Error and Variance Analysis in the Fourier Domain

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Render the Possibilities
SIGGRAPH2016

Outline

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- How error looks like in the Fourier domain

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- How error looks like in the Fourier domain
- Bias in the Fourier domain

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- Bias in the Fourier domain
- Variance in the Fourier domain

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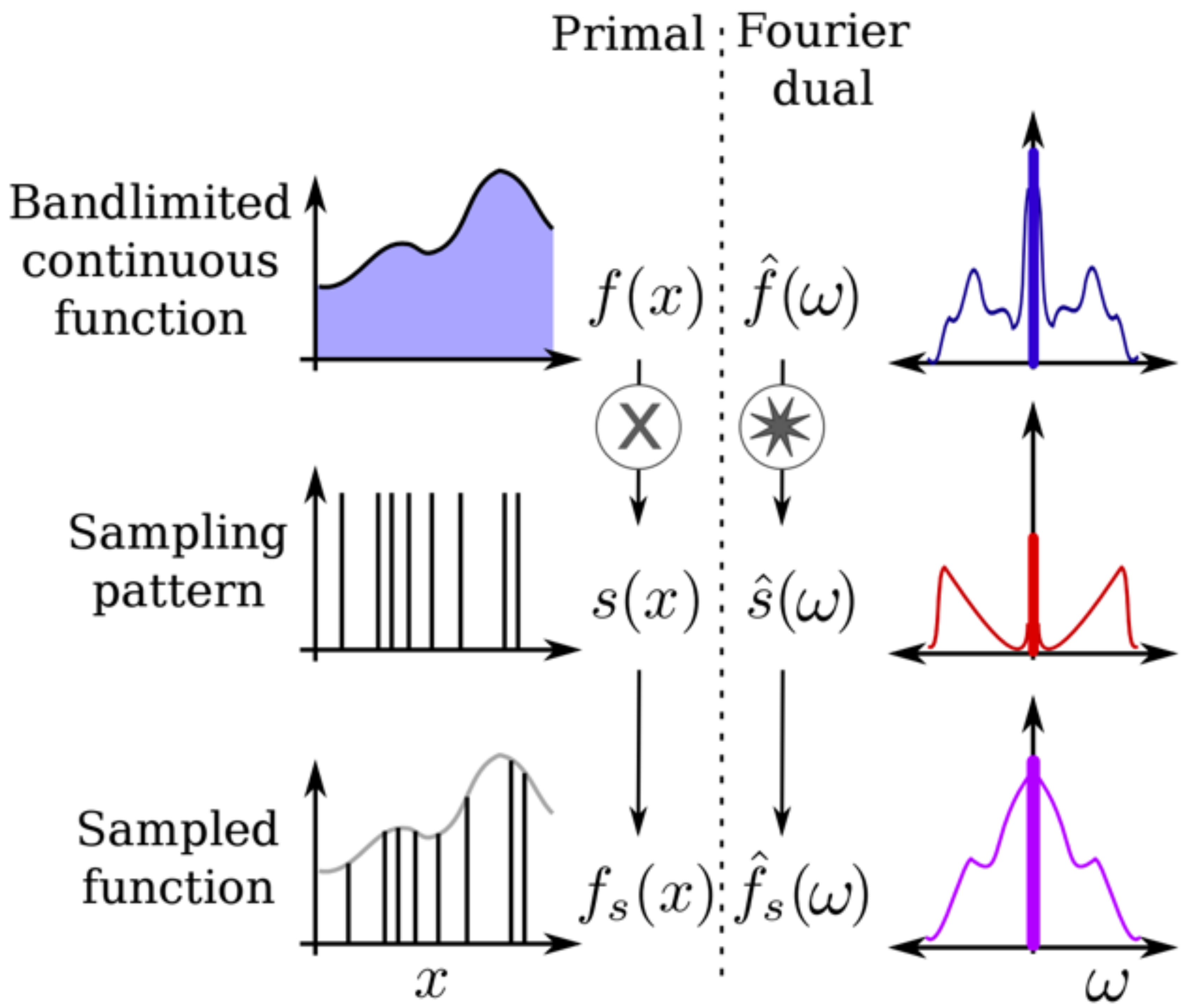
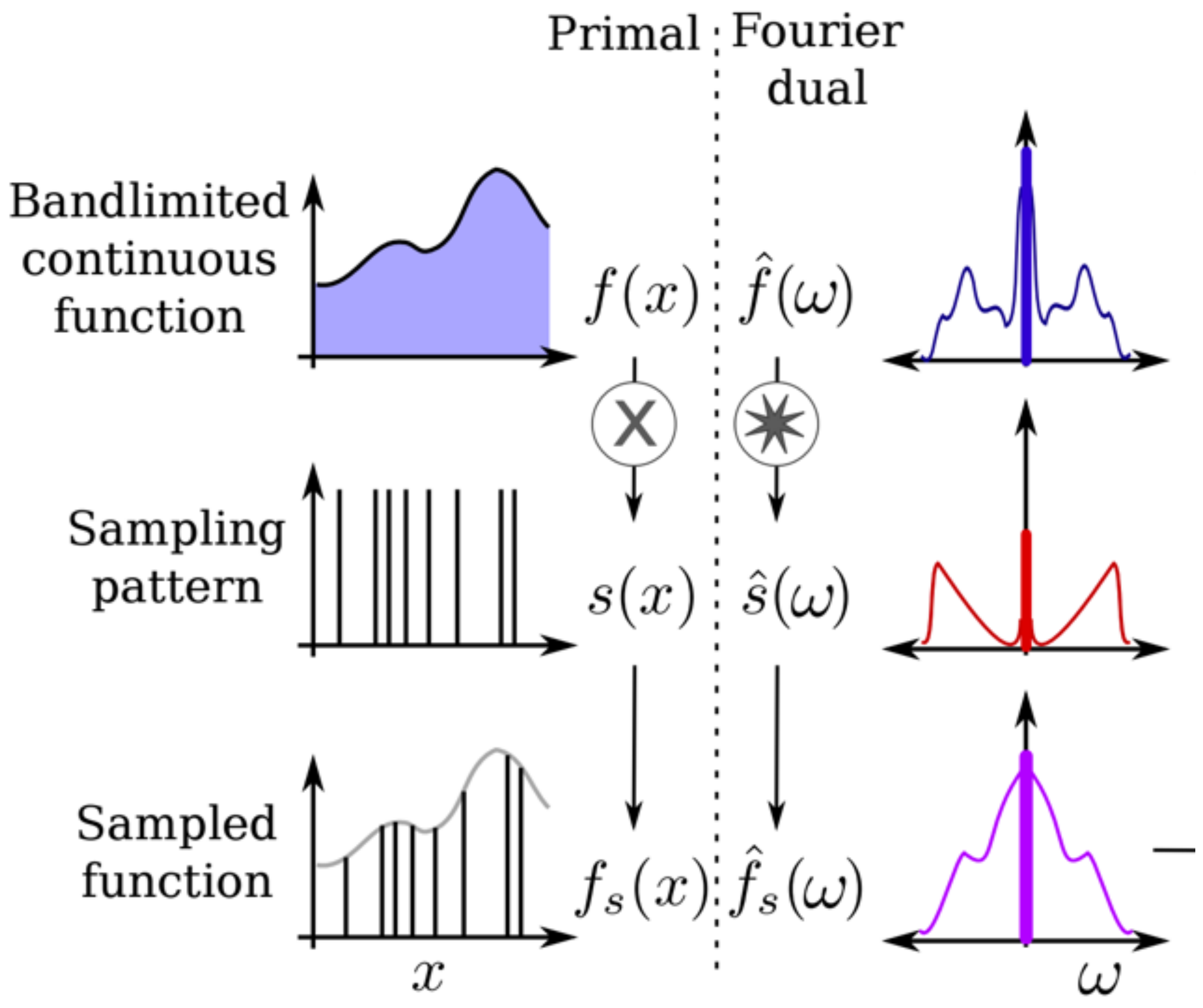
- Bias in the Fourier domain
- Variance in the Fourier domain
- Variance Convergence analysis of sampling patterns



Primal
Fourier dual
Bandlimited continuous function
$f(x)$
$\hat{f}(\omega)$
Sampling pattern
$s(x)$
$\hat{s}(\omega)$
Sampled function
$f_s(x)$
$\hat{f}_s(\omega)$
x
ω



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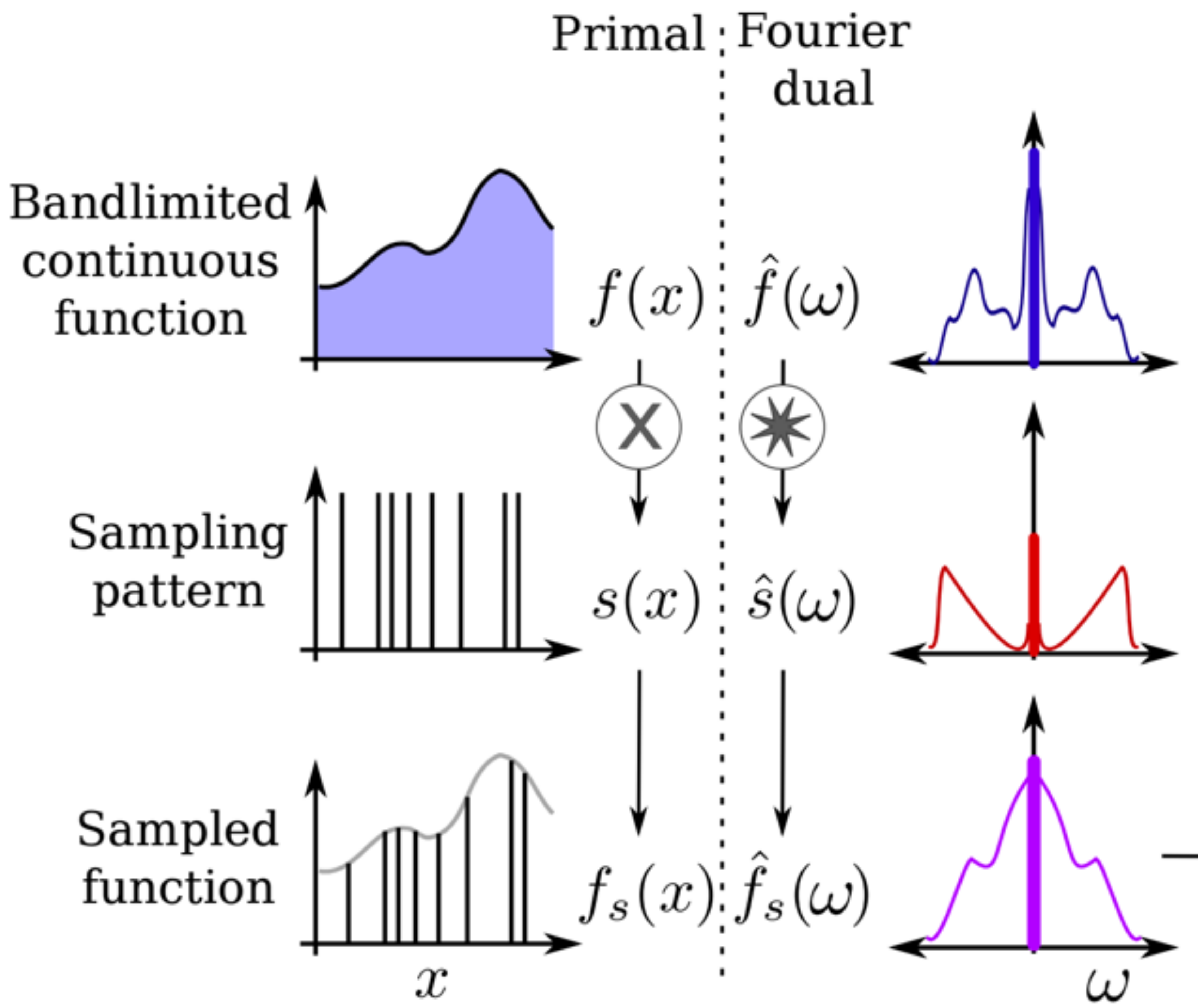
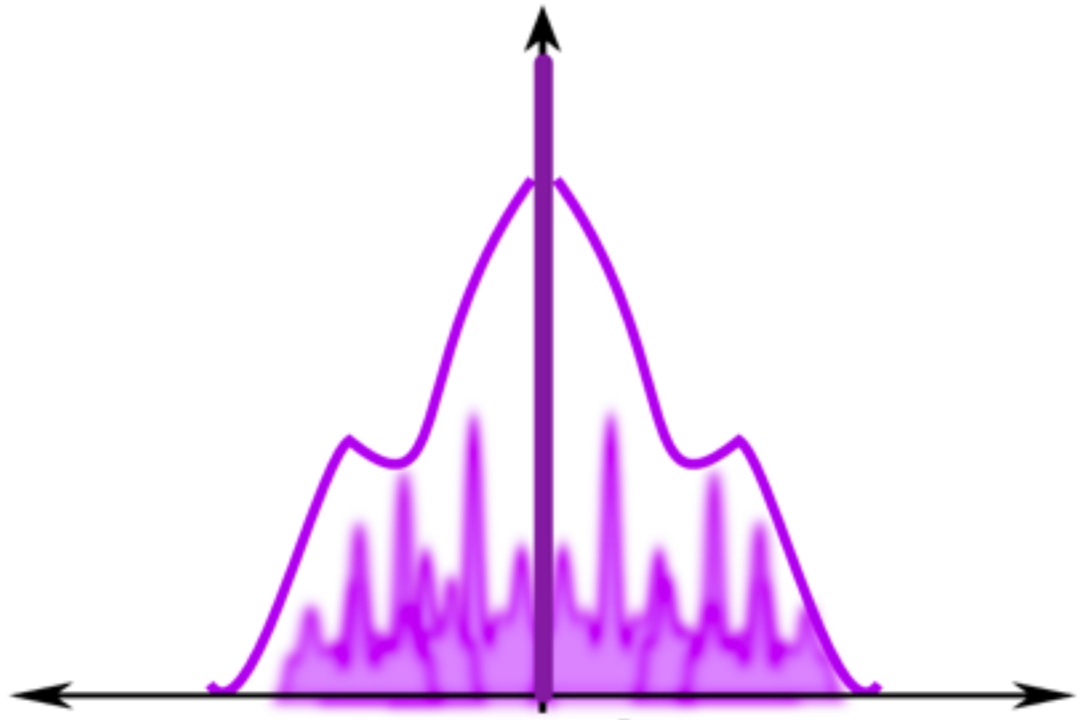
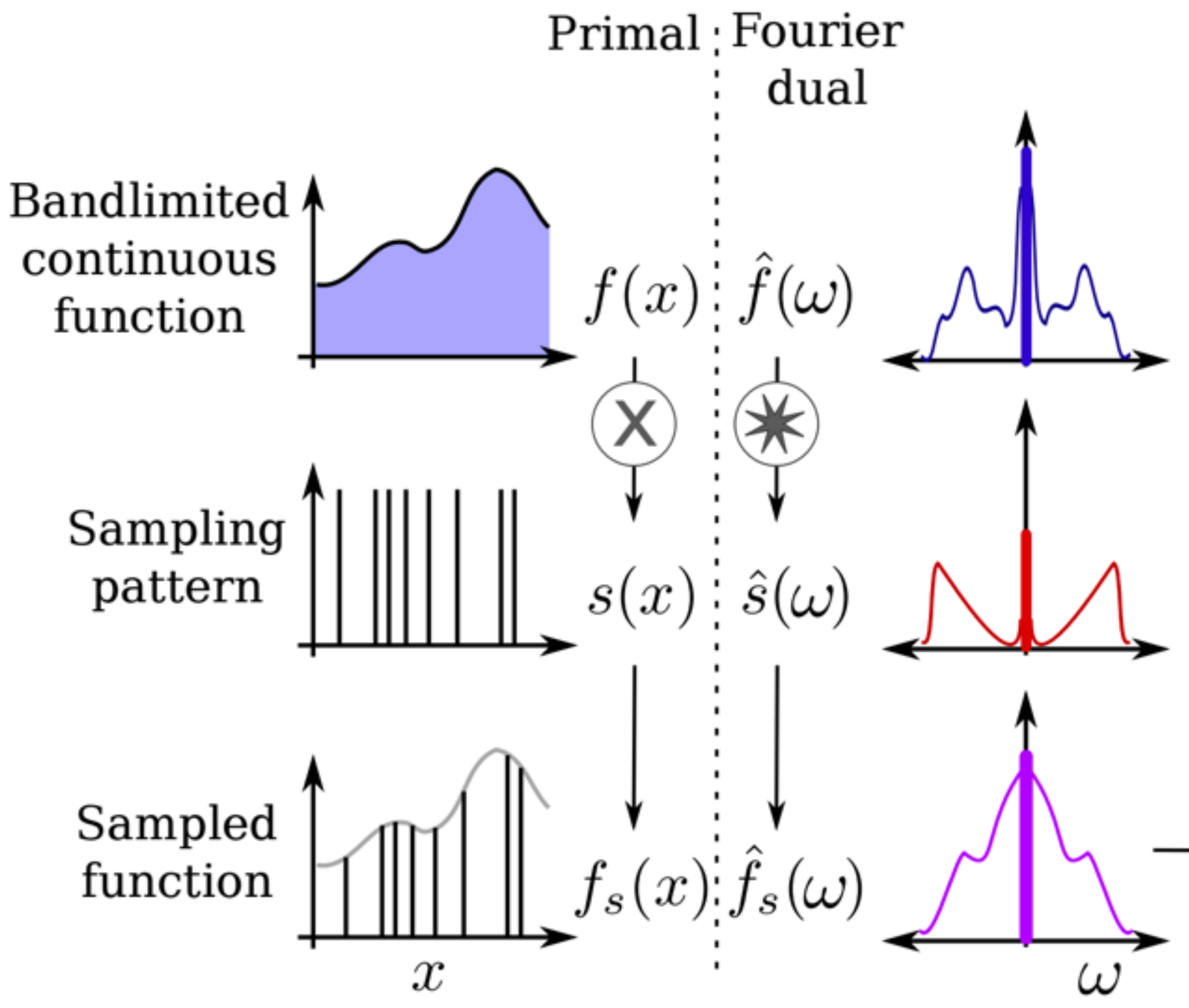


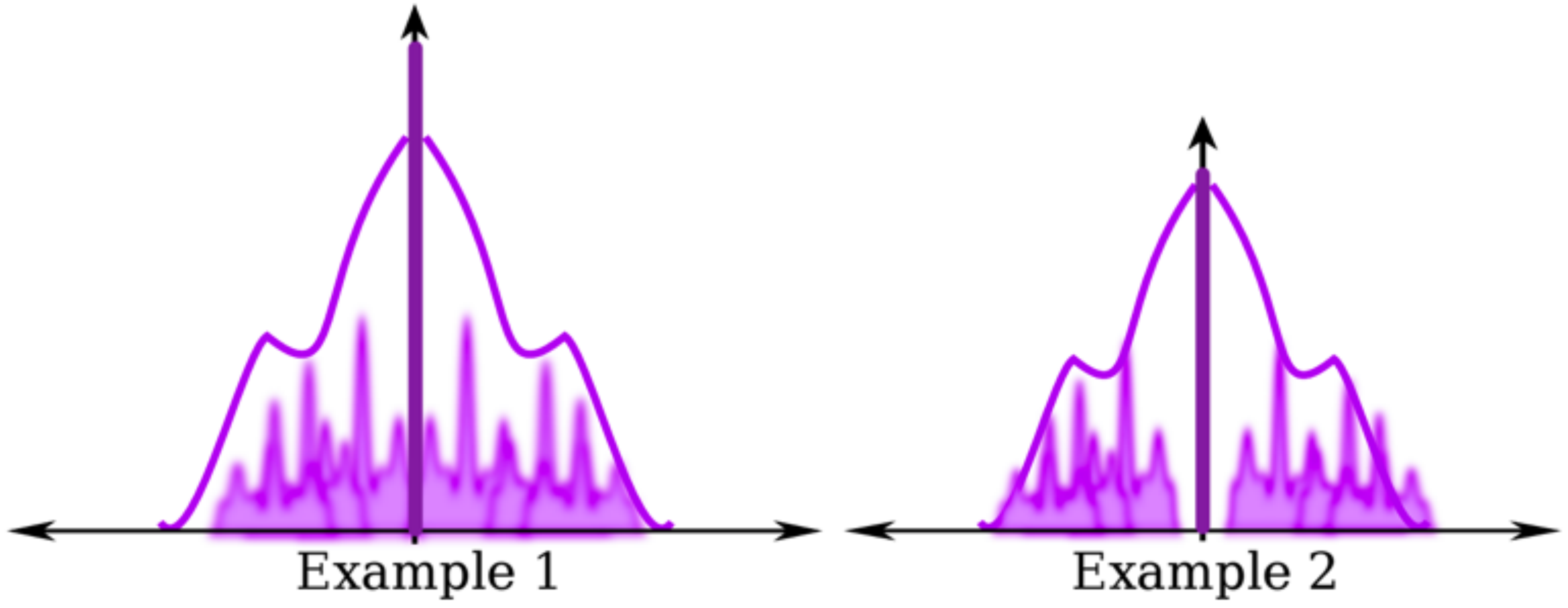
Aliases touch DC?	Yes
Aliases intersect C?	Yes
Reconstruction	ALIASED
Integration	ERROR

