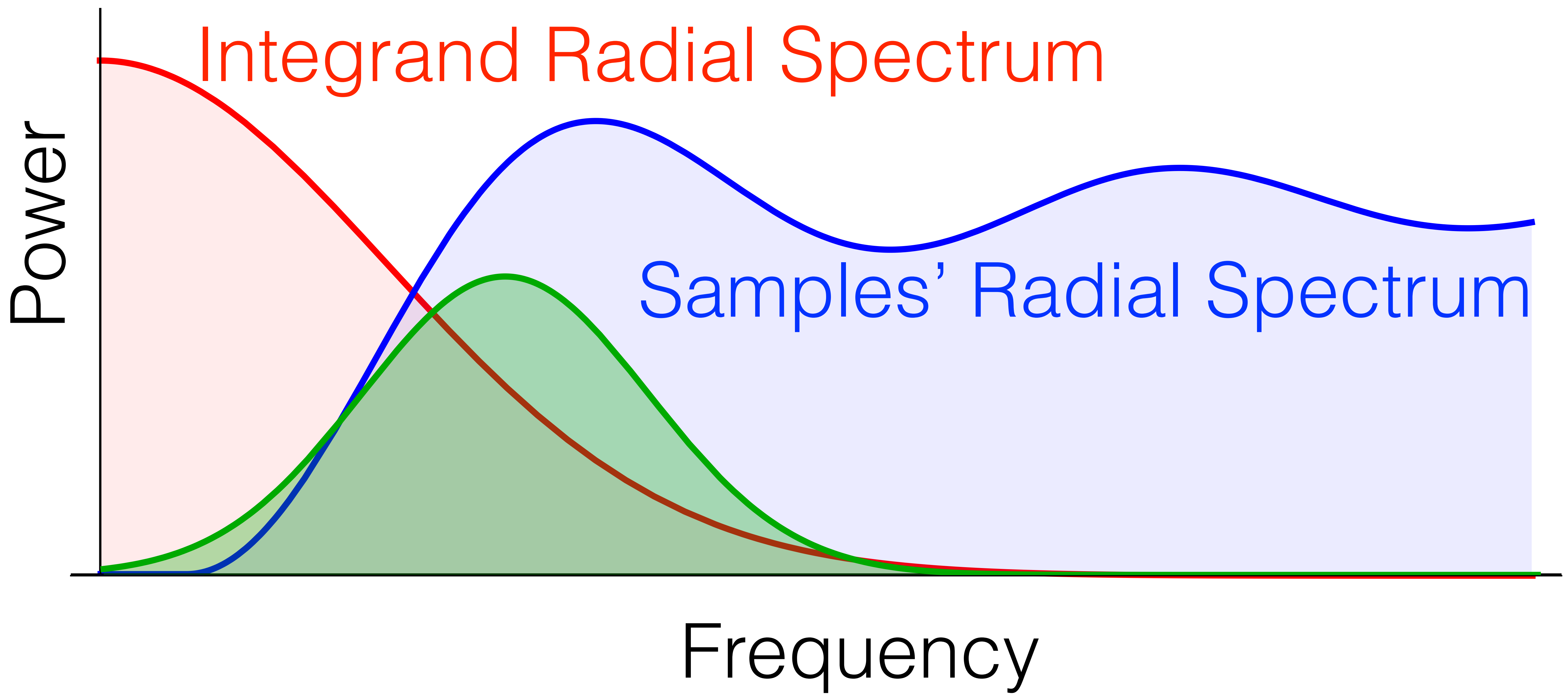
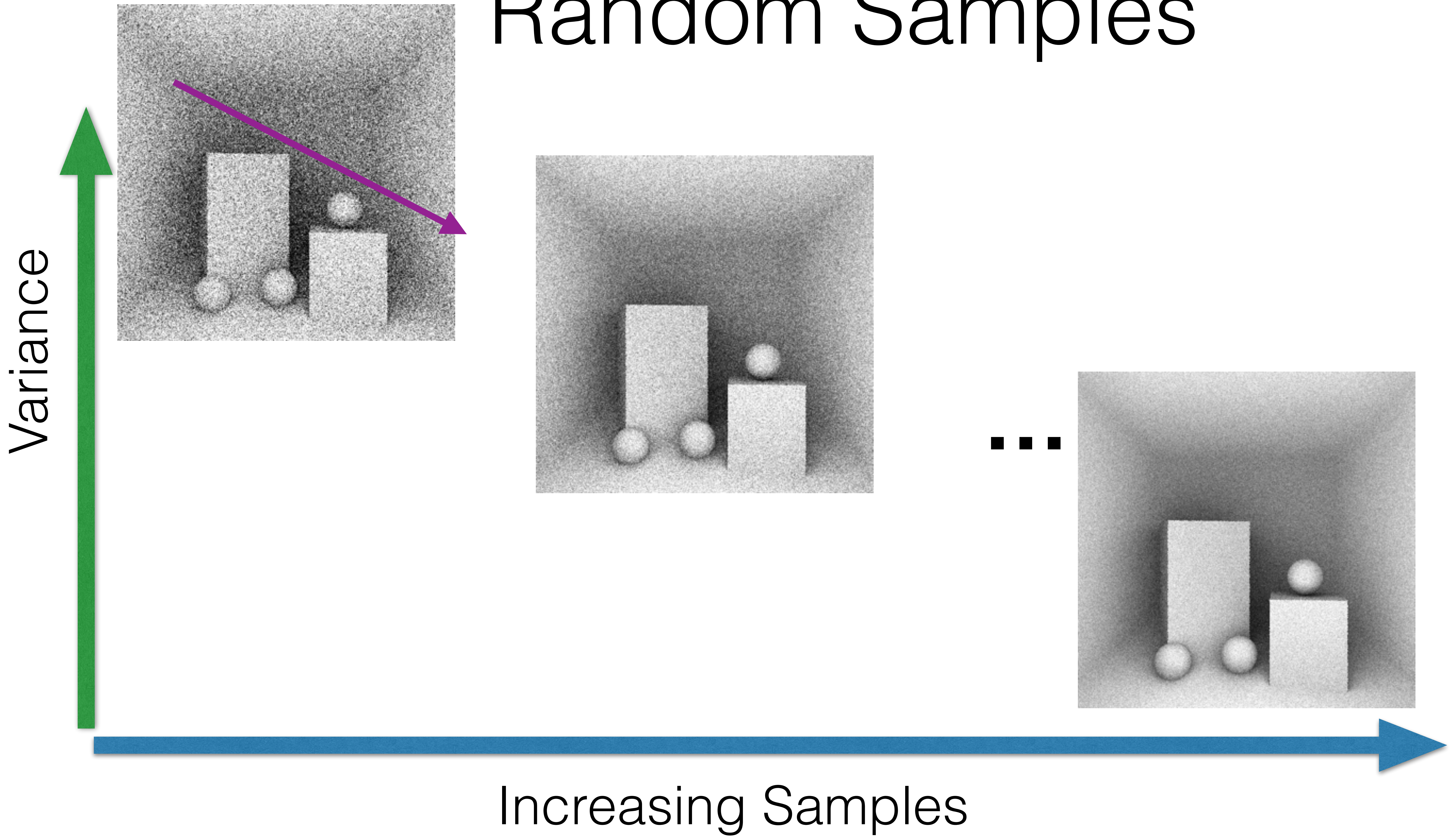
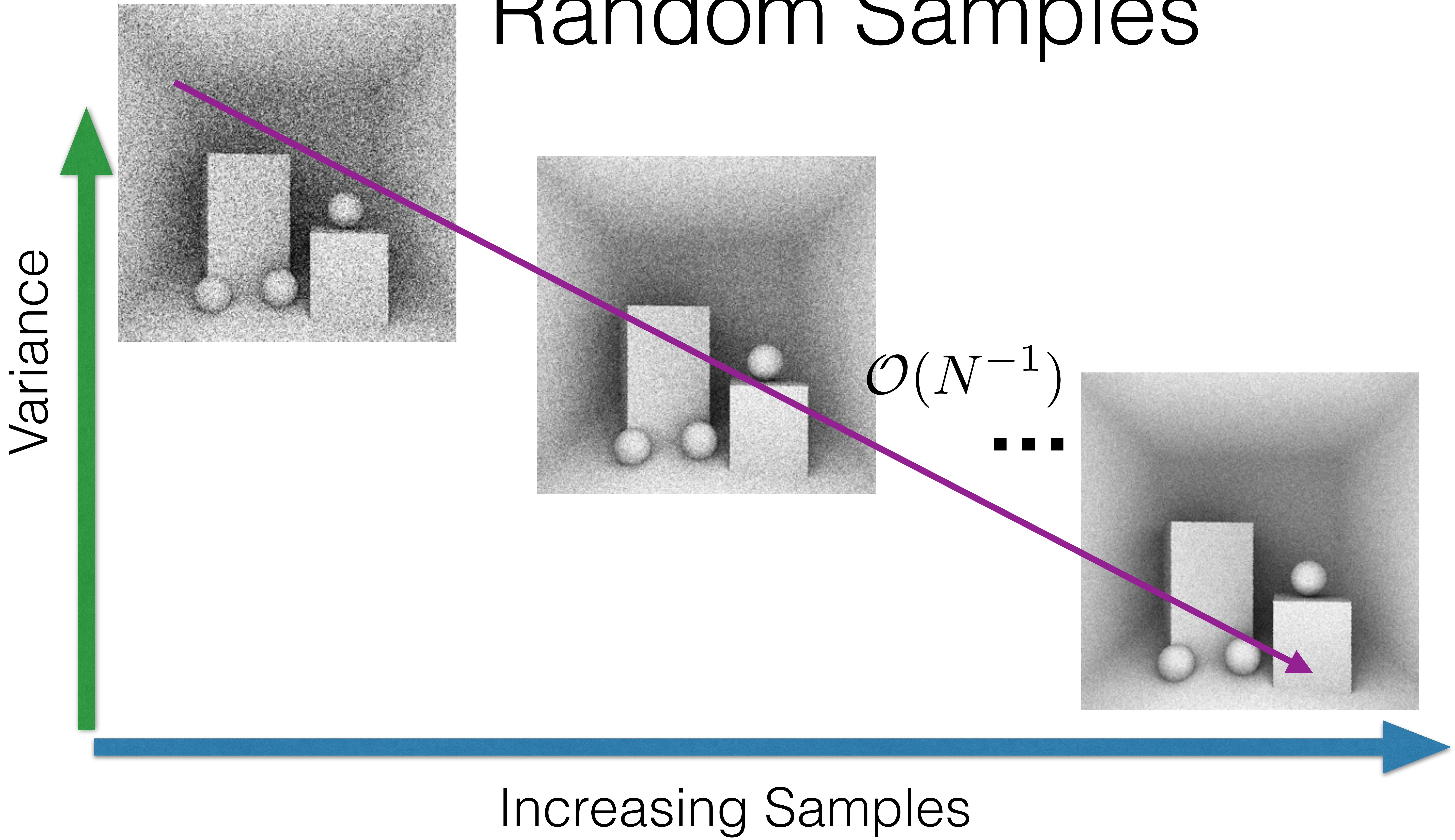
