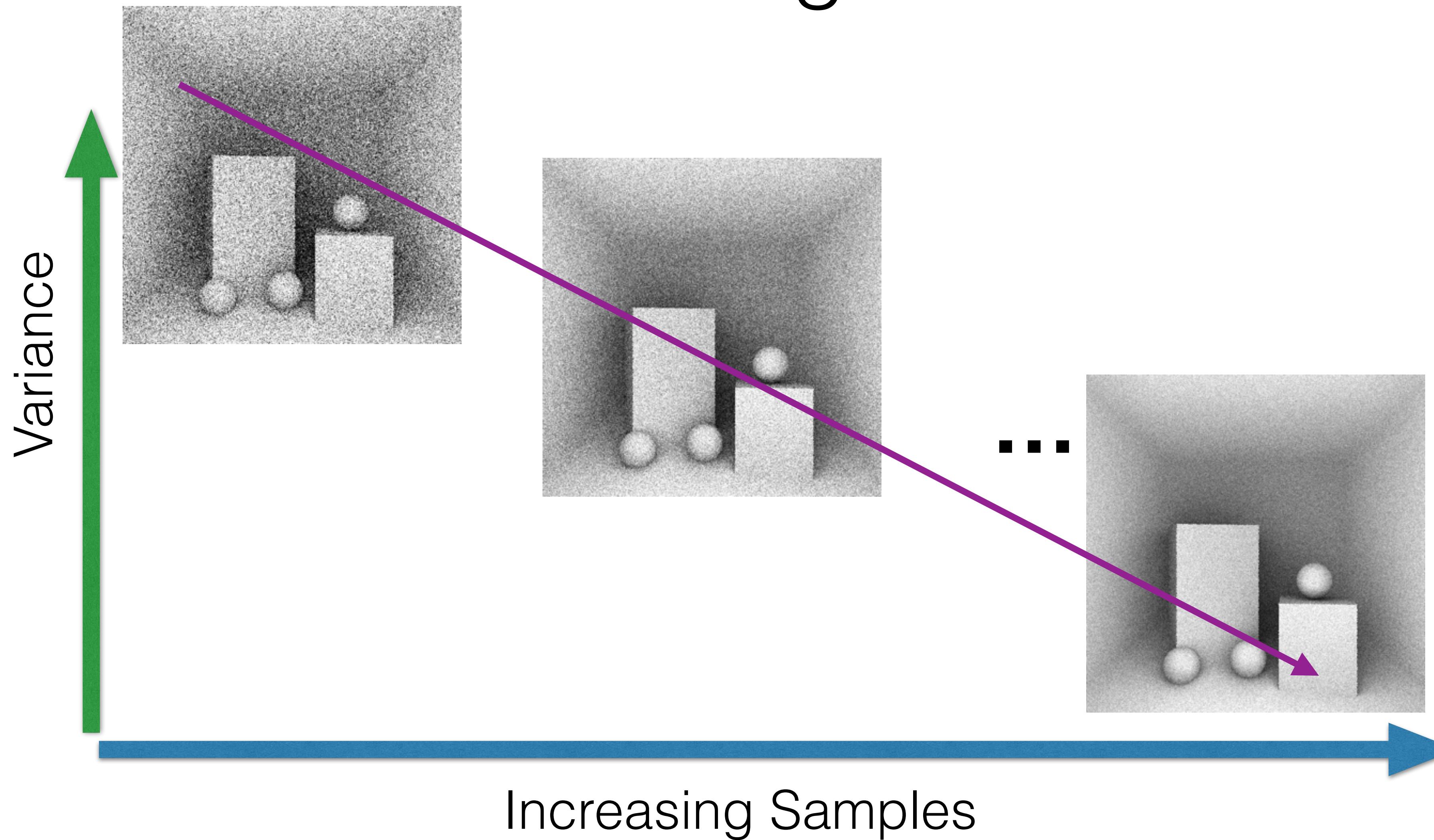
Convergence rate



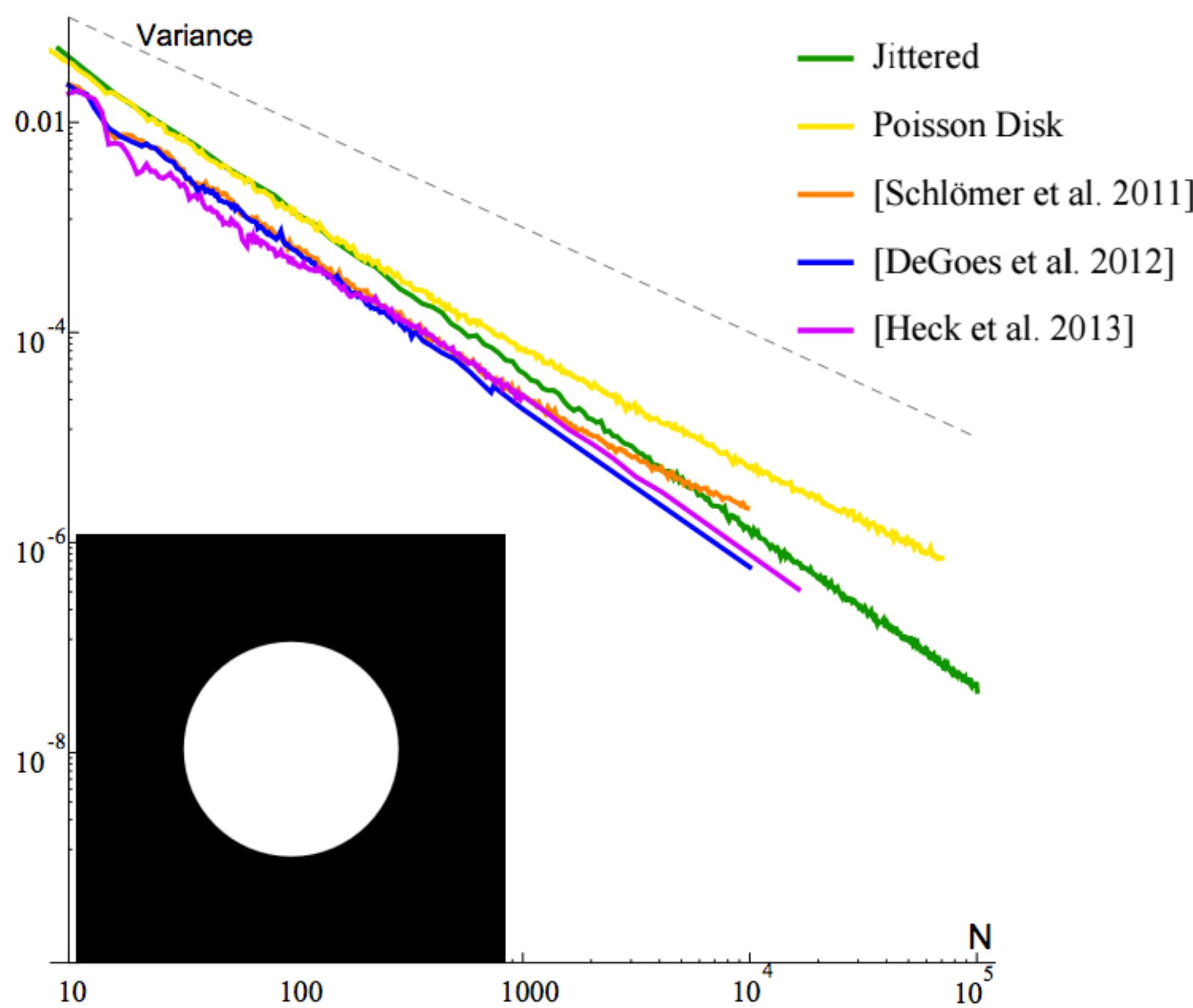
Convergence rate



