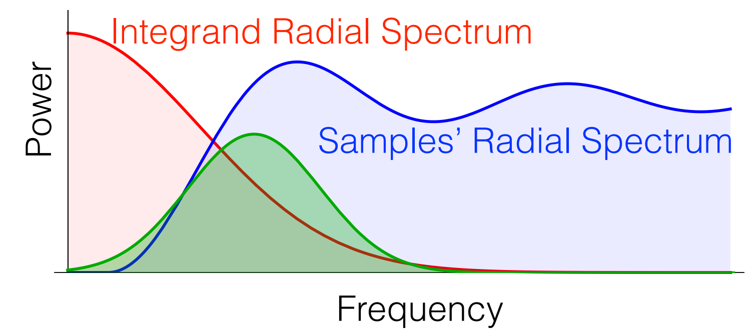
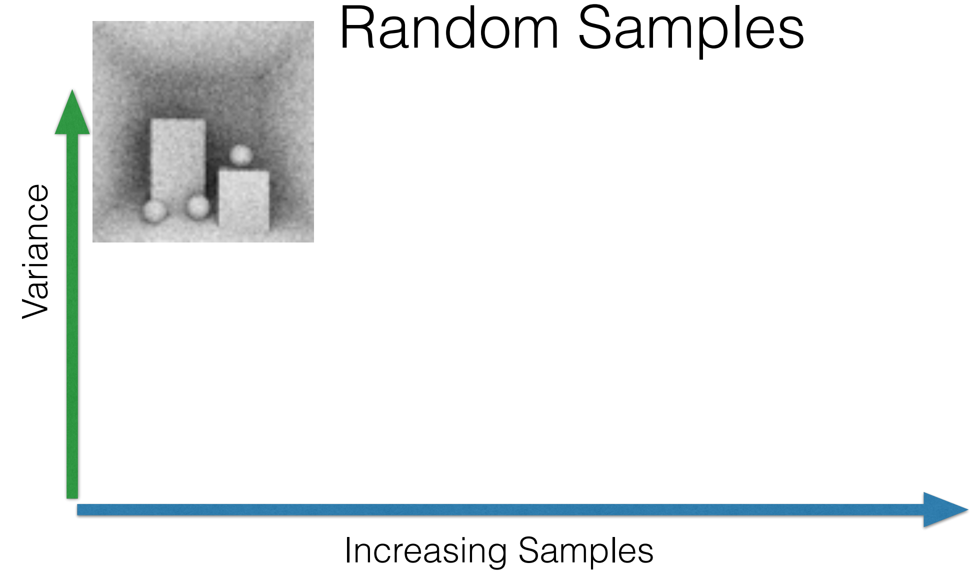
Part 3: Formal Treatment of MSE, Bias and Variance

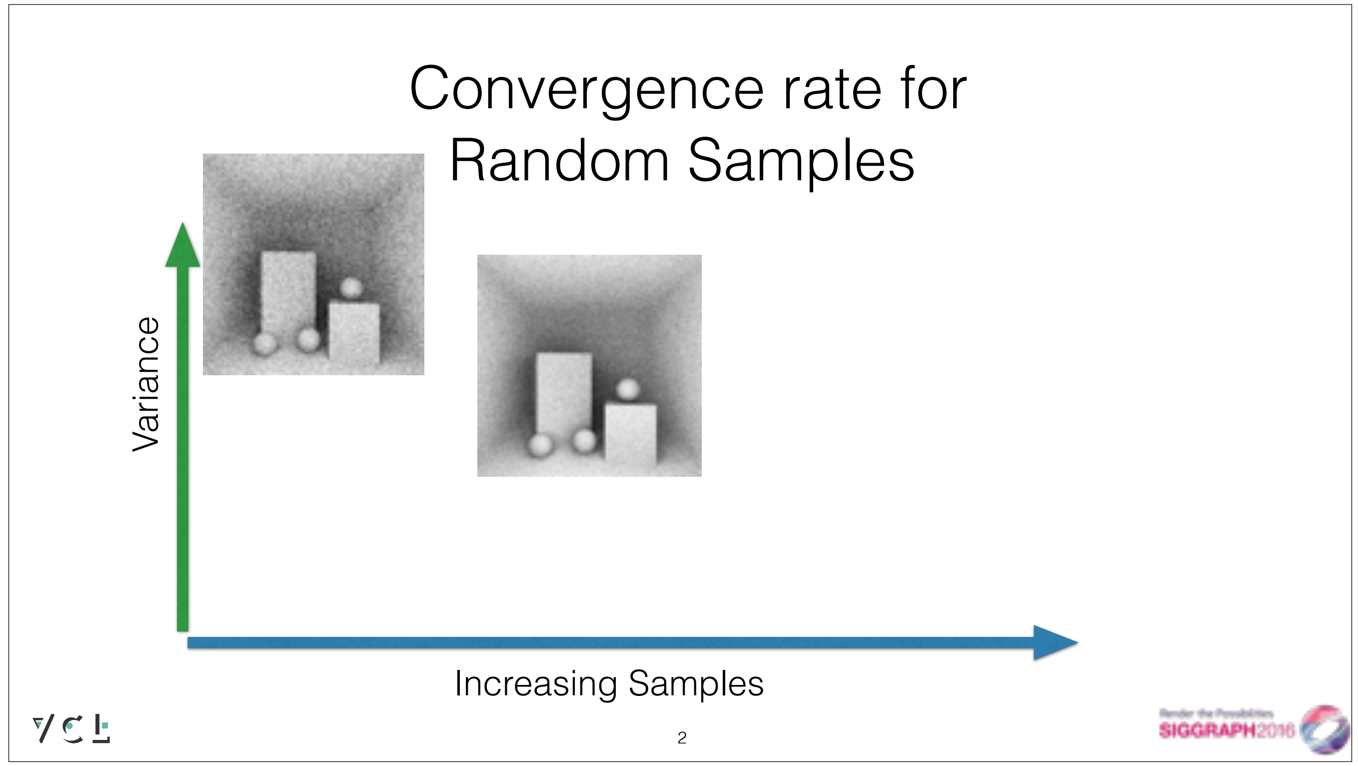


Hello everyone, in the previous part of the talk Wojciech explained us how different sampling patterns can be generated. We also saw how different sampling patterns can be represented in the Fourier domain by computing their expected power spectra. In this part of the talk, we will see how the error in Monte Carlo integration gets affected when using these different sampling patterns.

Convergence rate for Random Samples



Lets start by looking at this simple rendering of an ambient occlusion example which looks very noisy.
One way to reduce variance is by increasing the number of samples.
[click] As we increase the number of samples, [click][click] the noise or the variance in the image goes down.



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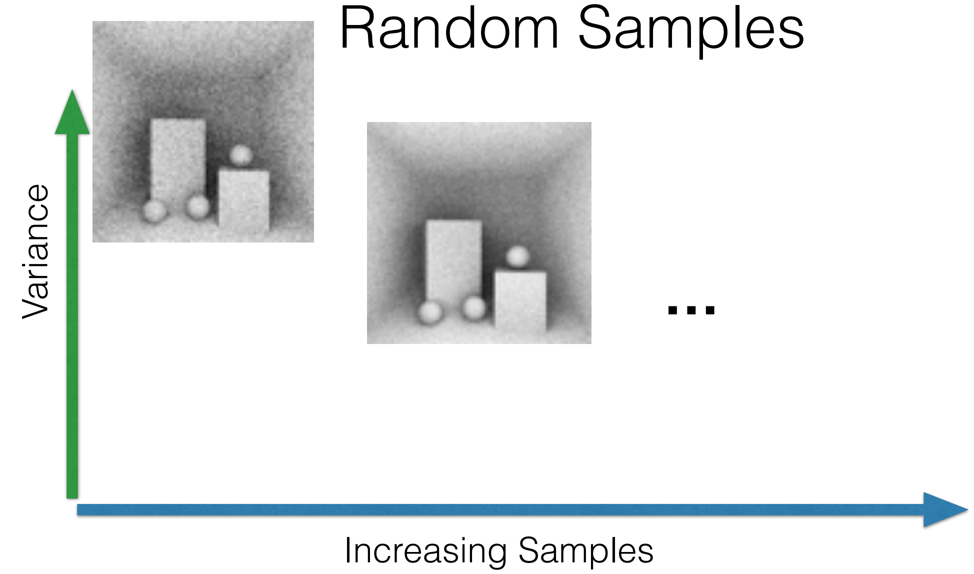

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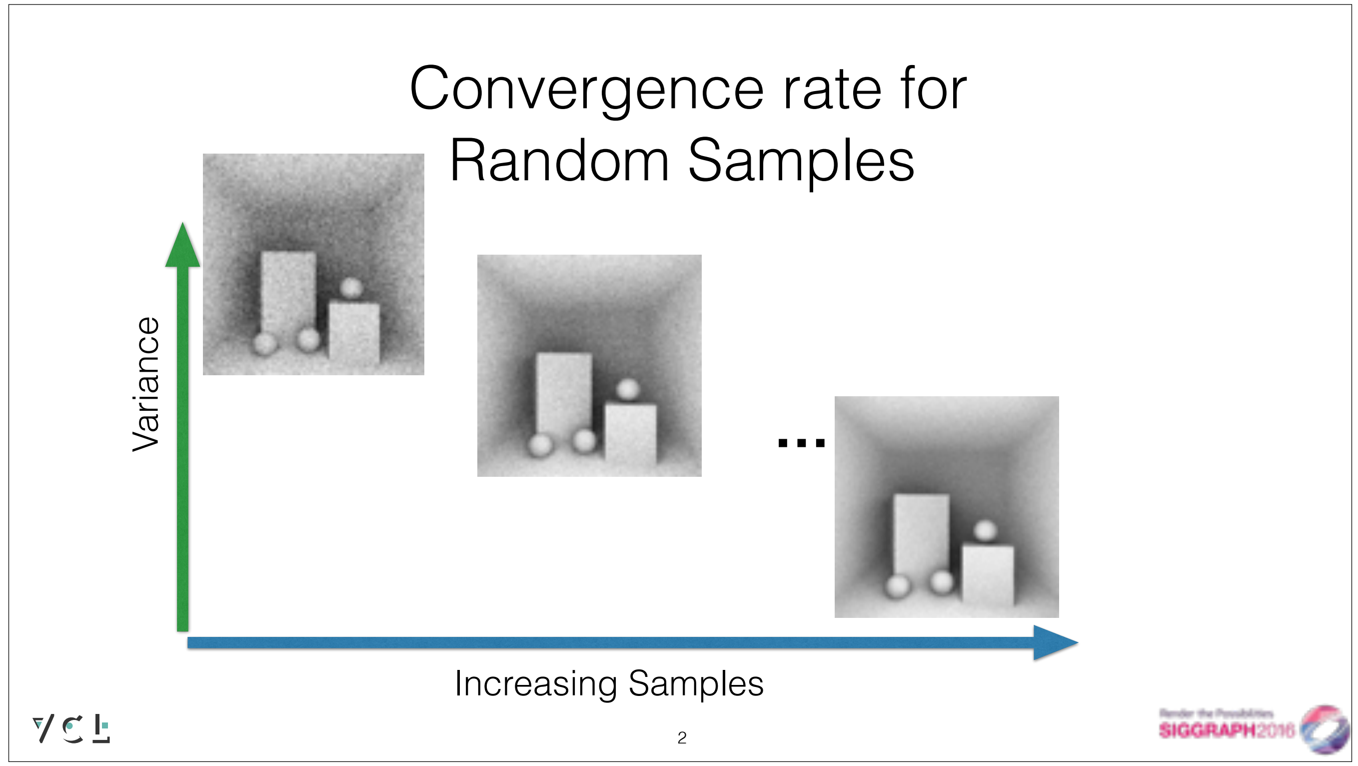

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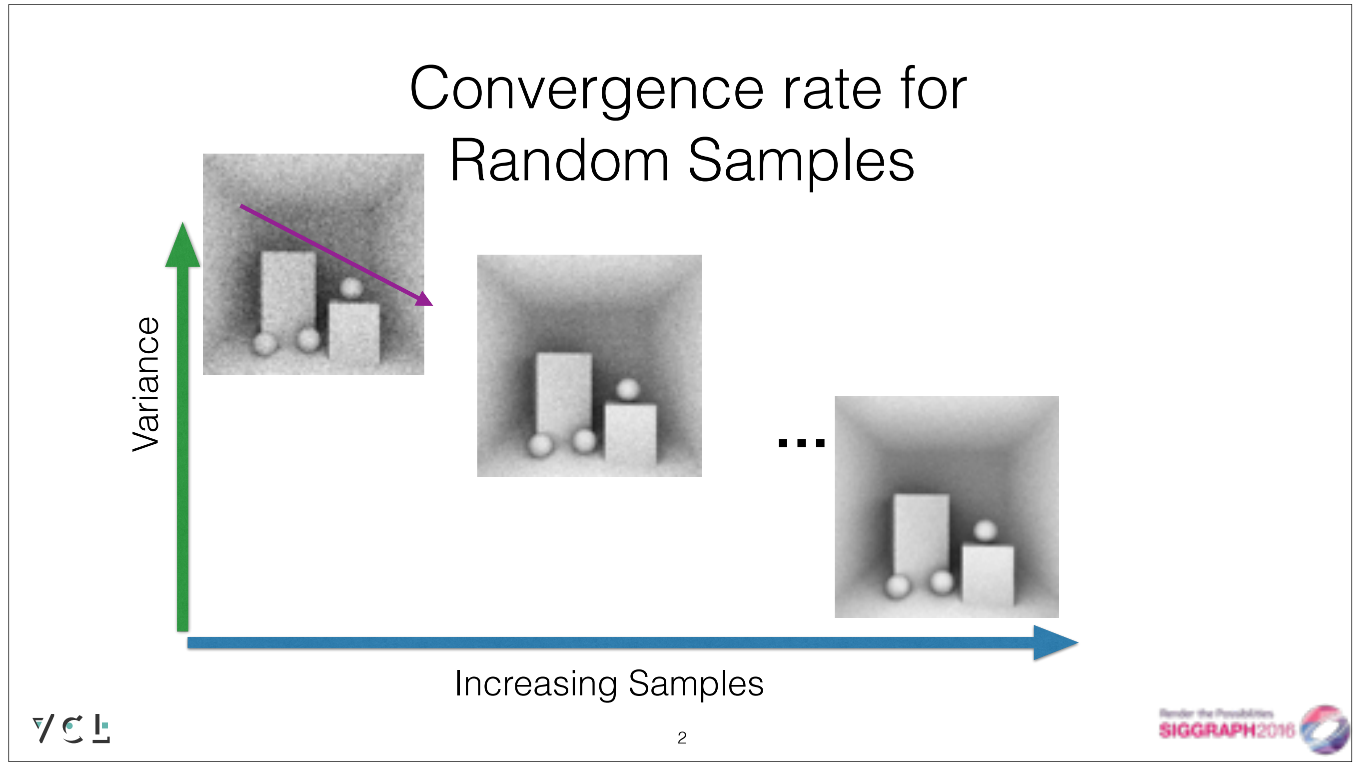

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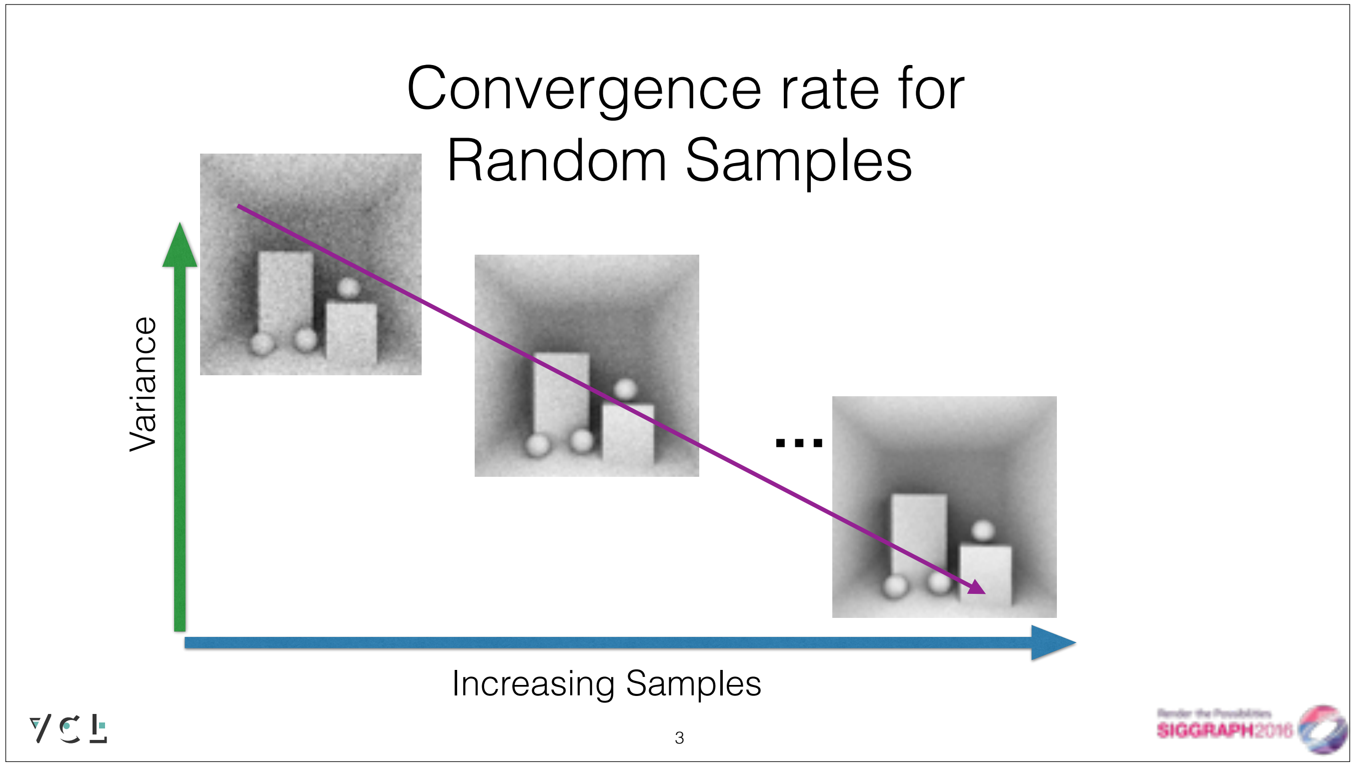
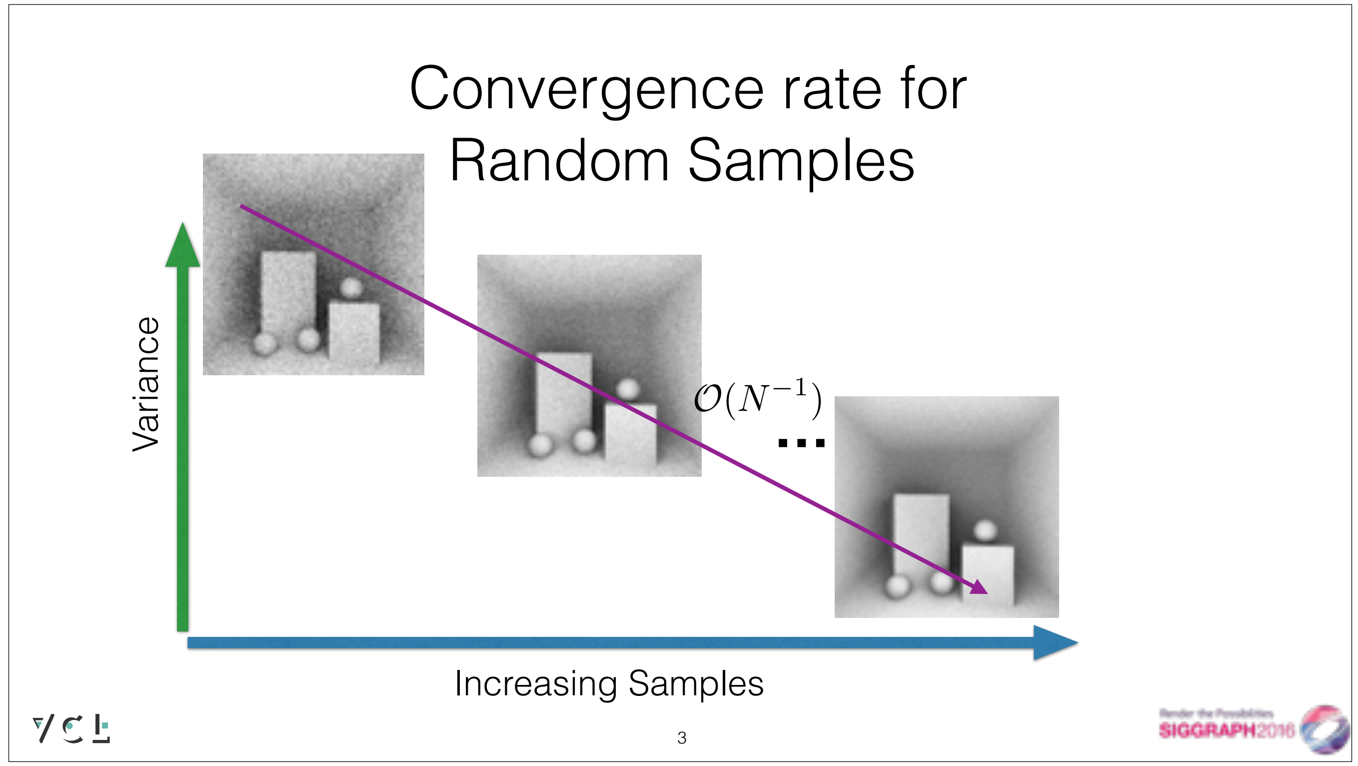


And we know that, for random samples, the rate at which the image converges to a reference value is given by [click] O(1/N).
That means, if we increase the number of samples by 2, the variance goes down by half.