# Part 3: Formal Treatment of MSE, Bias and Variance



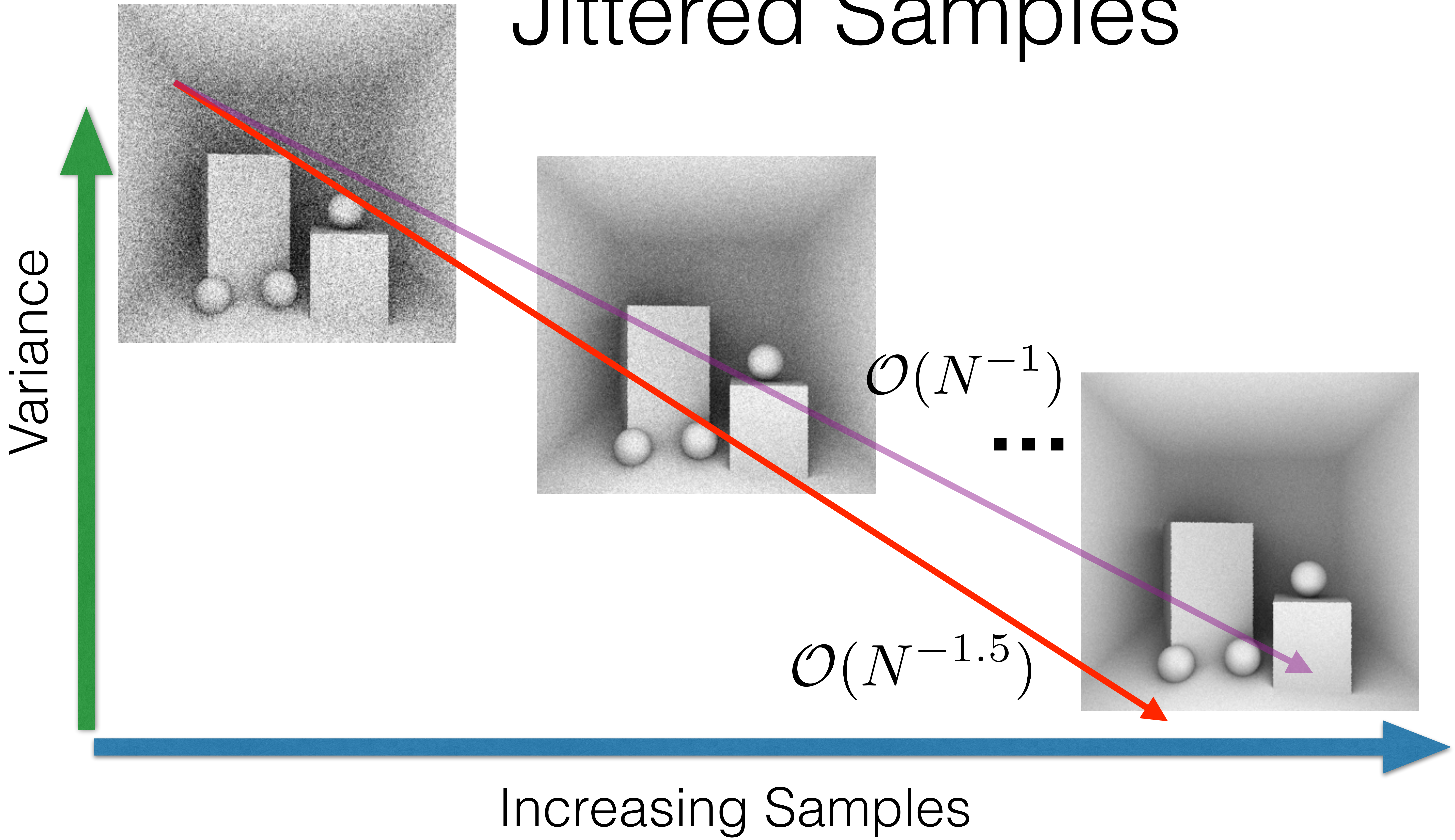
# Convergence rate