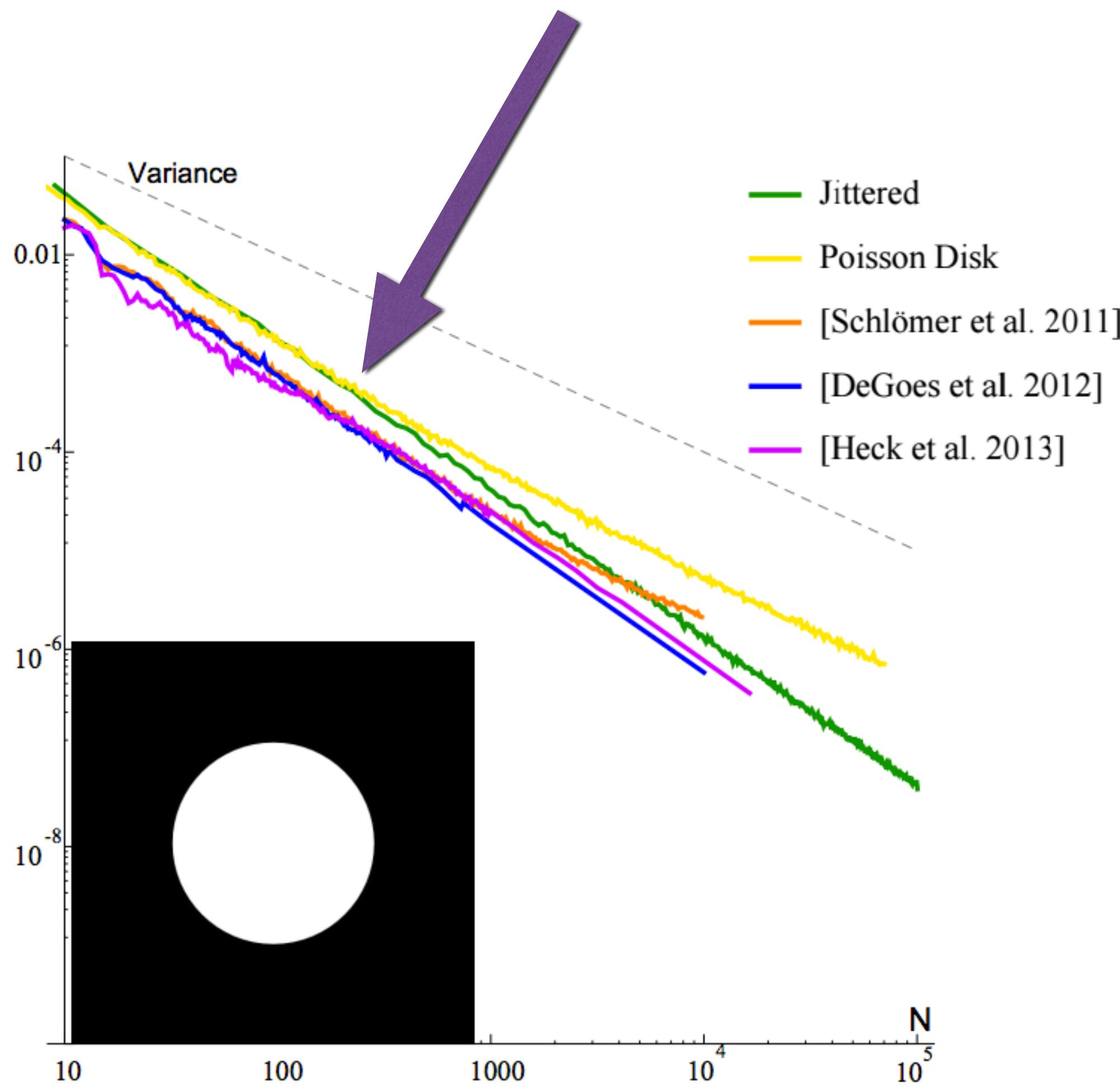
Convergence rate



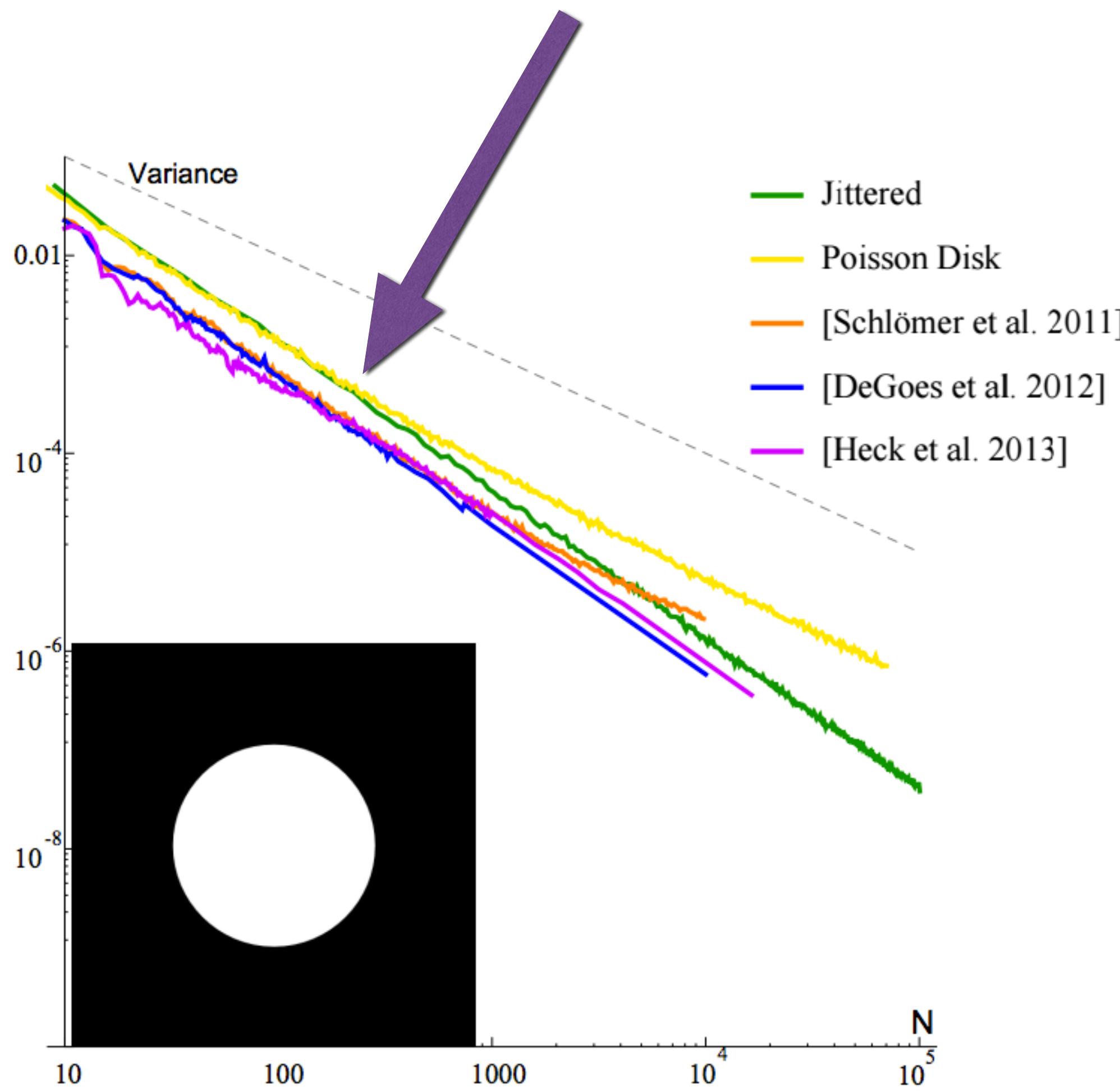