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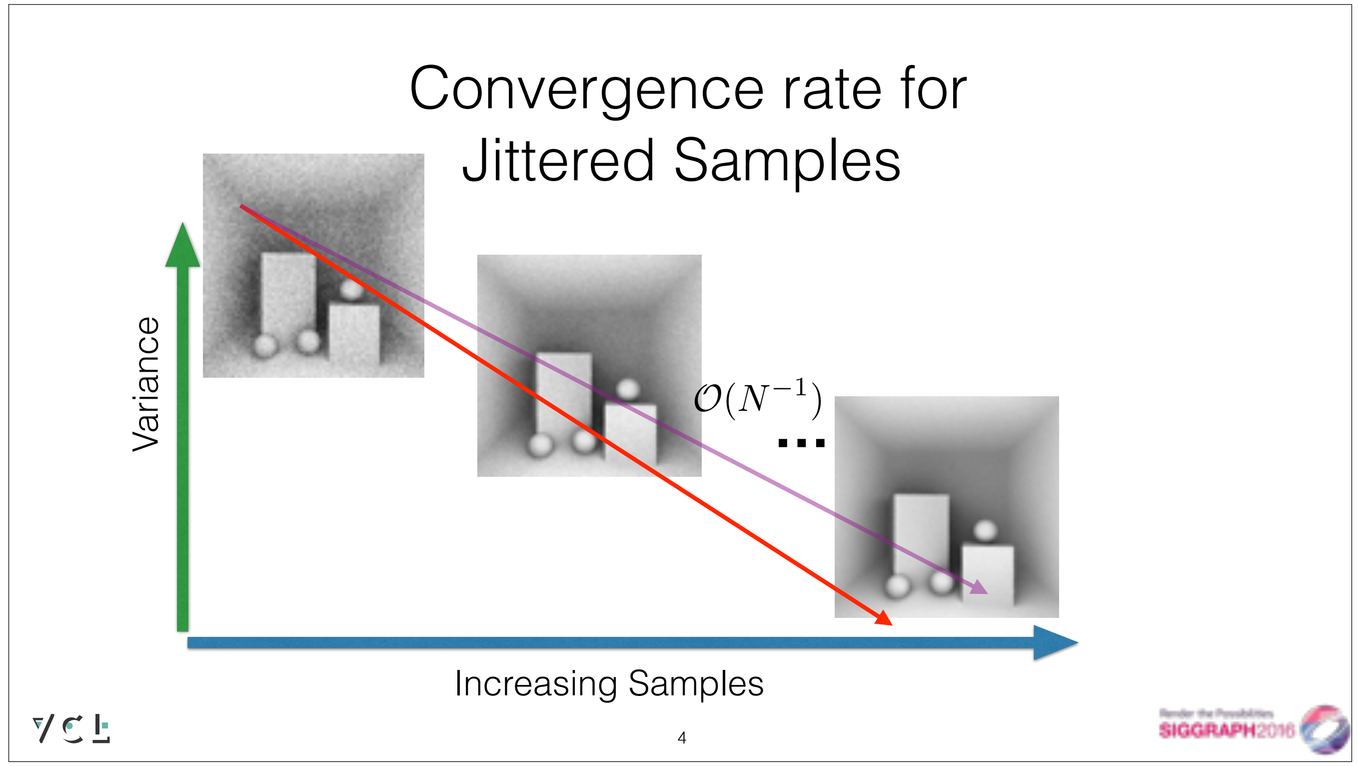
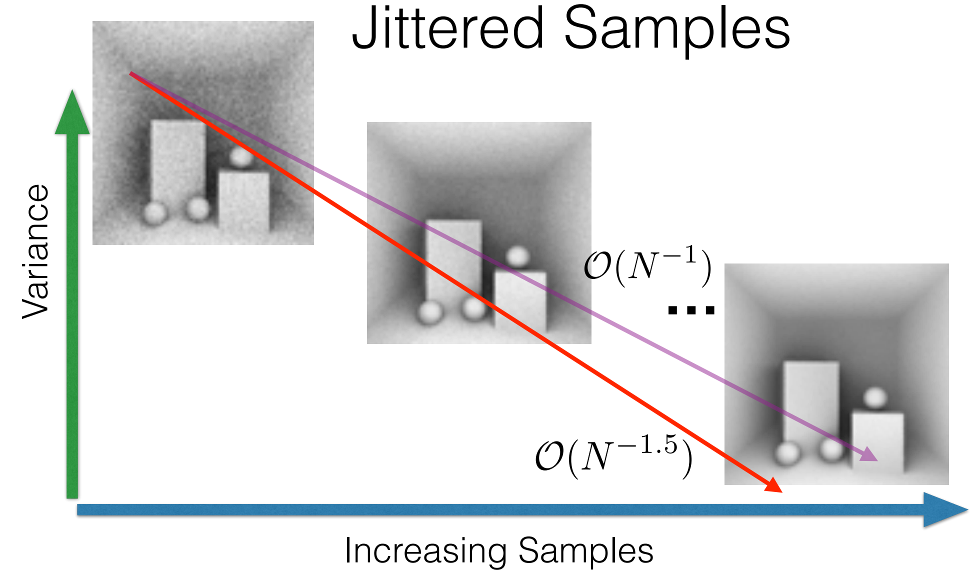


Another way to get faster convergence is by choosing some other sampler, like jittered [click], which has a much faster convergence rate compared to random samples.

Convergence rate for Jittered Samples



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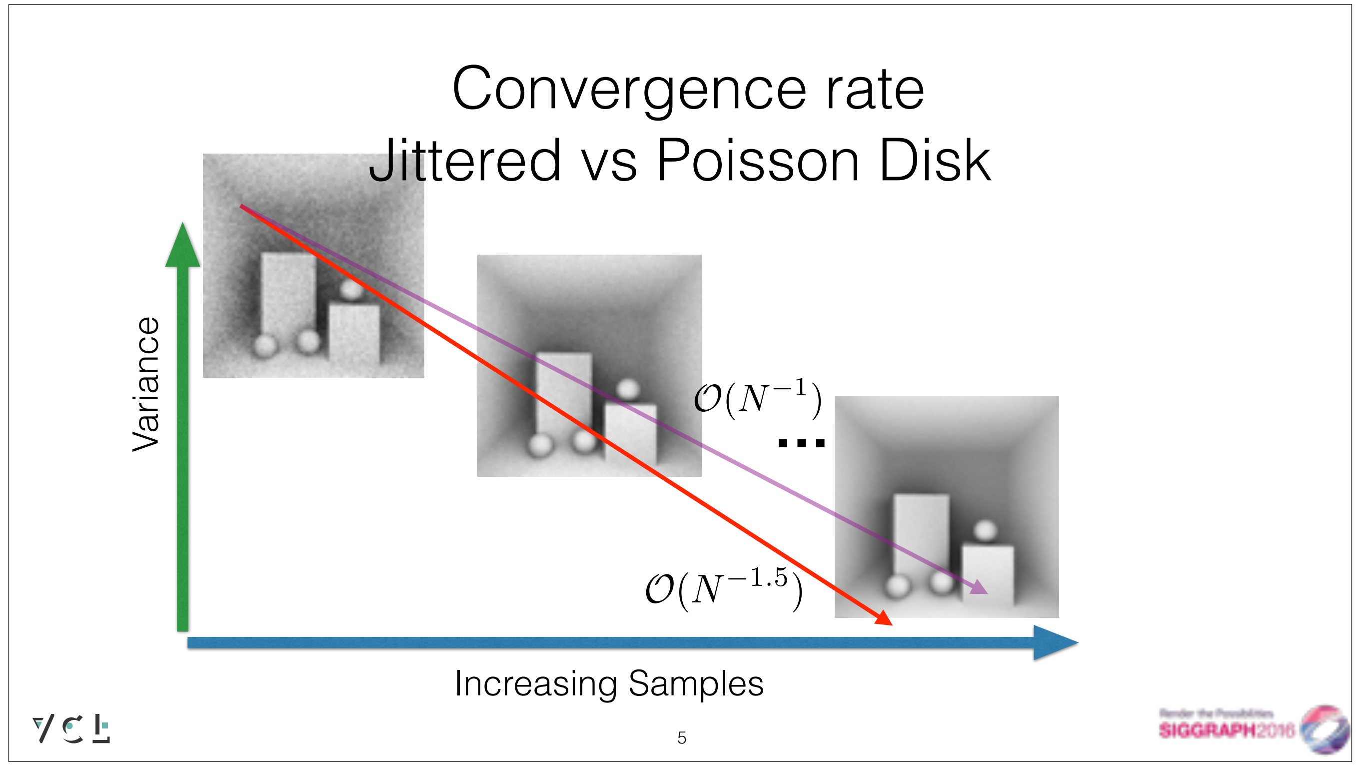
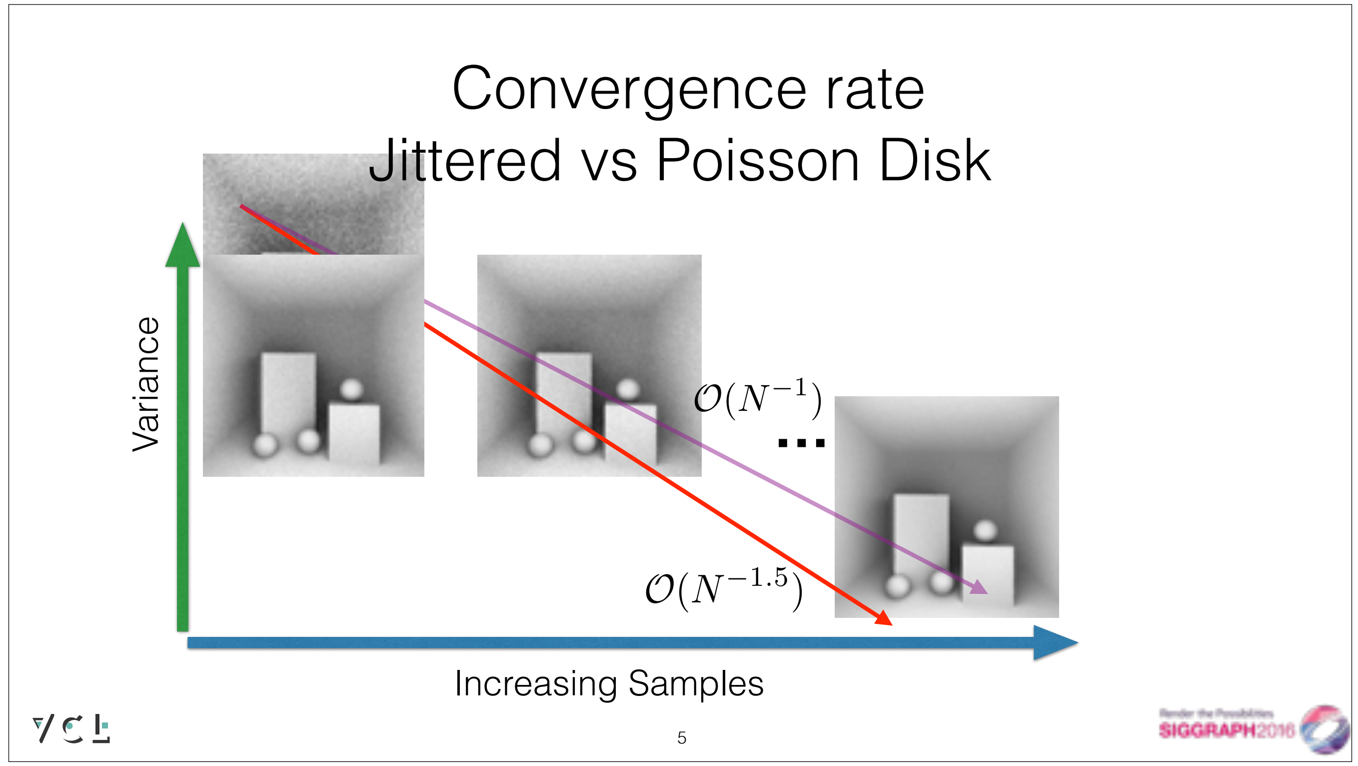


If the sampel budget is fixed, it is also possible to reduce noise or variance in the images by taking blue noise samples, like Poisson Disk [click], for which noise can be heavily reduced even with a small sample count. However [click]..



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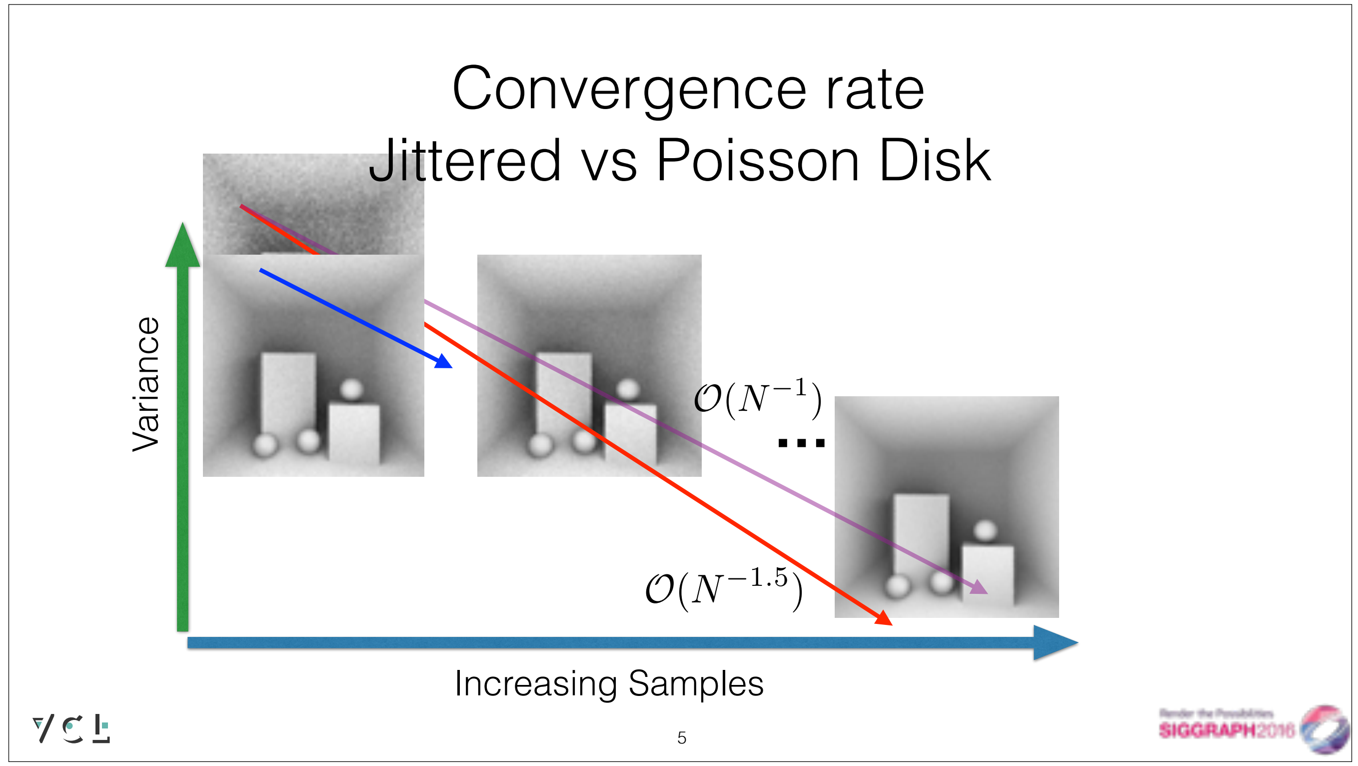

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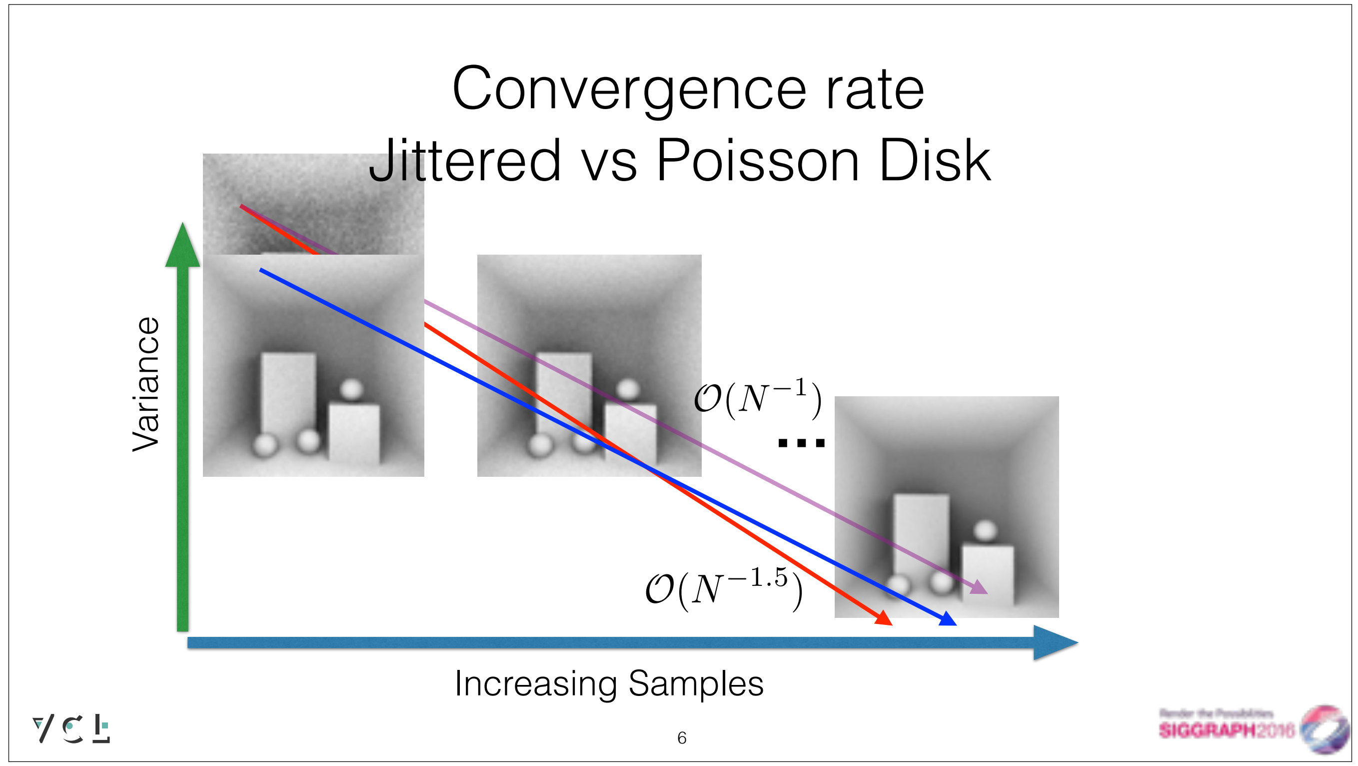
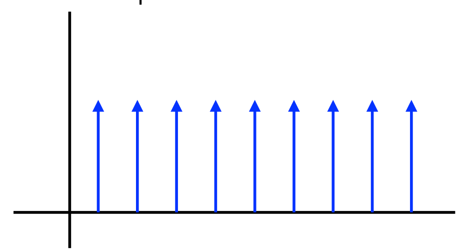


… as we increase the number of Poisson disk samples the convergence rate seems to follow the same behaviour as random samples. In this part of the course, we will try to understand why this happens. Lets start with a 1D example.