for Random Samples



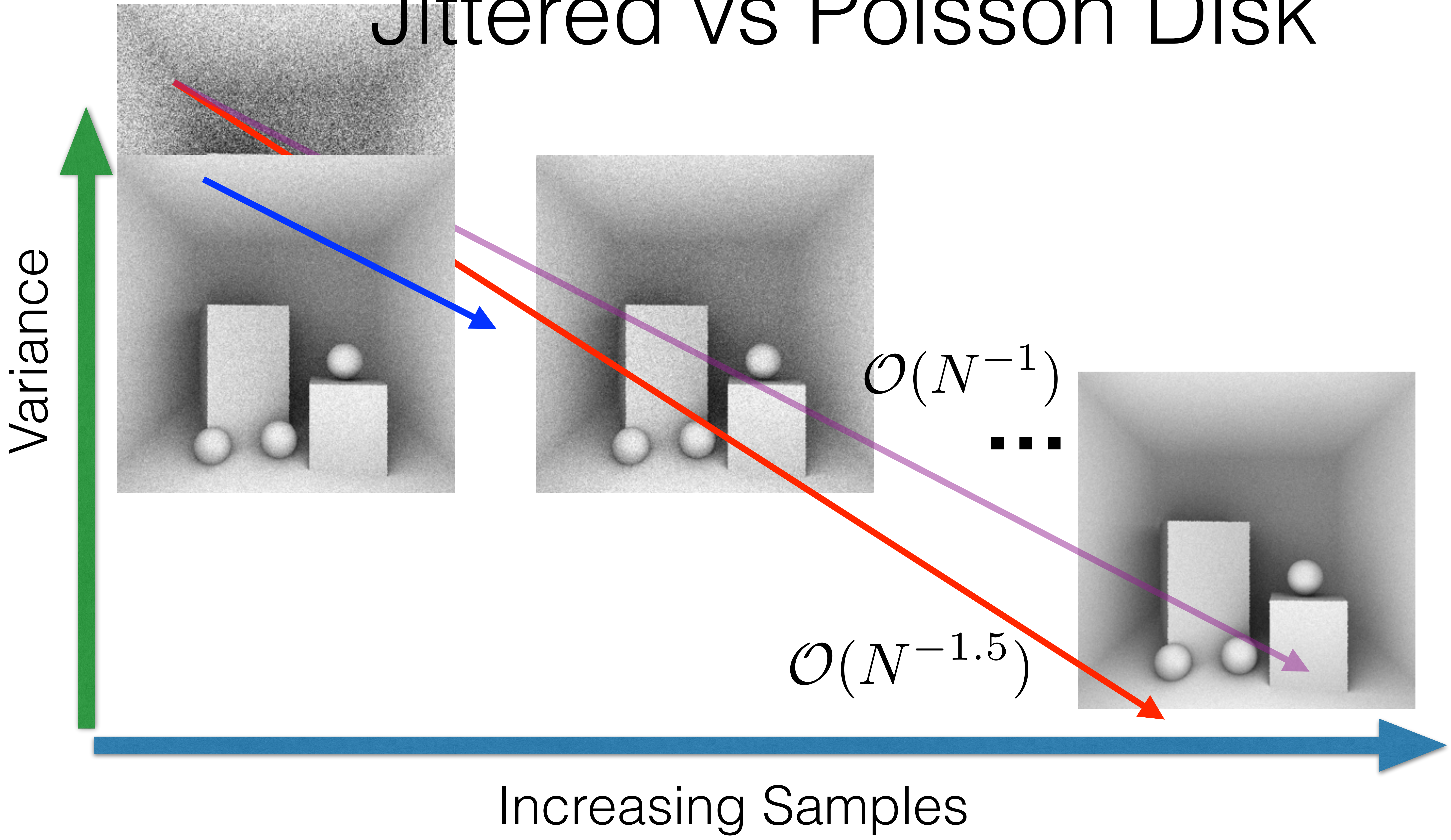
# Convergence rate for Random Samples