Disk Function as Worst Case



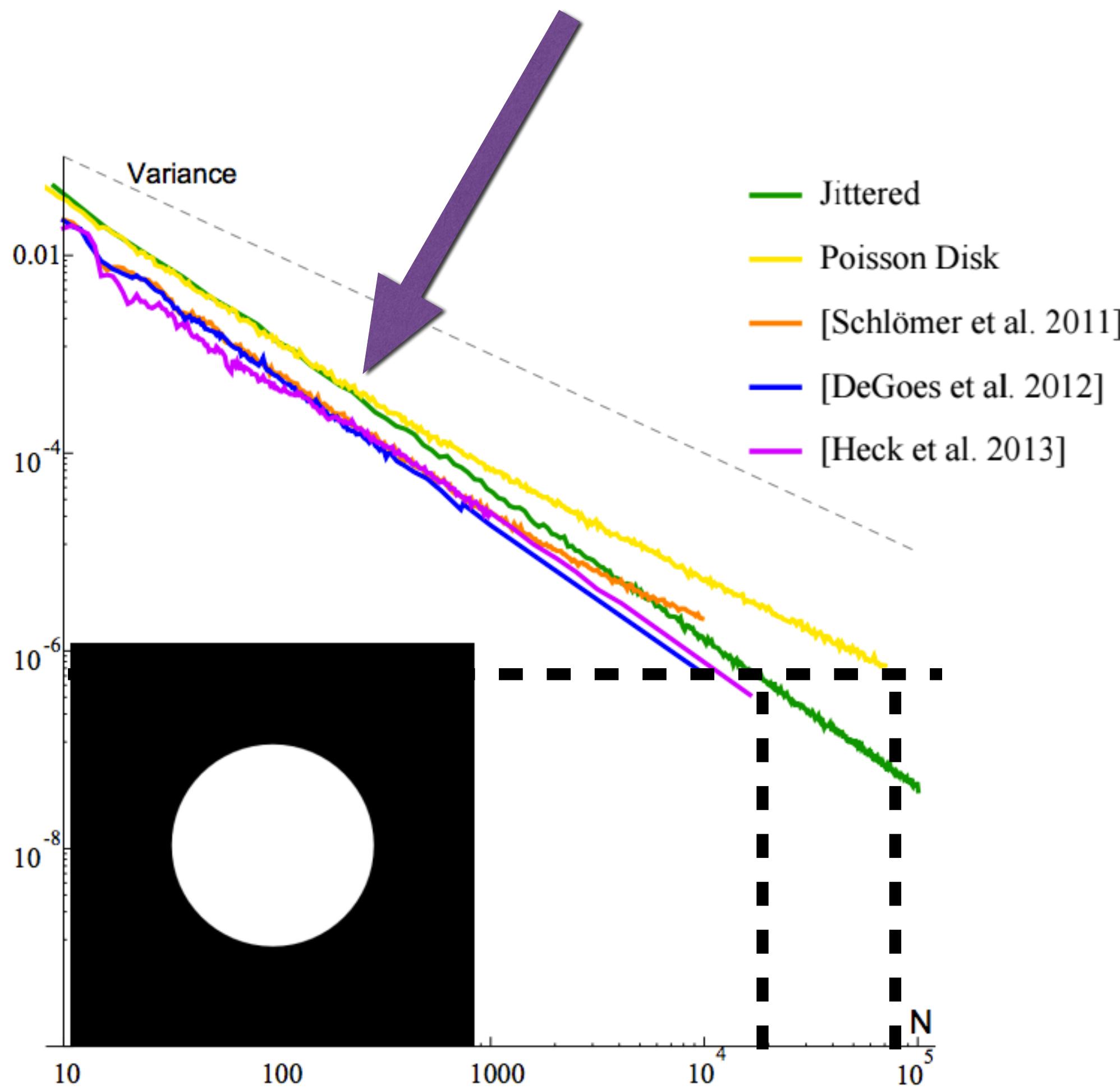
Disk Function as Worst Case