Samples and function in Fourier Domain

Spatial Domain

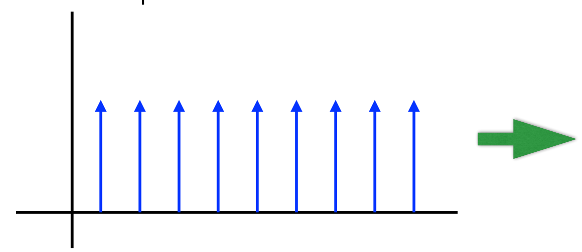


Fourier Domain

Here, we have a regular sample distribution, which is also [click] regularly distributed [click] in the Fourier domain
[click]: And we also consider a 1D function f(x)
[click]: with the following [click] Fourier spectrum.

Samples and function in Fourier Domain

Spatial Domain Fourier Domain



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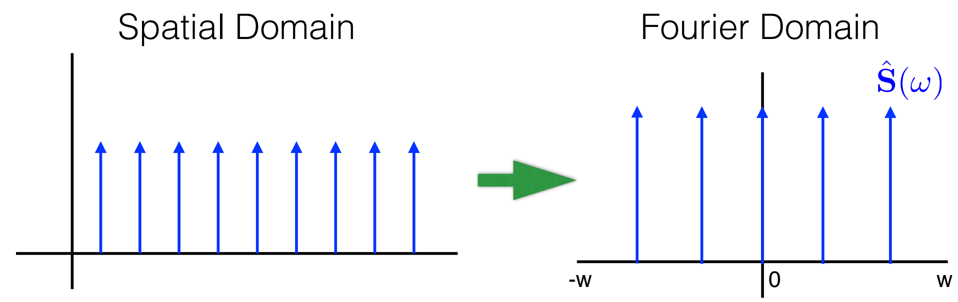

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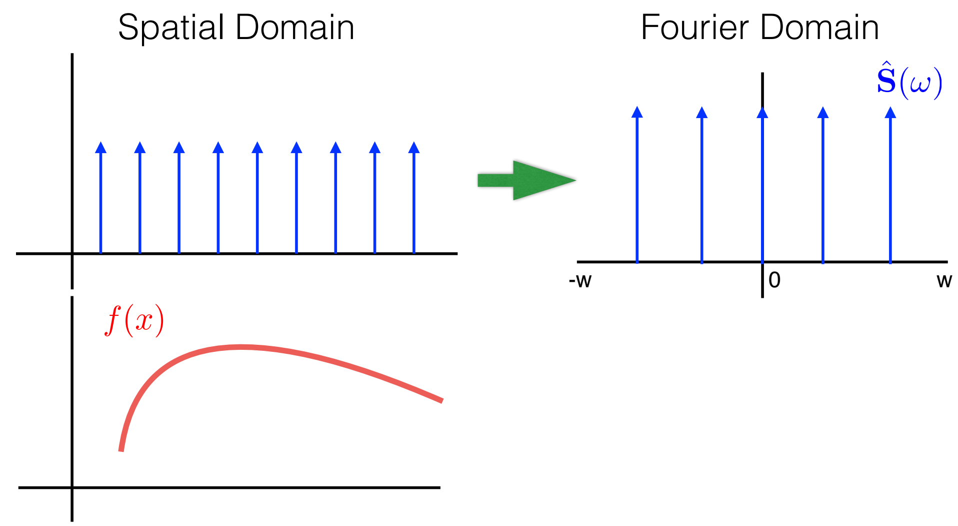

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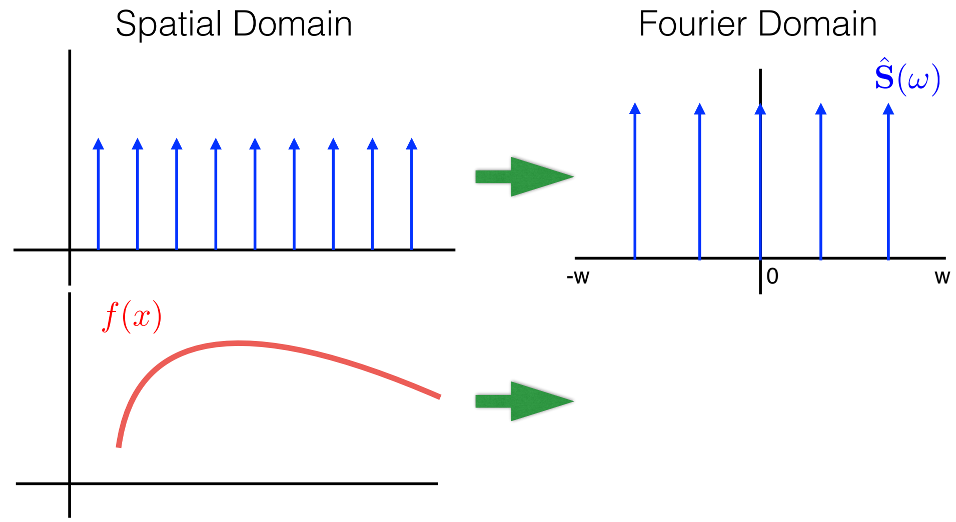

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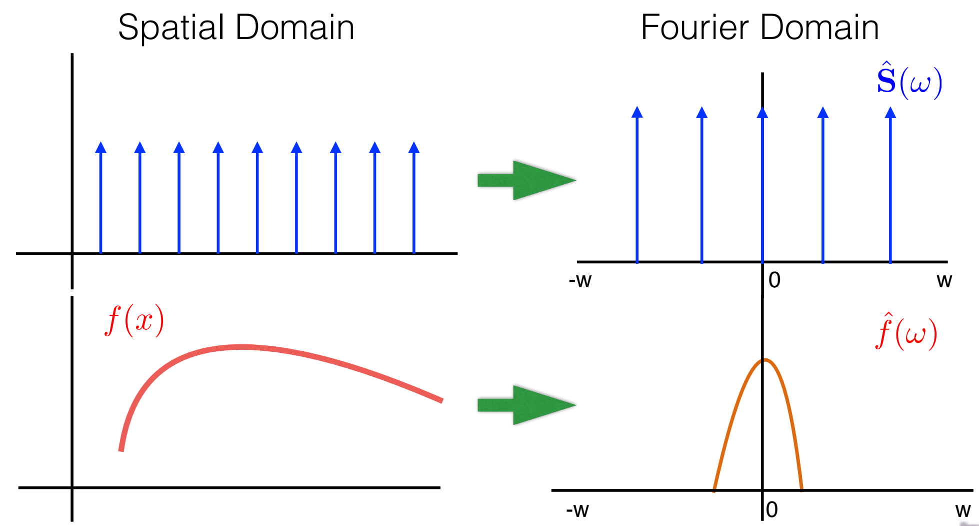

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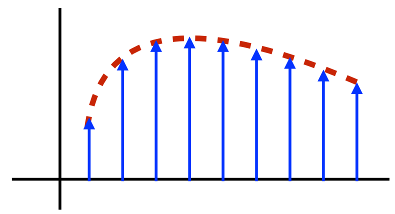
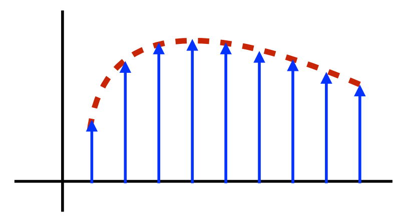
Sampling in Primal Domain is Convolution in Fourier Domain



$f(x)\,\mathbf{S}(x)$

Sampling a function in the Spatial or the Primal domain, involves multiplication of the function f with the sampling pattern S.
[click]:

Sampling in Primal Domain is Convolution in Fourier Domain



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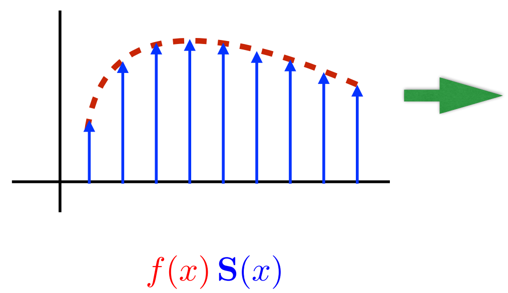

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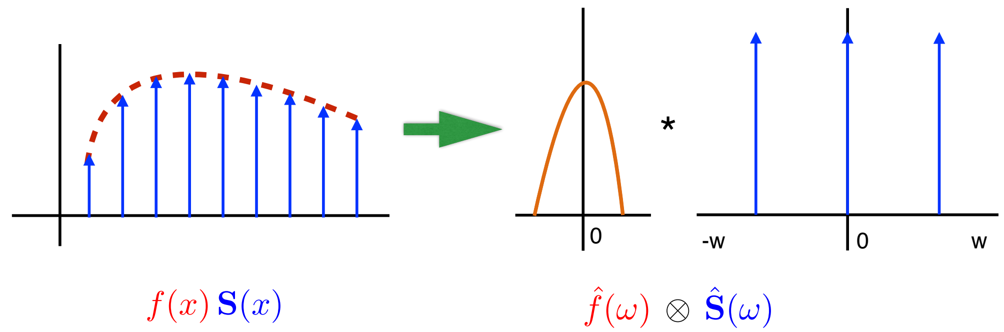

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[click]x3: which correspond to a convolution in the Fourier domain. Now if we look at these replicas…

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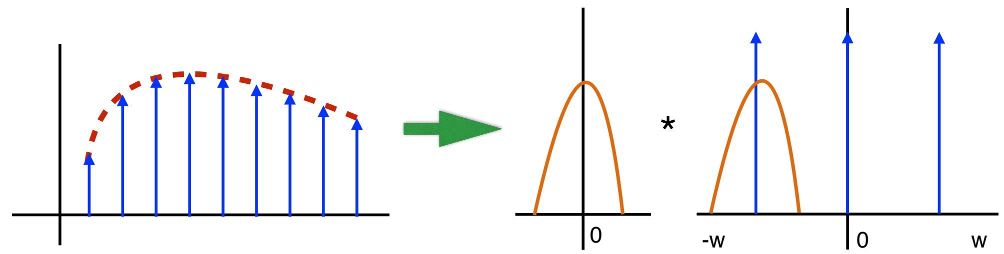

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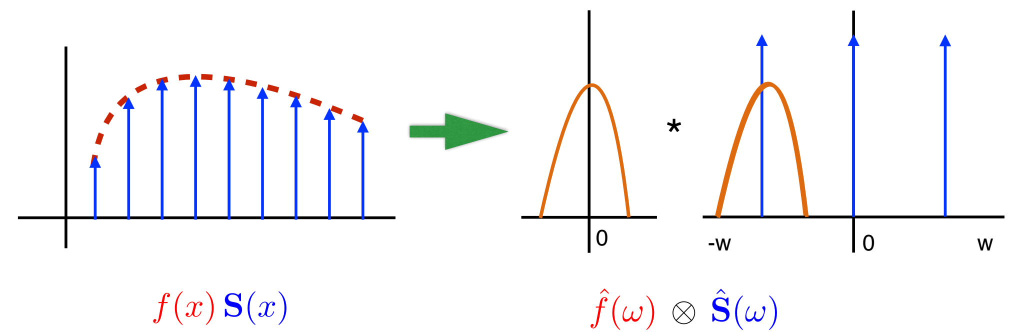

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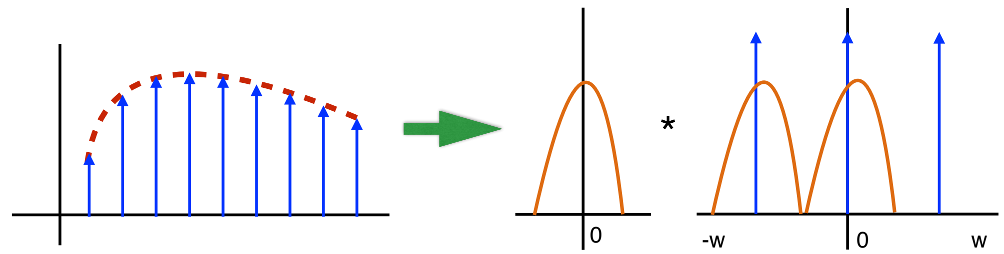

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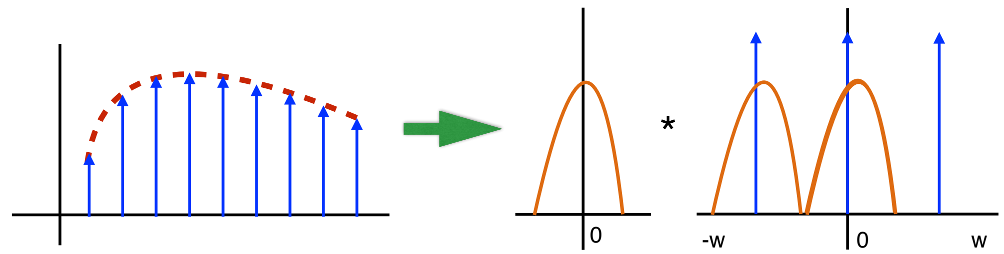

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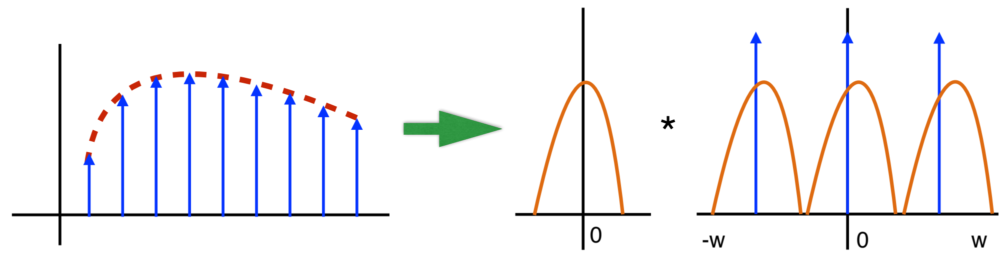

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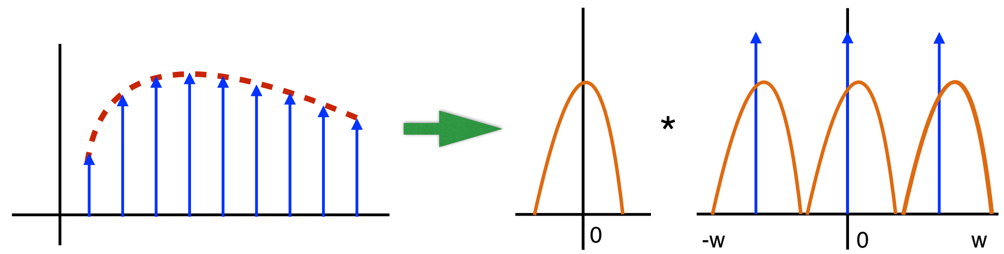

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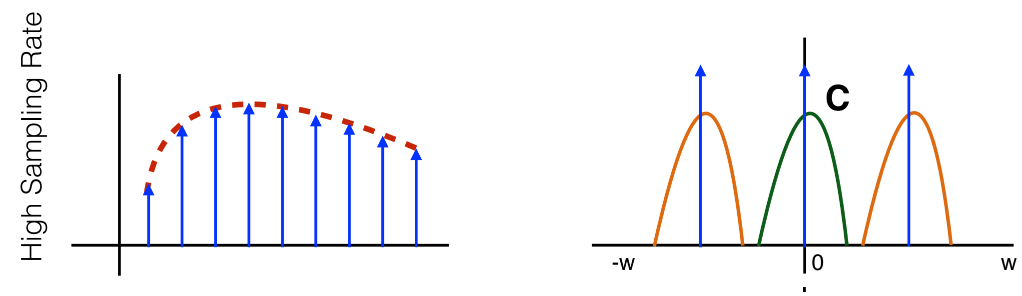
Aliasing in Reconstruction



the central replica C [click] shown in green, is of great interest in reconstruction.
[click]: When we don’t have enough samples
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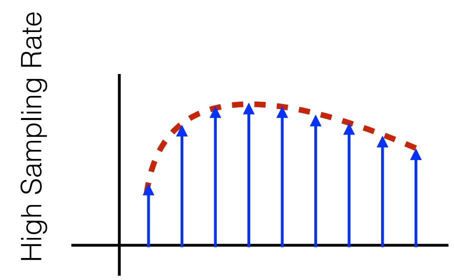

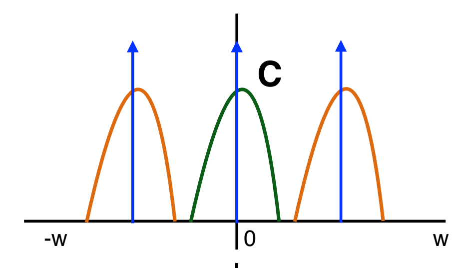

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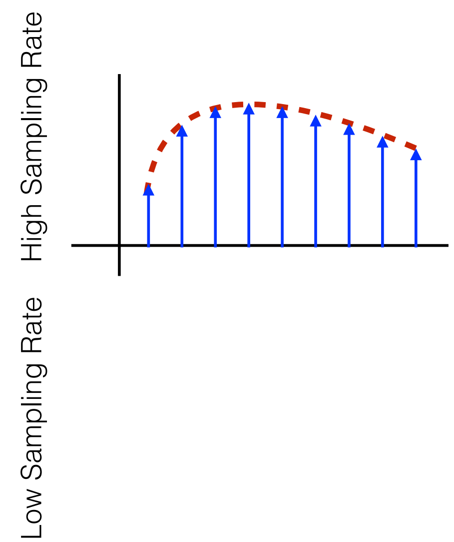