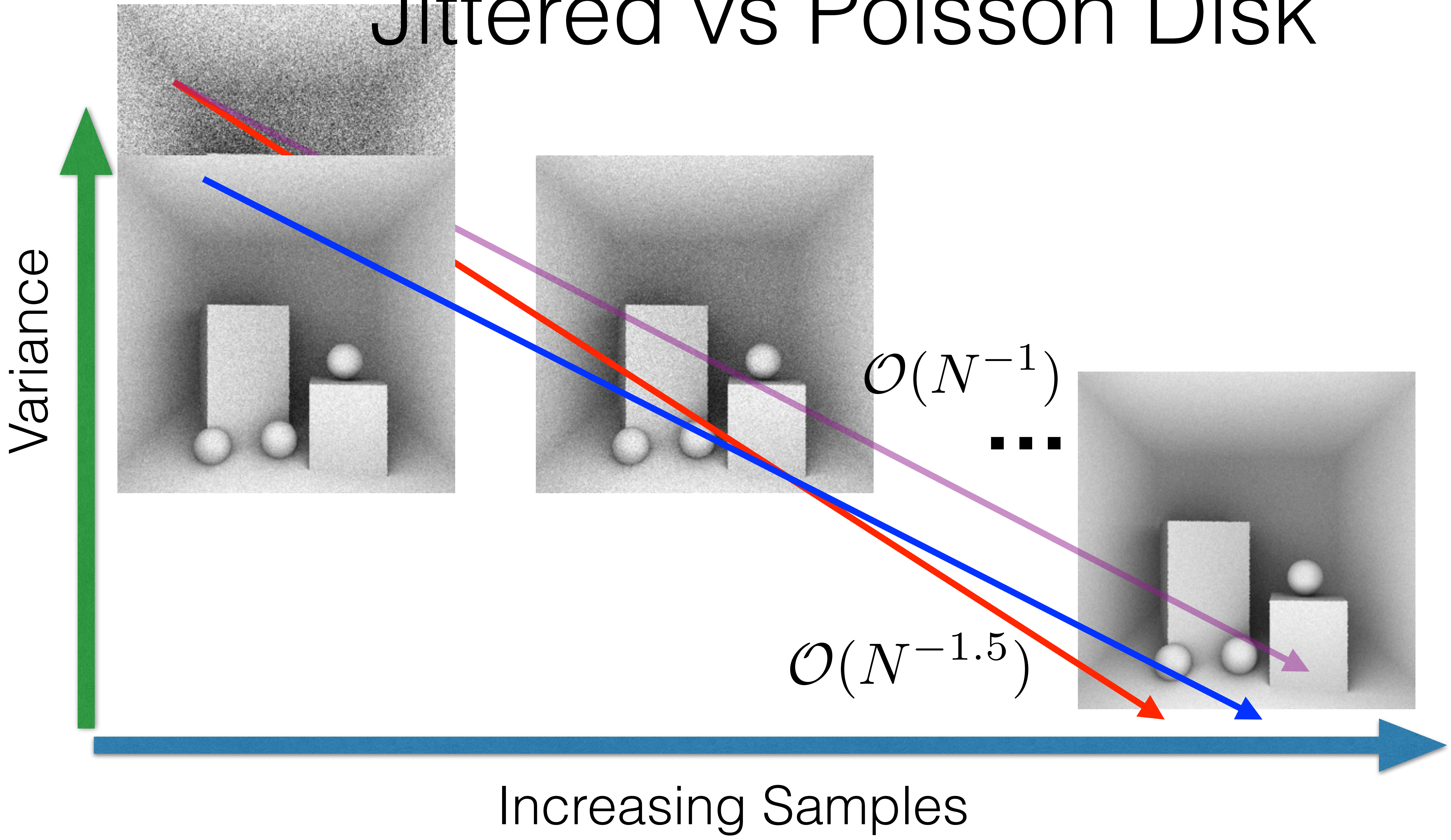
# Convergence rate for Jittered Samples



# Convergence rate
# Jittered vs Poisson Disk