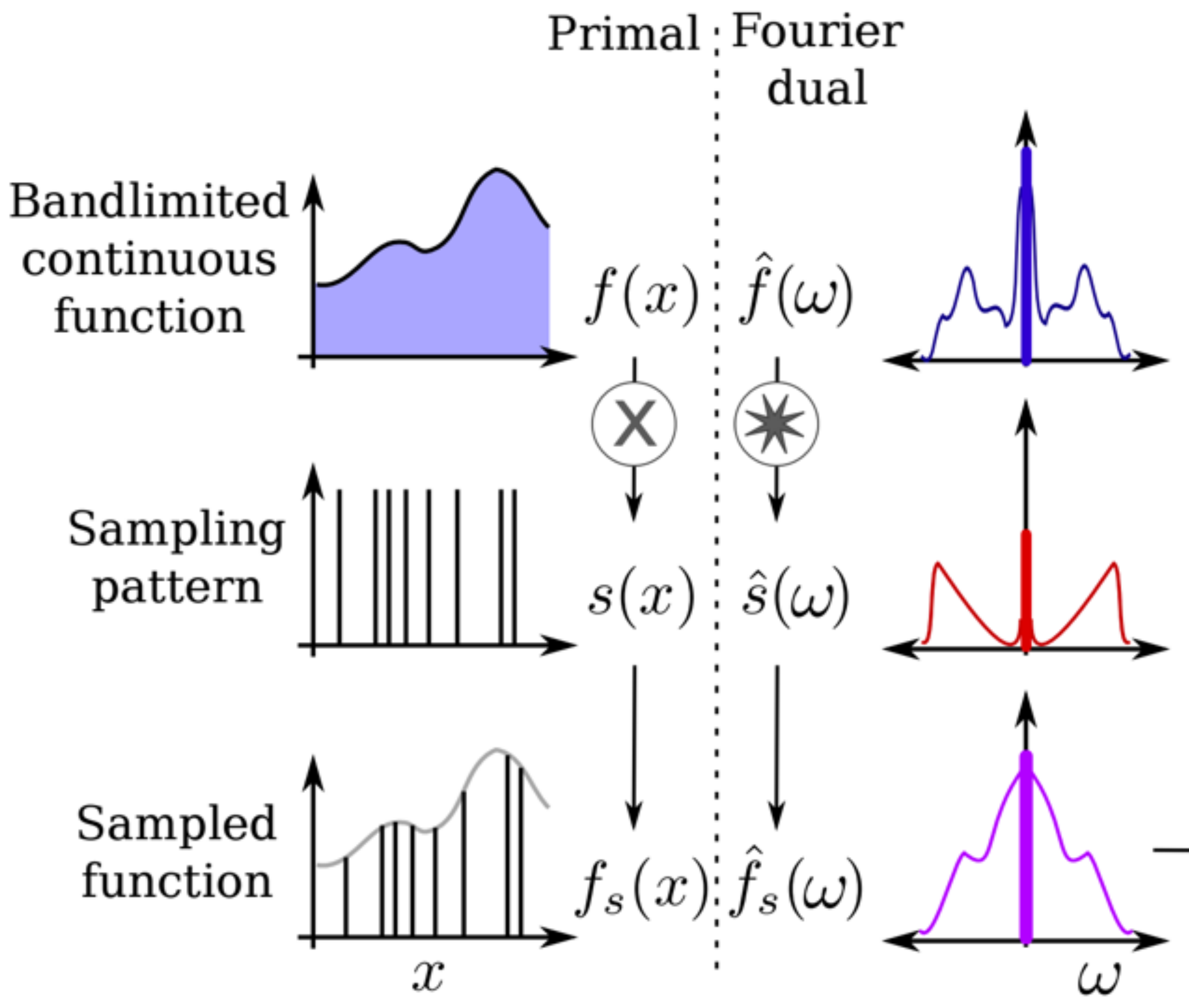


Example 1

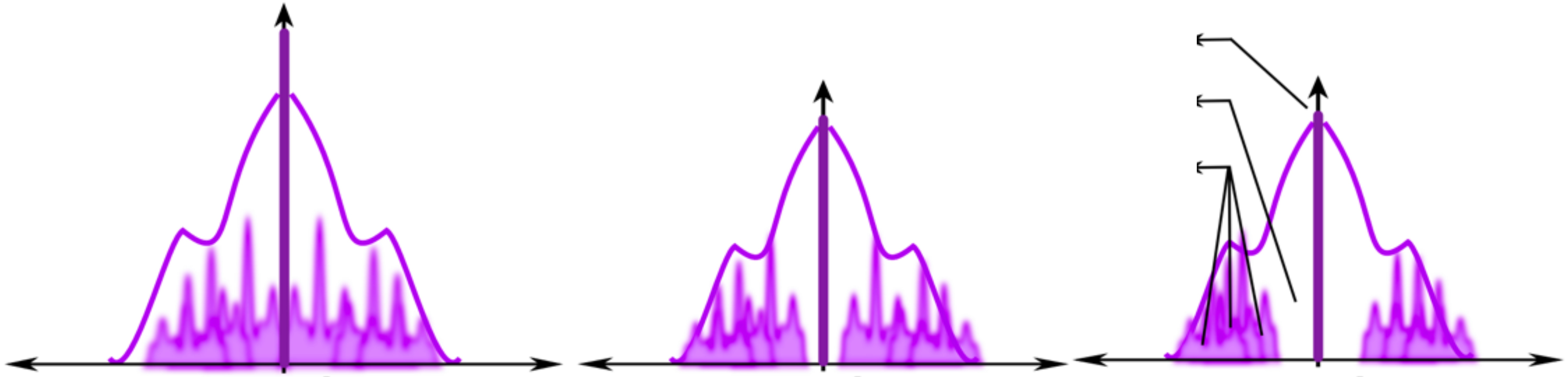


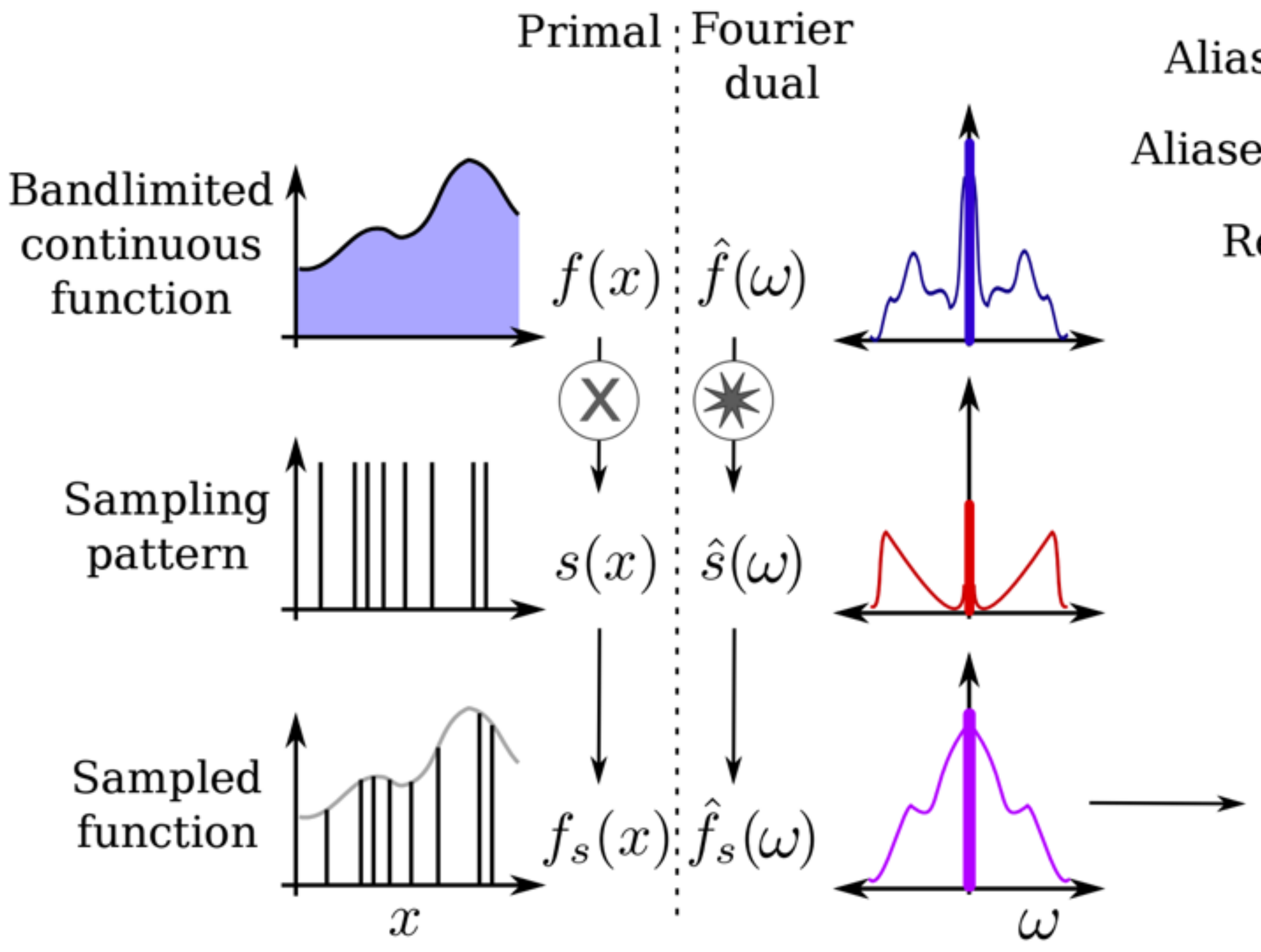
	Example 1	Example 2
Aliases touch DC?	Yes	No
Aliases intersect C?	Yes	Yes
Reconstruction	ALIASED	ALIASED
Integration	ERROR	NO ERROR





	Example 1	Example 2	Example 3
Aliases touch DC?	Yes	No	No
Aliases intersect C?	Yes	Yes	No
Reconstruction	ALIASED	ALIASED	ALIAS FREE
Integration	ERROR	NO ERROR	NO ERROR





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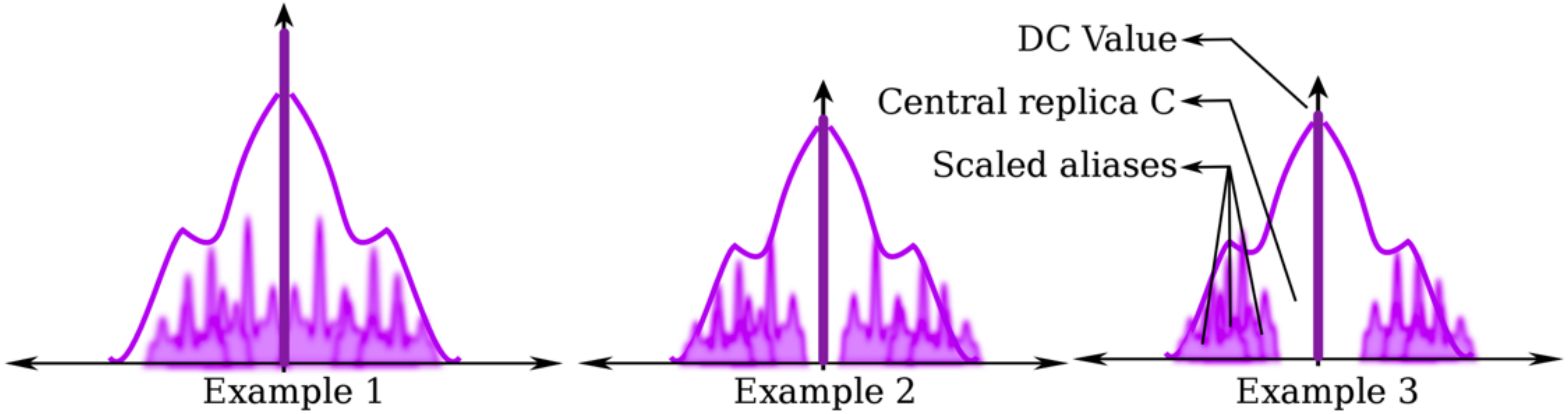

Monte Carlo Integration

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$$\mathbf{S}(x) = \frac{1}{N}\sum_{k=1}^{N}\delta(x - x_k)$$

Monte Carlo Integration

$$\tilde{\mu}_N = \int_D f(x)\mathbf{S}(x)dx = \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

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$$\hat{\mathbf{S}}(\omega) = \frac{1}{N}\sum_{k=1}^{N}e^{-i2\pi\omega x_k}$$

$$\text{Error} = \text{Bias}^2 + \text{Variance}$$

Error in Spatial Domain

$$I = \int_D f(x)dx$$

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To obtain an unbiased estimator:

Bias in Fourier Domain

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To obtain an unbiased estimator:

$$E[\hat{\mathbf{S}}(\omega)] = 0$$

for frequencies other than zero

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Bias in Fourier Domain

$$\int_\Omega E[\hat{f}^*(\omega)]E[\hat{\mathbf{S}}(\omega)]d\omega \; = \int_\Omega |E[\hat{f}^*(\omega)]|\,|E[\hat{\mathbf{S}}(\omega)]|\, e^{\Phi(E[\hat{f}^*(\omega)])}\, e^{\Phi(E[\hat{\mathbf{S}}(\omega)])} d\omega$$

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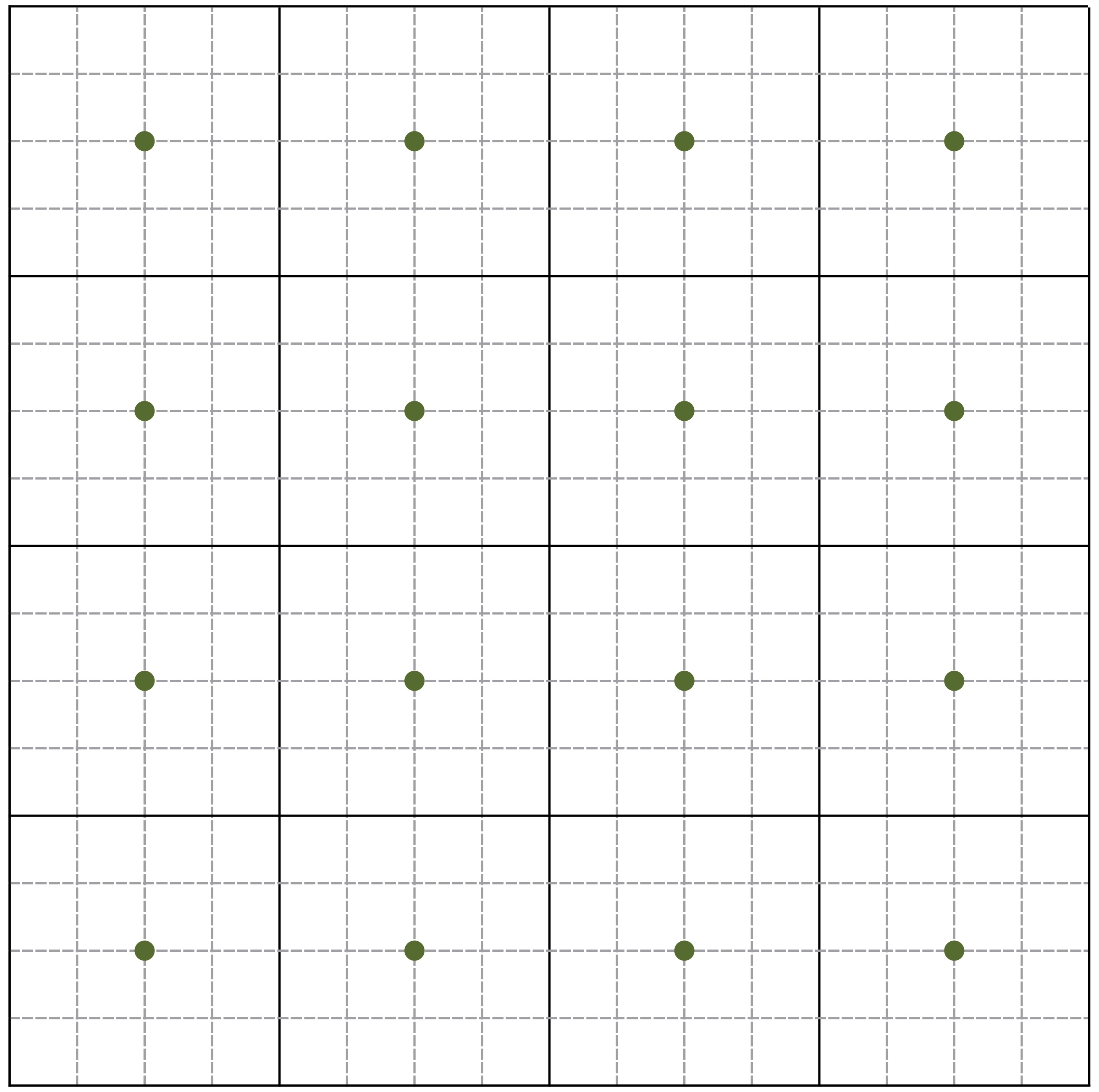
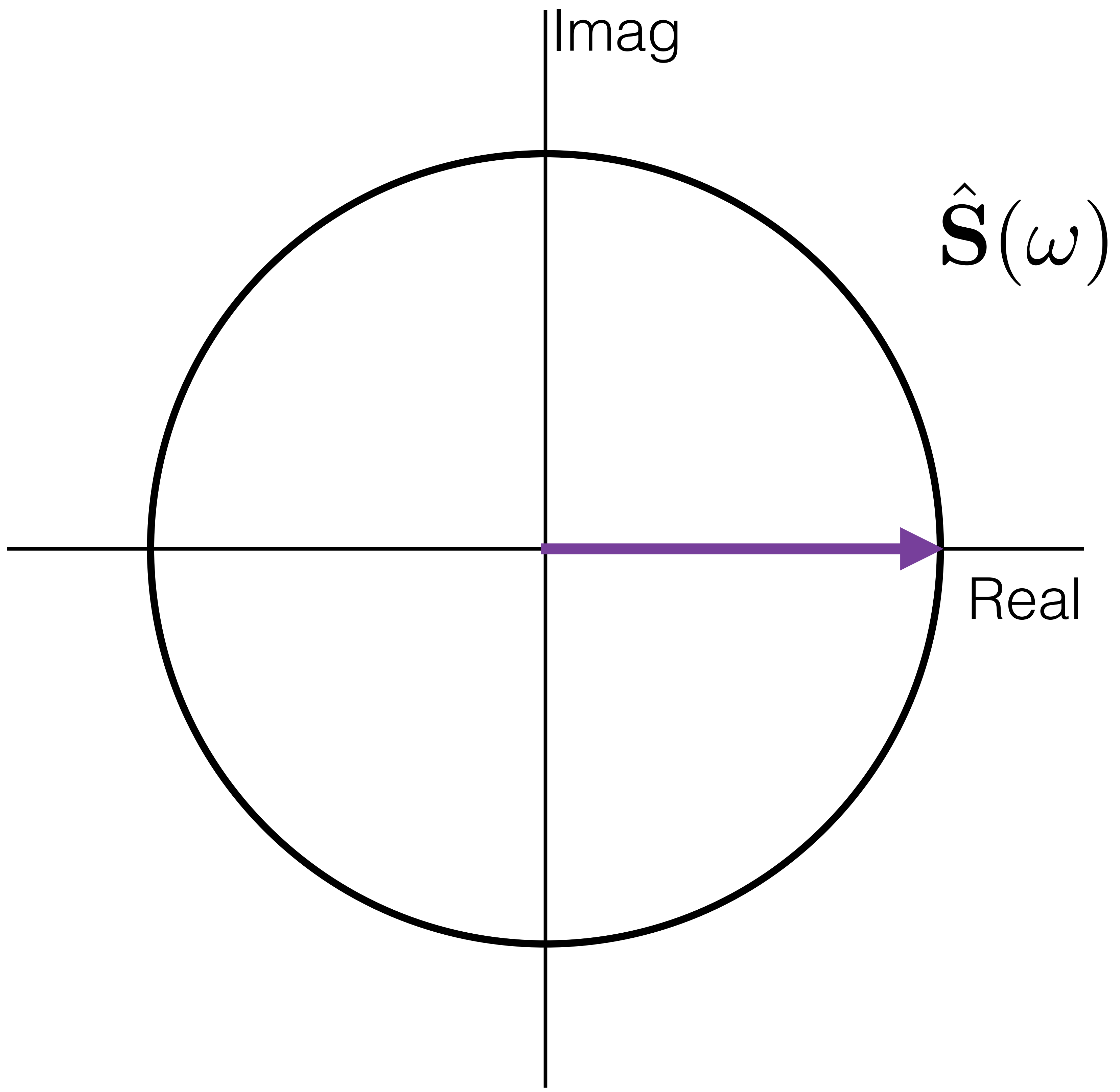
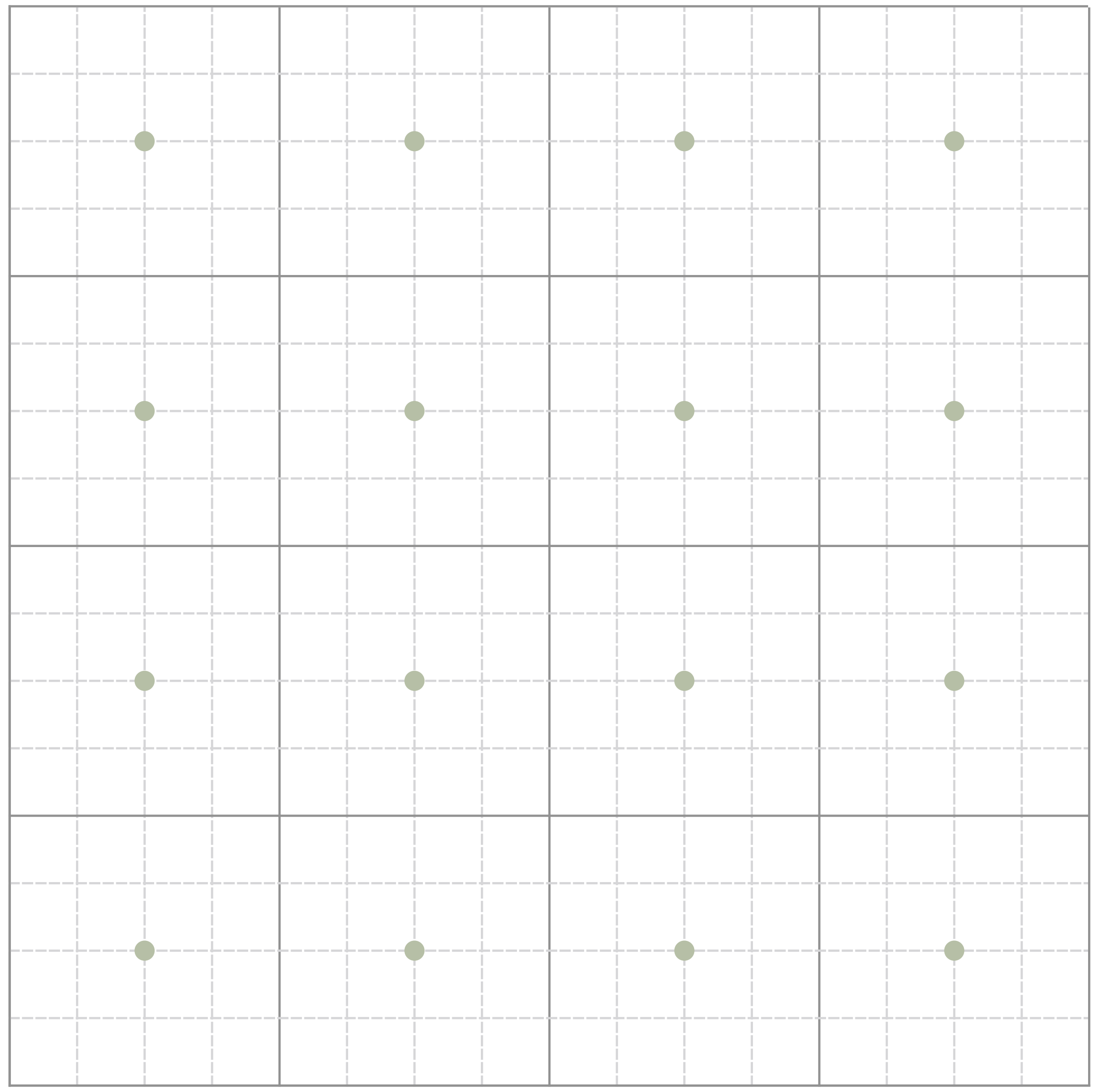
Phase Distribution