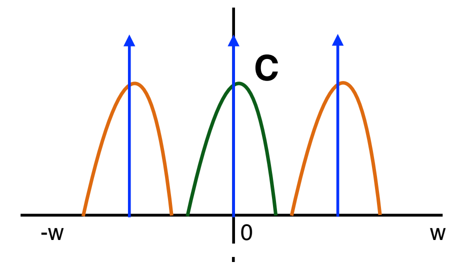

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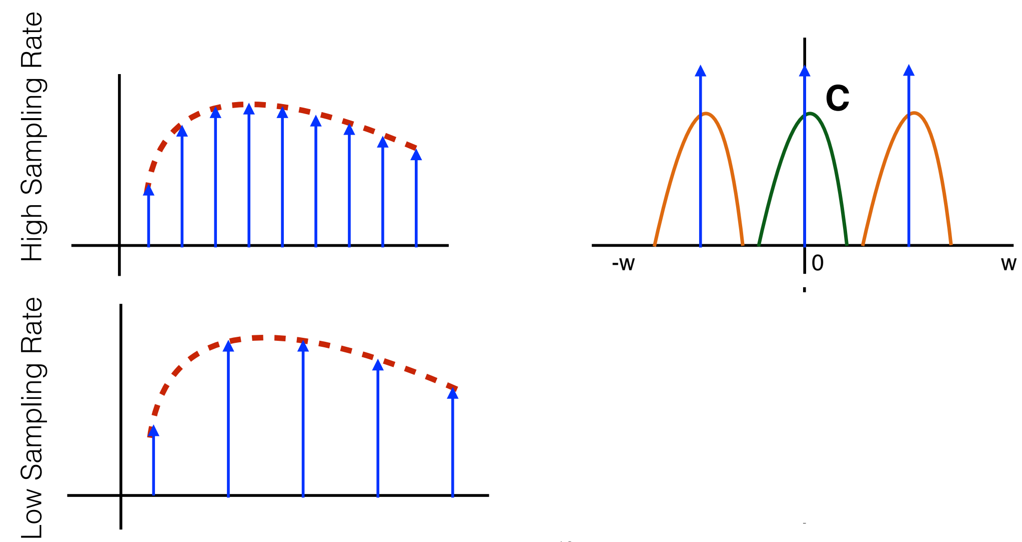

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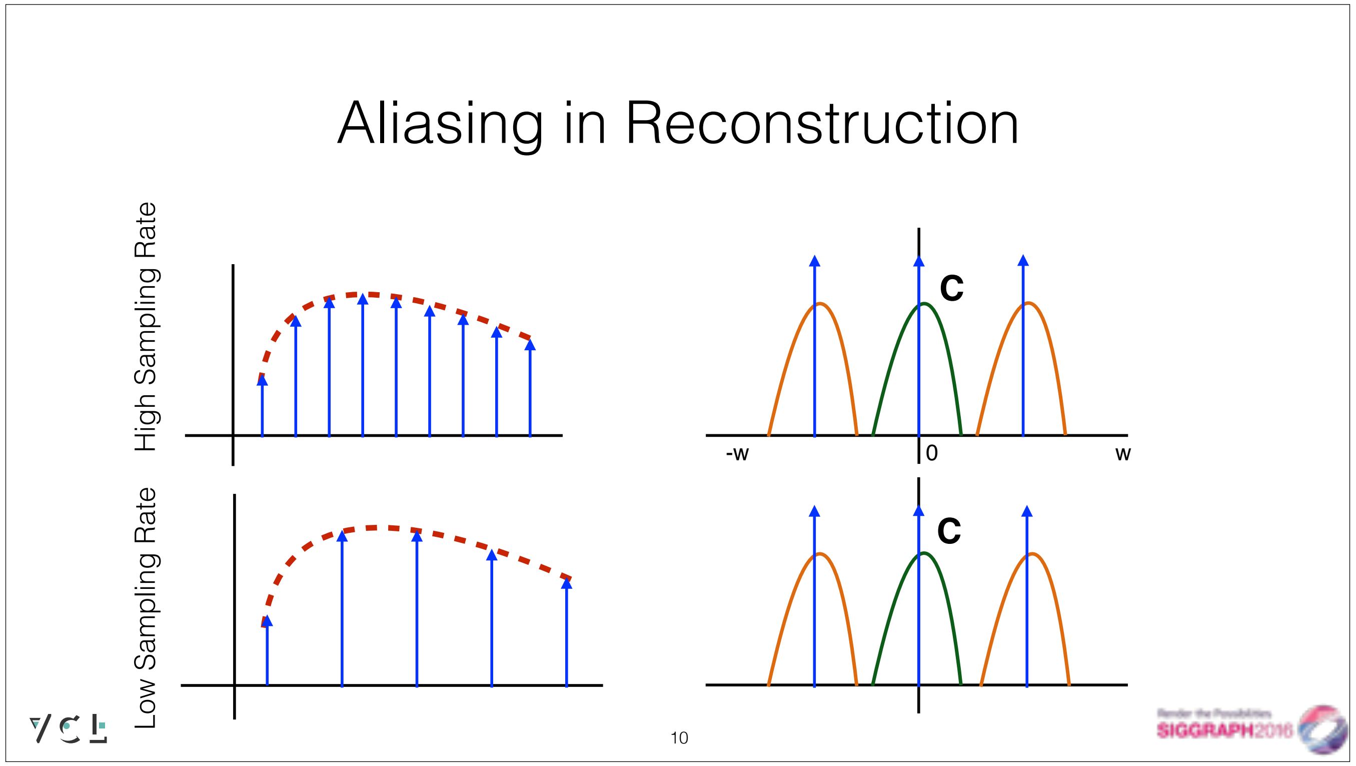

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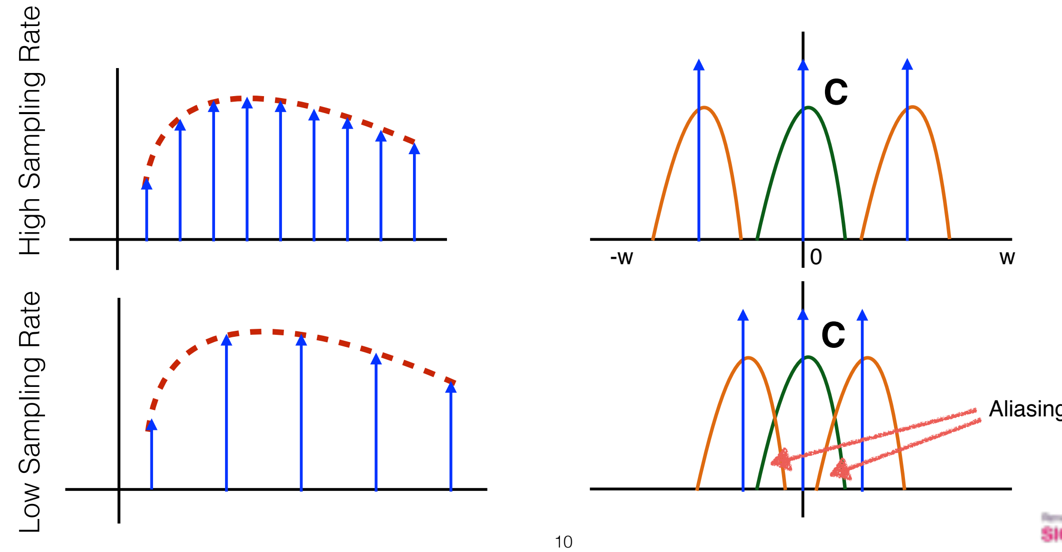

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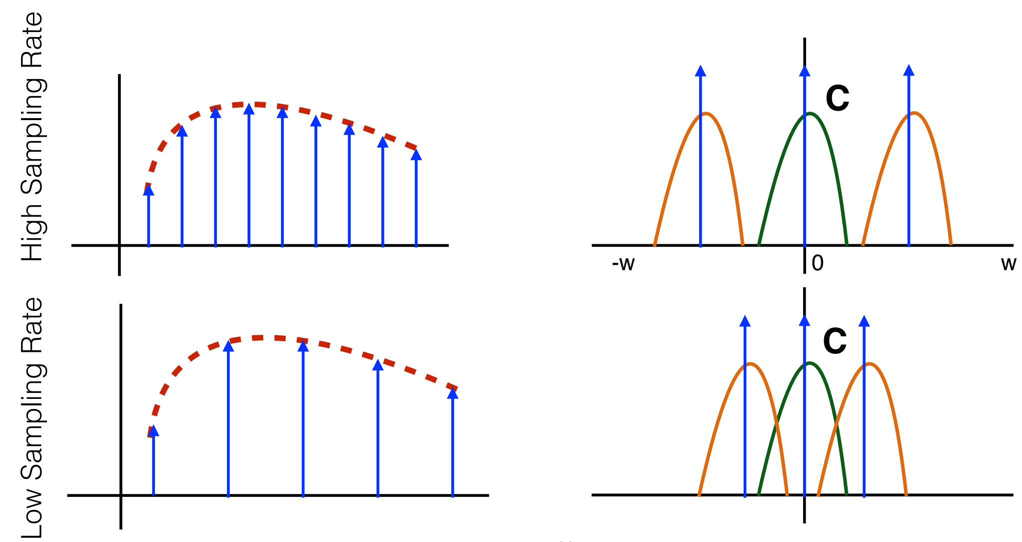
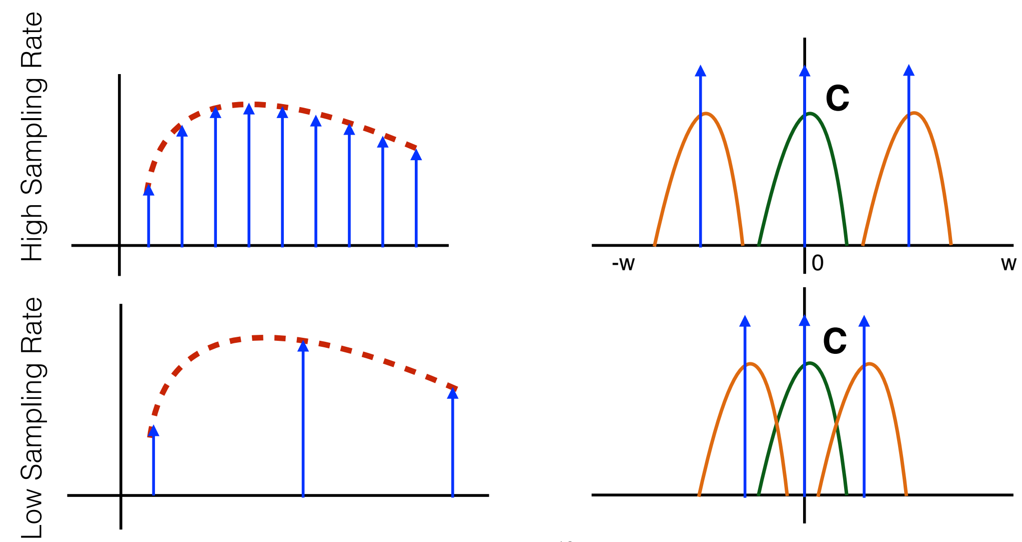


What if our sample budget is even lesser than that ??? [click]

Error in Monte Carlo Integration



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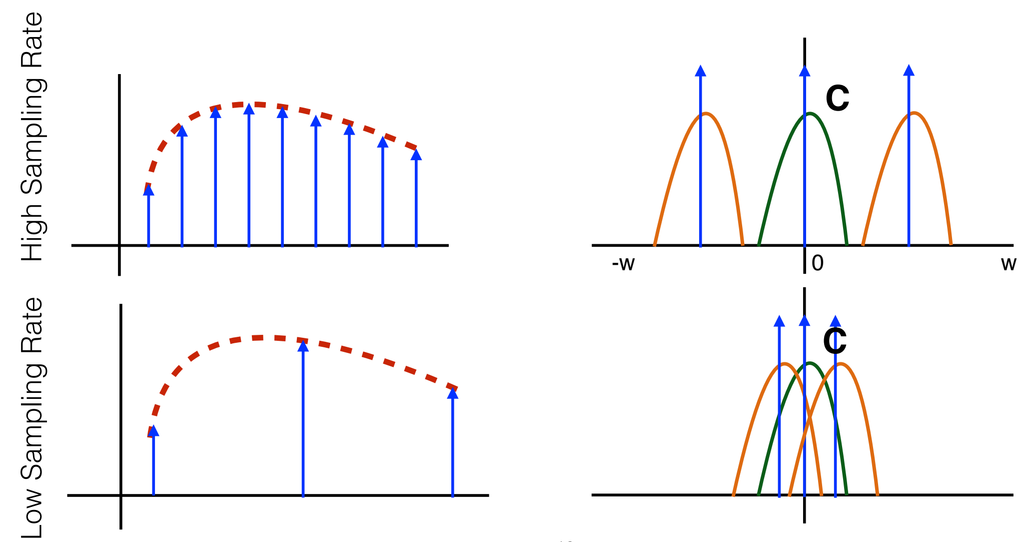

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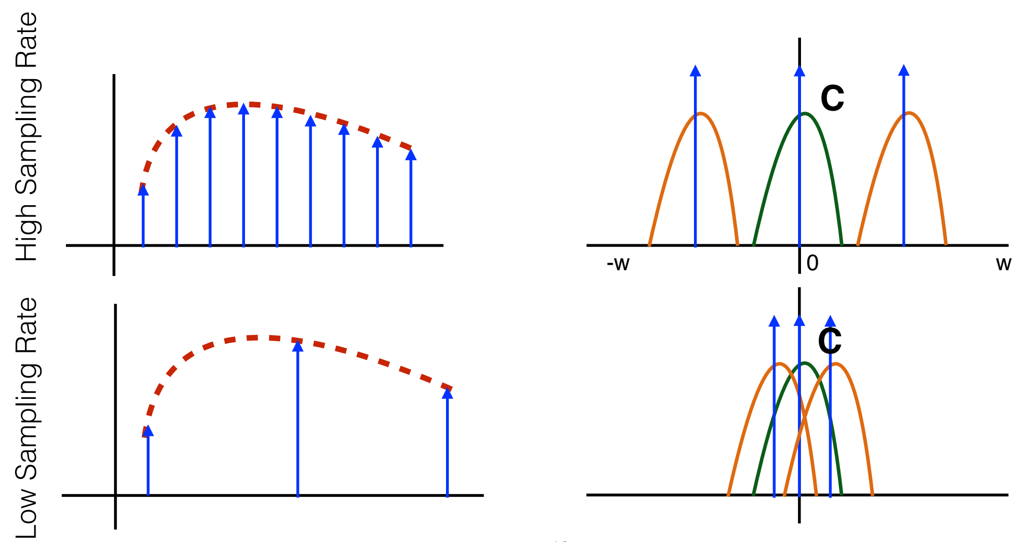

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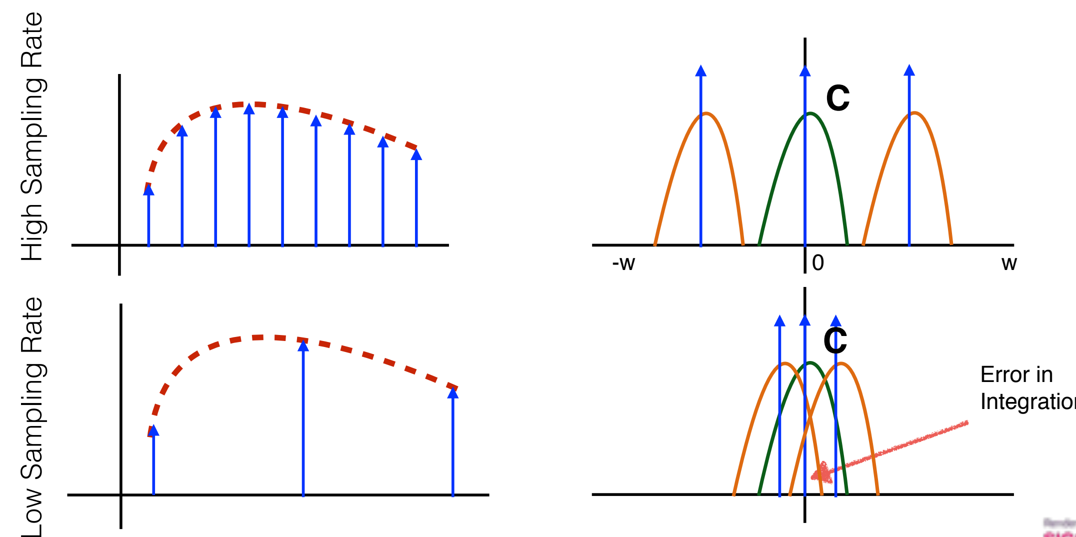

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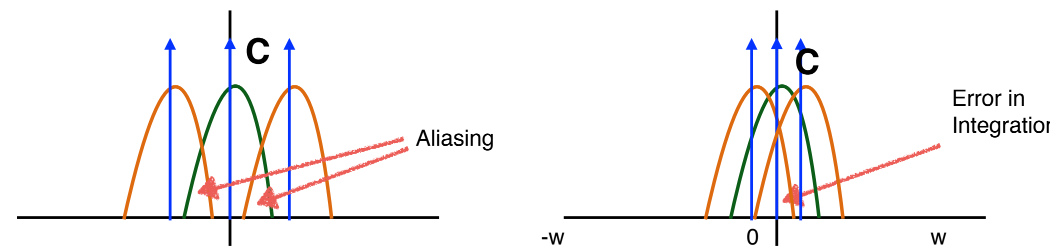
Aliasing (Reconstruction) vs. Error (Integration)



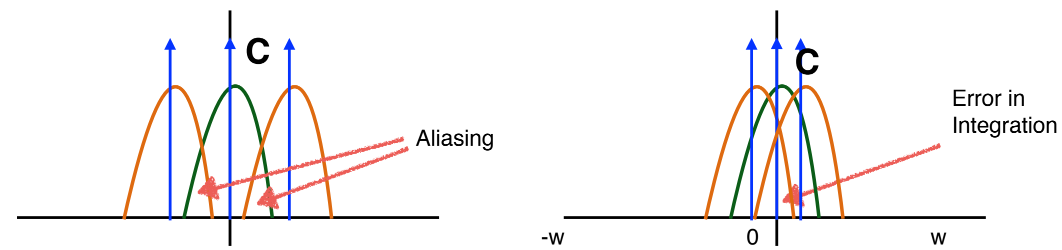
Fredo Durand in his tech report pointed out that, a function can be perfectly integrated if it is sampled at at least its maximum frequency. This is a factor of two compared to Nyquist and Shannon.

That is, as long as the replicas do not pollute the DC component, we can reduce the sampling rate without introducing error in the integration.

Lets now introduce formally the Monte Carlo integration to further study this error

Aliasing (Reconstruction) vs. Error (Integration)

Fredo Durand [2011]
Belcour et al. [2013]



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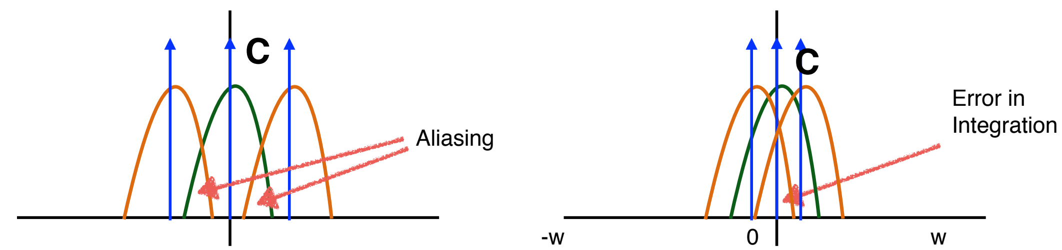

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Integration in the Fourier Domain

Integration is the DC term in the Fourier Domain

Spatial Domain:

$$I = \int_D f(x)dx$$

Our goal is to solve this integral of f(x) which could be of any dimension.