# Convergence rate
# Jittered vs Poisson Disk



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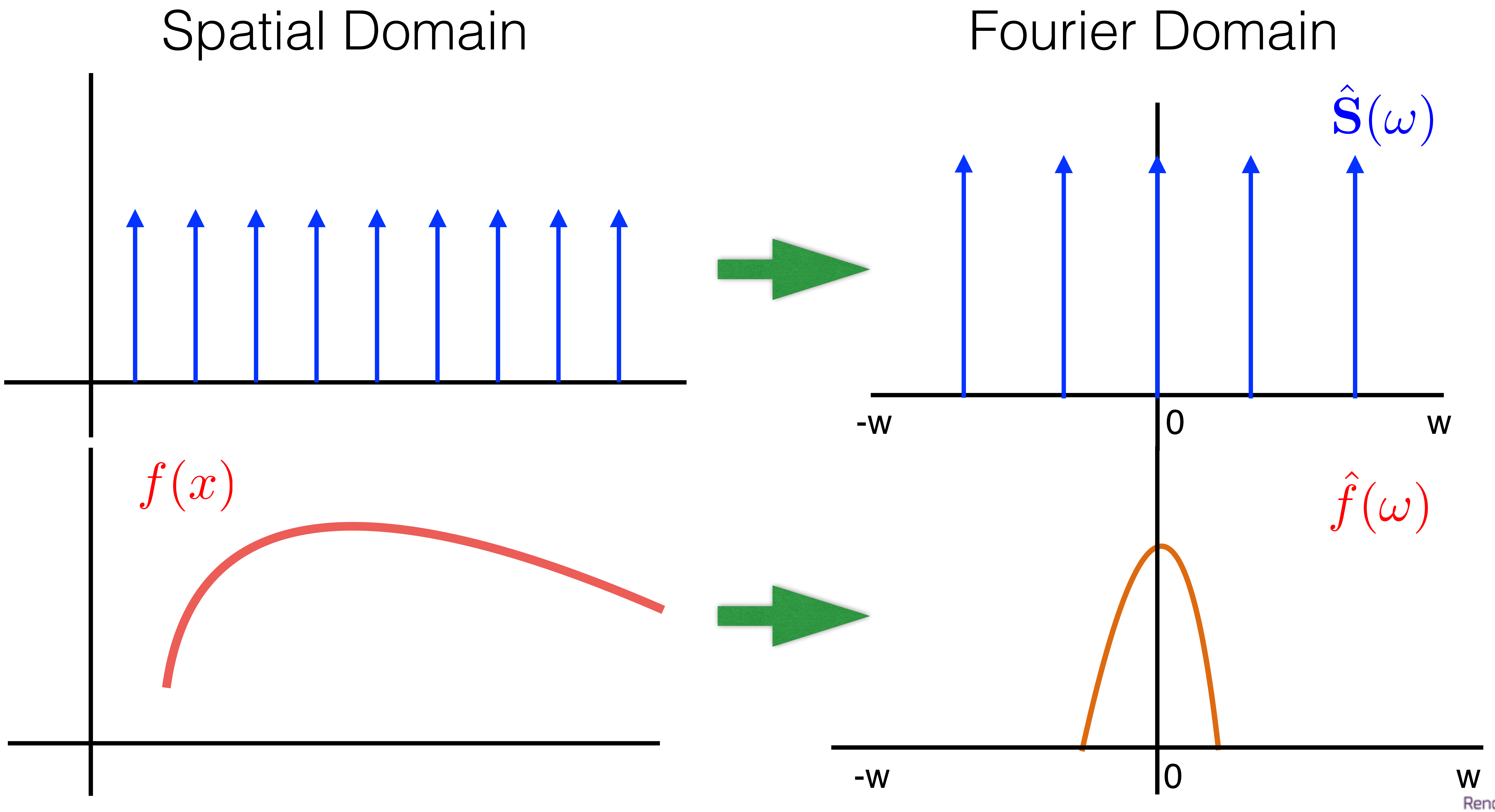