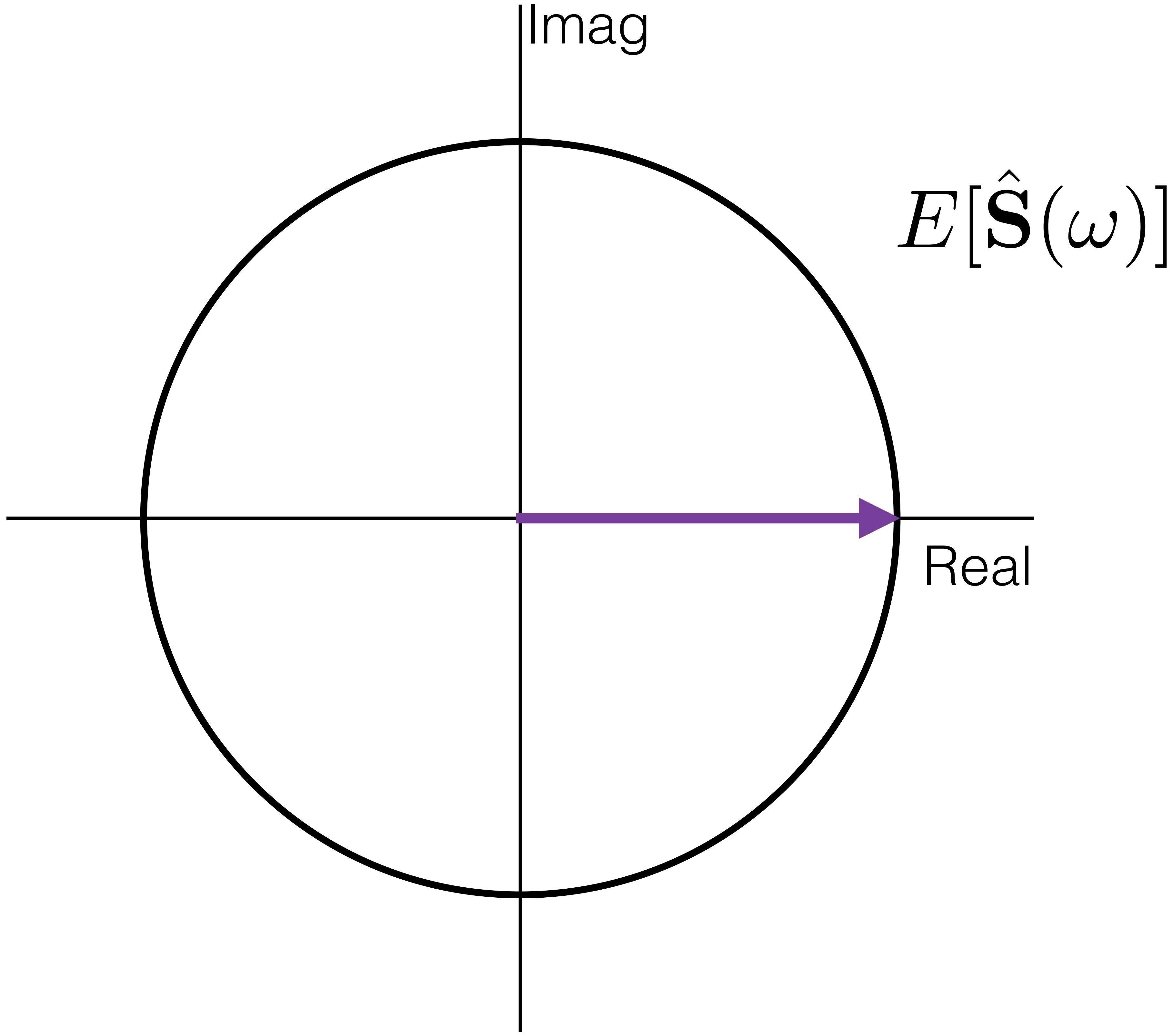




Phase Distribution

Multiple realizations





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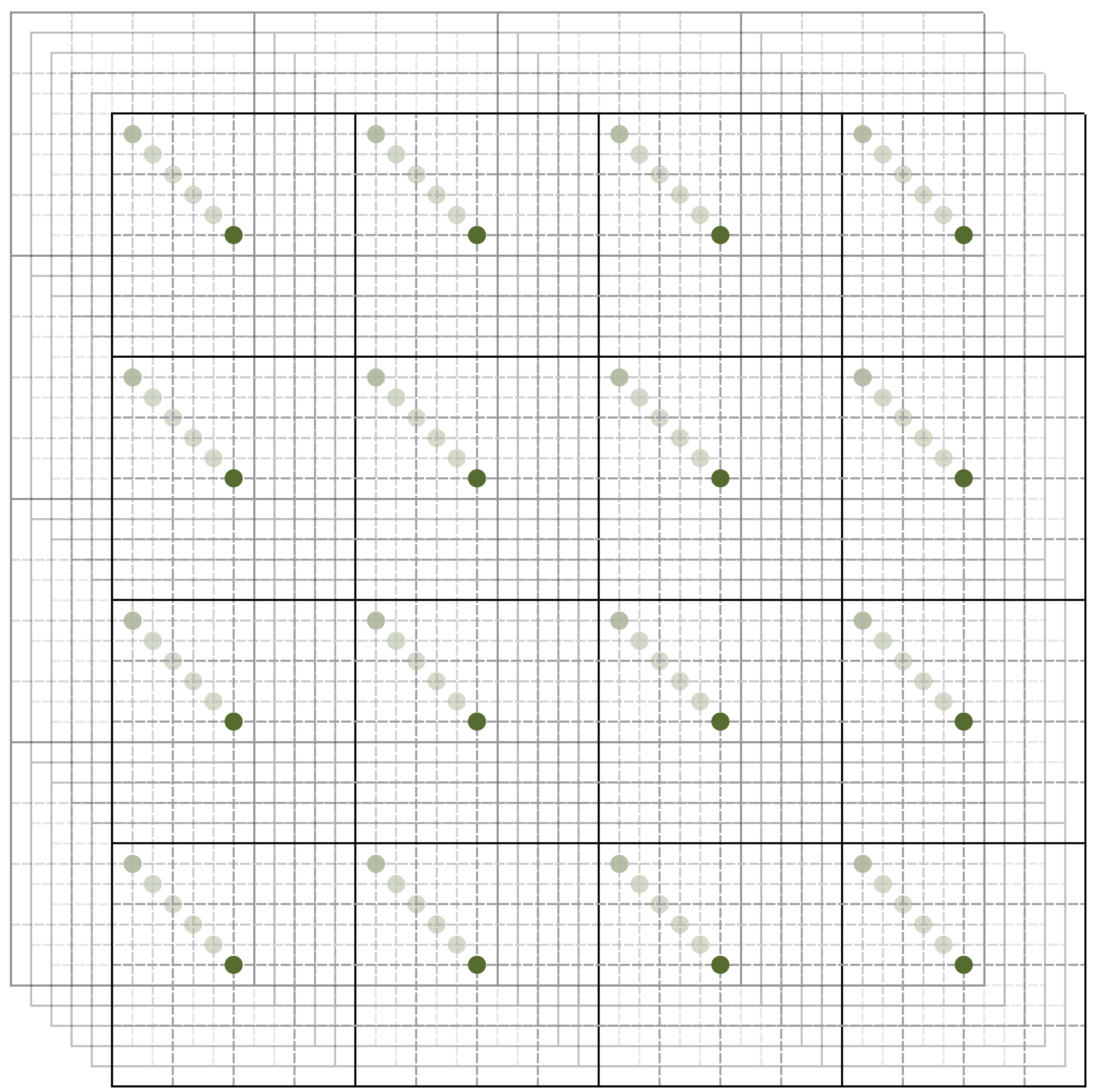

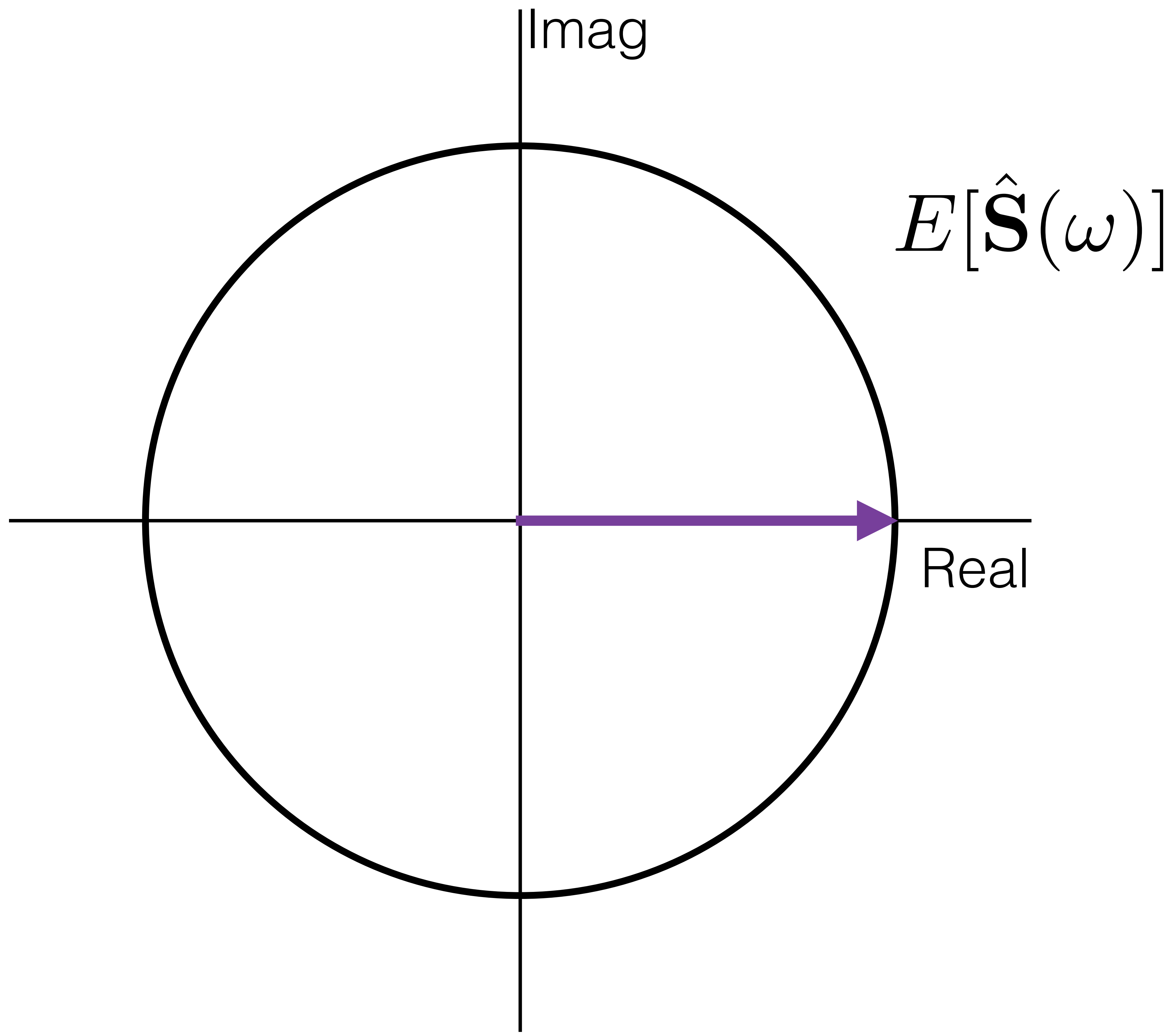

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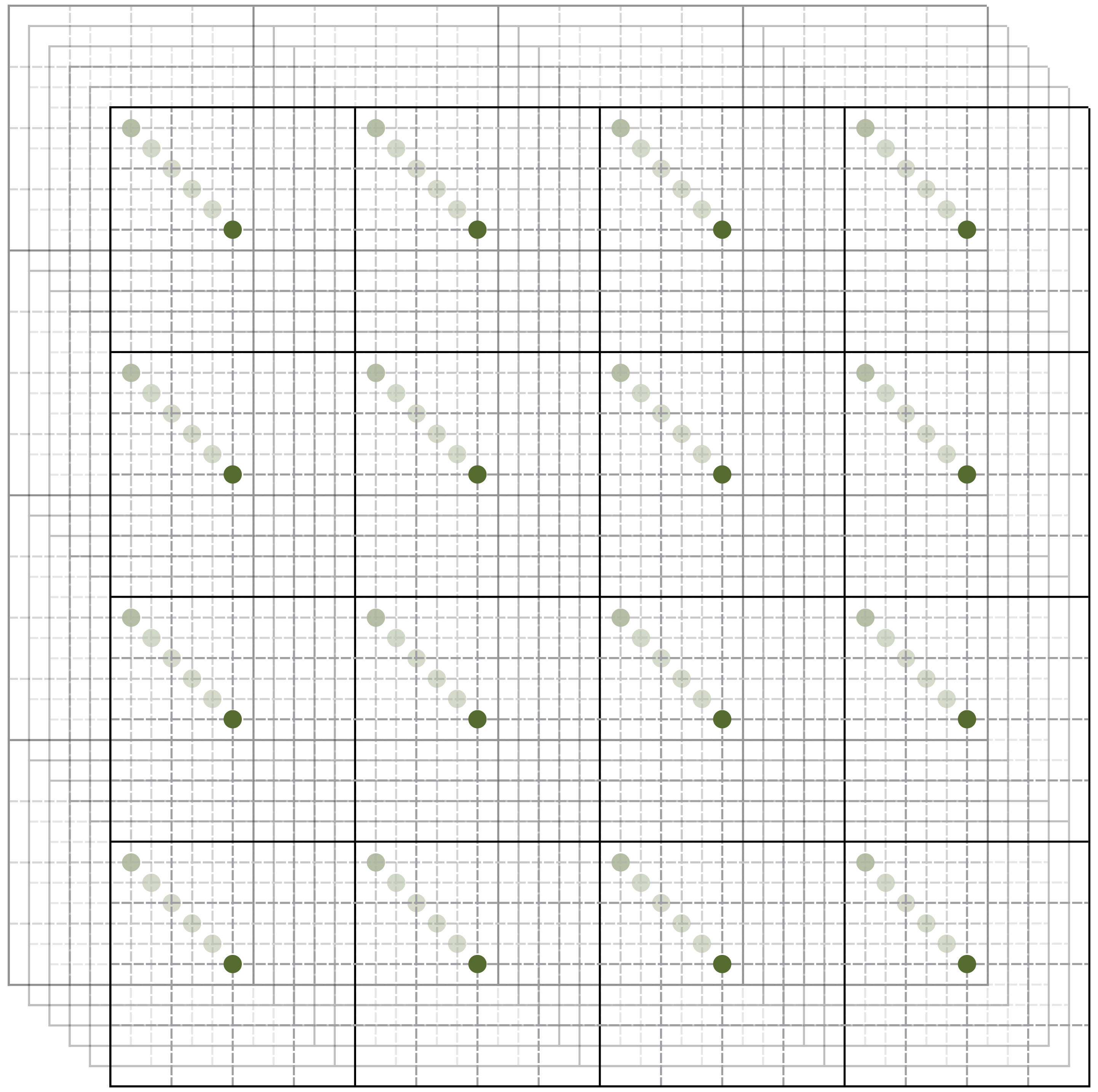

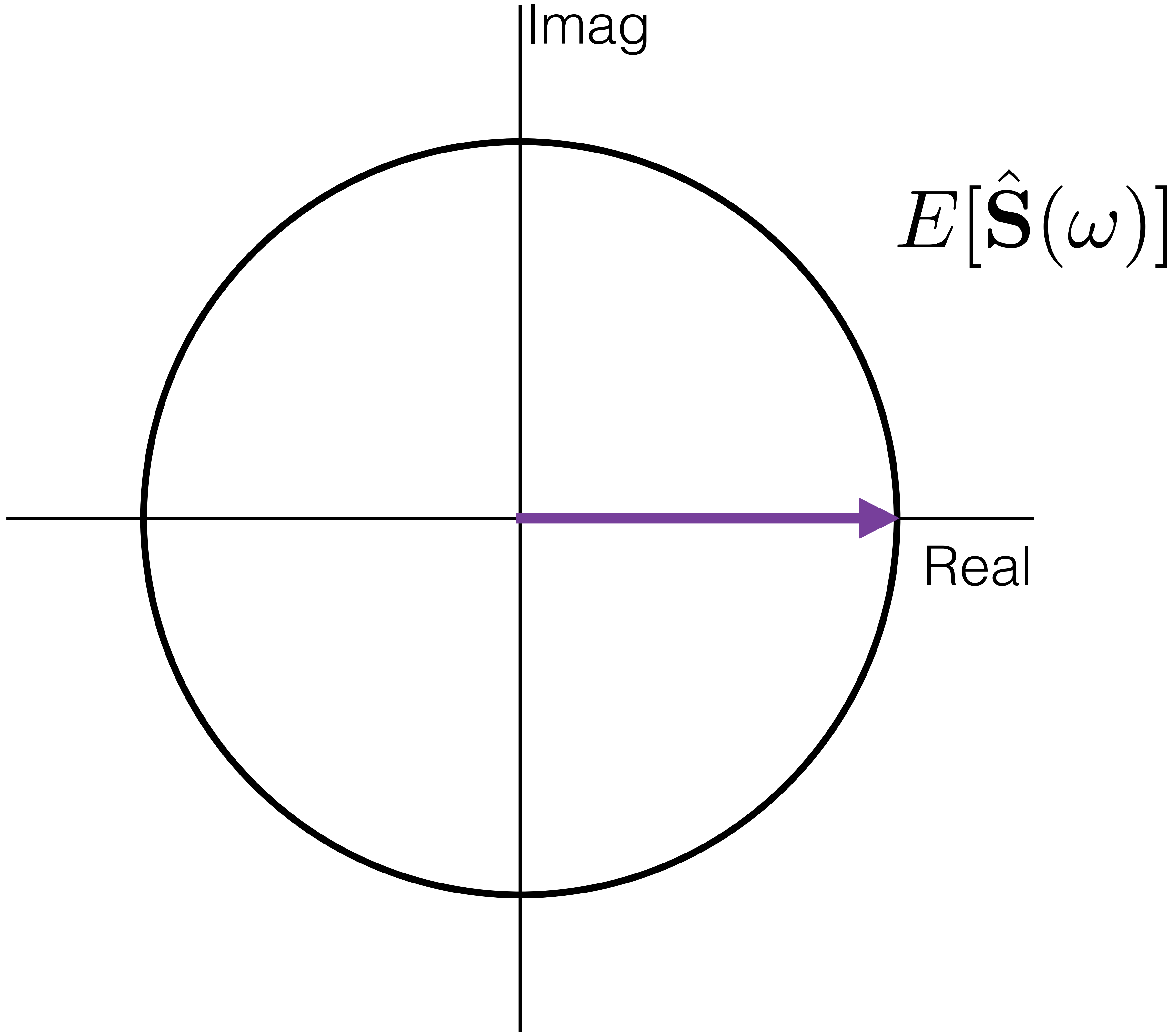

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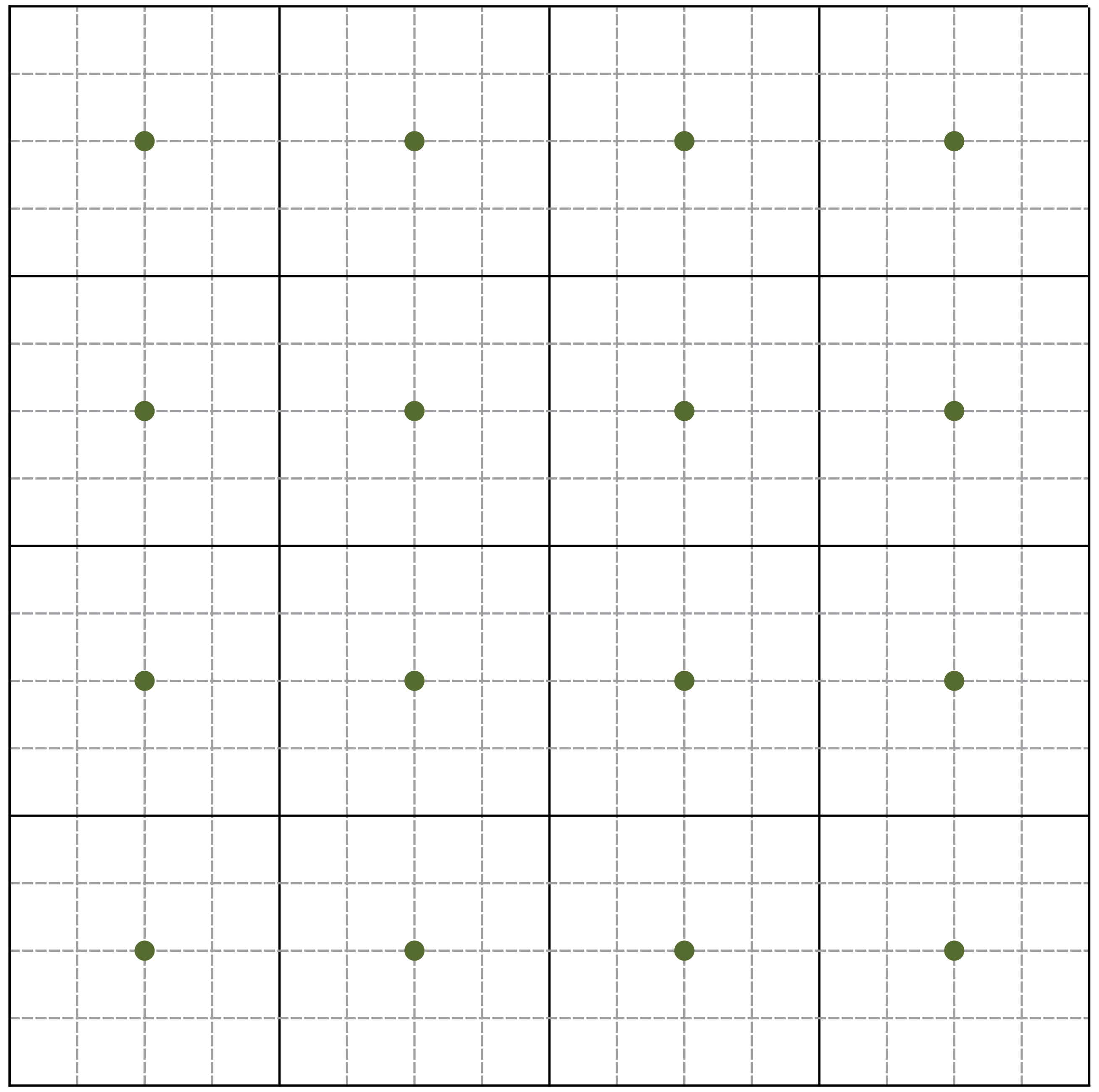