[click]: This, by definition, is the DC component, the zero frequency, of the Fourier transform of f

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$$\hat{f}(0)$$

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Monte Carlo Estimator in Spatial Domain

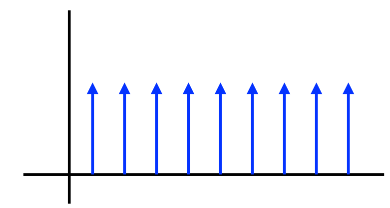
$$\tilde{\mu}_N = \int_D f(x)\mathbf{S}(x)dx$$

When we cannot solve the integral of f(x) analytically, we sort to the Monte Carlo integration where we sample the integrand with points that can be written in the continuous form as follows:

Monte Carlo Estimator in Spatial Domain

$$\tilde{\mu}_N = \int_D f(x)\mathbf{S}(x)dx$$

$$\mathbf{S}(x) = \frac{1}{N}\sum_{k=1}^{N}\delta(x - x_k)$$



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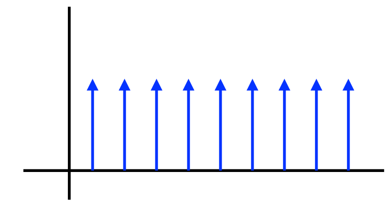

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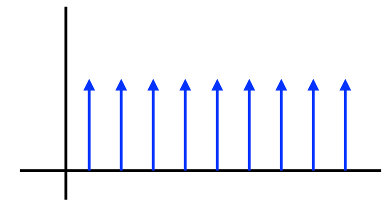

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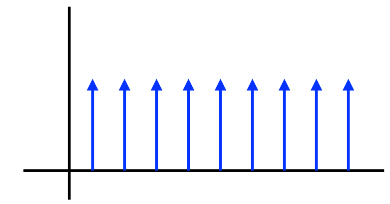

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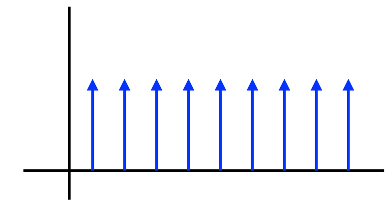

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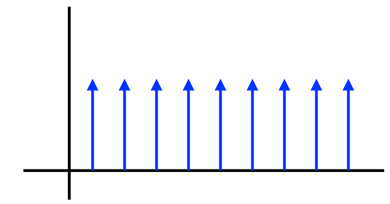

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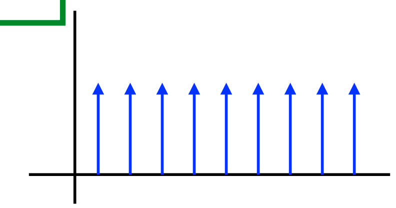

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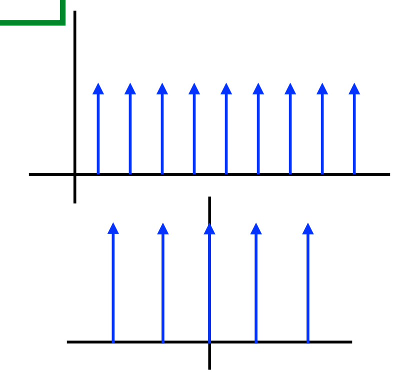

Note that, S(omega) would be different for different sampling patterns. Here for illustration purposes I am keeping it as a regular pattern.

How to Formulate Error in Fourier Domain ?

$I = \hat{f}(0)$

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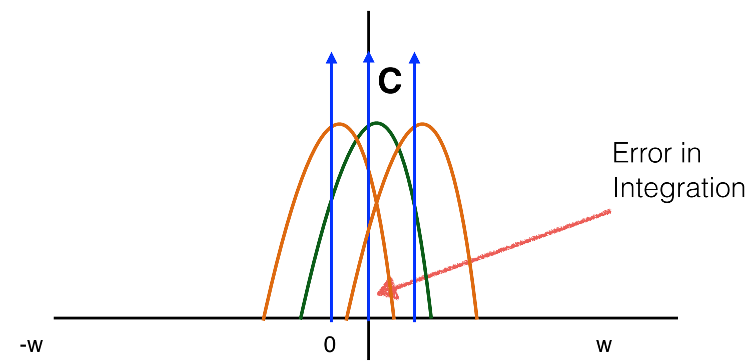

Fredo Durand [2011]

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One thing we have learnt so far is that Error in integration can be expressed in the Fourier domain as the aliasing caused by the sampling pattern at the DC. Let us first see how we would write error in the spatial domain.

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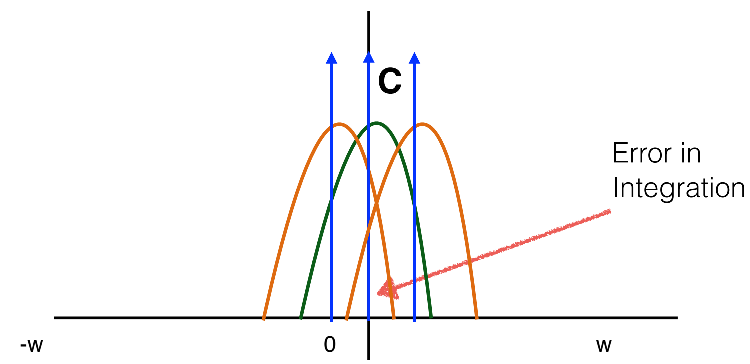

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In the spatial domain, error is the difference between the true integral [click] and the Monte Carlo estimator [click] of the function f
We can represent this equation in the Fourier domain by using the above two equations on top of this slide as follows:

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Fredo Durand [2011]

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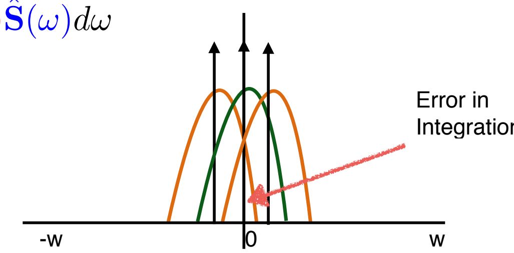

Fredo Durand [2011]

We know that error [click]

$$\text{Error} = \text{Bias}^2 + \text{Variance}$$

…, by definition, is composed of the variance and the Bias square terms.
We would like to represent each of these terms in there Fourier domain. and see what insights we can gain from this.

Properties of Error

- Bias
- Variance

By definition, Bias is the [click] expected value of the error, where we represent the expectation operator with these angular brackets.

Similarly, By applying variance operator on the error term we can gain some insights here. These both terms are very well explored by Kartic Subr and Jan Kautz in 2013. Let’s look at the bias first.

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Bias in the
Monte Carlo Estimator

Lets first look at the bias:

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Error: $$I - \tilde{\mu}_N = \hat{f}(0) - \int_{\Omega} \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

We start from the Error equation that we have seen before
Note that, here, only S is the random variable.

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This summarises the bias term in the Fourier domain.
Note that, [click] if we can make the equation inside the box exactly the same as f^hat (0), we can get rid of the bias.
This would also mean that we can get an unbiased estimator.
In practice, we never know for sure about the Fourier spectrum of the function we are trying to integrate. However, we do have some information about the Fourier spectrum of a sampling function S.

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for frequencies other than zero

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How to obtain $\langle \hat{\mathbf{S}}(\omega) \rangle = 0$?

Now the question is:

Complex form in Amplitude and Phase

$$\langle \hat{\mathbf{S}}(\omega) \rangle = |\langle \hat{\mathbf{S}}(\omega) \rangle| \, e^{-\Phi(\langle \hat{\mathbf{S}}(\omega) \rangle)}$$

We know one thing for sure here that any complex form has an Amplitude and phase.
Let us try to look at how phase is distributed for a regular sampler.

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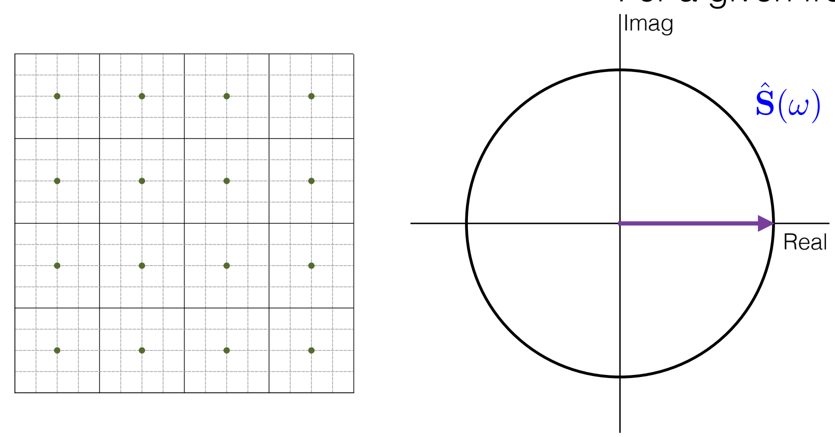
Phase change due to Random Shift

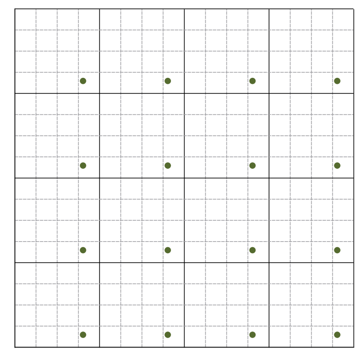
For a given frequency ω

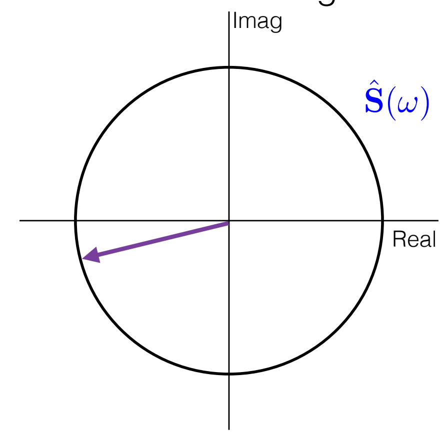


Here we compute this complex vector for a given frequency by taking the Fourier transform of the distribution shown on one side. However, if we uniformly randomly shift the whole point set at each realization,
[click]

Phase change due to Random Shift

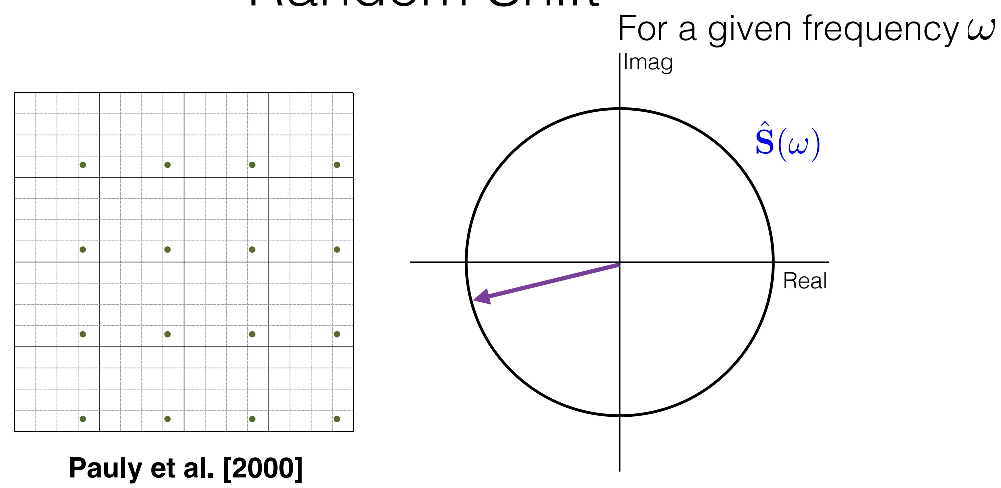
For a given frequency ω





[click], the phase also randomly shifts over the complex plane at a given frequency

Phase change due to Random Shift



Pauly et al. [2000]
Ramamoorthi et al. [2012]

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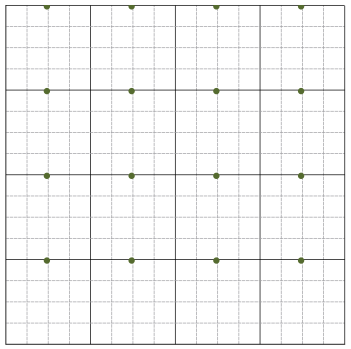
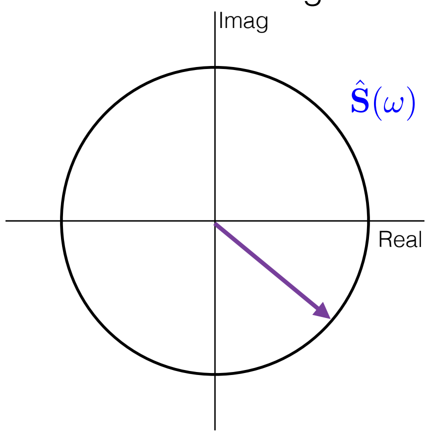


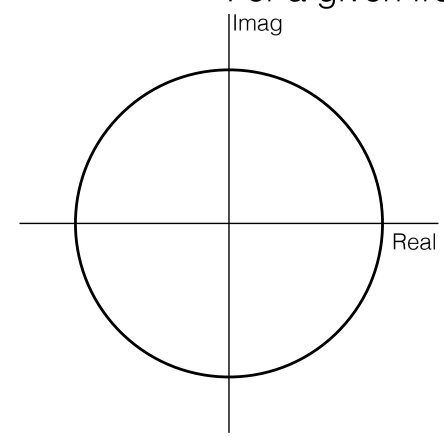


[click]…

Phase change due to Random Shift

Multiple realizations

For a given frequency ω



and over multiple realisations, we get the phase uniformly distributed over the complex plane and this would result in an expected value of S to be zero [click] for all omega other than zero.

Phase change due to Random Shift

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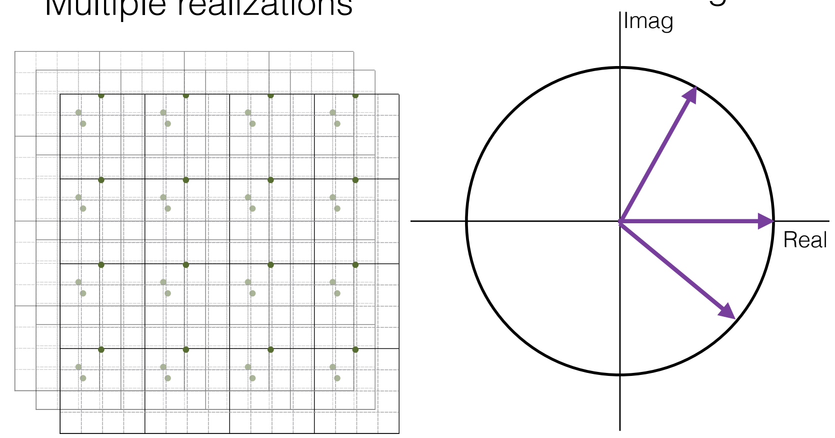

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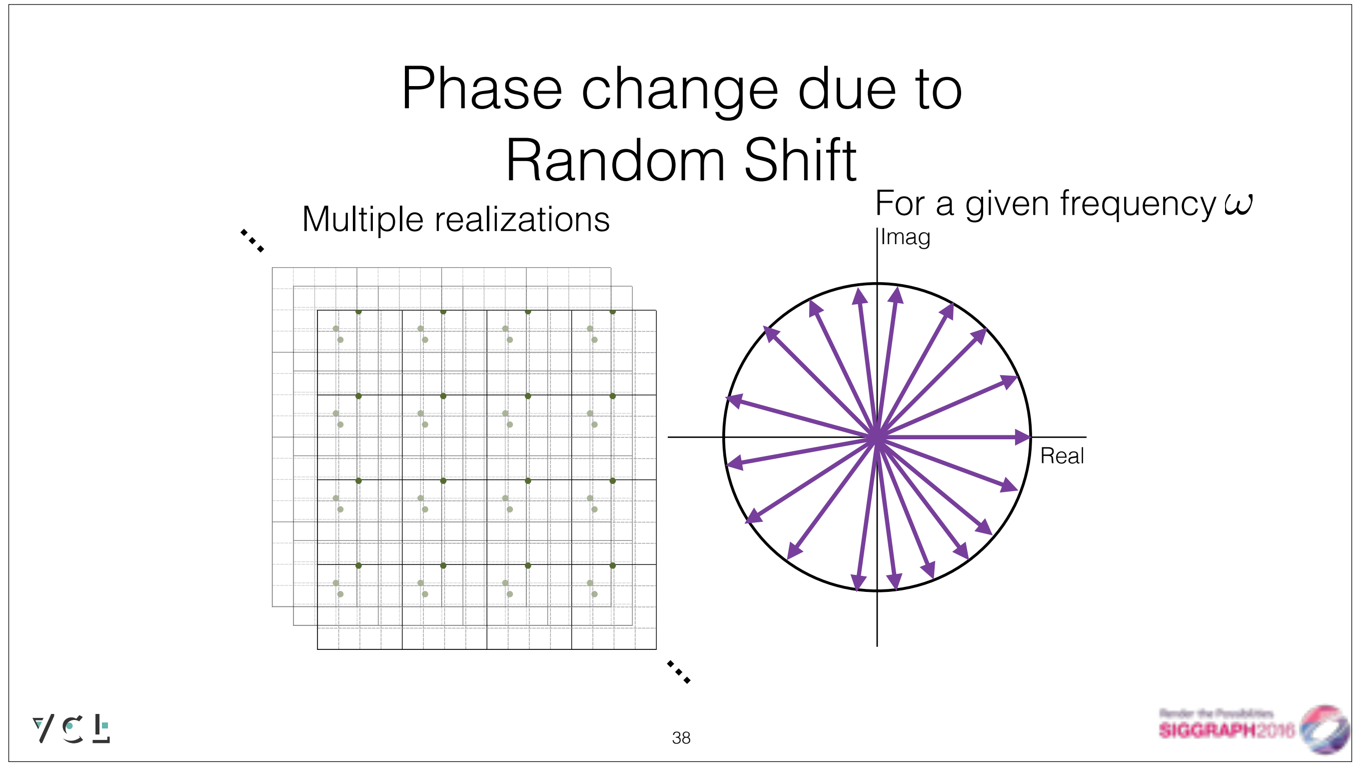

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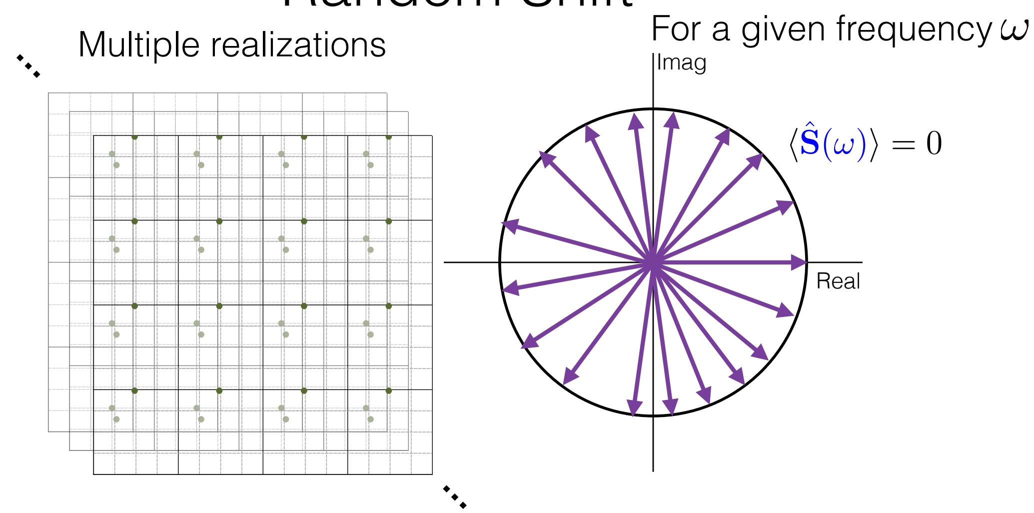