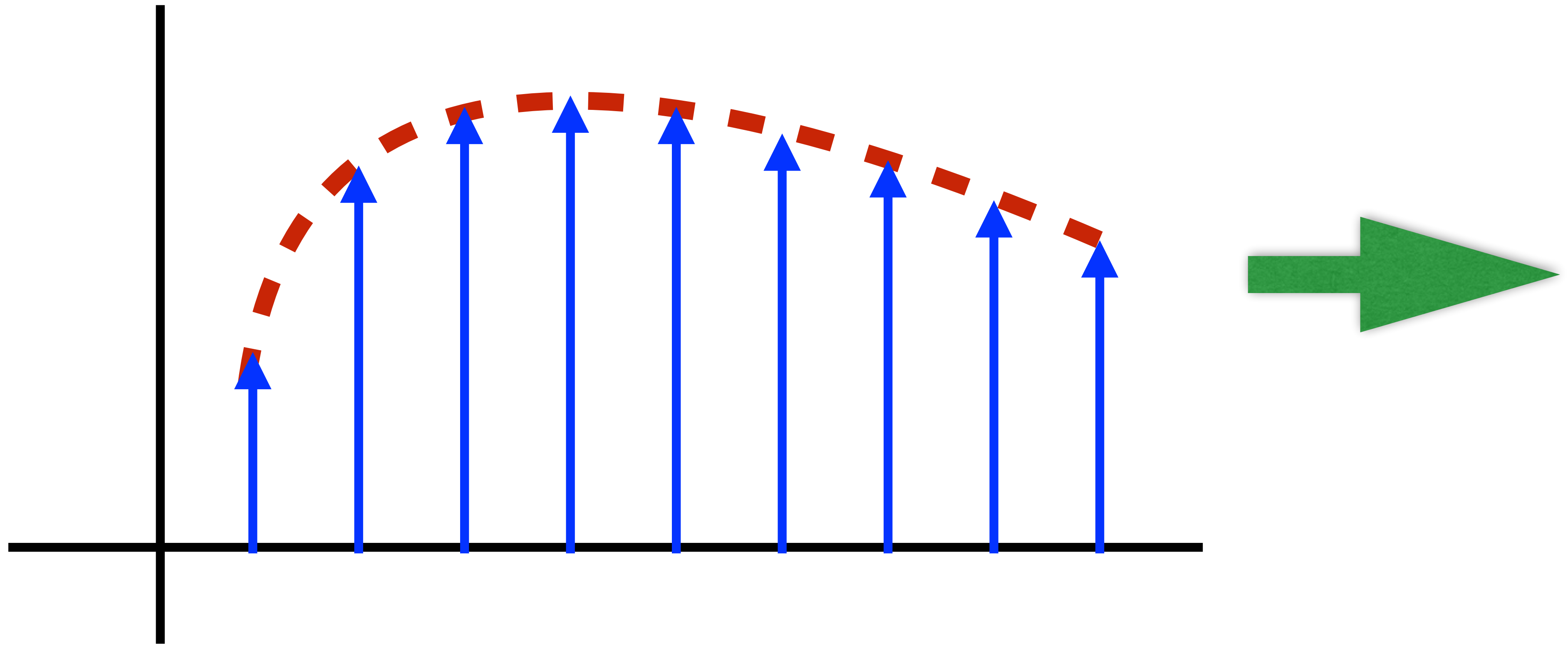
# Samples and function in Fourier Domain