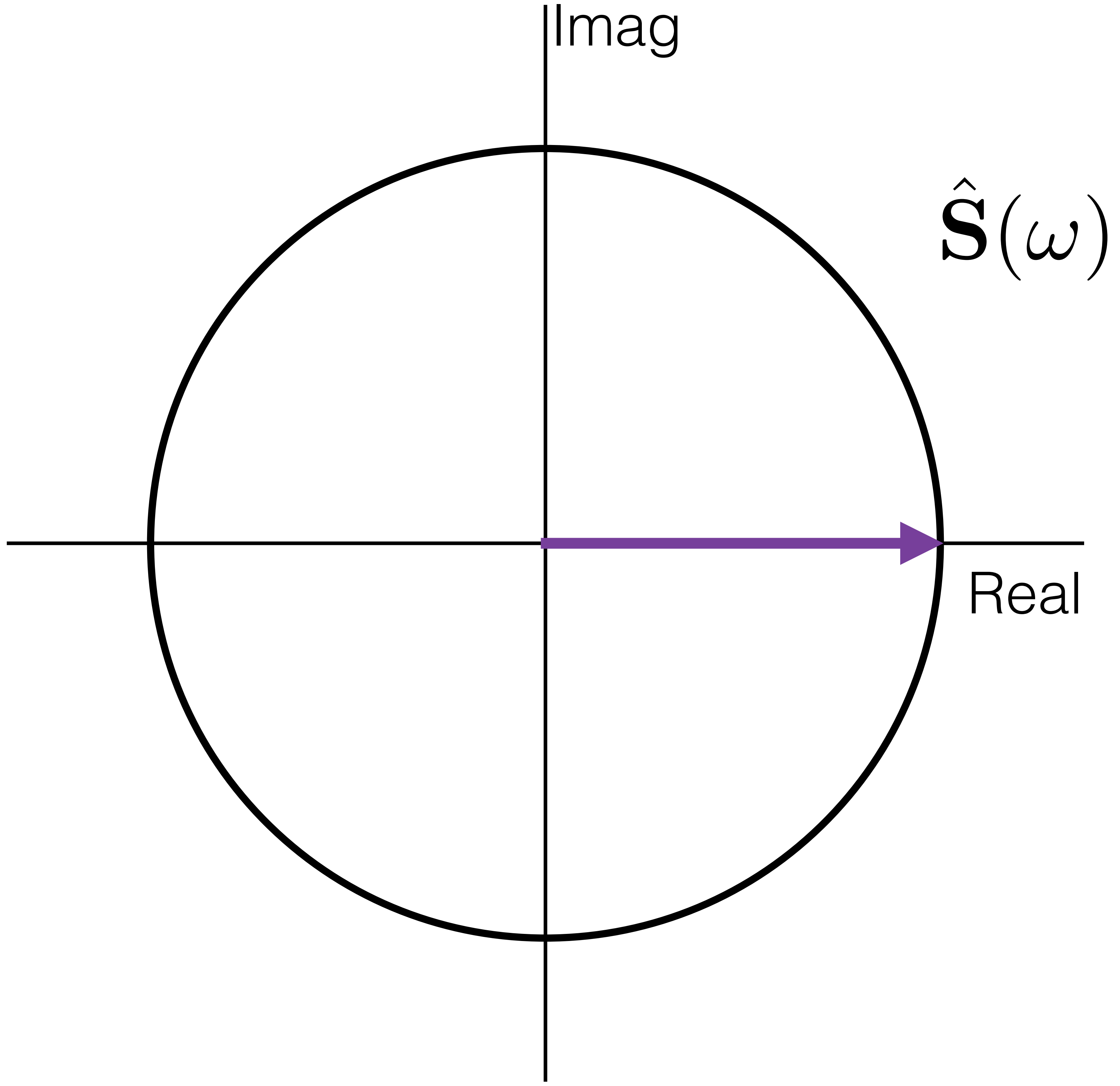

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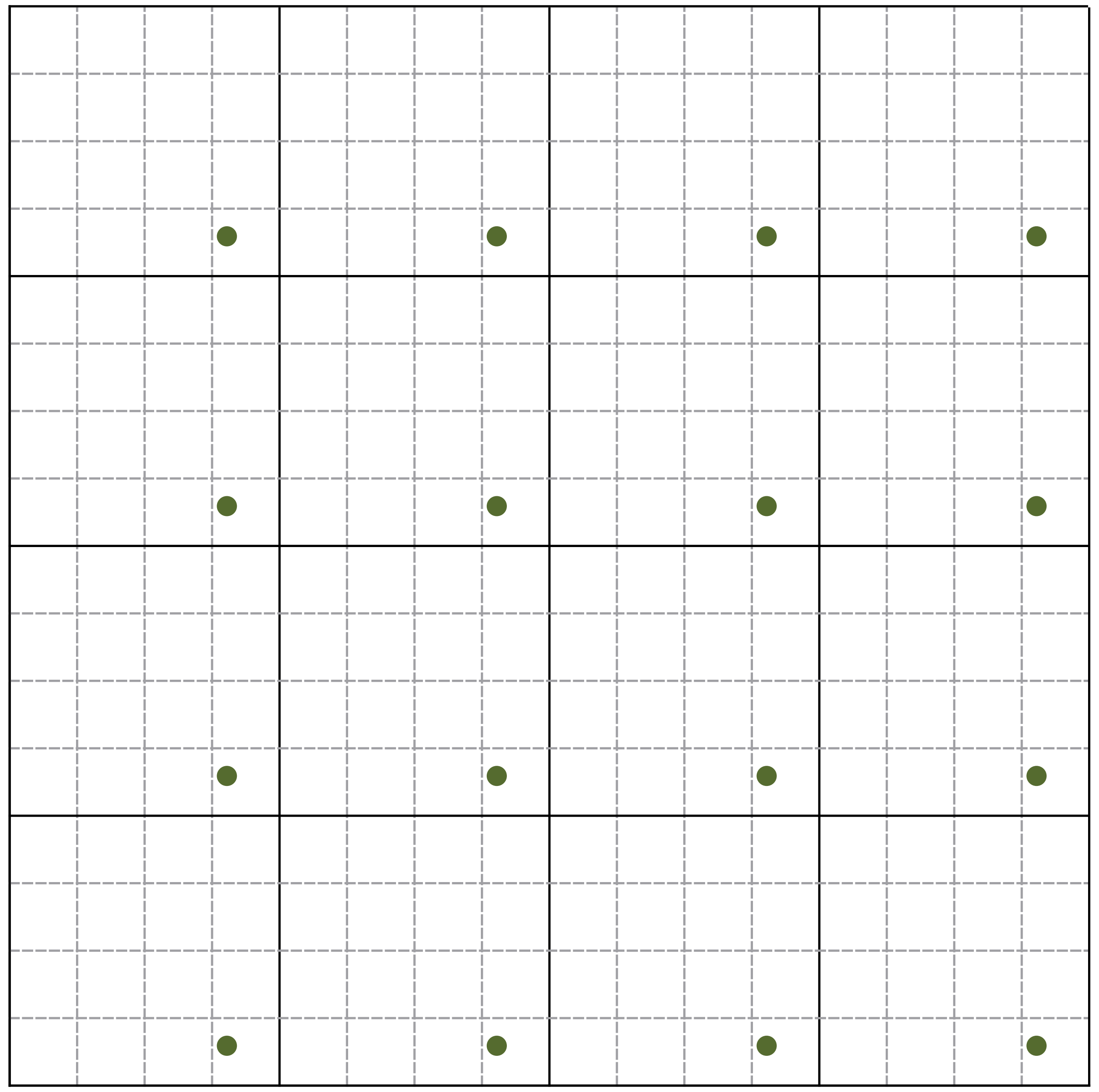

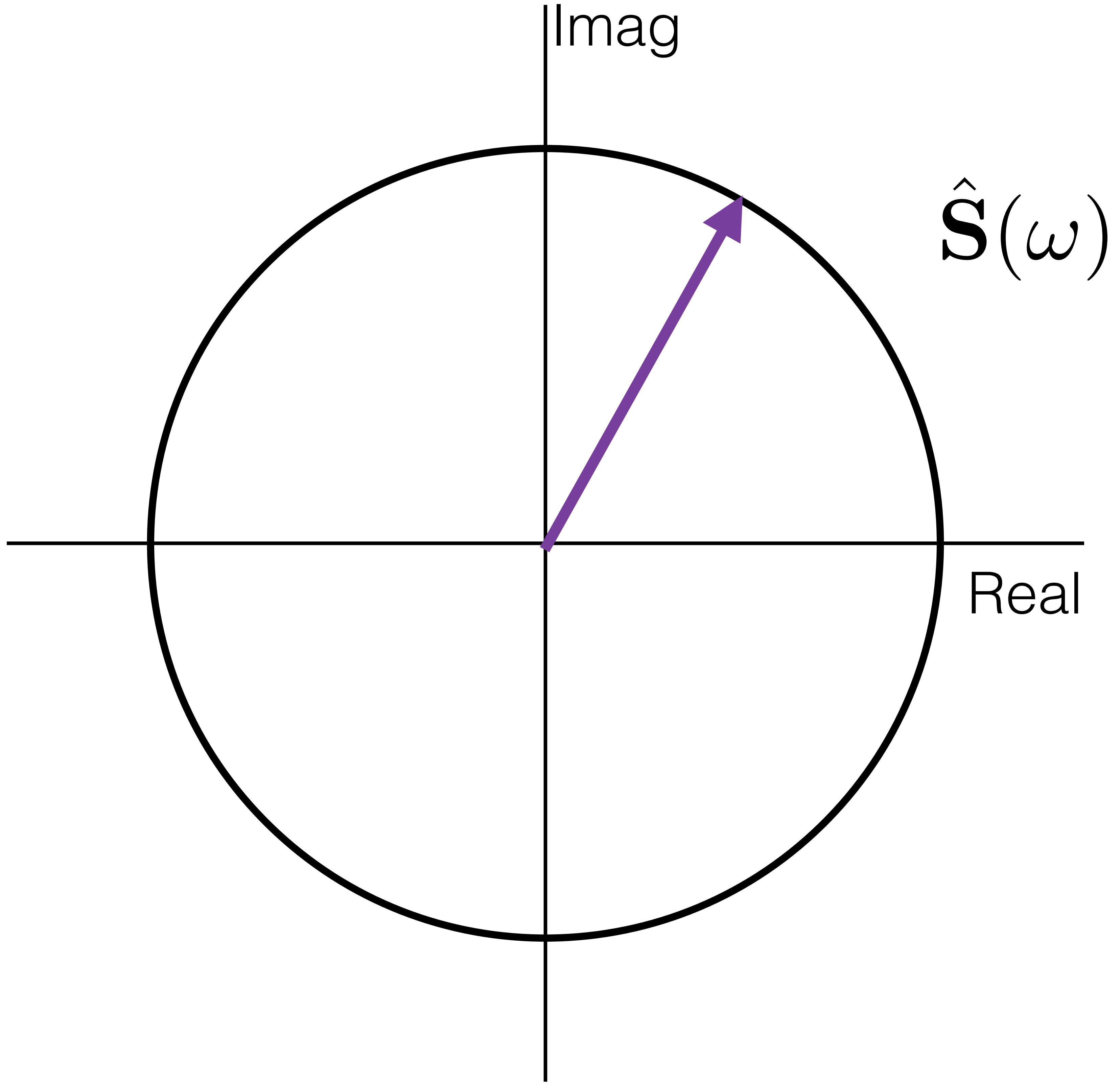

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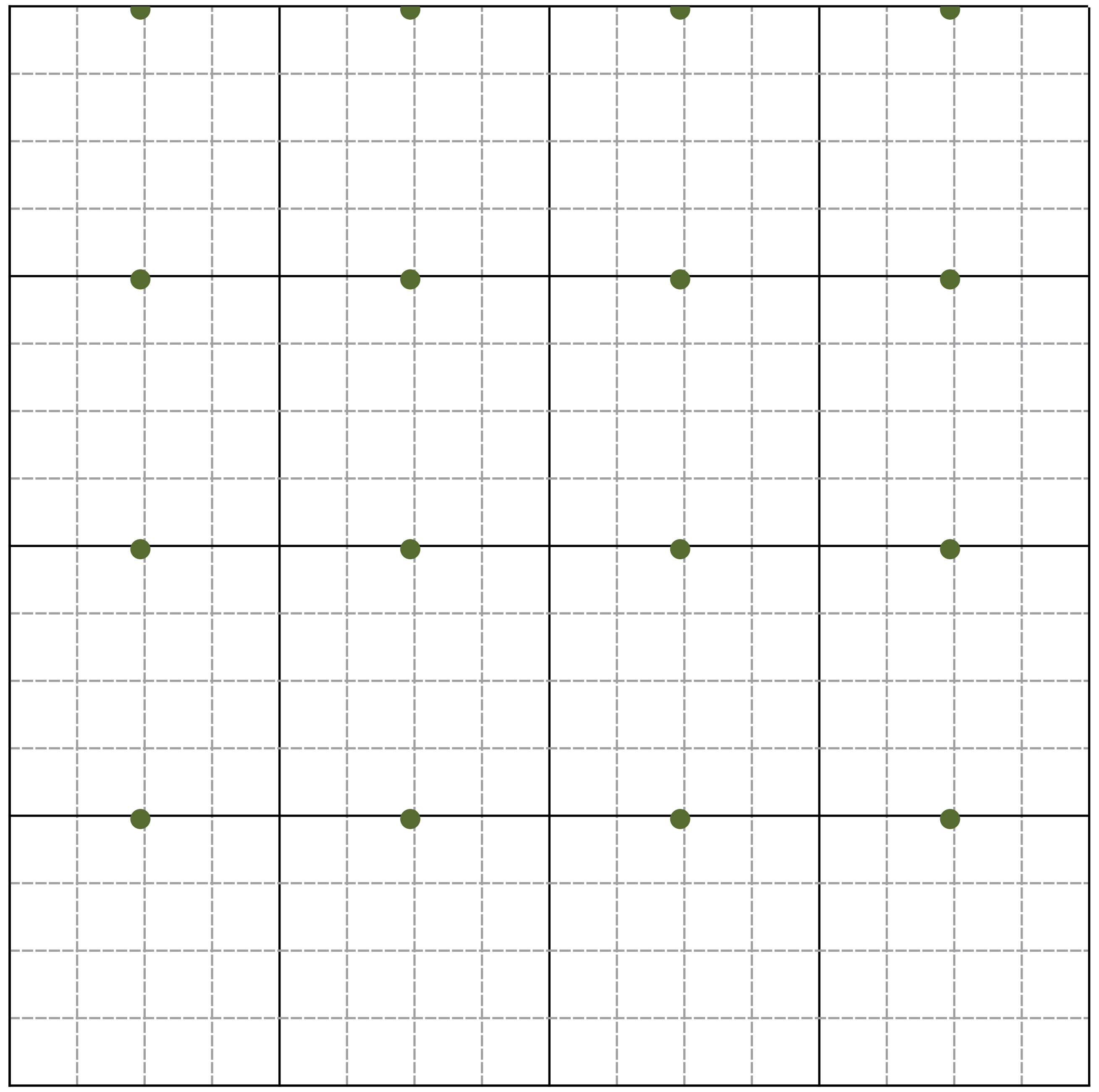

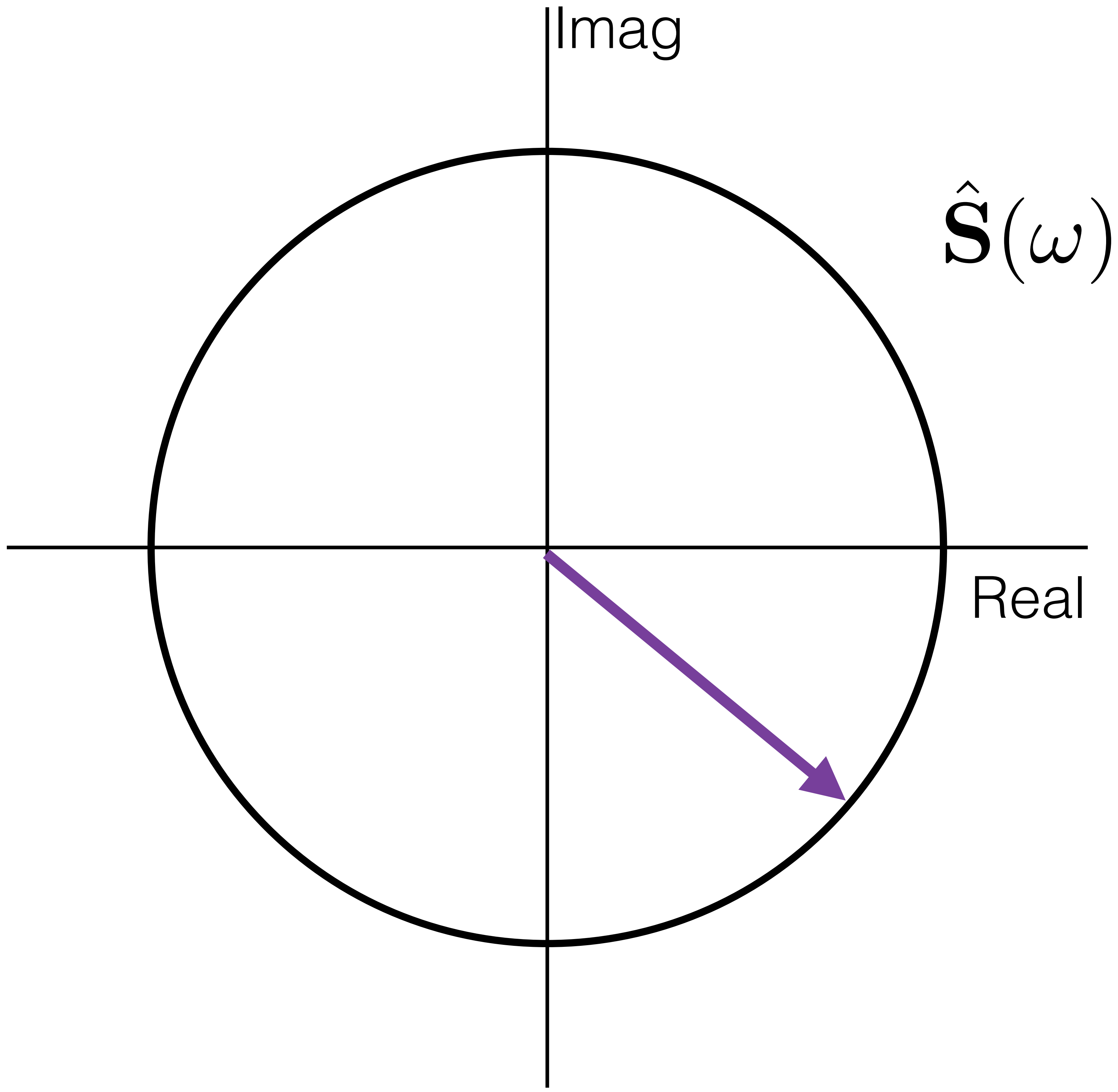

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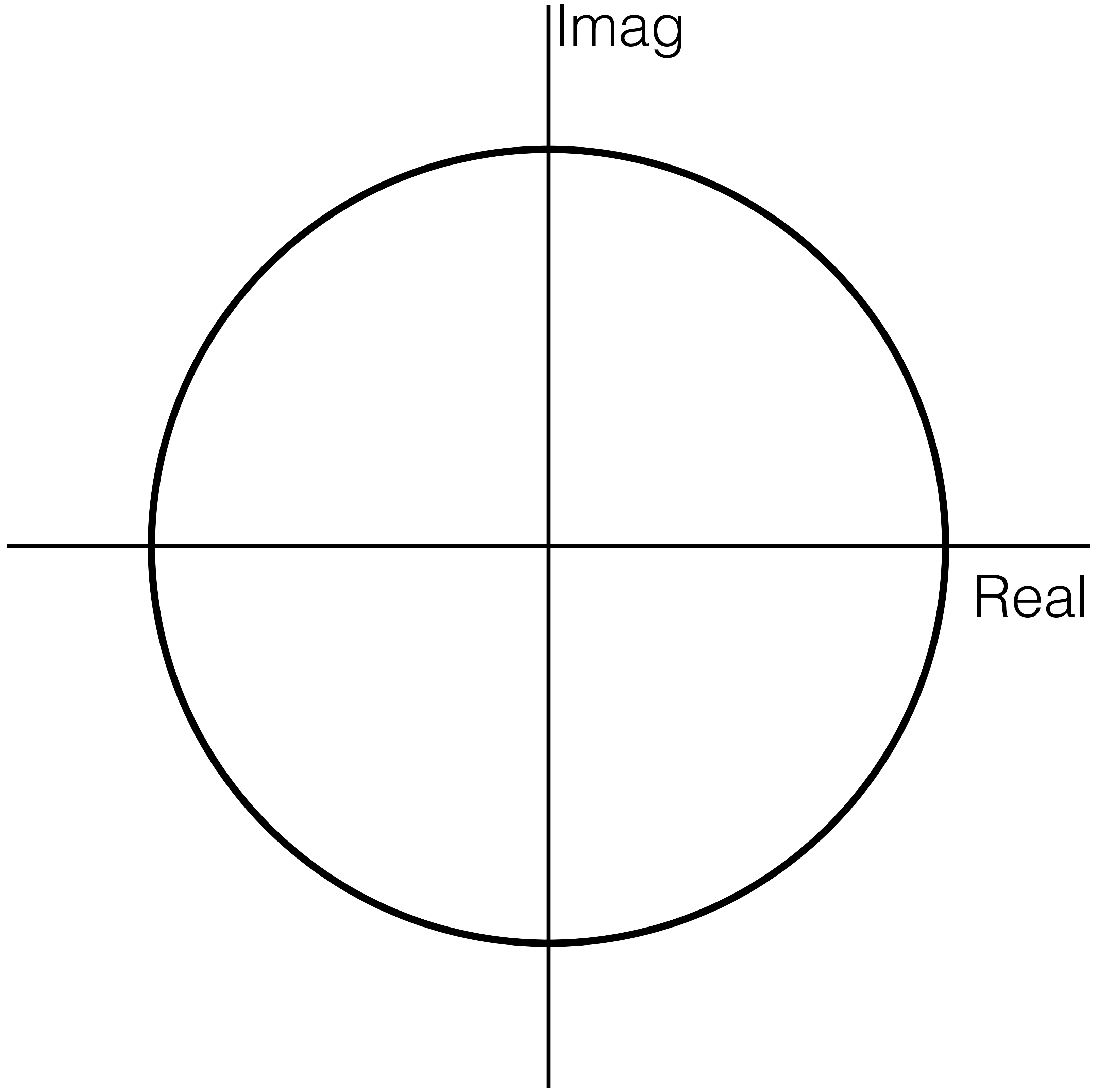

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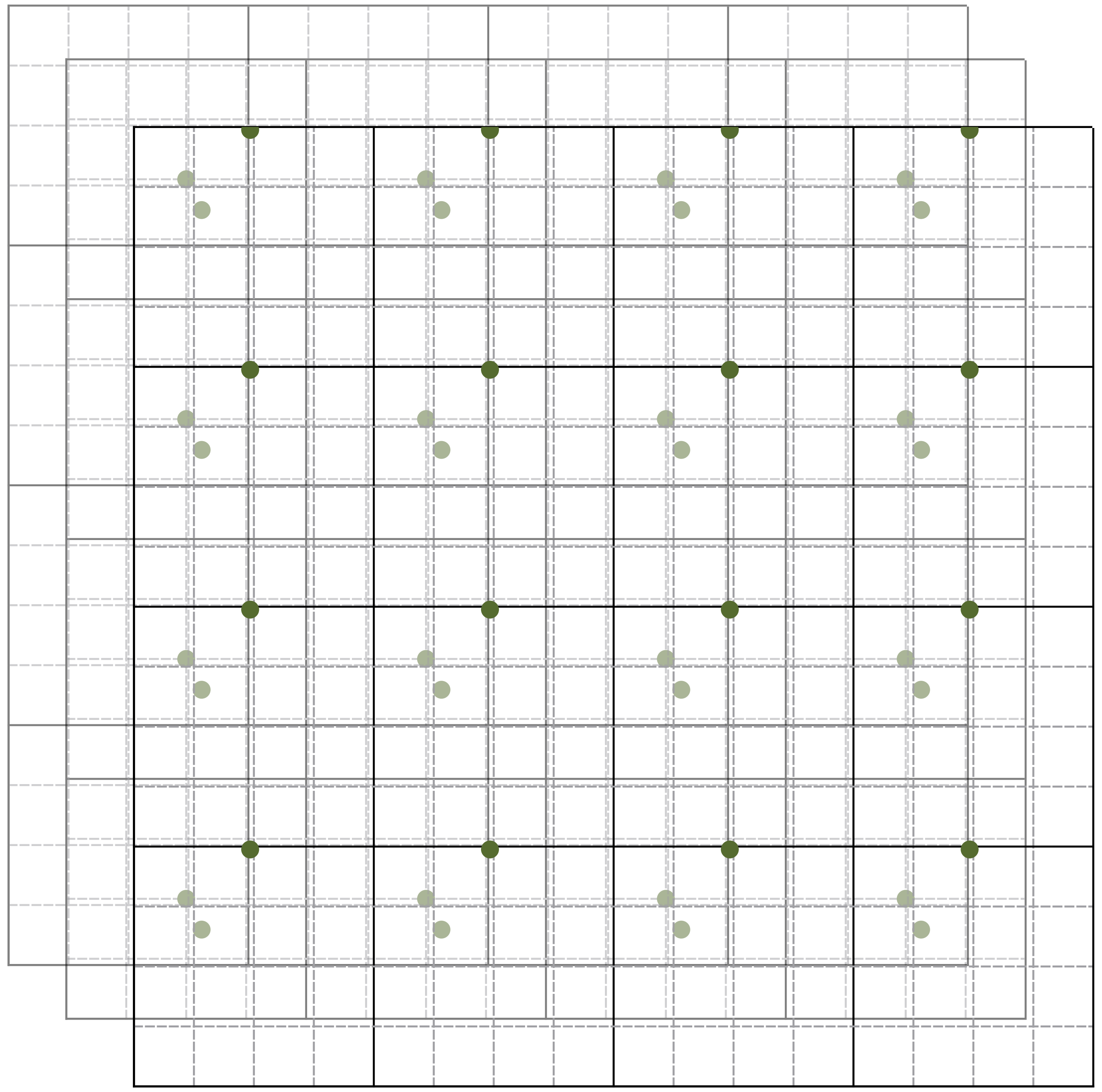