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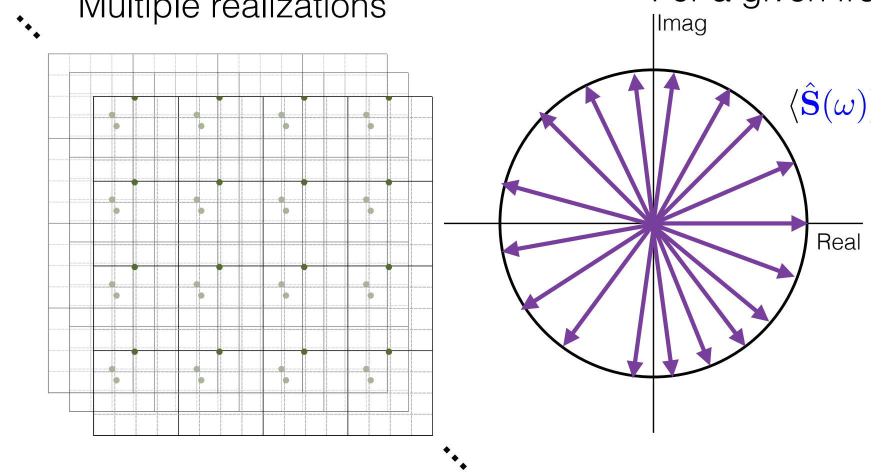

$$\langle \hat{\mathbf{S}}(\omega) \rangle = 0 \ \ \forall \omega \neq 0$$

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-Homogenisation also allows representation of error only in terms of variance.
-We can take any sampling pattern and homogenise it to make the corresponding Monte Carlo estimator unbiased.
-We will use this idea of homogenization to simplify our formulation later on.
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-We can take any sampling pattern and homogenise it to make the corresponding Monte Carlo estimator unbiased.
-We will use this idea of homogenization to simplify our formulation later on.
Now lets look at the [click]:

Variance in the Fourier domain

variance in the fourier domain

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Error: $$I - \tilde{\mu}_N = \hat{f}(0) - \int_{\Omega} \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

We start from the Error equation that we have seen before:
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[click]x2

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$$P_f(\omega) = |\hat{f}^*(\omega)|^2 \quad \text{Power Spectrum}$$

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This Variance was derived by Subr and Kautz in 2013.
Read text

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This is a general form, both for homogenised as well as non-homogenised sampling patterns

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Now the question is can we represent this equation in something that we already know???
For purely random samples, however, Fredo derived the variance of the Monte Carlo estimator in the [click] following form where the variance is represented in the form of expected Power spectrum of the random samples. This is great! because we have seen in the previous part of the talk the expected Power spectrum of random samples.
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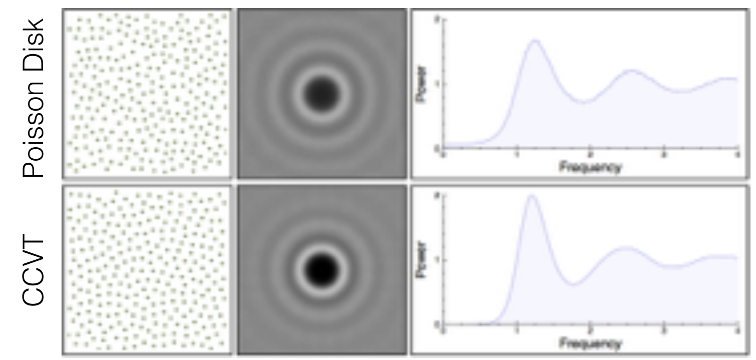

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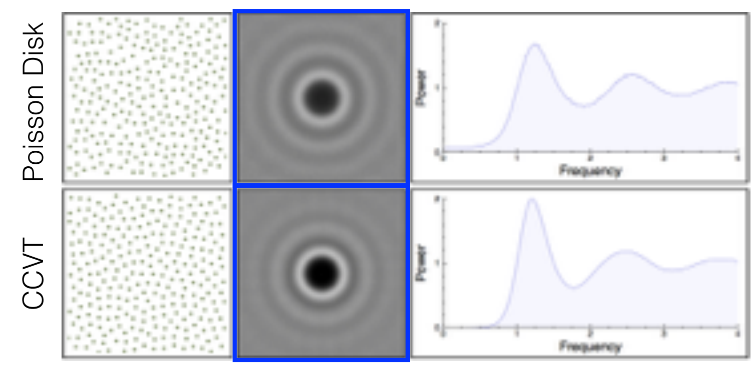

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Now, we have two integrals here in this variance form. If we consider only the [click] isotropic sampling power spectra, [click] for which we get the same energy [click] at a given radius [radius] from the DC.

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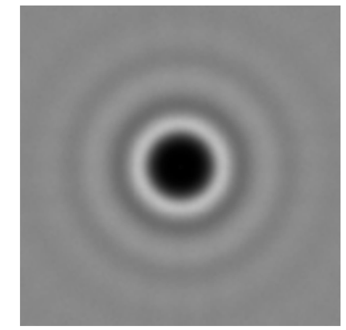

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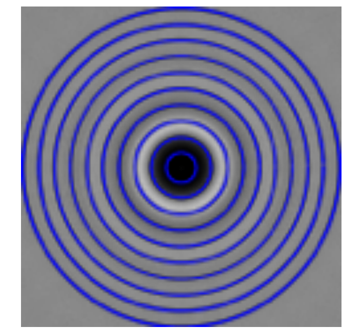

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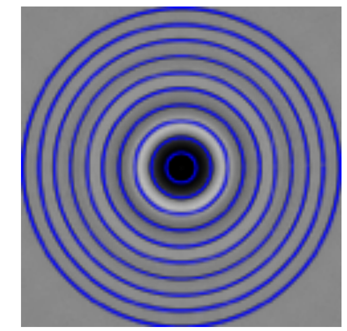

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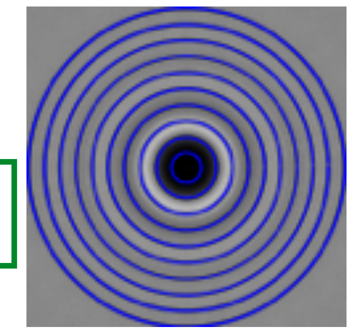

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Lets now look at what this Ps\rho written in blue correspond to.

Variance in terms of 1-dimensional Power Spectra

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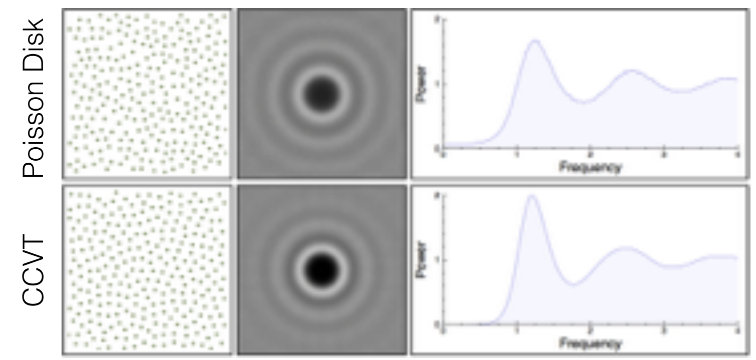

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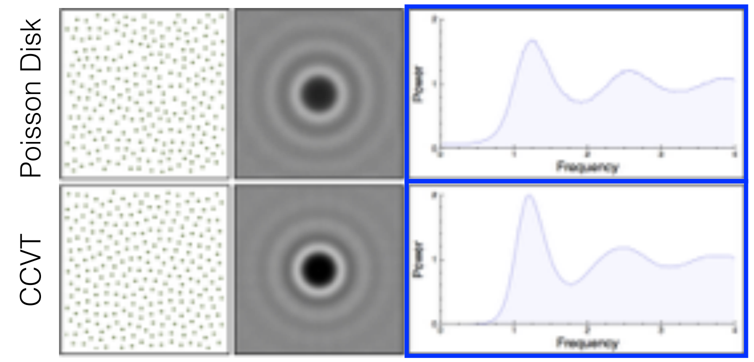

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Now, if we look at this variance equation, it is an integral over the product of samples' and the function power spectra.

Lets try to visualise this product. In red you see the integrand f Radial spectrum and in bue you see the sampling radial power spectrum.

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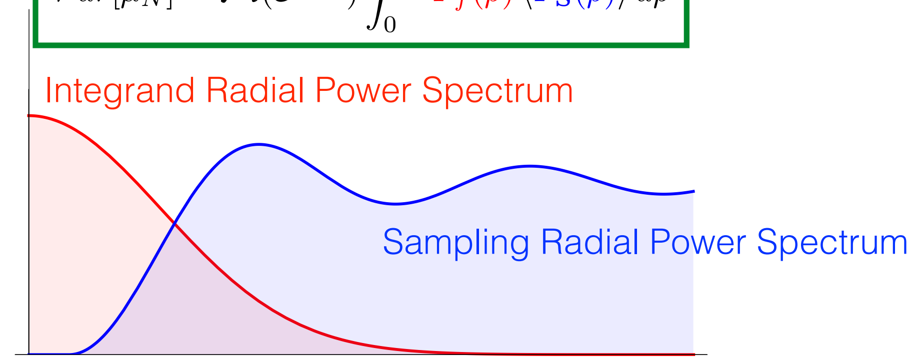

For given number of Samples

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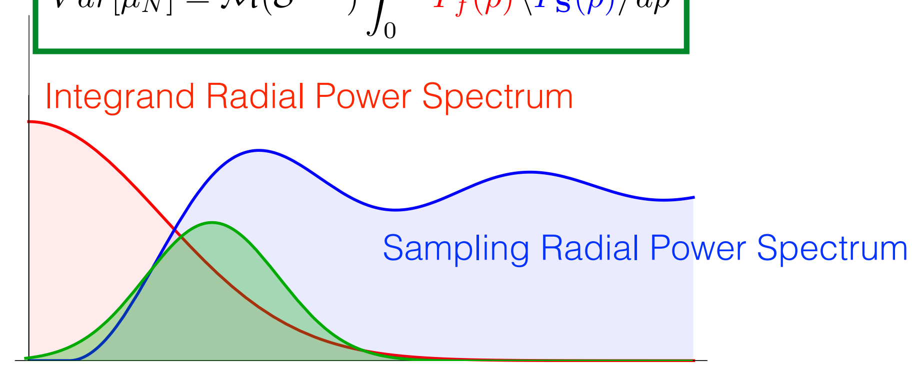

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The product of these two curves is shown in green and the area under this curve actually represents the variance.
Reducing variance actually correspond to reducing the area under this green curve. One way to obtain low variance is by increasing the number of samples. In the fourier domain, increasing sample count corresponds to …

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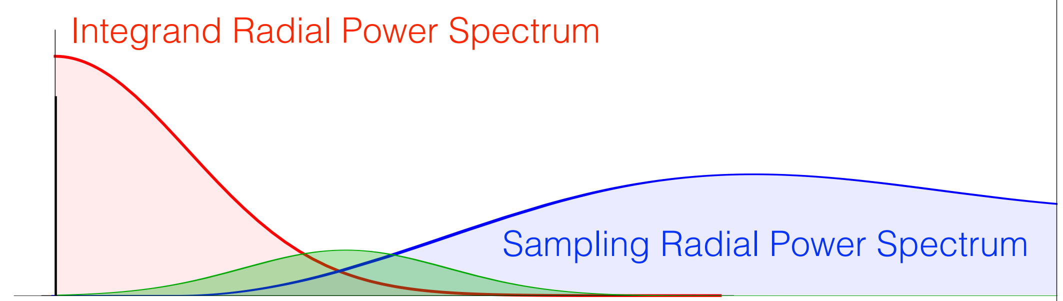

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scaling the sampling power spectrum towards higher frequency.

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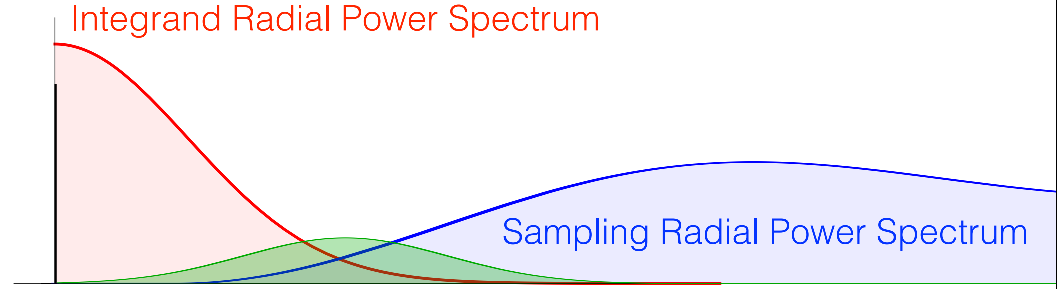

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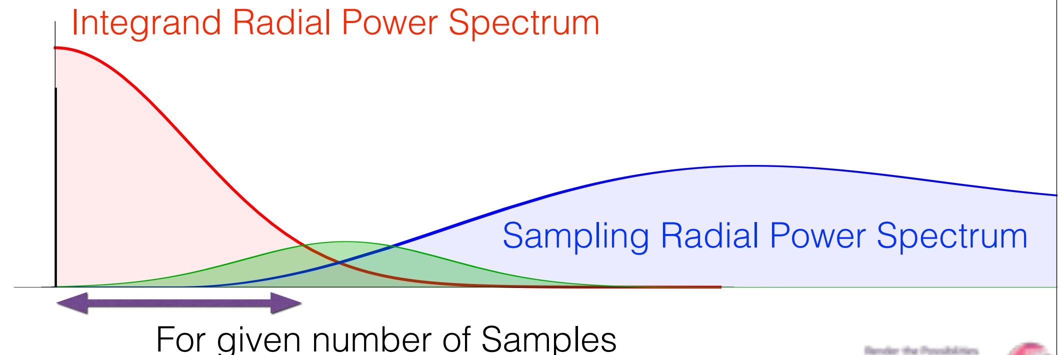

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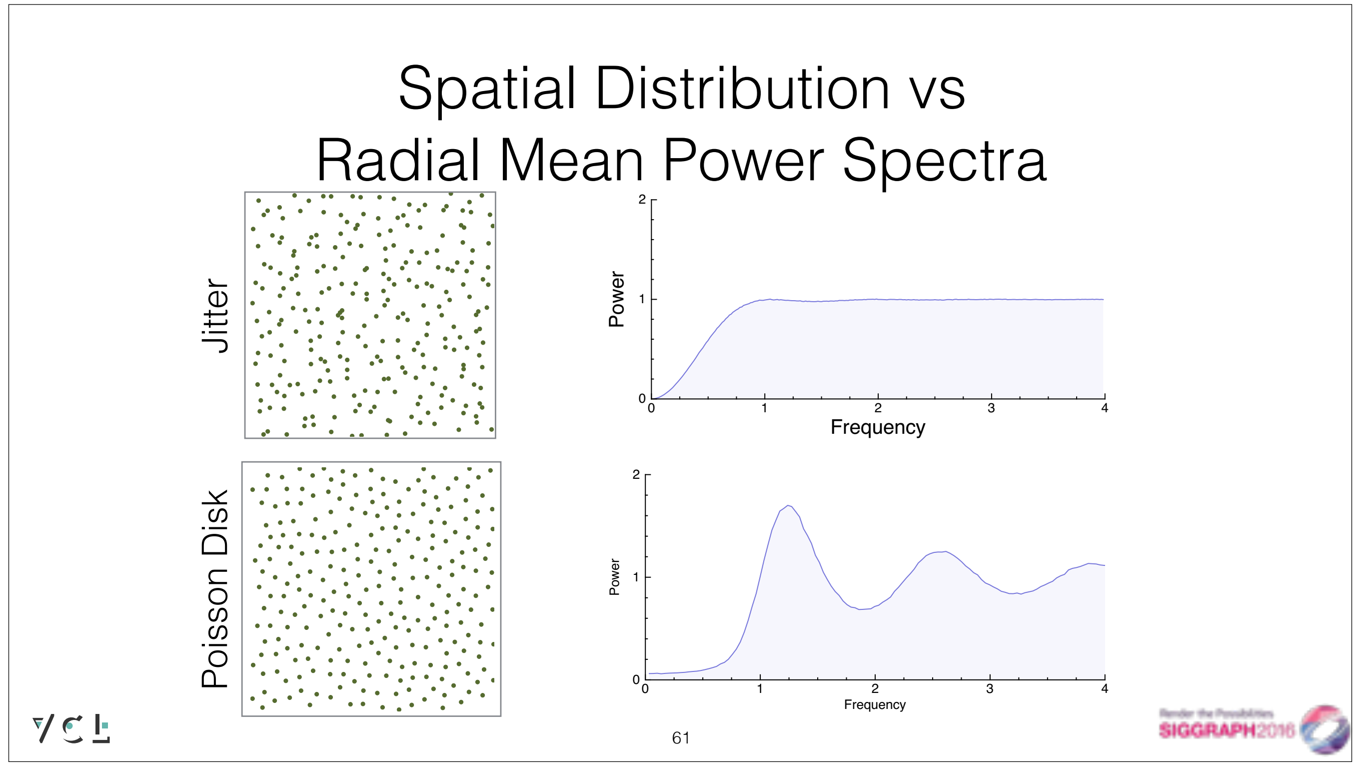

Lets look at the radial mean power spectrum of Jitter and Poisson disk.
In the spatial domain, Poisson disk seems to have more uniform distribution compared to the jittered samples. However …

For 2-dimensions

Samplers	Worst Case	Best Case
Random		
Jitter		
Poisson Disk		
CCVT		

Pilleboue et al. [2015]

VCL

If we look at the theoretical convergence rates obtained using the variance formulation we get the following results:
[click] For random we always O(1/N) convergence rate [click]
[click] For jitter, we get a better convergence rate that further improves [click] for integrands without discontinuities
[click] For Poisson disk samples, which is a blue noise distribution, we obtain the convergence rate of O(1/N) for any kind of integrand.
[click] For CCVT, which is also a blue noise sampling distribution we obtain convergence rates much better than random and jittered samples.