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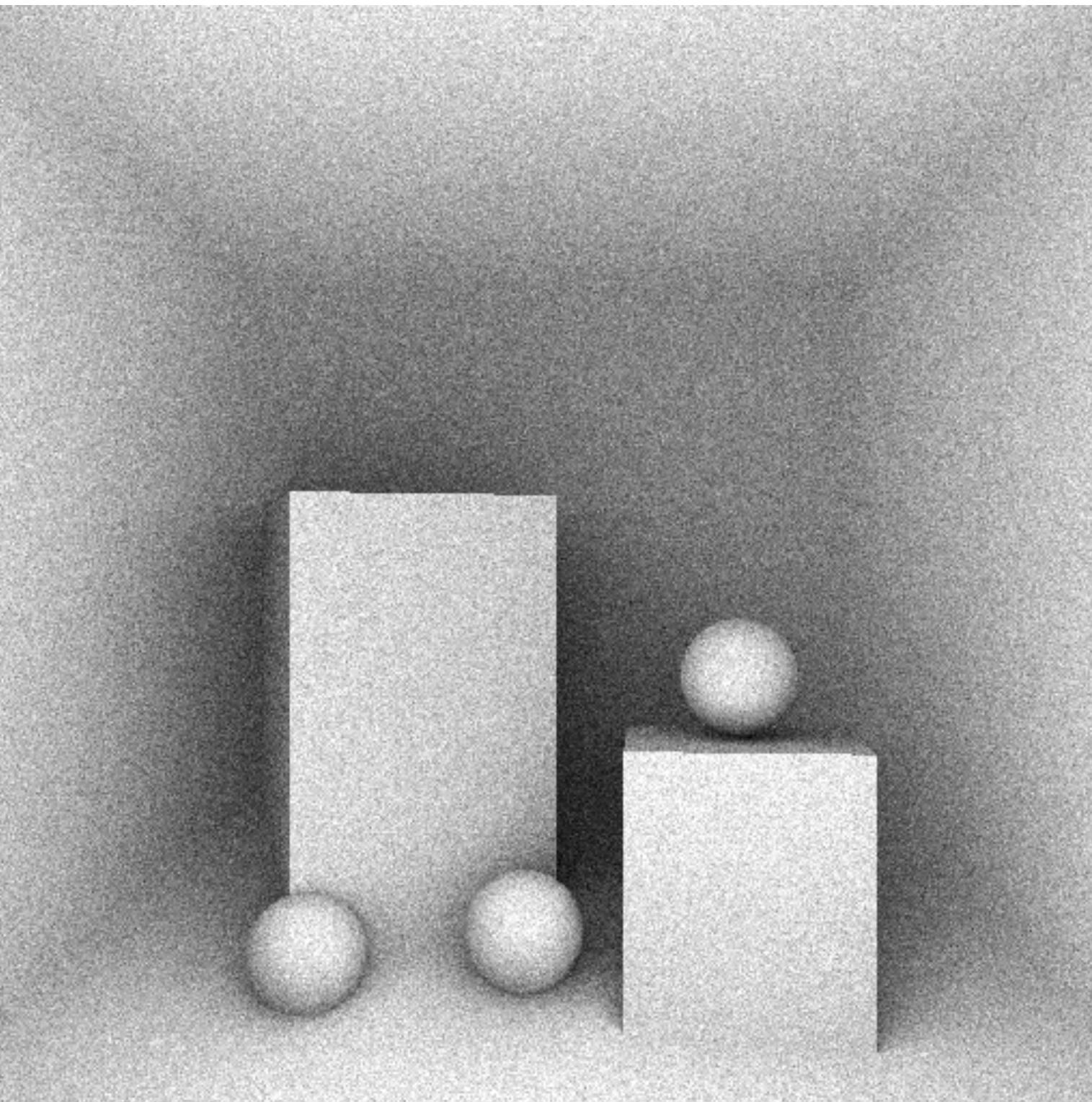
Disk Function as Worst Case