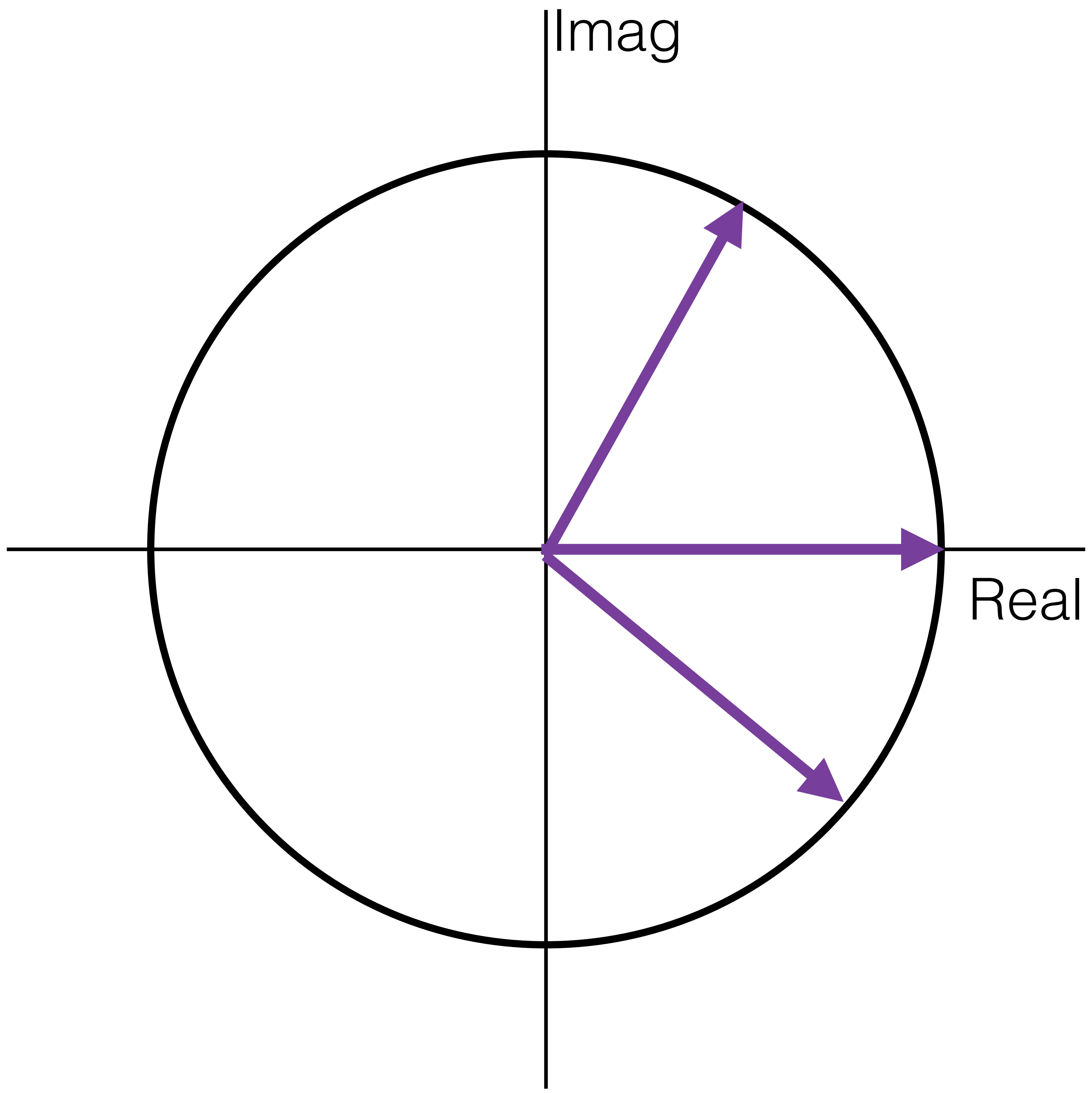

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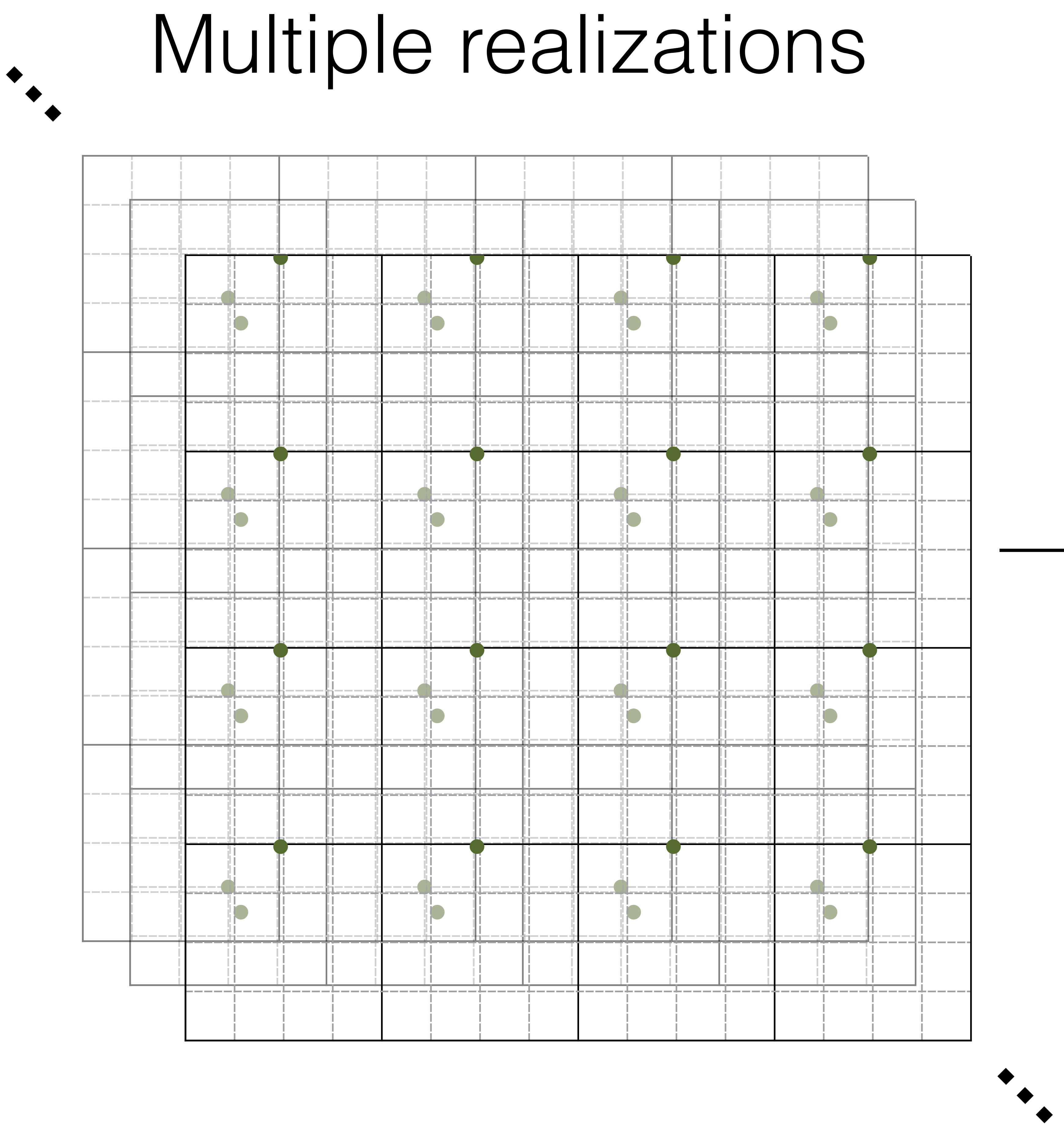

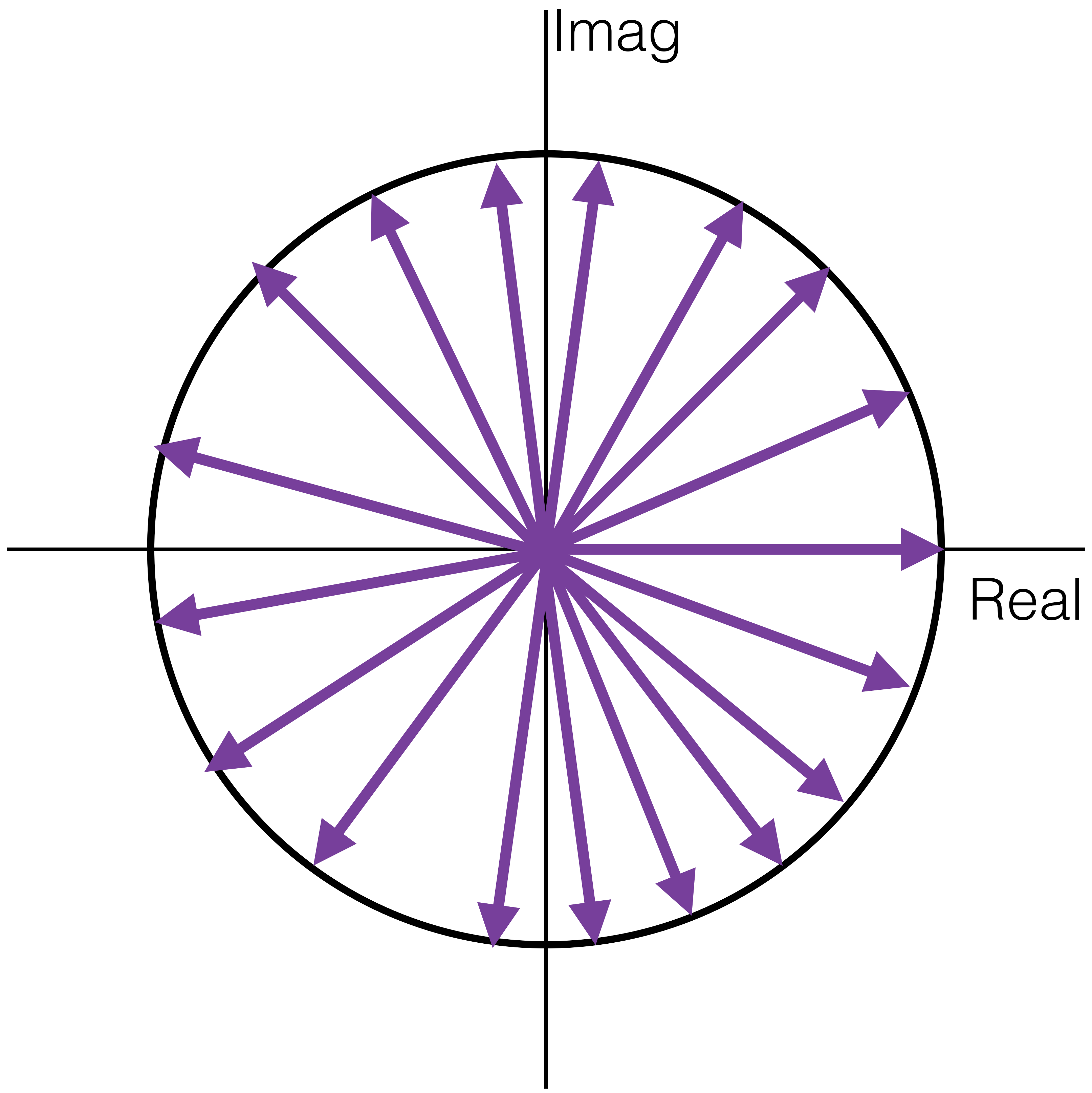

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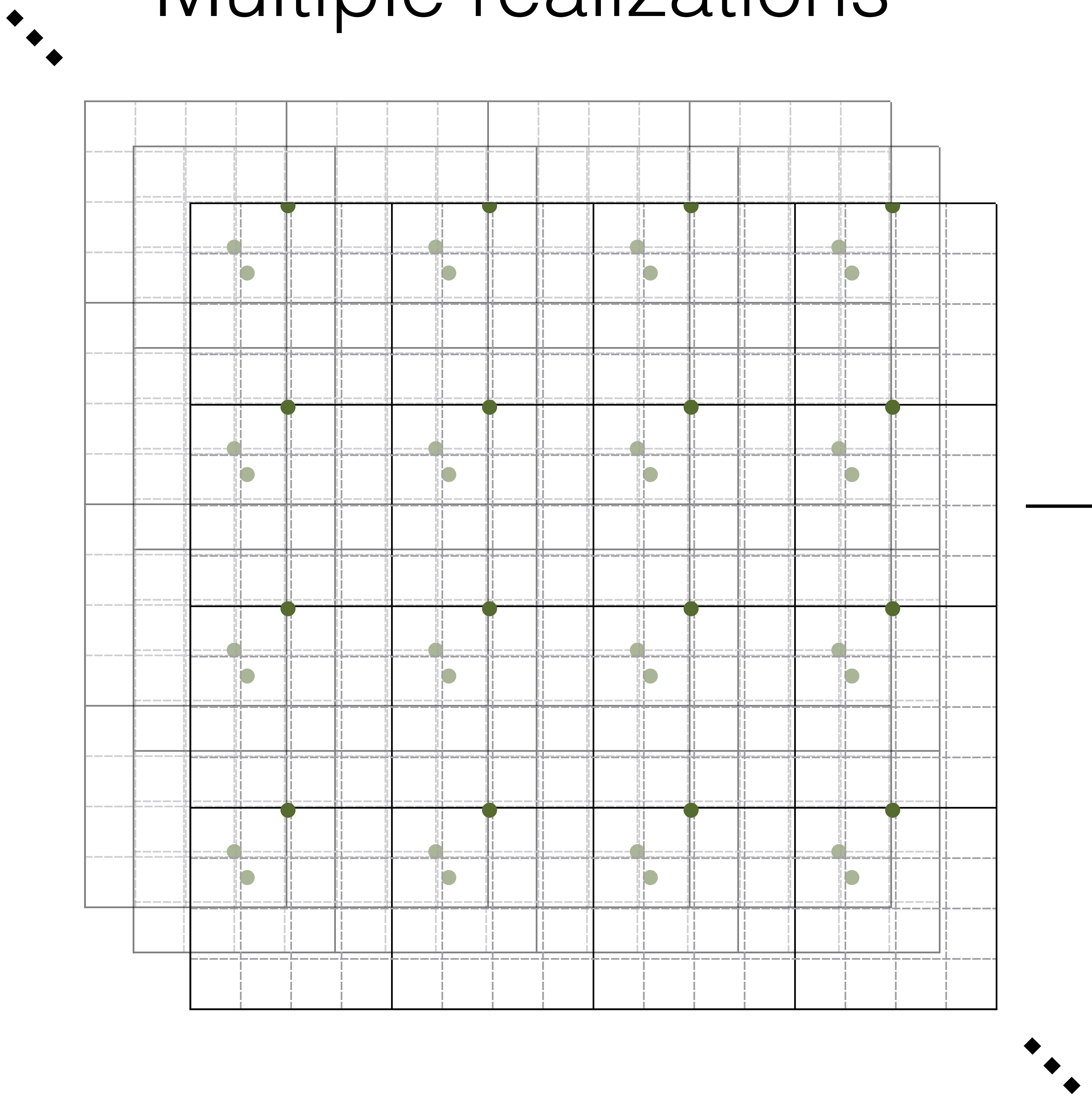

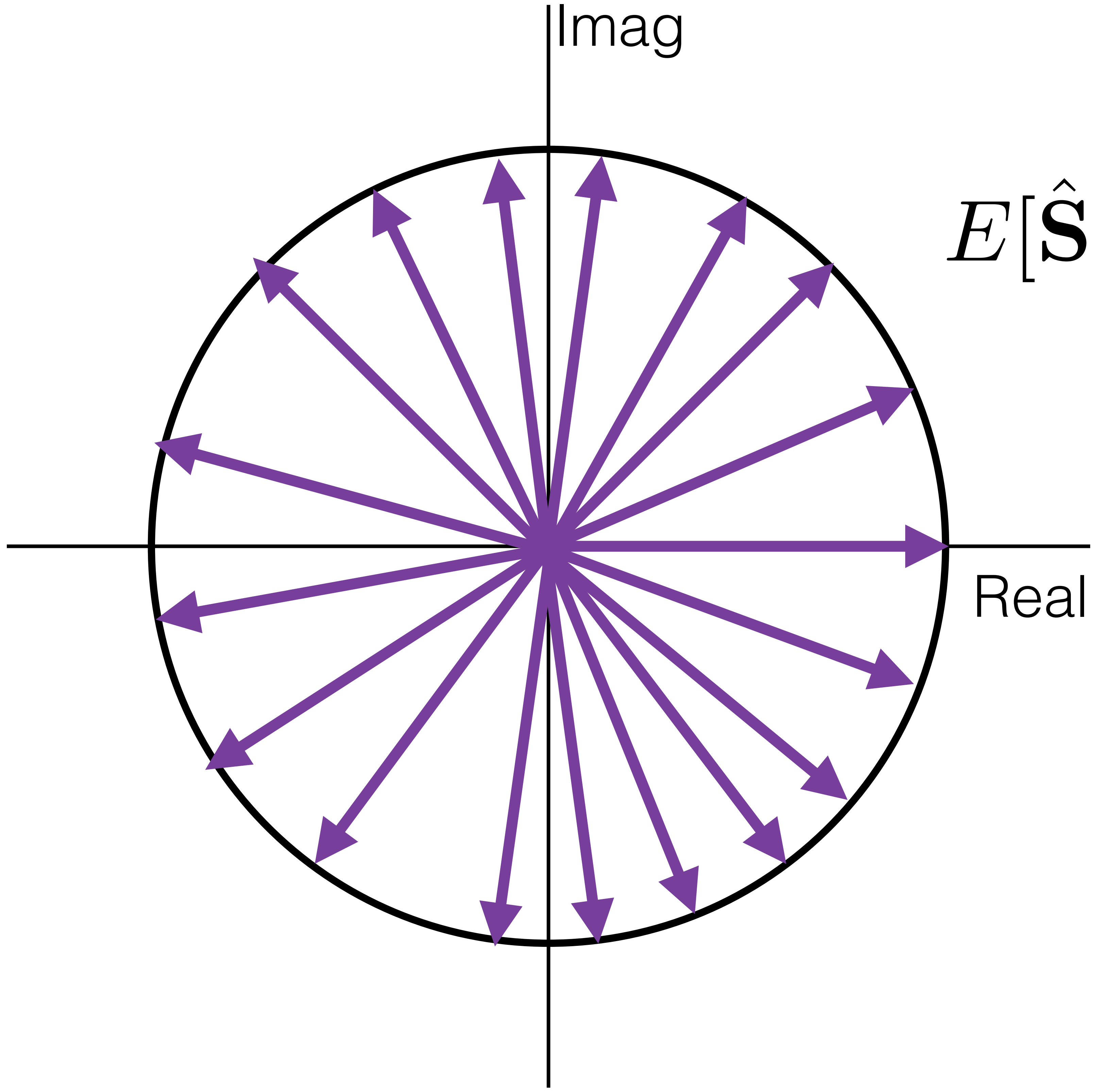

$$E[\hat{\mathbf{S}}(\omega)] = 0$$

Phase Distribution

- Uniform random shifting of samples change the phase of the samples' Fourier spectrum

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- Zero expected Fourier spectrum of a sampling pattern implies translation invariant sample distribution

Phase Distribution

- Uniform random shifting of samples change the phase of the samples' Fourier spectrum
- Uniform random shifting does not affect the amplitude
- Zero expected Fourier spectrum of a sampling pattern implies translation invariant sample distribution
- The process is named Homogenization

$$\text{Error} = \text{Bias}^2 + \text{Variance}$$

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Homogenization allows representation of error only in the variance form

Variance in the Fourier domain

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$$I - \tilde{\mu}_N = \hat{f}(0) - \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

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$$Var[\tilde{\mu}_N] = Var[\int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega]$$

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$$Var[\tilde{\mu}_N] = \int_\Omega |\hat{f}^*(\omega)|^2 Var[\hat{\mathbf{S}}(\omega)]d\omega$$

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This is a general form both for homogenised as well as non-homogenised sampling patterns

Variance in the Fourier domain

$$\boxed{Var[\tilde{\mu}_N] = \int_\Omega |\hat{f}^*(\omega)|^2 Var[\hat{\mathbf{S}}(\omega)] d\omega}$$

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For homogenised sampling pattern:

Variance in the Fourier domain

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For homogenised sampling pattern:

$$Var[\tilde{\mu}_N] = \int_\Omega |\hat{f}^*(\omega)|^2 E[|\hat{\mathbf{S}}(\omega)|^2] d\omega$$