For 2-dimensions

Samplers	Worst Case	Best Case
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Jitter		
Poisson Disk		
CCVT		

Pilleboue et al. [2015]

If we look at the theoretical convergence rates obtained using the variance formulation we get the following results:
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For 2-dimensions

Samplers	Worst Case	Best Case
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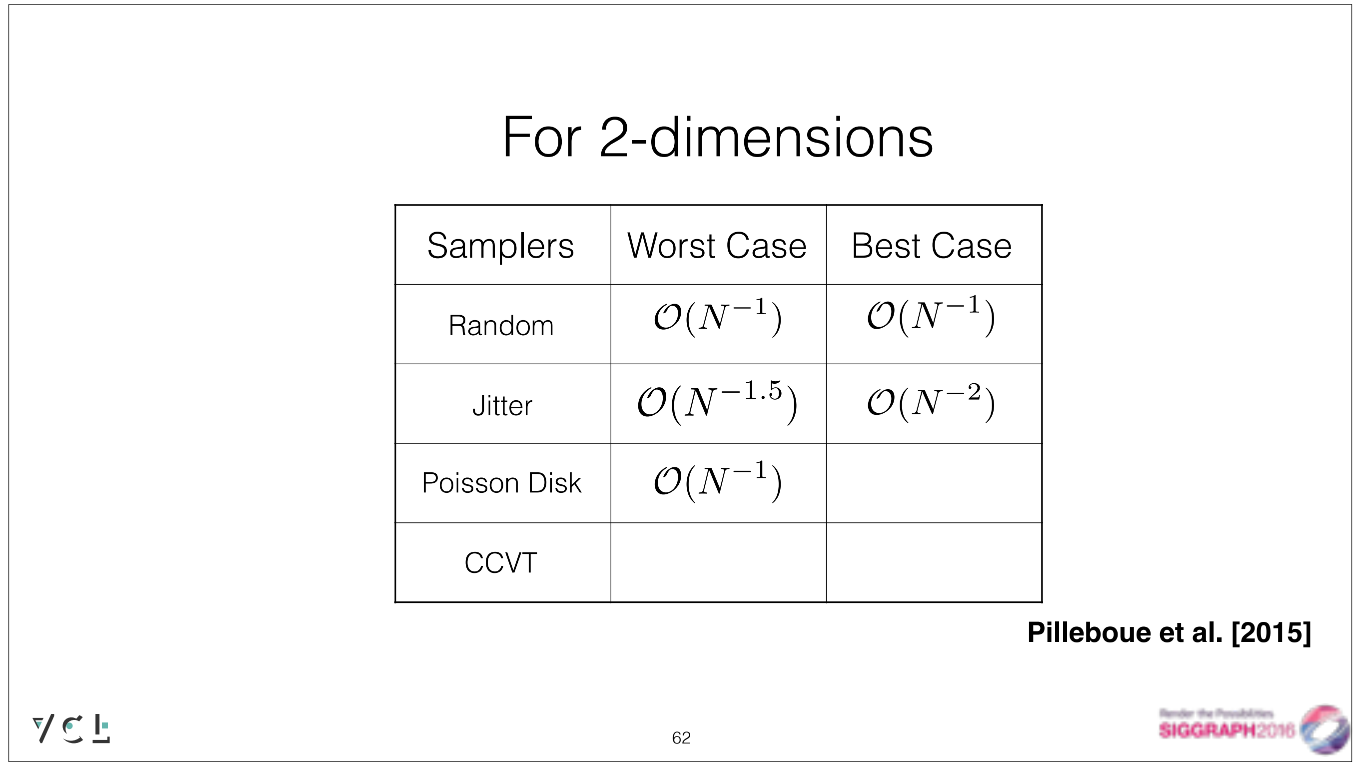

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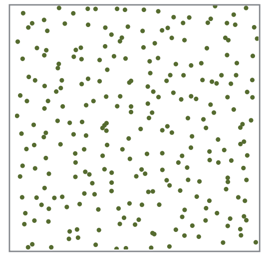
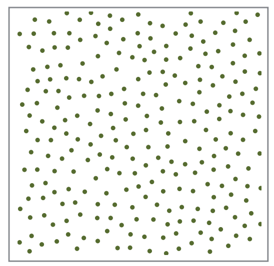
For 2-dimensions

Jitter



Poisson Disk



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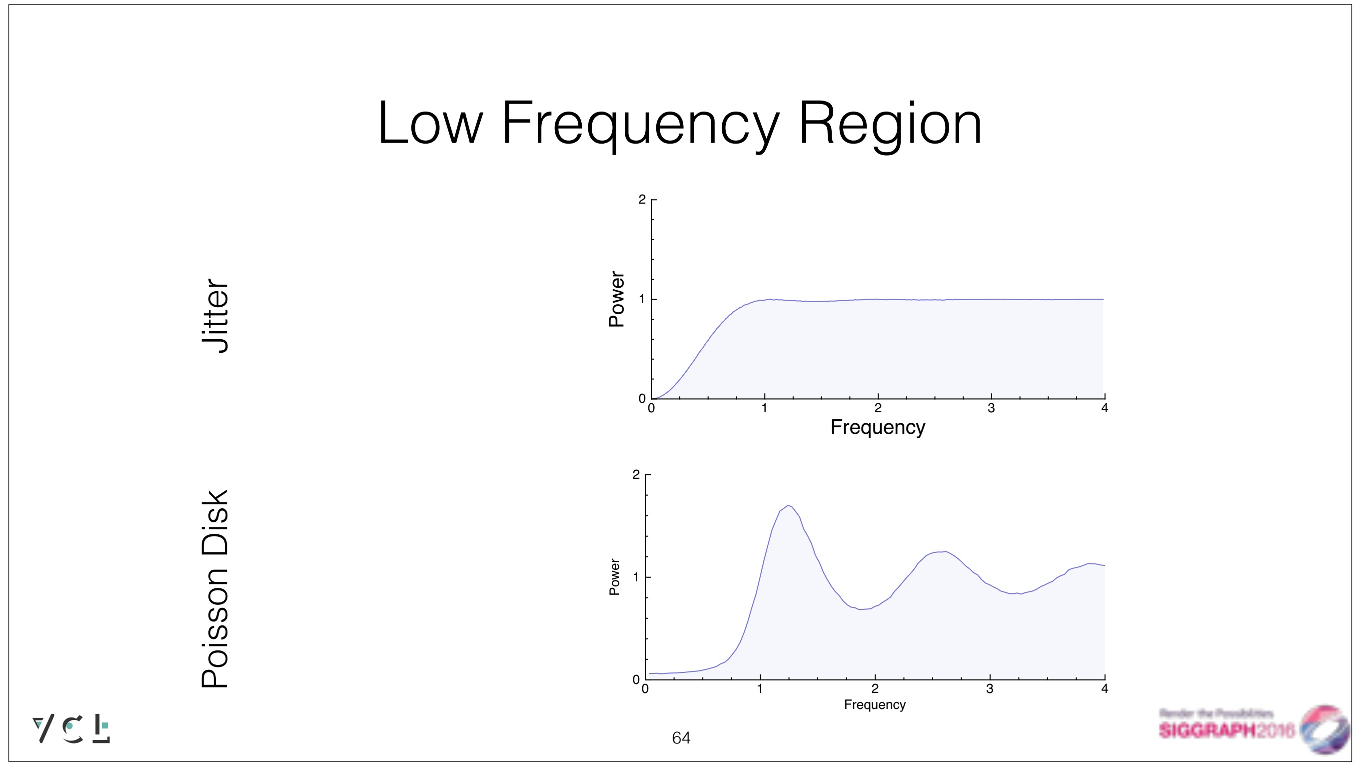
Pilleboue et al. [2015]

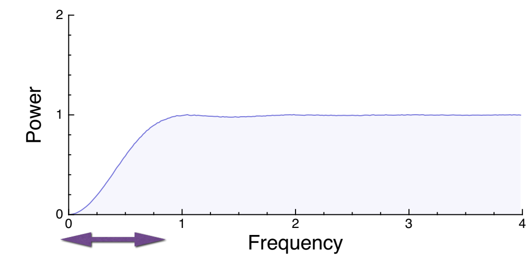
If we look at the distribution of jitter and Poisson disk samples, Poisson disk is more uniform. Then, what is making the convergence rate of Poisson disk bad ???



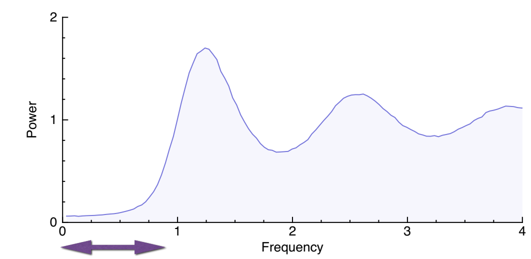
If we focus [click] on the low frequency region [click], we observe that in the case of Jitter, the power is going all the way to zero near the DC frequency. However, for Poisson disk, there is an offset in the low frequency region. And this is explaining a lot of things.

Low Frequency Region

Jitter



Poisson Disk



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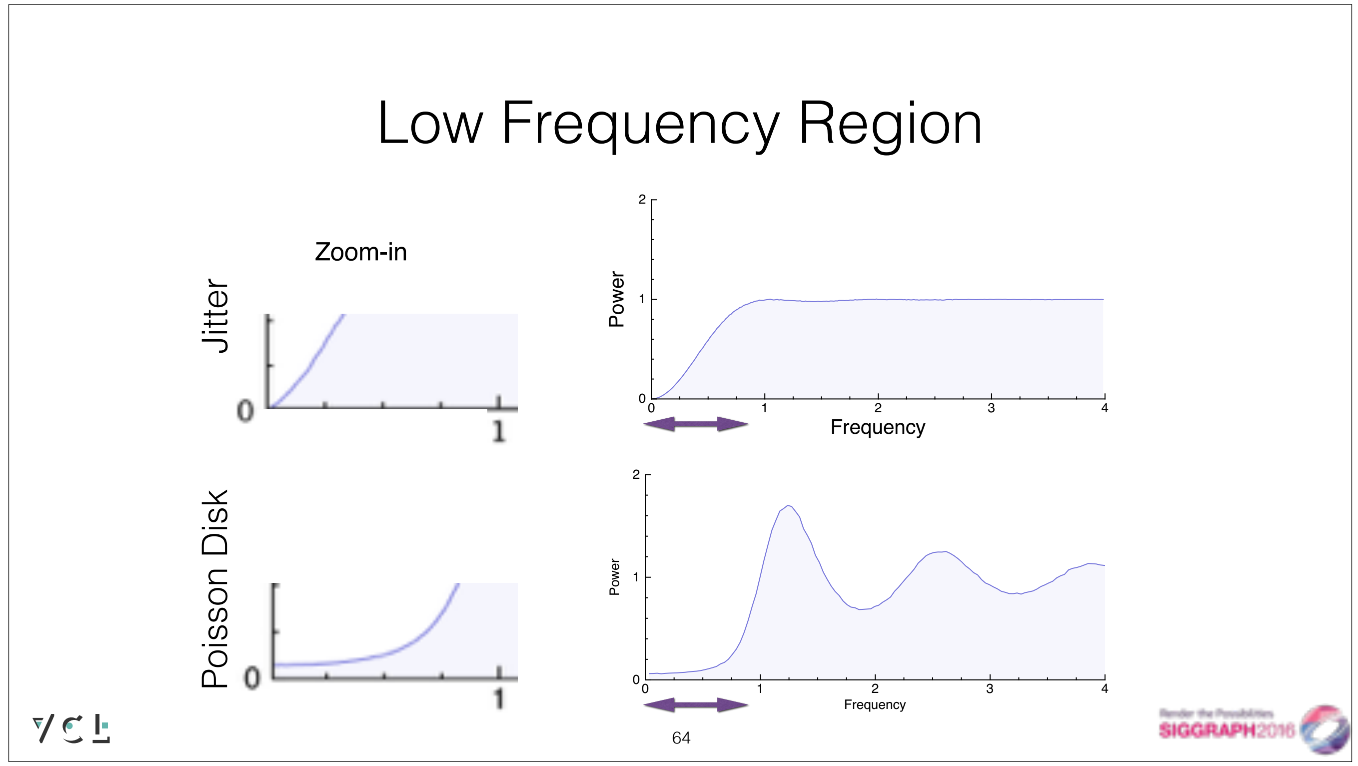

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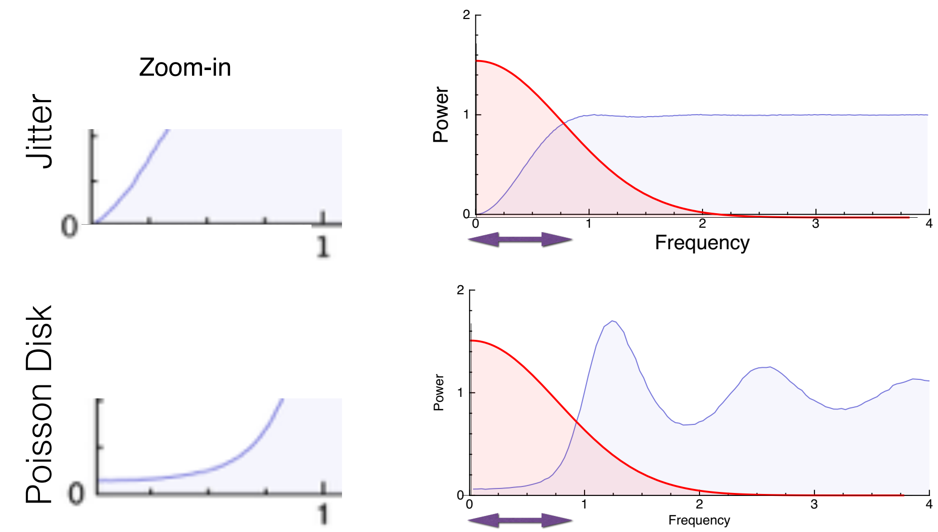
Variance for Low Sample Count



For any given integrand, we can see that the overlap is higher in case of jitter compared to thee Poisson disk radial power spectrum. [click] The corresponding product, shown in green, has greater area under the curve in case of jitter compared to Poisson disk. This explains that for a low sample count, Poisson disk would give low variance compared to Jitter. However, …

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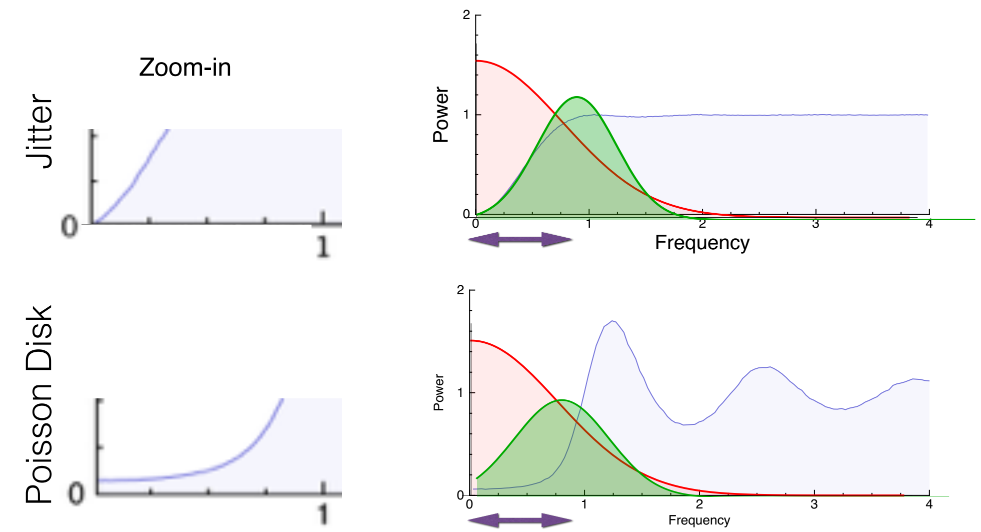

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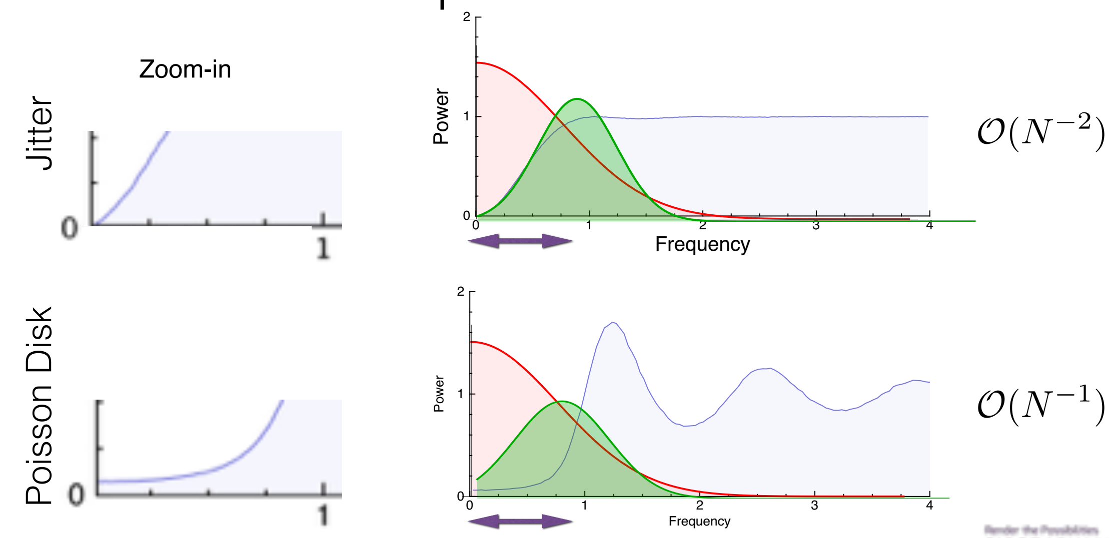
Variance for Increasing Sample Count

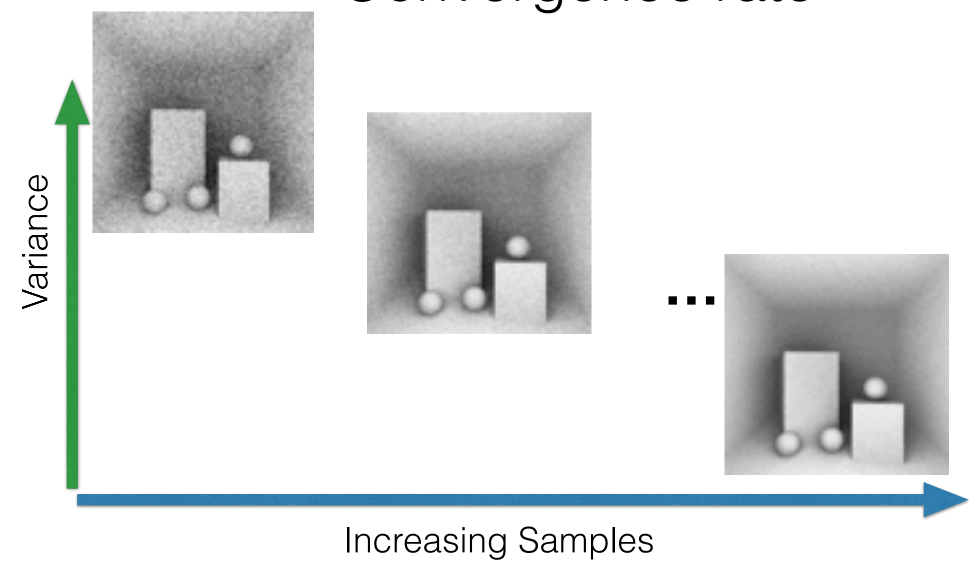


In terms of convergence, Jitter is still better because the green curve tends to zero near the DC frequency, where as for the Poisson Disk there is a non-zero offset, that we can see in the green curve at the DC which reflects in terms of bad convergence rate.