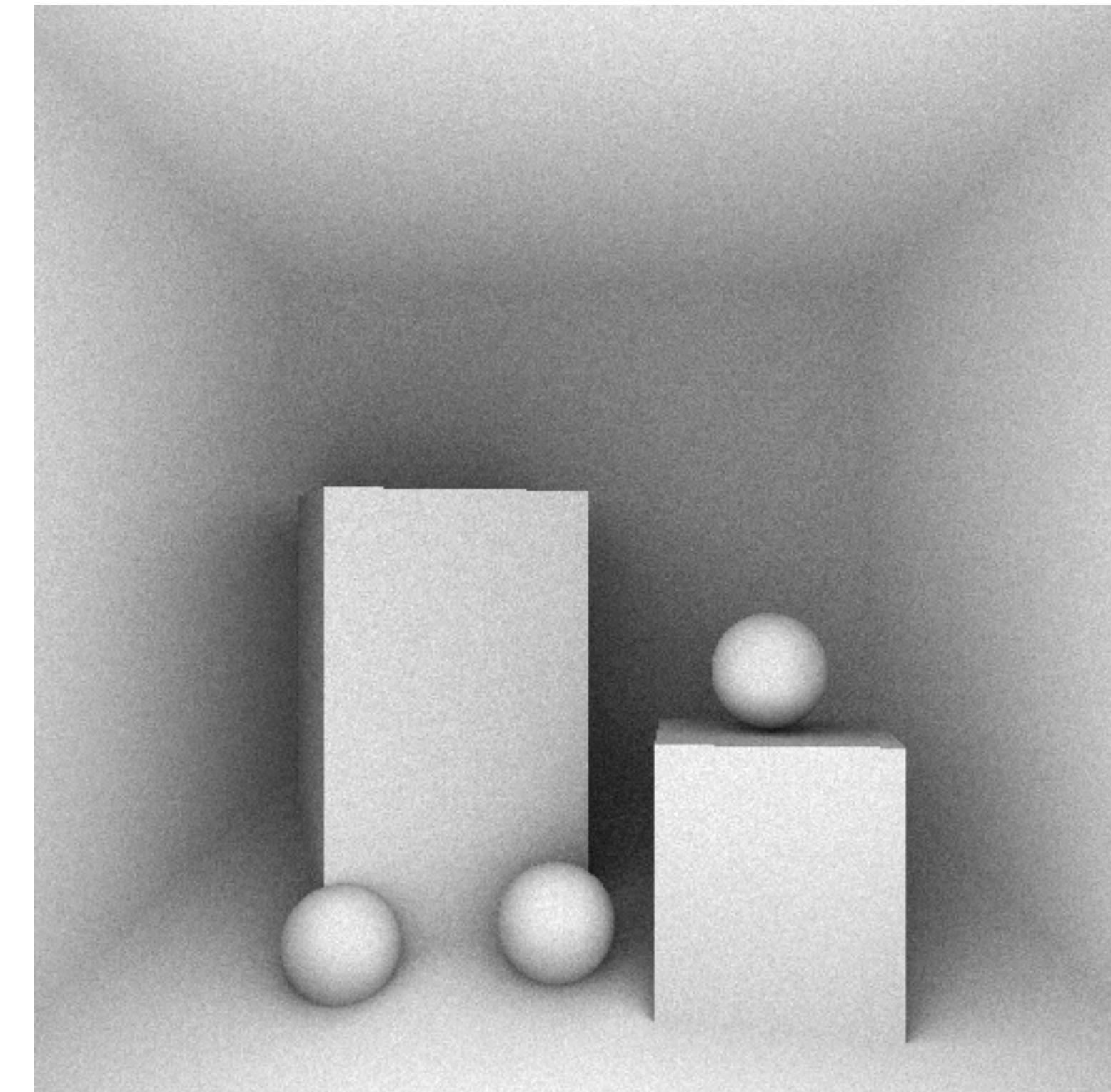
Ambient Occlusion Examples

Random vs Jittered

96 Secondary Rays



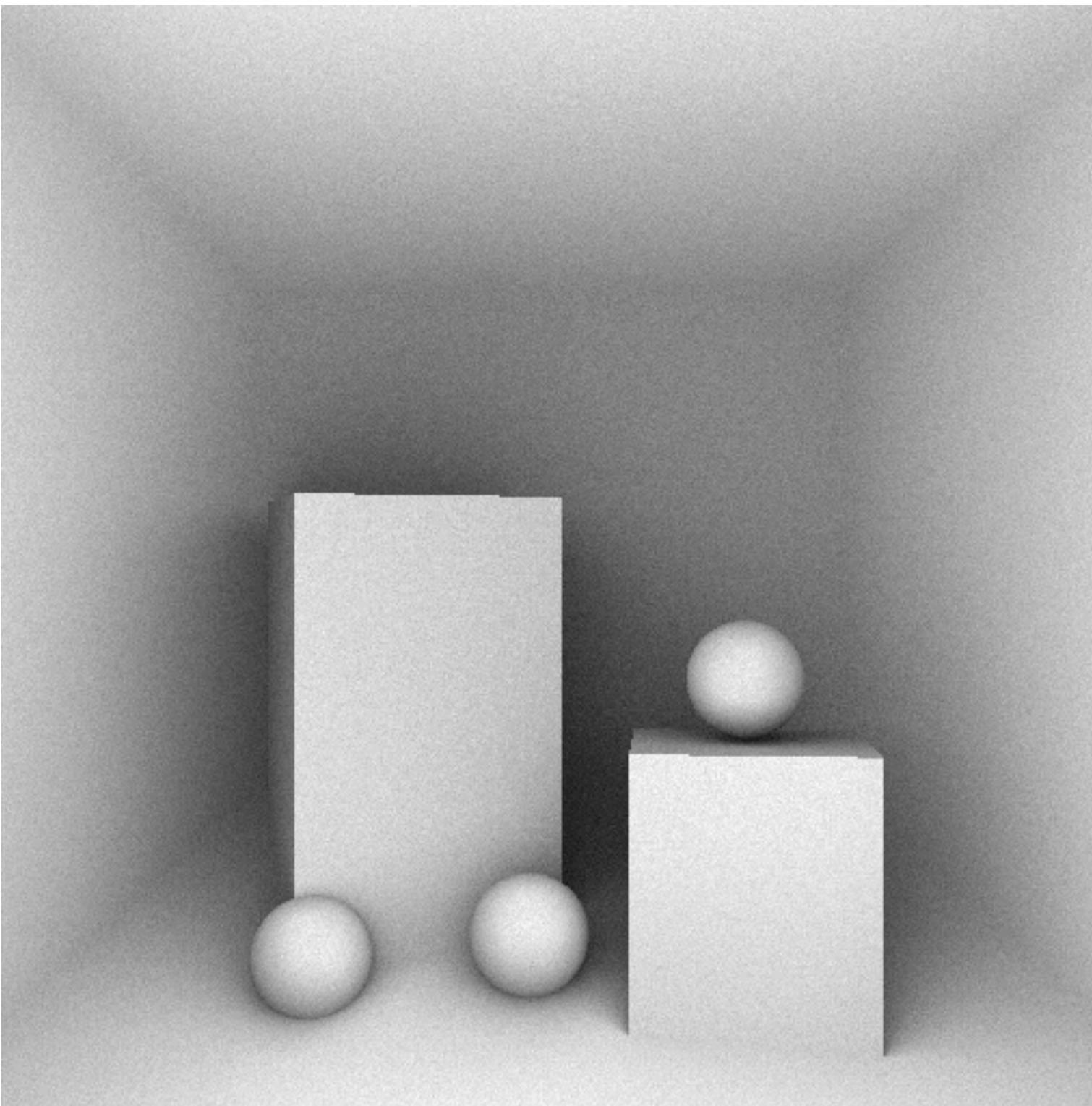
MSE: 4.74×10^{-3}



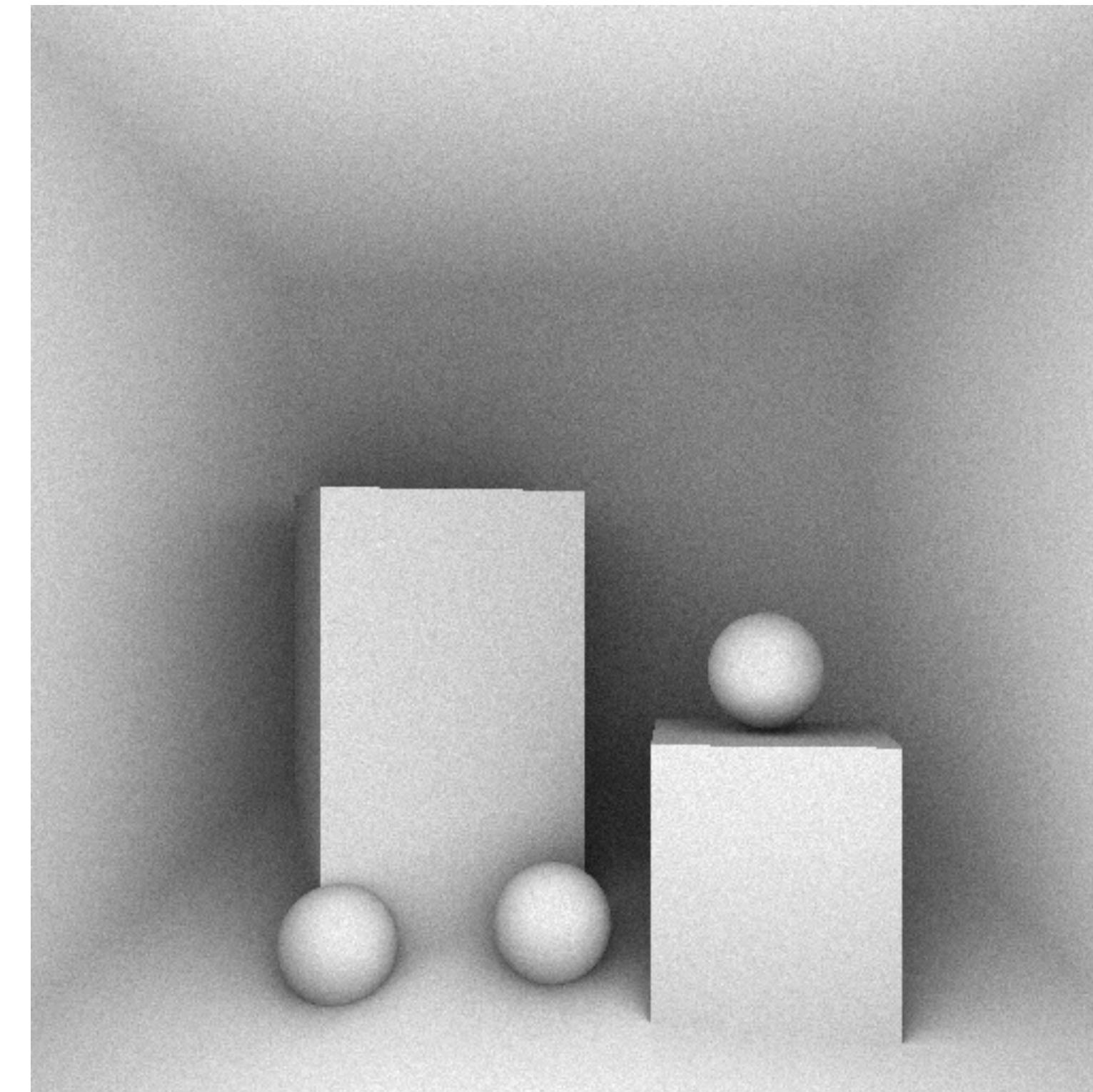