Variance in the Fourier domain

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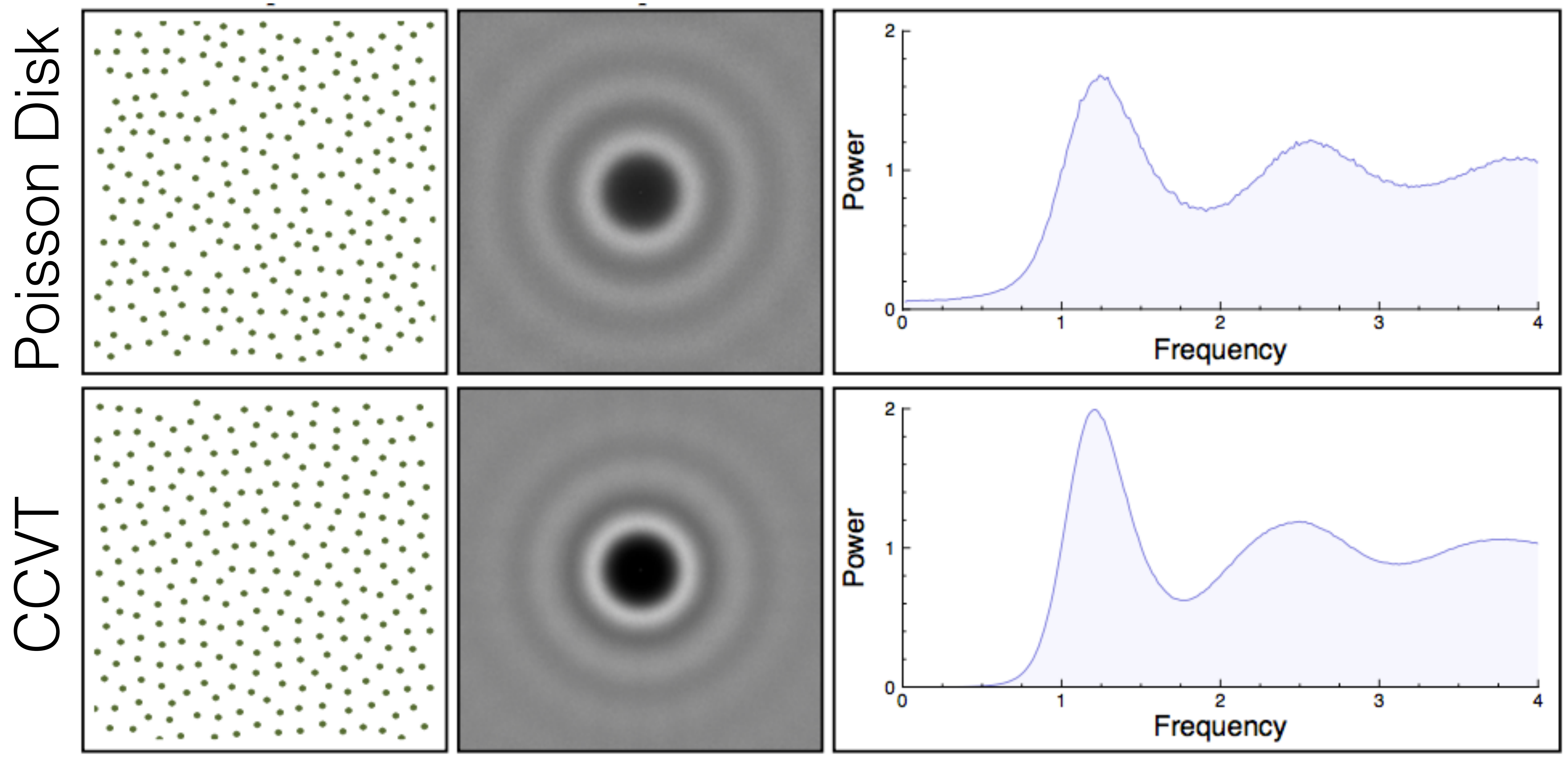

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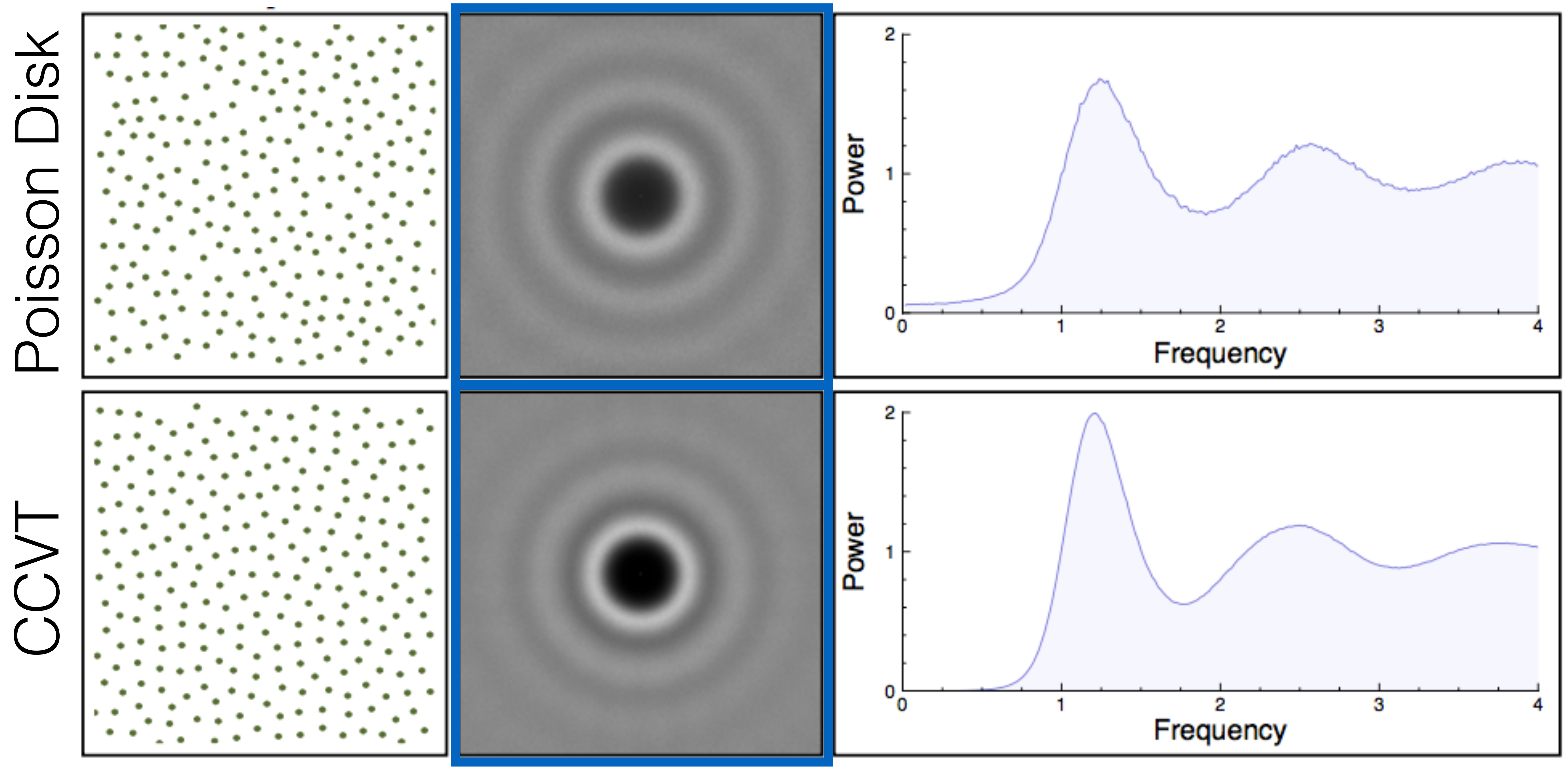

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$$Var[\tilde{\mu}_N] = \int_\Omega P_f(\omega) E[P_{\mathbf{S}}(\omega)] d\omega$$

Variance in the Fourier domain

$$Var[\tilde{\mu}_N] = \int_{\Omega} |\hat{f}^*(\omega)|^2 E[|\hat{\mathbf{S}}(\omega)|^2] d\omega$$

$$Var[\tilde{\mu}_N] = \int_{\Omega} P_f(\omega) E[P_{\mathbf{S}}(\omega)] d\omega$$

In polar coordinates:

Variance in the Fourier domain

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$$Var[\tilde{\mu}_N] = \int_{\Omega} P_f(\omega) E[P_{\mathbf{S}}(\omega)] d\omega$$

In polar coordinates:

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^{\infty} \int_{\mathcal{S}^{d-1}} P_f(\rho\mathbf{n}) E[P_{\mathbf{S}}(\rho\mathbf{n})] d\mathbf{n} d\rho$$

Variance in the Fourier domain

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \int_{\mathcal{S}^{d-1}} P_f(\rho \mathbf{n}) E[P_{\mathbf{S}}(\rho \mathbf{n})] d\mathbf{n} d\rho$$

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For isotropic power spectra:

Variance in the Fourier domain

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \int_{\mathcal{S}^{d-1}} P_f(\rho\mathbf{n}) E[P_{\mathbf{S}}(\rho\mathbf{n})] d\mathbf{n} d\rho$$

For isotropic power spectra:

$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) E[\tilde{P}_{\mathbf{S}}(\rho)] d\rho}$$

Variance in the Fourier domain

$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) E[\tilde{P}_{\mathbf{S}}(\rho)] d\rho}$$

Variance in the Fourier domain

$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^{\infty} \tilde{P}_f(\rho) E[\tilde{P}_{\mathbf{S}}(\rho)] d\rho}$$

Variance in the Fourier domain

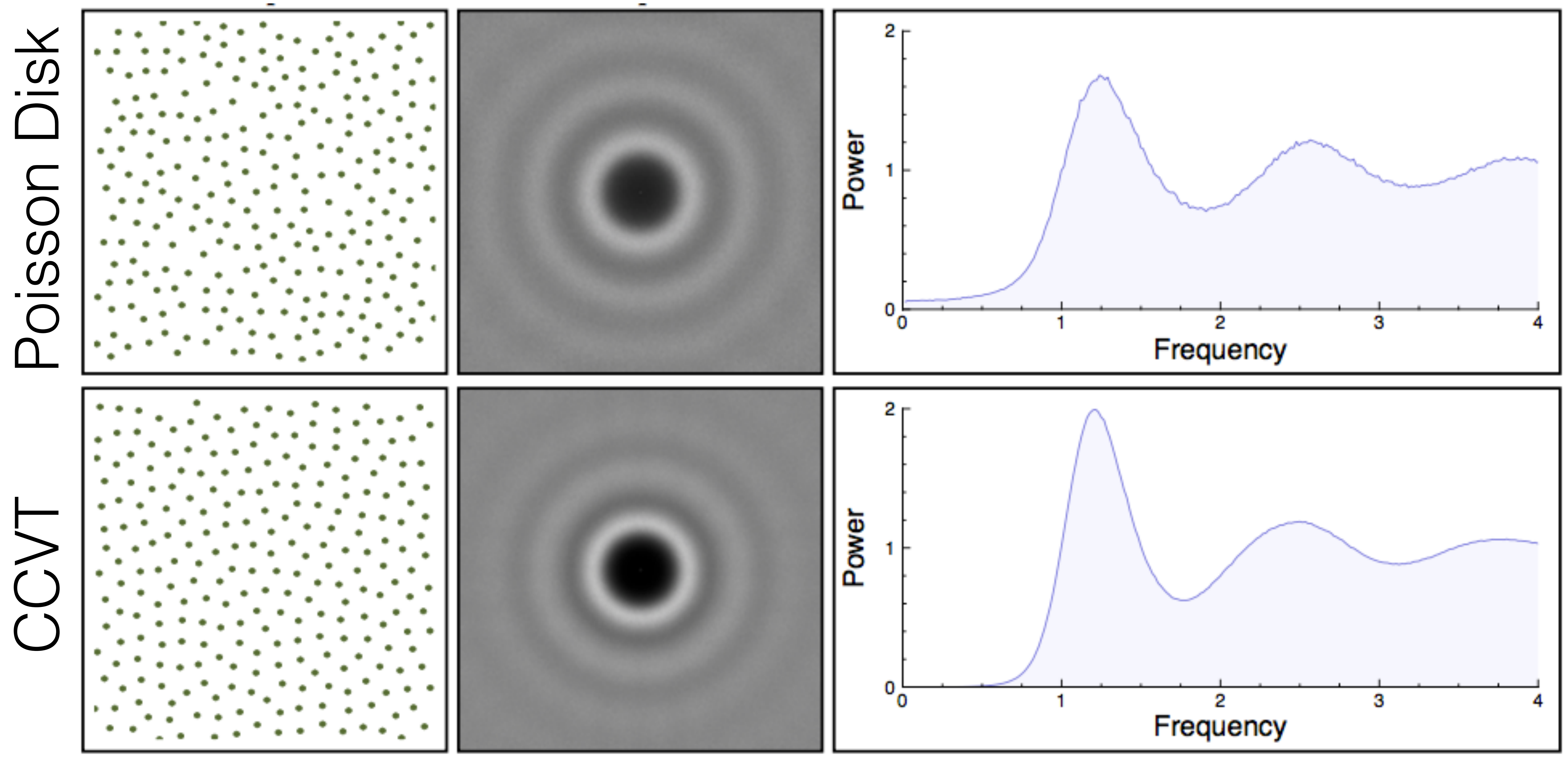
$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) E[\tilde{P}_{\mathbf{S}}(\rho)] d\rho}$$

Variance in the Fourier domain

$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) E[\tilde{P}_{\mathbf{S}}(\rho)] d\rho}$$

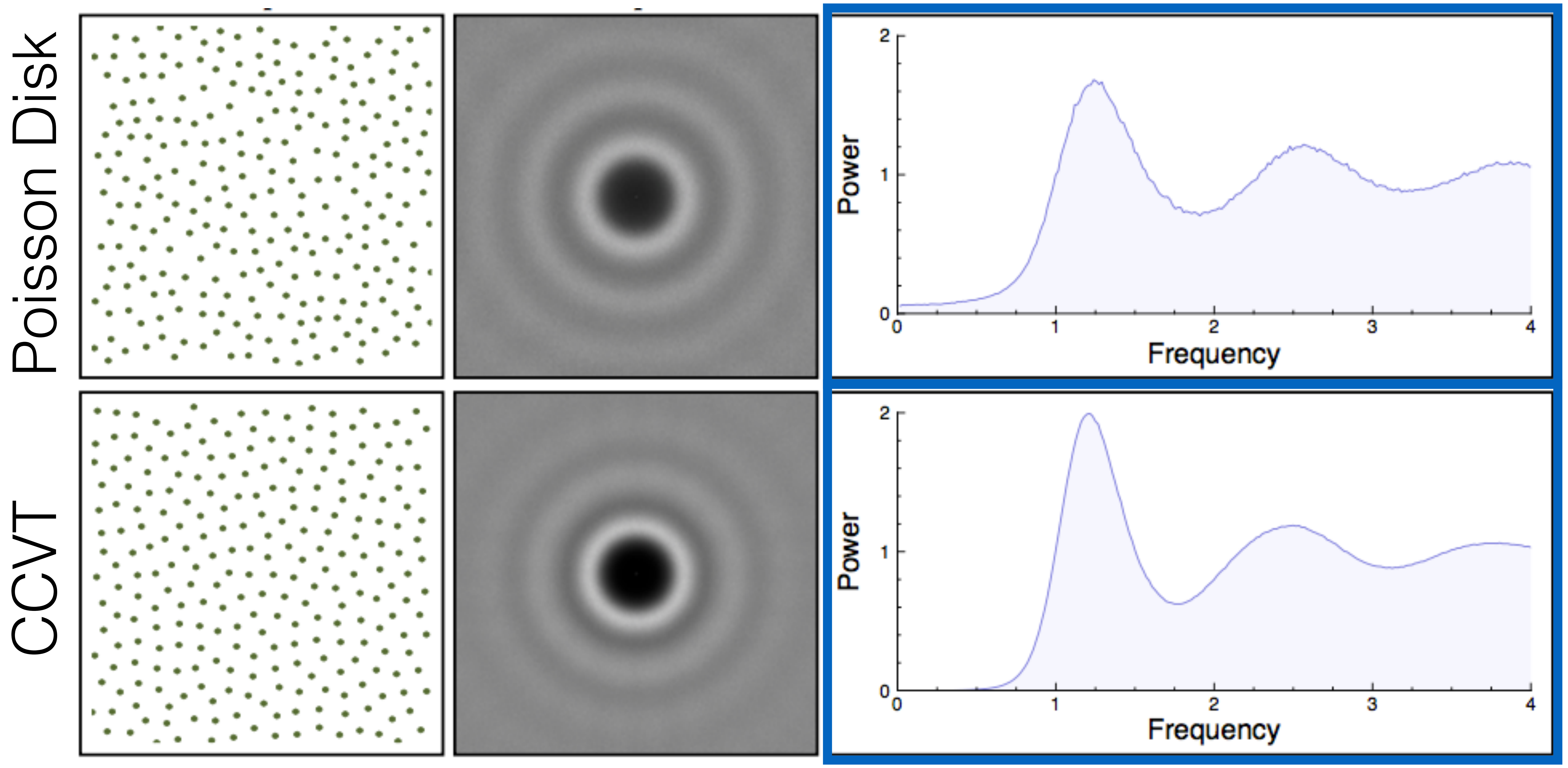
Variance in the Fourier domain

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \boxed{E[\tilde{P}_{\mathbf{S}}(\rho)]} d\rho$$



Variance in the Fourier domain

$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) E[\tilde{P}_{\mathbf{S}}(\rho)] d\rho}$$



Variance Convergence Analysis

Power Spectra Product

$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^{\infty} \tilde{P}_f(\rho) E[\tilde{P}_{\mathbf{S}}(\rho)] d\rho}$$

Power Spectra Product

$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) E[\tilde{P}_{\mathbf{S}}(\rho)] d\rho}$$

Power Spectra Product

$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) E[\tilde{P}_{\mathbf{S}}(\rho)] d\rho}$$

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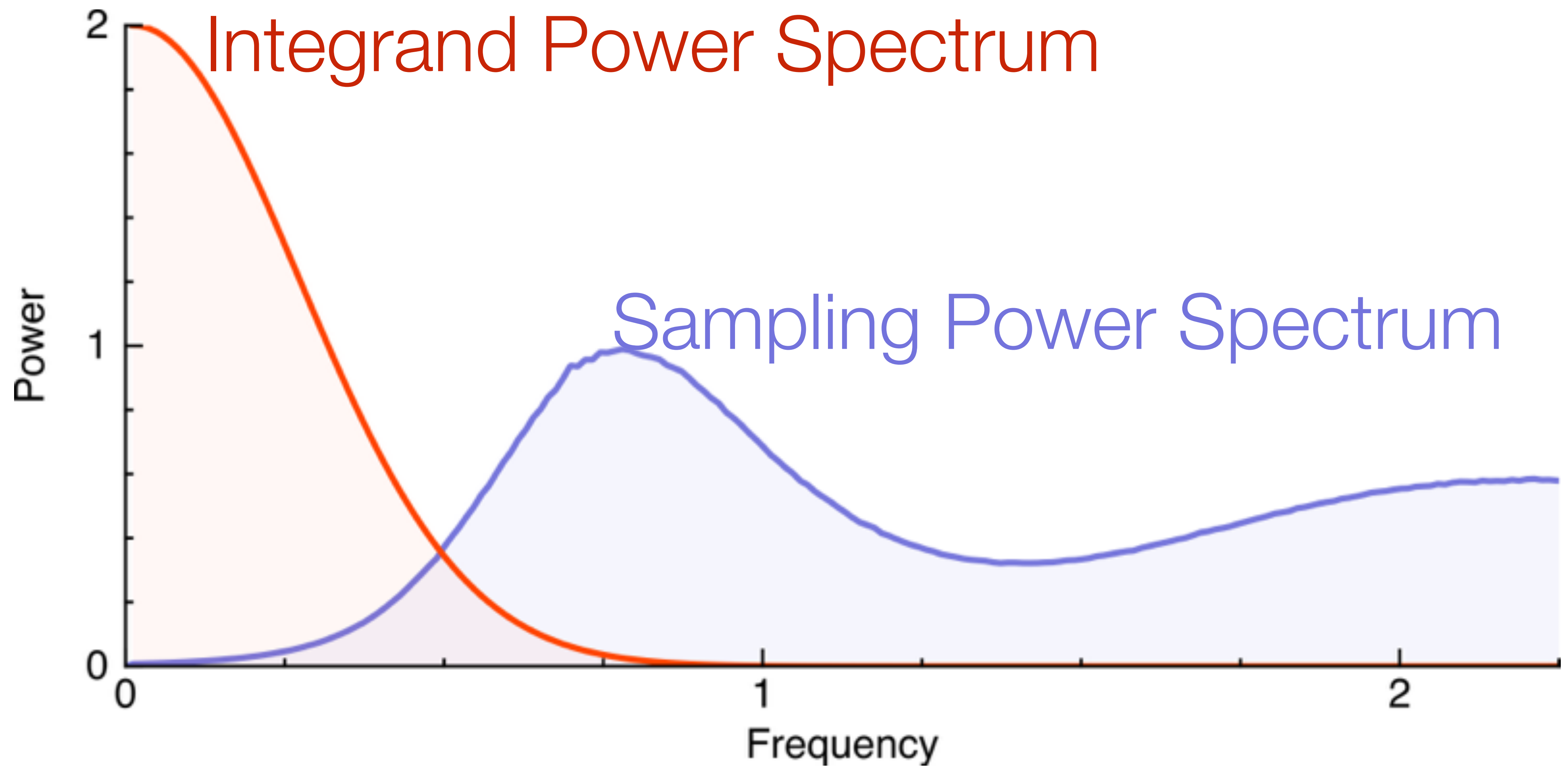
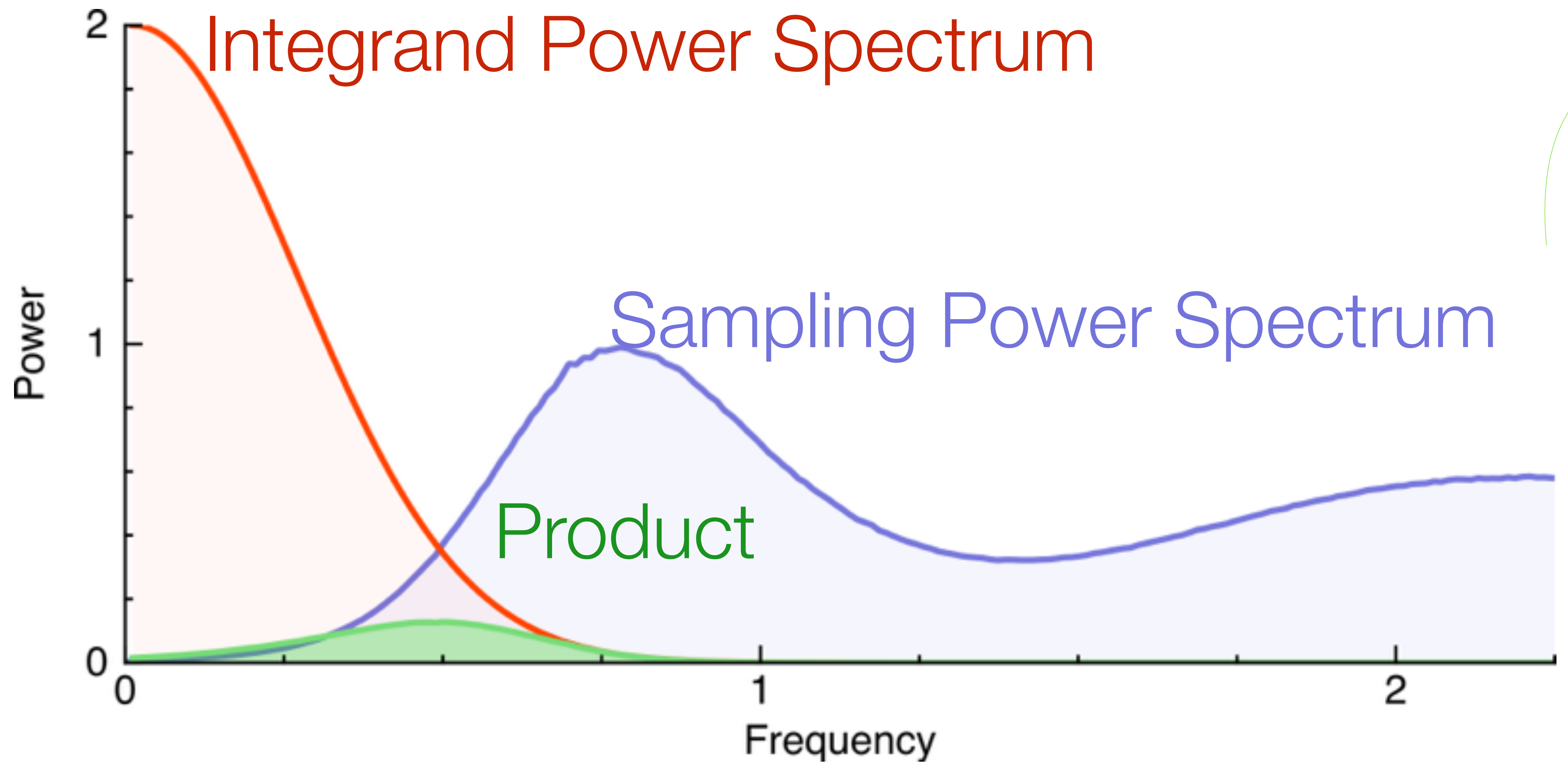