Experimental Verification

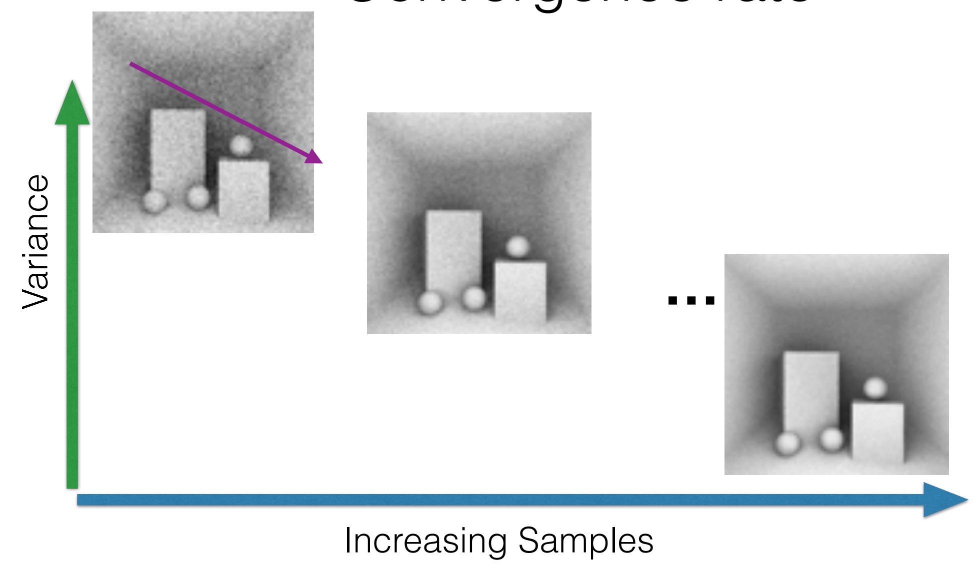
Lets now verify these convergence rates with some experiments.

Convergence rate



We first define the convergence rate, which is the slope [click][click]…

Convergence rate



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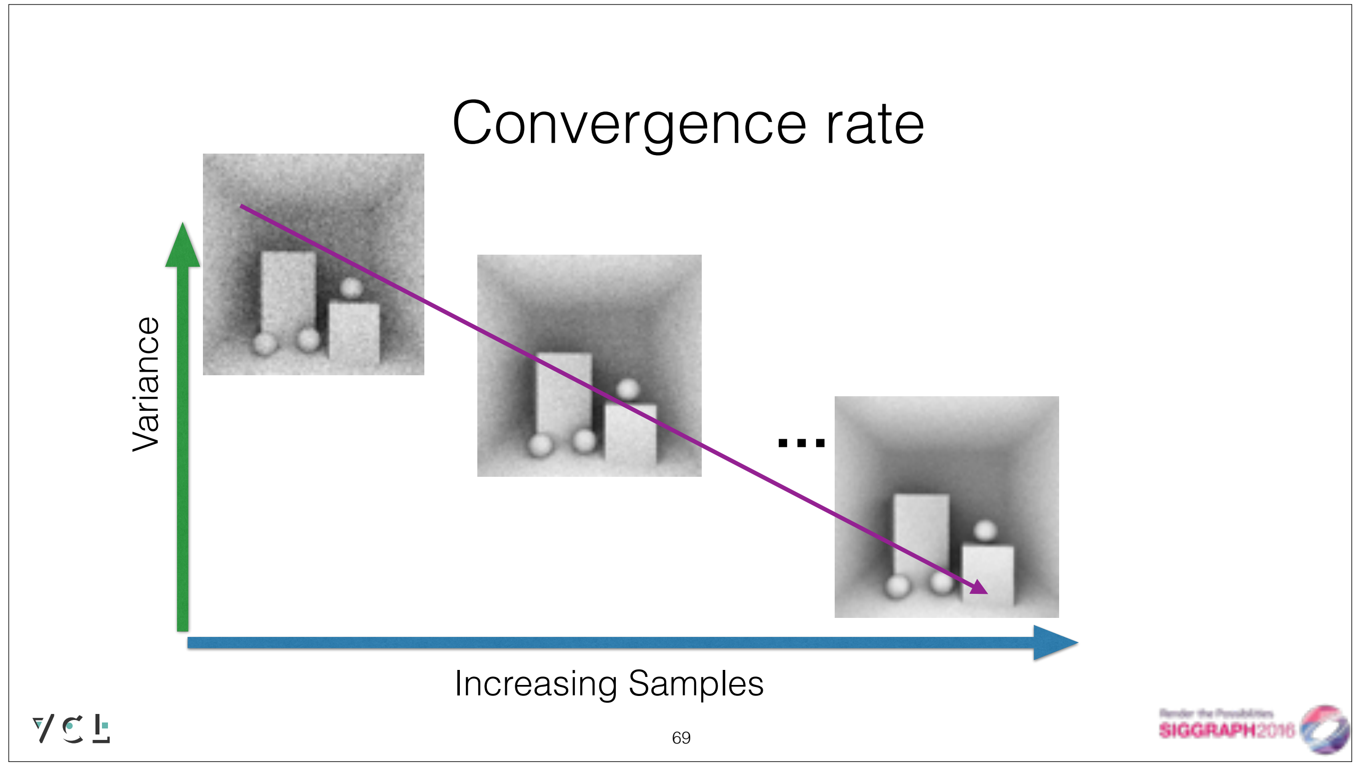
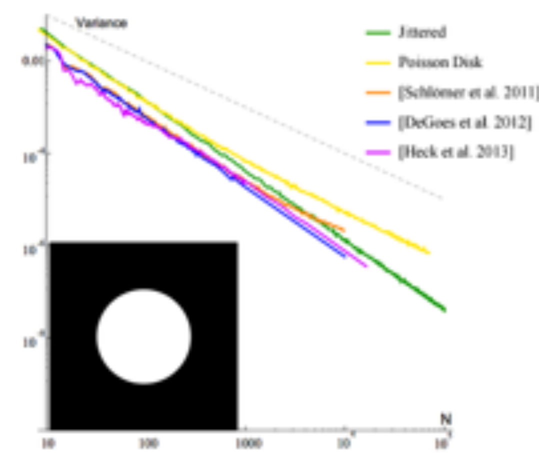


… in the logarithmic plot of variance vs the Number of samples. Lets first see some toy examples for which the power spectra is analytically known [click].

Disk Function as Worst Case



Here we are showing some experiments with the Disk function. We can focus our attention on the green and Yellow curve here [click], As you can see, [click] the variance curves for jitter crosses Poisson disk at a small sample count

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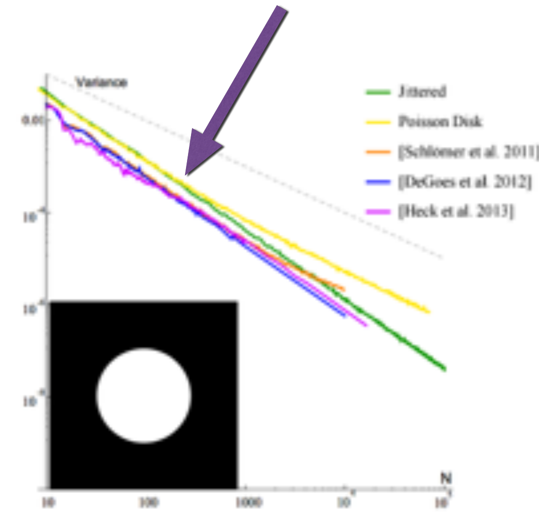

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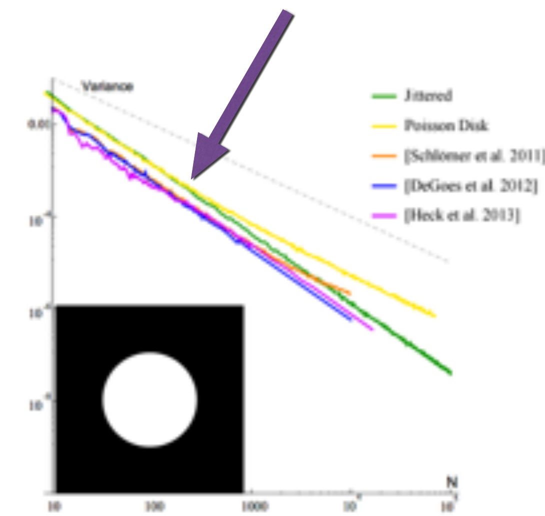

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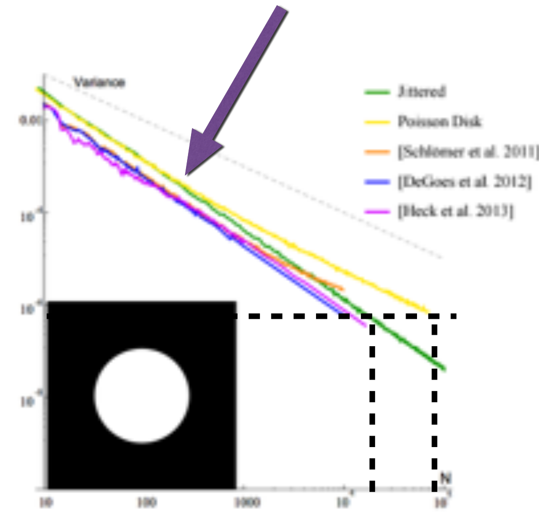
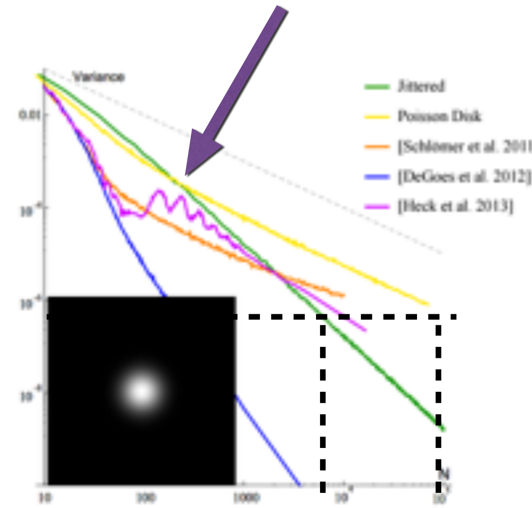


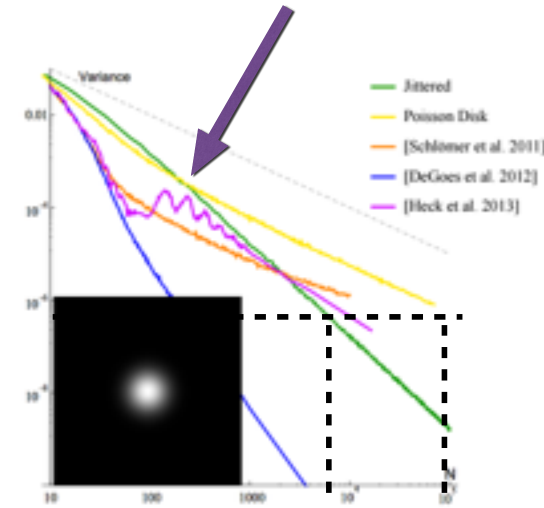
and after that its jitter sampling pattern that gives low variance for a given number of samples. Now lets look at some rendering examples for the Ambient occlusion scene

Gaussian as Best Case



Similar behviour is also observed in the case of a Gaussian function that does not have any discontinuity.

Gaussian as Best Case



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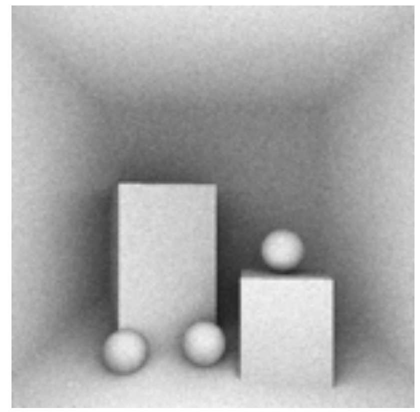
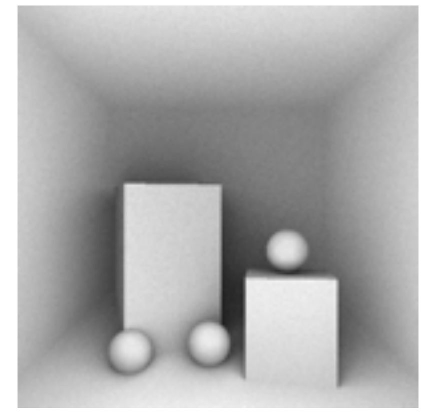
Ambient Occlusion Examples

Random vs Jittered

96 Secondary Rays



MSE: 4.74 x 10e-3

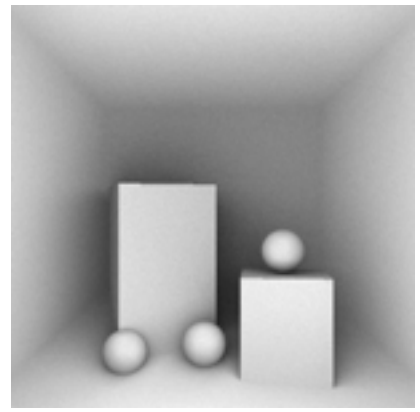


MSE: 8.56 x 10e-4

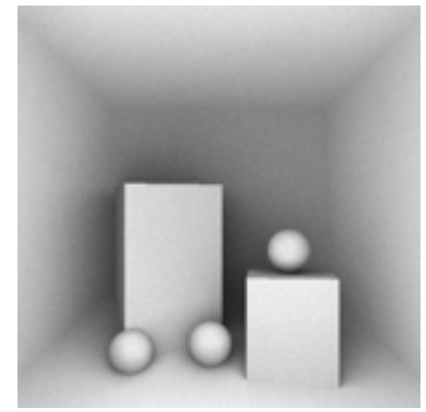
here we render the cornell box scene with random and jitter sampling pattern. You can observe that Jitter samples give less variance compared to random samples for a given sample count.

CCVT vs. Poisson Disk

96 Secondary Rays



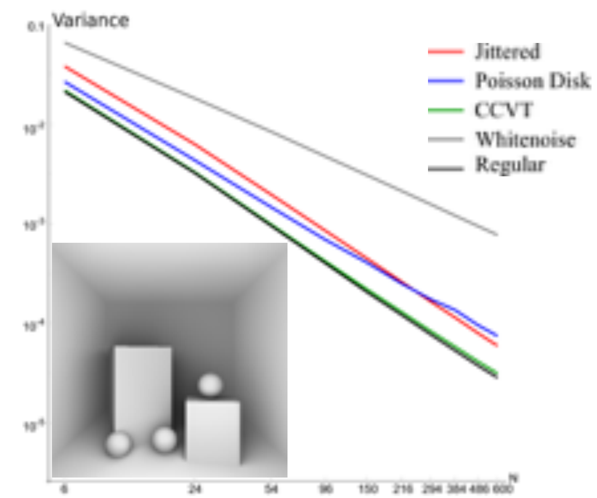
MSE: 4.24 x 10e-4



MSE: 6.95 x 10e-4

Similarly, for blue noise samples like CCVT and Poisson disk we can observe that the error value for Poisson disk is a little higher compared to the CCVT.

Convergence rates



As you can see here, if we again compare Poisson disk with the Jittered sampling pattern [click]

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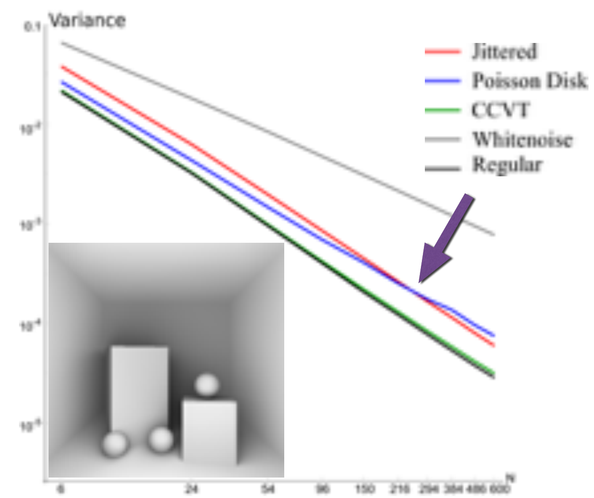

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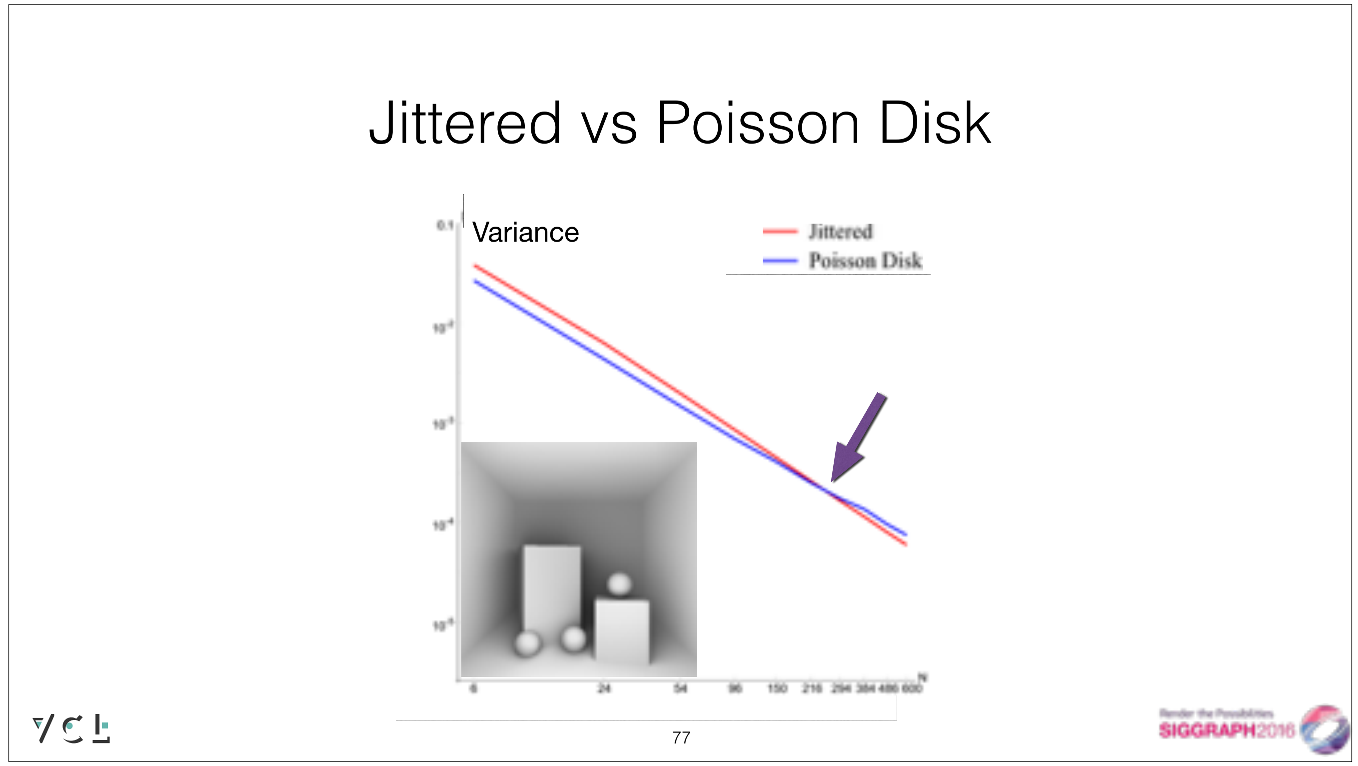

We can see that the jittered variance curve intersects the Poisson disk samples at a sample count of around 400 samples, which is a relatively small number.

What are the benefits of this analysis ?

NOW, I would like to summarise my talk by answering this question: What are the benefits of this analysis ???:
[click] For offline rendering, it is very important to know which sampler would converge faster. By simply using a right sampler, a lot of time can be saved.
[click] For real time rendering, where the target is to get noise free images as quickly as possible, blue noise samplers can provide much higher benefits compared to random or jittered samples.

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