MSE: 8.56×10^{-4}

CCVT vs. Poisson Disk

96 Secondary Rays

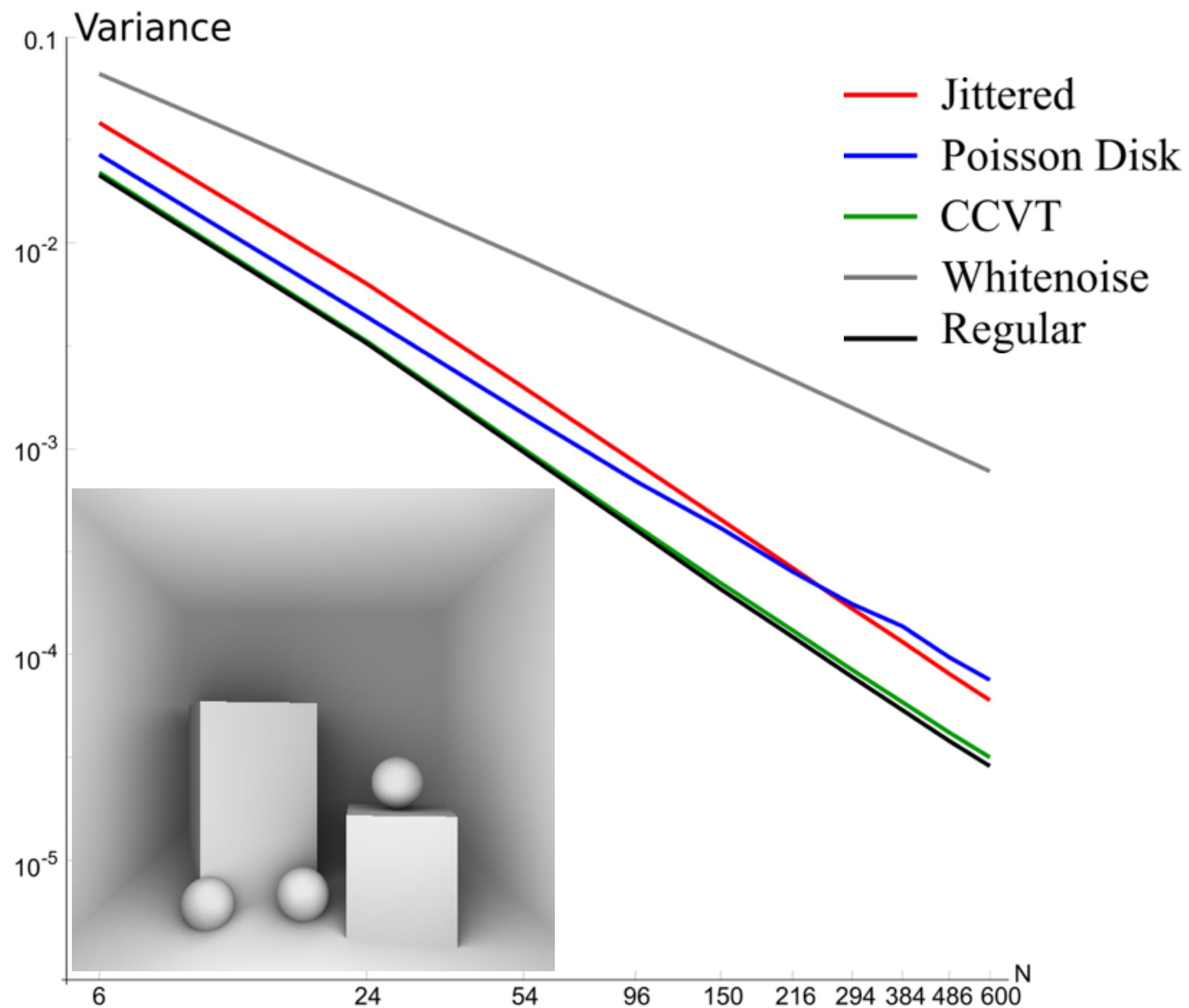


MSE: 4.24×10^{-4}

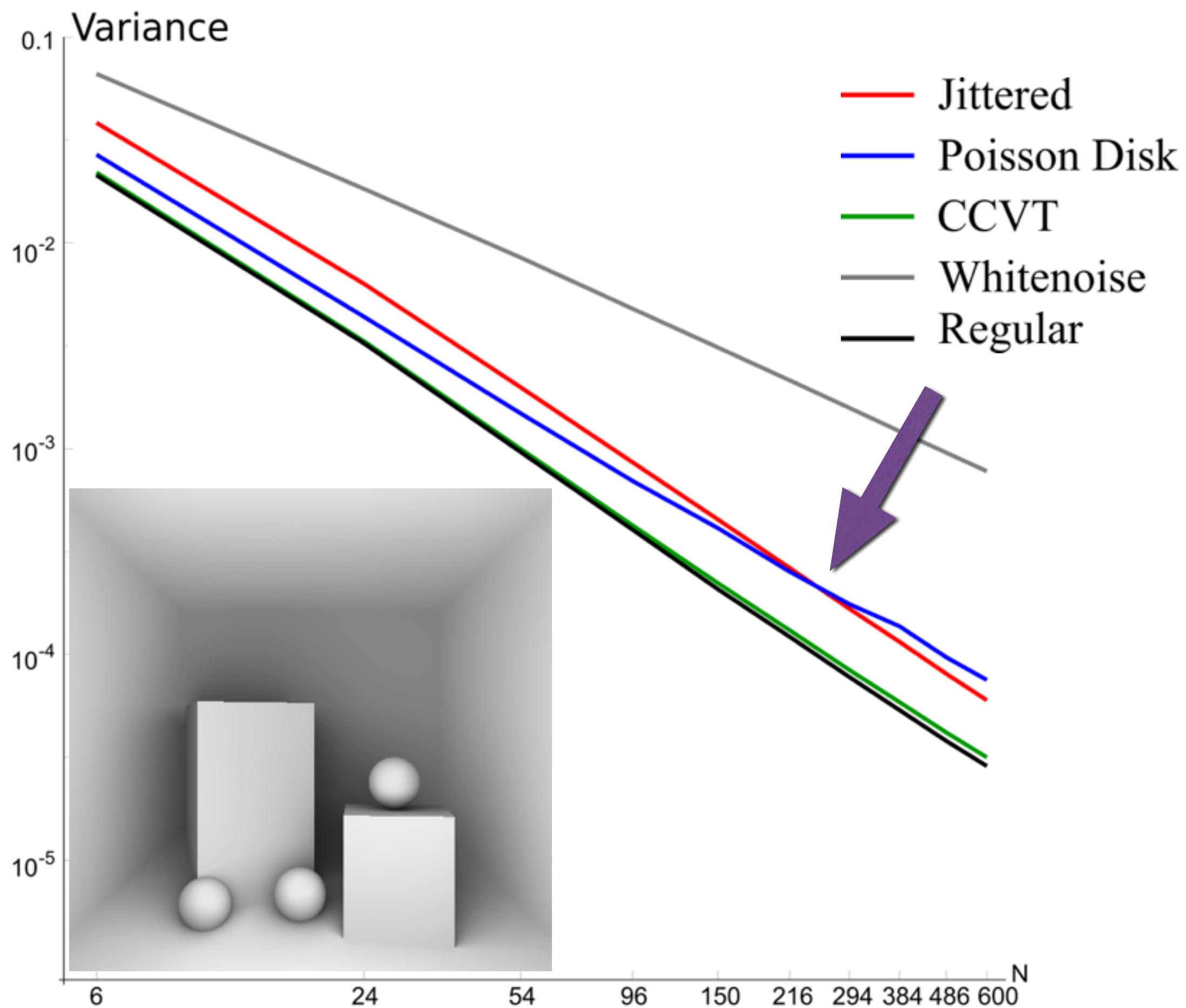


MSE: 6.95×10^{-4}