For given number of Samples

Power Spectra Product

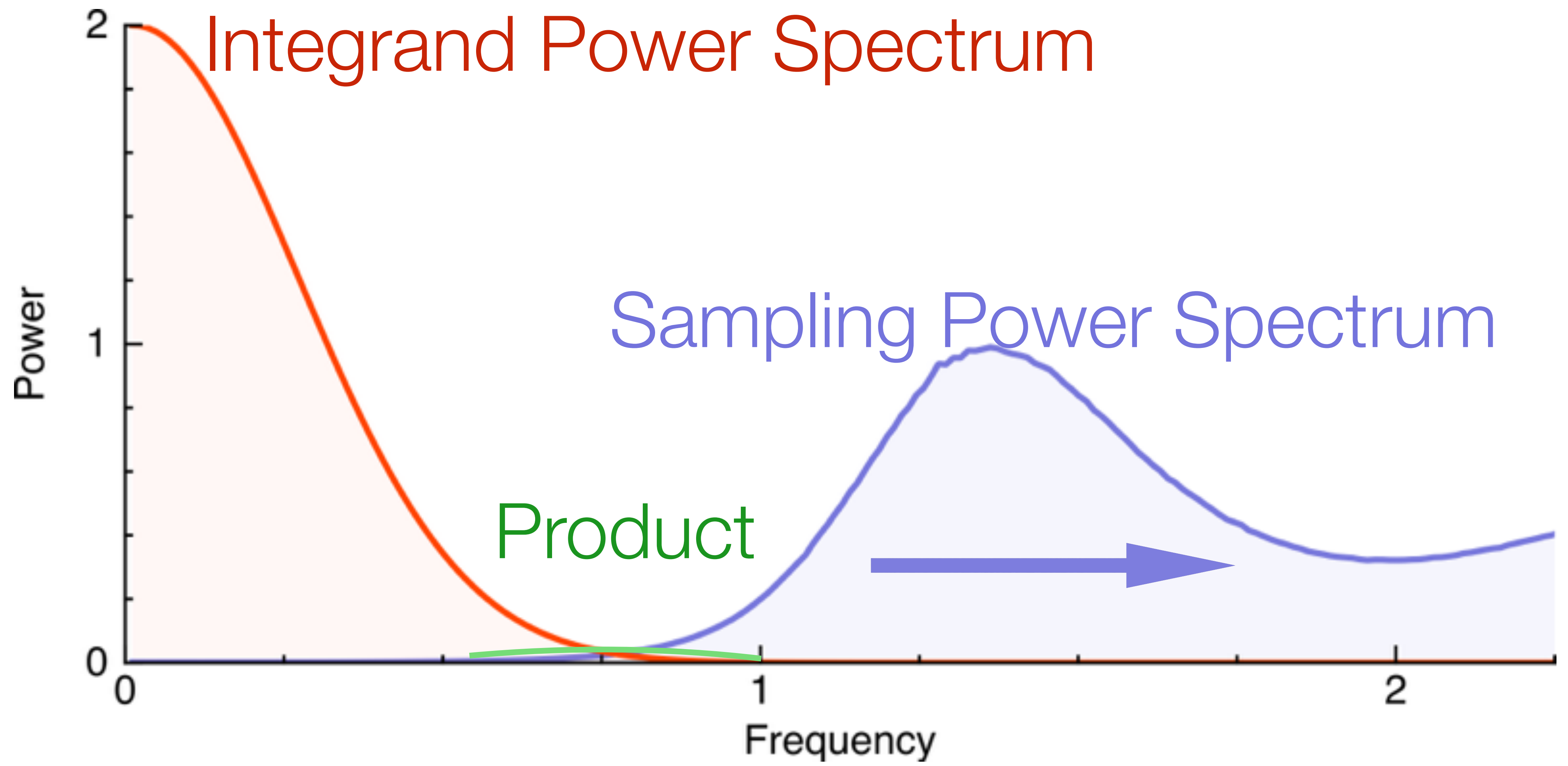
$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) E[\tilde{P}_{\mathbf{S}}(\rho)] d\rho}$$



For given number of Samples

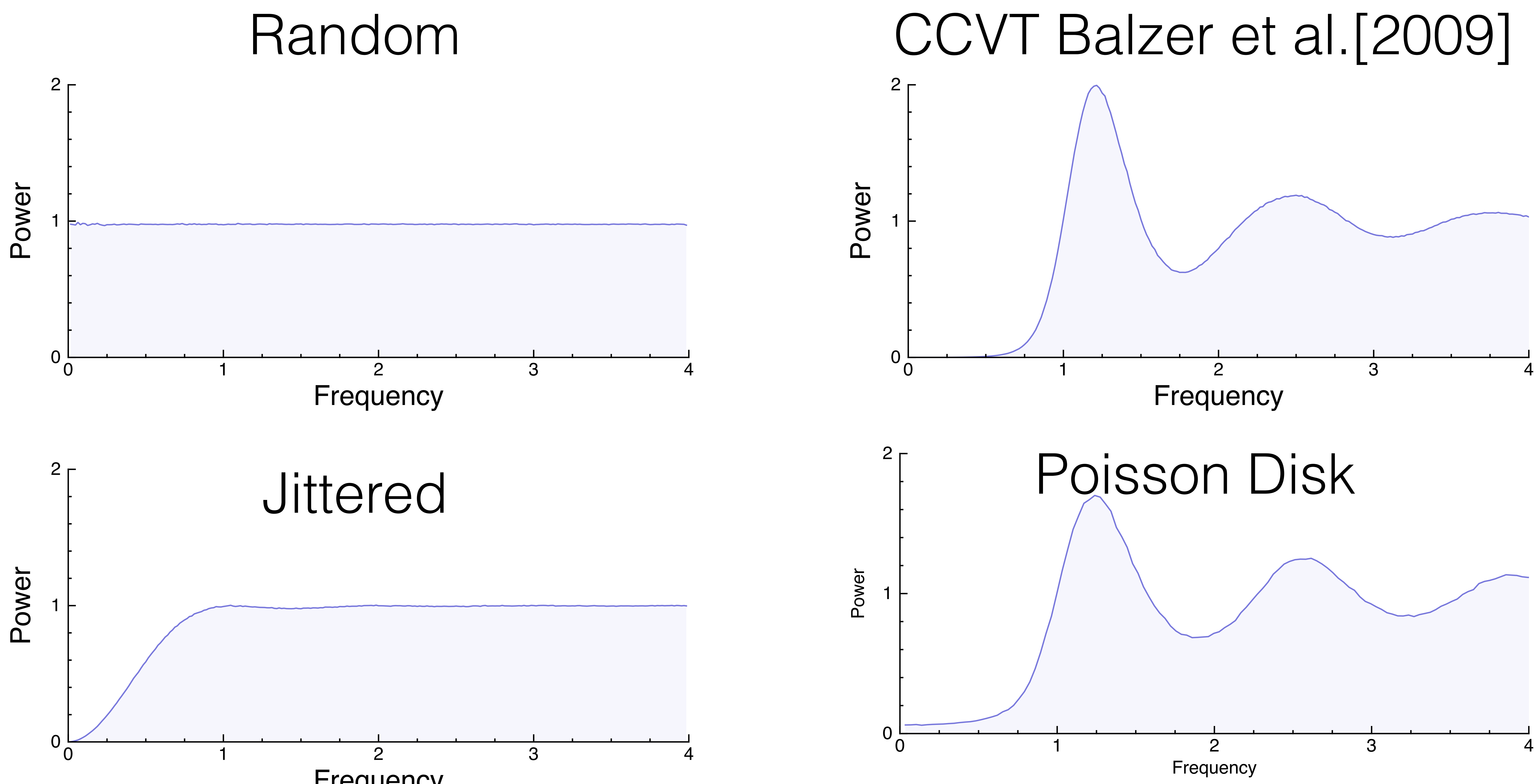
Power Spectra Product

$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) E[\tilde{P}_{\mathbf{S}}(\rho)] d\rho}$$



As we increase the number of samples

Radial Mean Power Spectra

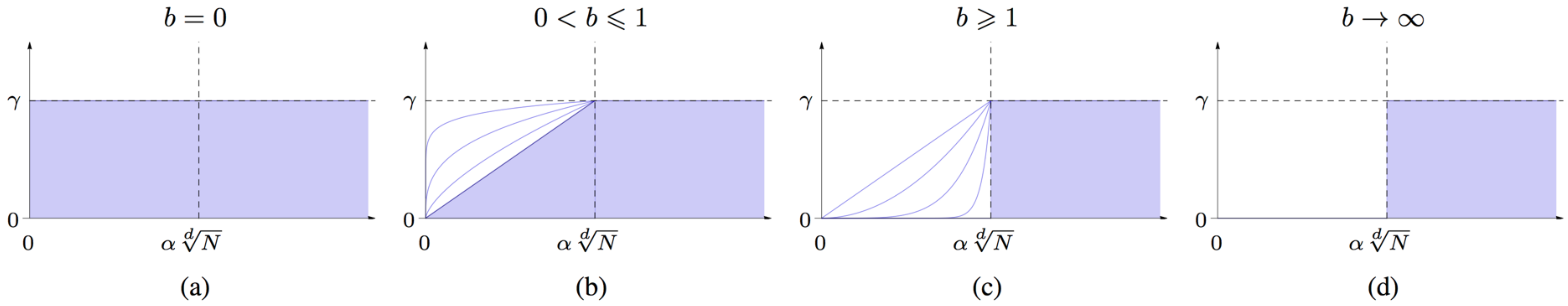


Samples' Radial Power Spectra Profiles

$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) E[\tilde{P}_{\mathbf{S}}(\rho)] d\rho}$$

Samples'
Radial Power Spectra Profiles

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) E[\tilde{P}_{\mathbf{S}}(\rho)] d\rho$$



Samples'
Radial Power Spectra Profiles

$$\tilde{P}_{\mathbf{s}}(\rho)_N = \begin{cases} \gamma \left(\frac{\rho}{\alpha \sqrt[d]{N}} \right)^b & \rho < \sqrt[d]{N} \\ \gamma & \text{otherwise} \end{cases}$$

Samples' Radial Power Spectra Profiles

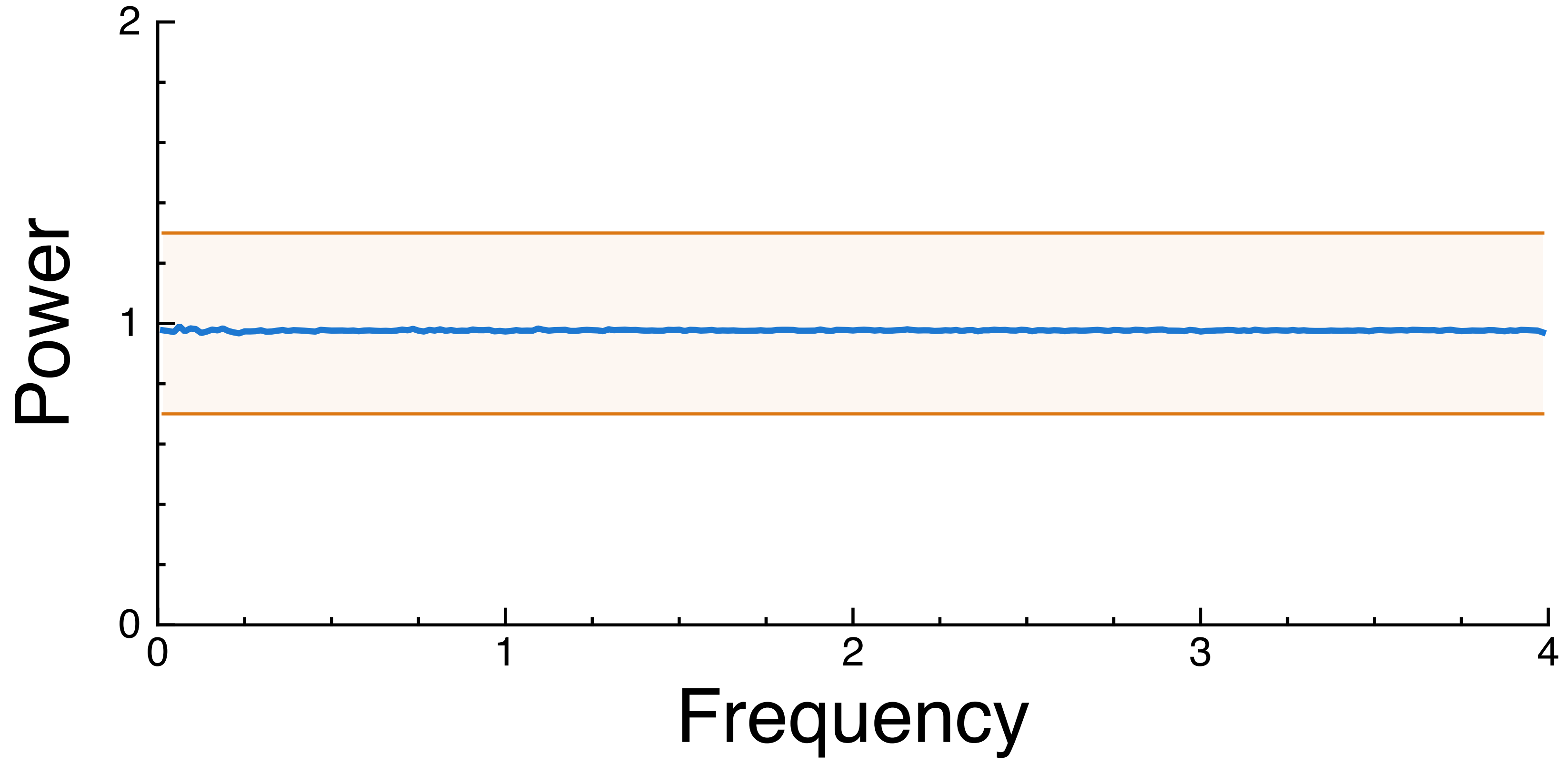
$$\tilde{P}_{\mathbf{S}}(\rho)_N = \begin{cases} \gamma \left(\frac{\rho}{\alpha \sqrt[d]{N}} \right)^b & \rho < \sqrt[d]{N} \\ \gamma & \text{otherwise} \end{cases}$$

where,

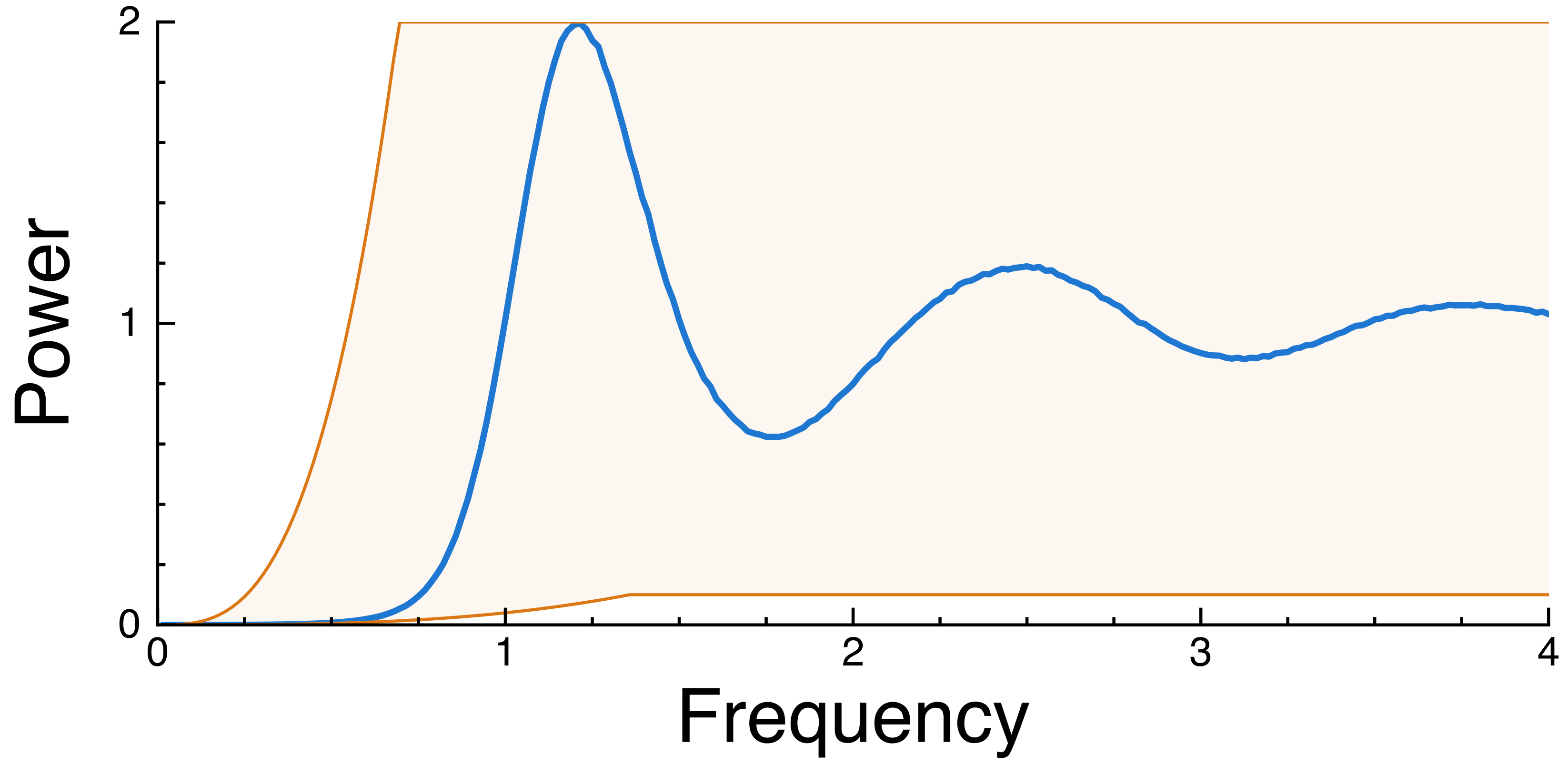
$\tilde{P}_{\mathbf{S}}(\rho)_N = N \times \tilde{P}_{\mathbf{S}}(\rho)$	Normalized radial power spectra
$\gamma, \alpha \in \mathcal{R}^+/0$	Positive non-zero constant
d	Dimensions
N	Number of Samples

Bounds for Radial Mean Power Spectra

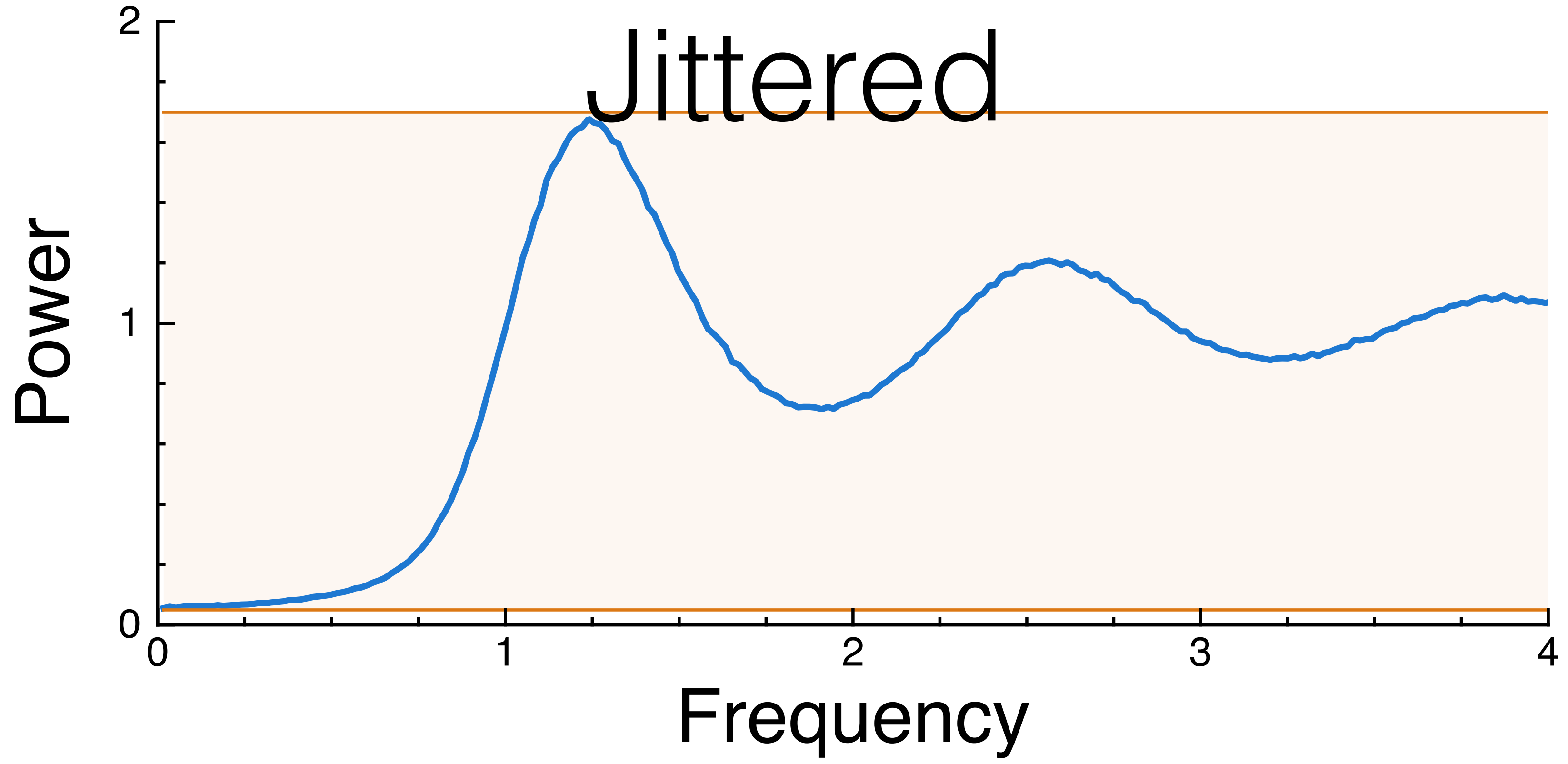
Random



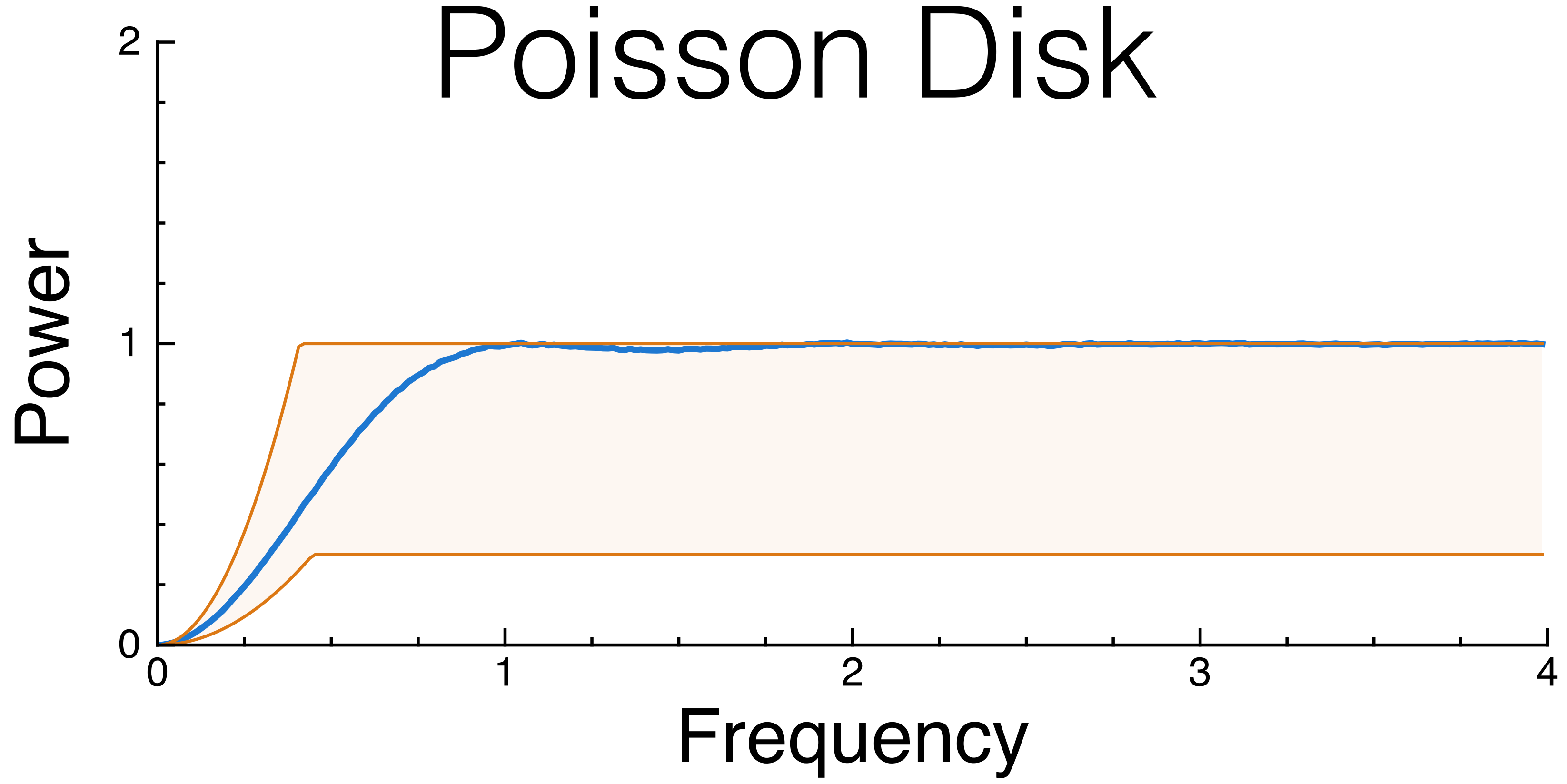
CCVT Balzer et al.[2009]



Jittered



Poisson Disk



Analytical Classification of Power Spectra

$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) E[\tilde{P}_{\mathbf{S}}(\rho)] d\rho}$$

Analytical Classification of Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) E[\tilde{P}_{\mathbf{S}}(\rho)] d\rho$$

Analytical Classification of Power Spectra

$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) E[\tilde{P}_{\mathbf{S}}(\rho)] d\rho}$$

Constant profiles

Quadratic profiles

Other profiles

Analytical Classification of Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) E[\tilde{P}_{\mathbf{S}}(\rho)] d\rho$$

Constant profiles

Quadratic profiles

Other profiles

Analytical Classification of Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) E[\tilde{P}_{\mathbf{S}}(\rho)] d\rho$$

Best possible Integrand Power spectrum

⋮

Worst possible Integrand Power spectrum

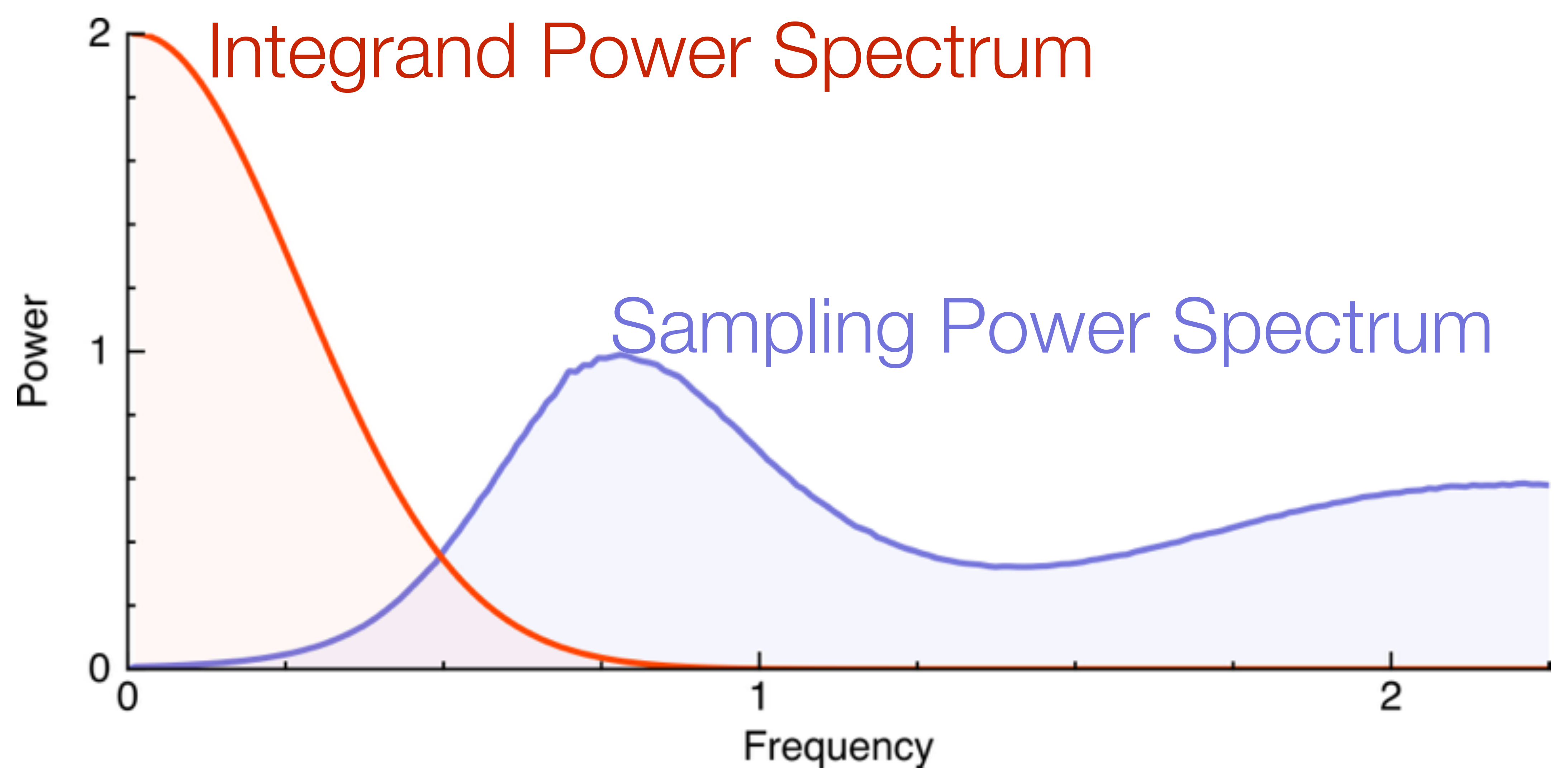
Constant profiles

Quadratic profiles

Other profiles

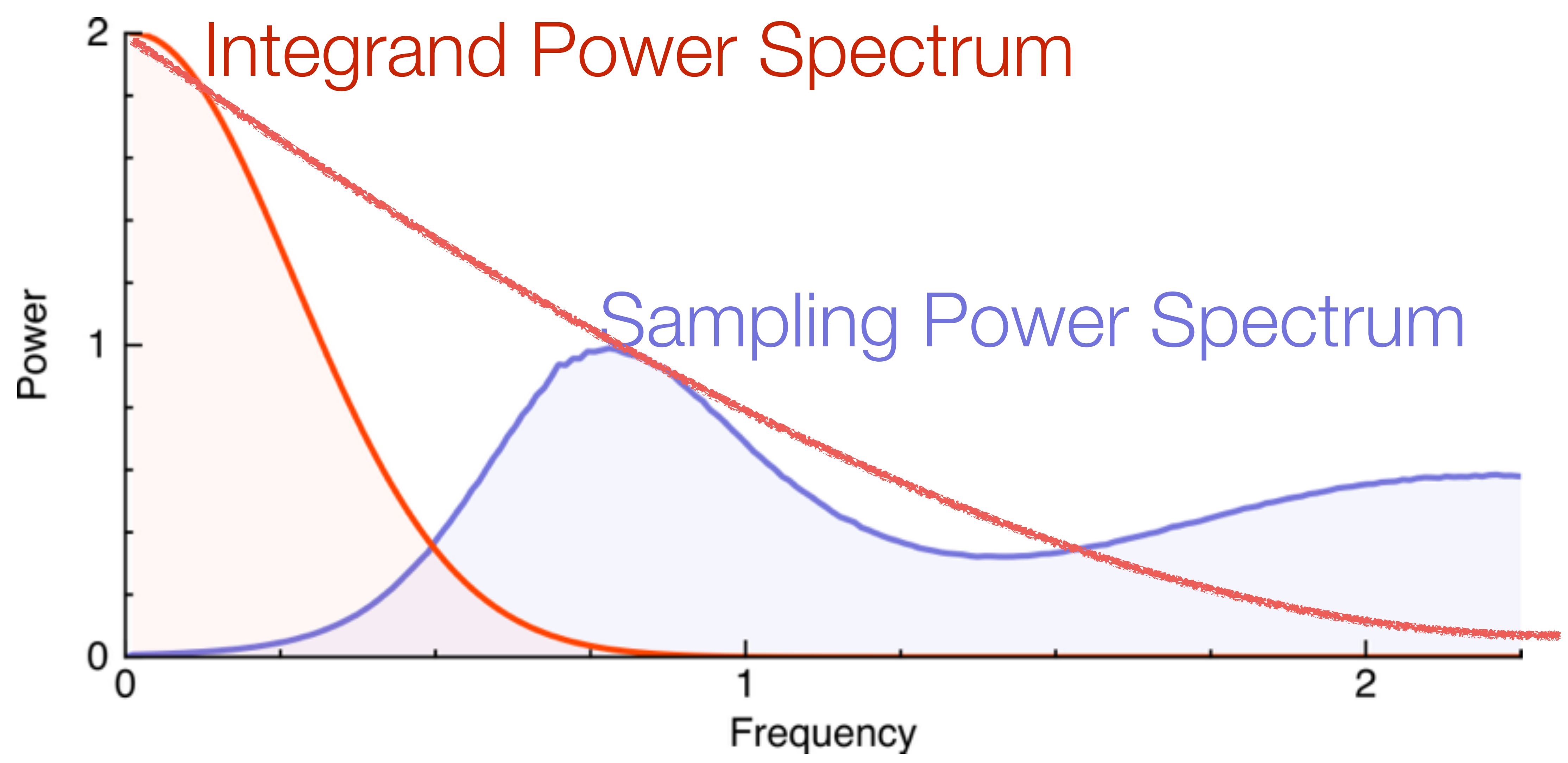
Best and Worst Cases

$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^{\infty} \tilde{P}_f(\rho) E[\tilde{P}_{\mathbf{S}}(\rho)] d\rho}$$



Best and Worst Cases

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) E[\tilde{P}_{\mathbf{S}}(\rho)] d\rho$$



Best and Worst Cases

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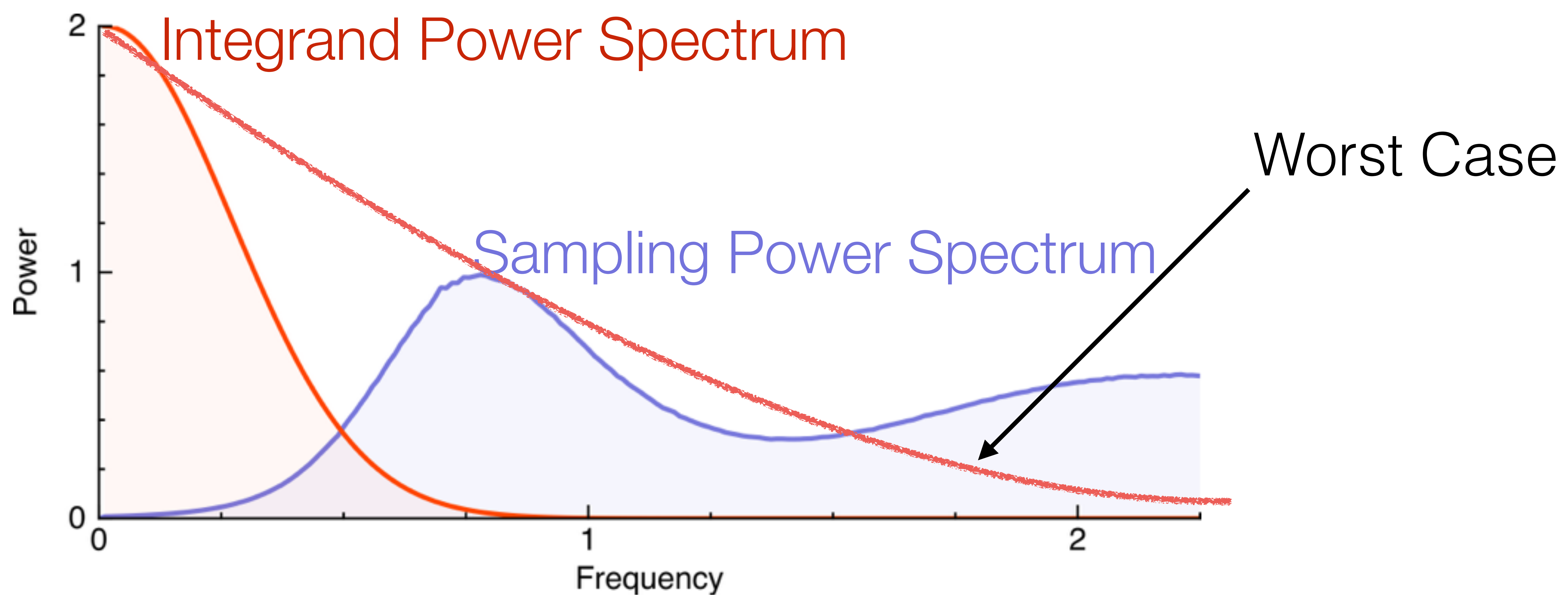
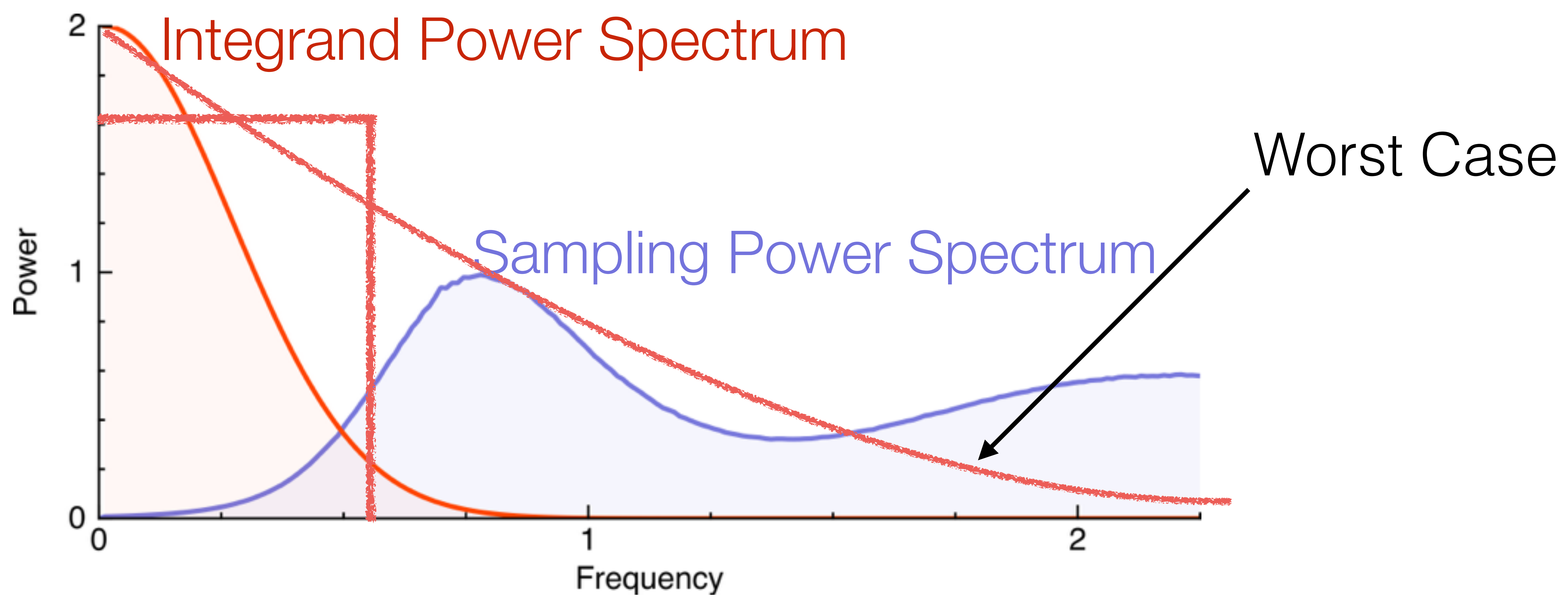


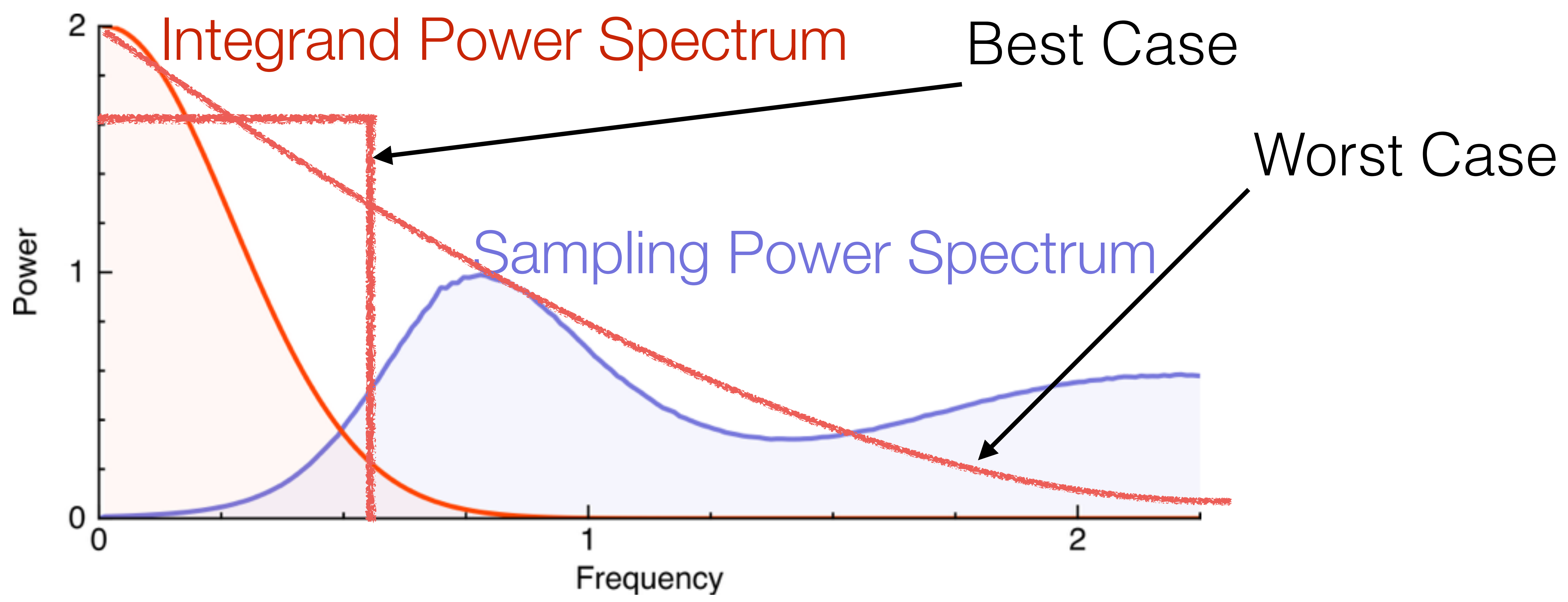
Best and Worst Cases

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) E[\tilde{P}_{\mathbf{S}}(\rho)] d\rho$$



Best and Worst Cases

$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) E[\tilde{P}_{\mathbf{S}}(\rho)] d\rho}$$



Best and Worst Cases

Worst Case:

$$\tilde{P}_f(\rho) = \begin{cases} c_f & \rho < \rho_0 \\ \rho^{-d-1} & \text{otherwise} \end{cases}$$

Brandolini et al. [2001]

Best Case:

$$\tilde{P}_f(\rho) = \begin{cases} c_f & \rho < \rho_0 \\ 0 & \text{otherwise} \end{cases}$$

Convergence Tool

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) E[\tilde{P}_{\mathbf{S}}(\rho)] d\rho$$

$$\tilde{P}_f(\rho) = \begin{cases} c_f & \rho < \rho_0 \\ \rho^{-d-1} & \text{otherwise} \end{cases}$$

$$\tilde{P}_f(\rho) = \begin{cases} c_f & \rho < \rho_0 \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{P}_{\mathbf{S}}(\rho)_N = \begin{cases} \gamma \left(\frac{\rho}{\alpha \sqrt[d]{N}} \right)^b & \rho < \sqrt[d]{N} \\ \gamma & \text{otherwise} \end{cases}$$

Convergence rates

(a) Best-case: $O\left(\frac{1}{N}\right)$ Worst-case: $O\left(\frac{1}{N}\right)$

(b) Best-case: $O\left(\frac{1}{N\sqrt[d]{N}^b}\right)$ Worst-case: $O\left(\frac{1}{N\sqrt[d]{N}^b}\right)$

(c) Best-case: $O\left(\frac{1}{N\sqrt[d]{N}^b}\right)$ Worst-case: $O\left(\frac{1}{N\sqrt[d]{N}}\right)$

(d) Best-case: 0 Worst-case: $O\left(\frac{1}{N\sqrt[d]{N}}\right)$