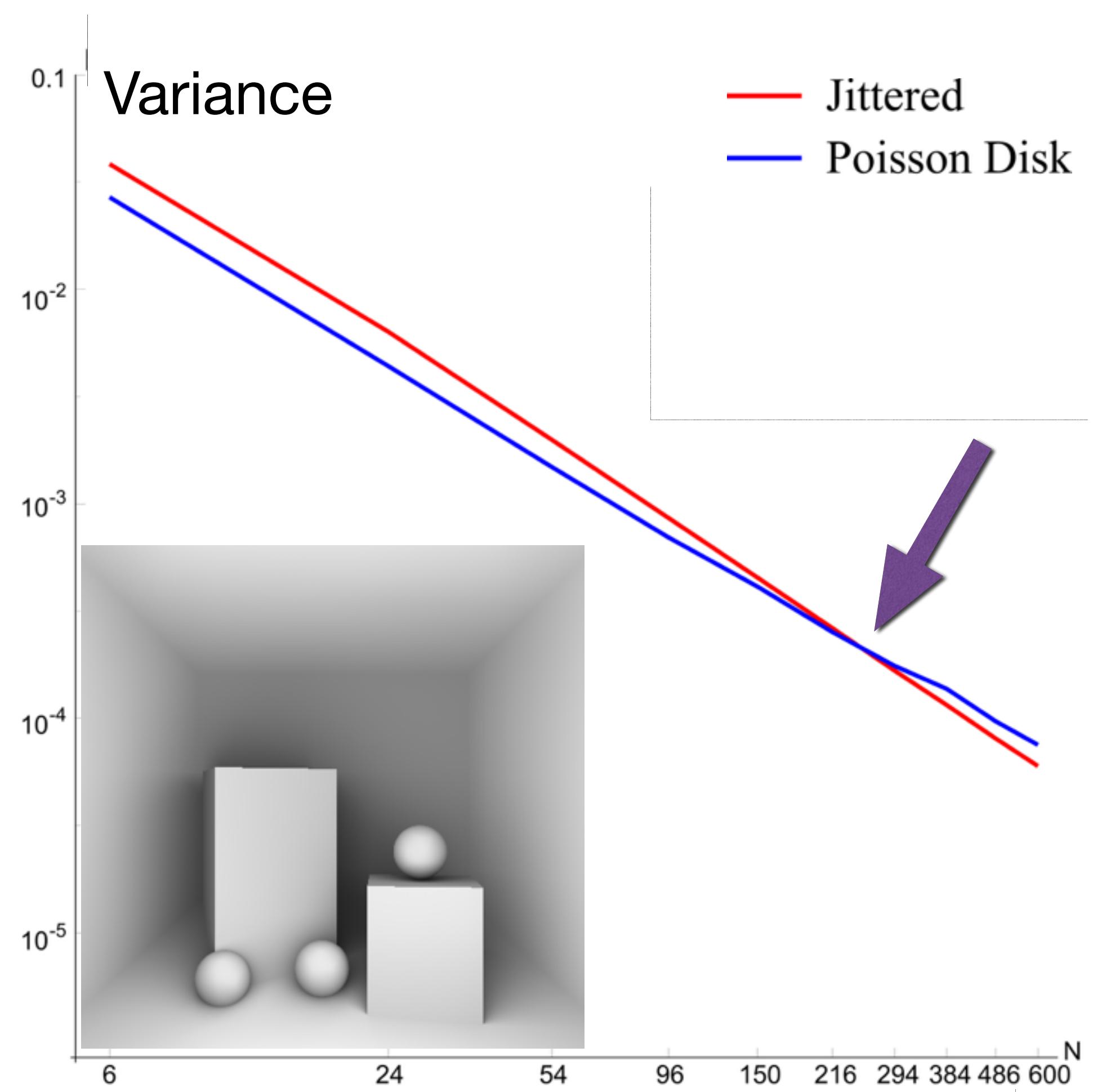
Convergence rates



Convergence rates



Jittered vs Poisson Disk



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- For offline rendering, analysis tells which samplers would converge faster.

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- For real time rendering, blue noise samples are more effective in reducing variance for a given number of samples