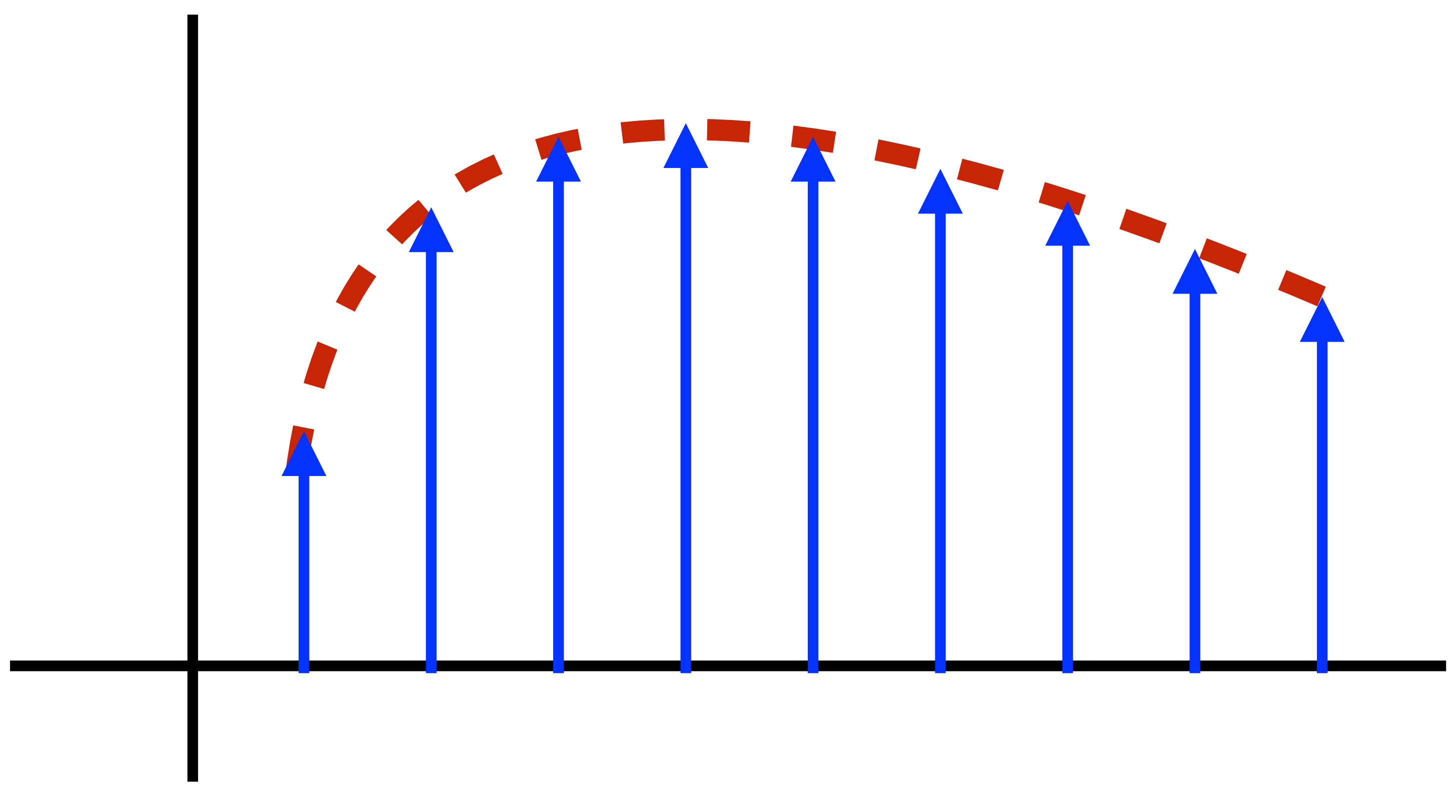
# Sampling in Primal Domain is Convolution in Fourier Domain

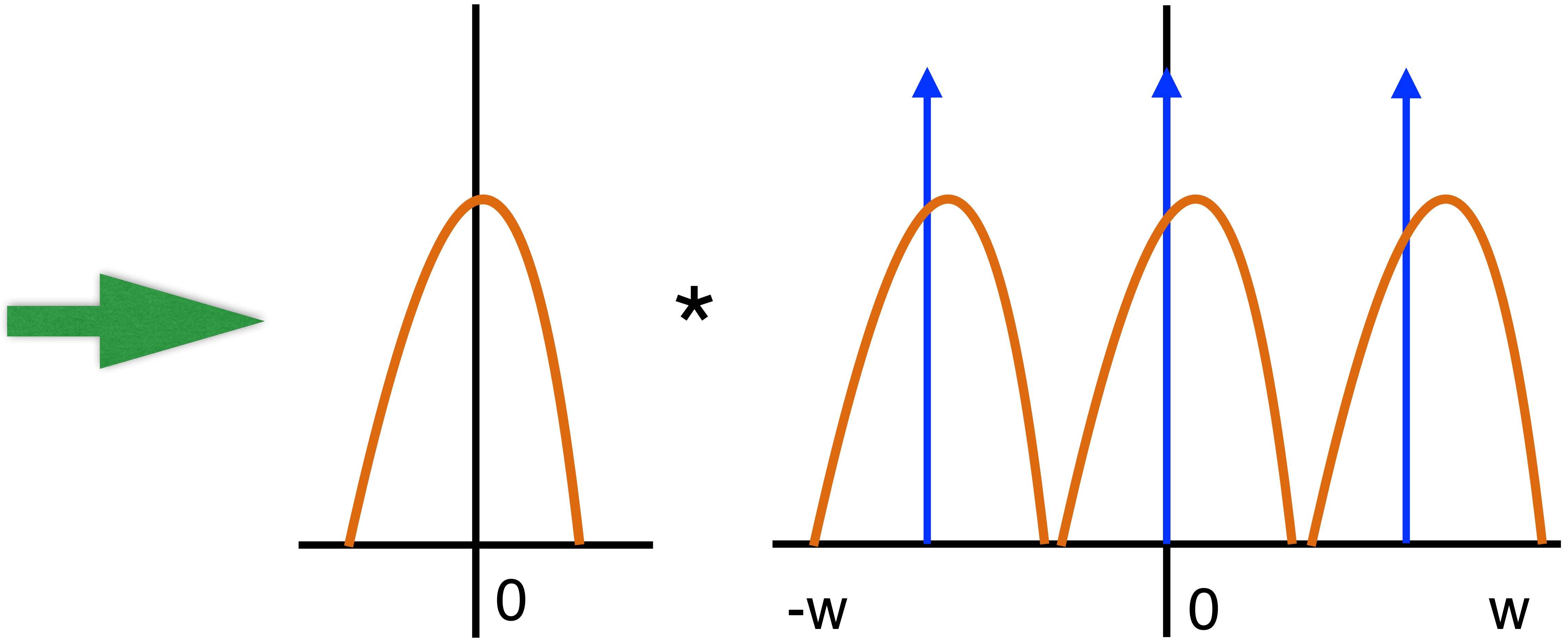


$f(x)\,\mathbf{S}(x)$

**Fredo Durand [2011]**

# Sampling in Primal Domain is Convolution in Fourier Domain



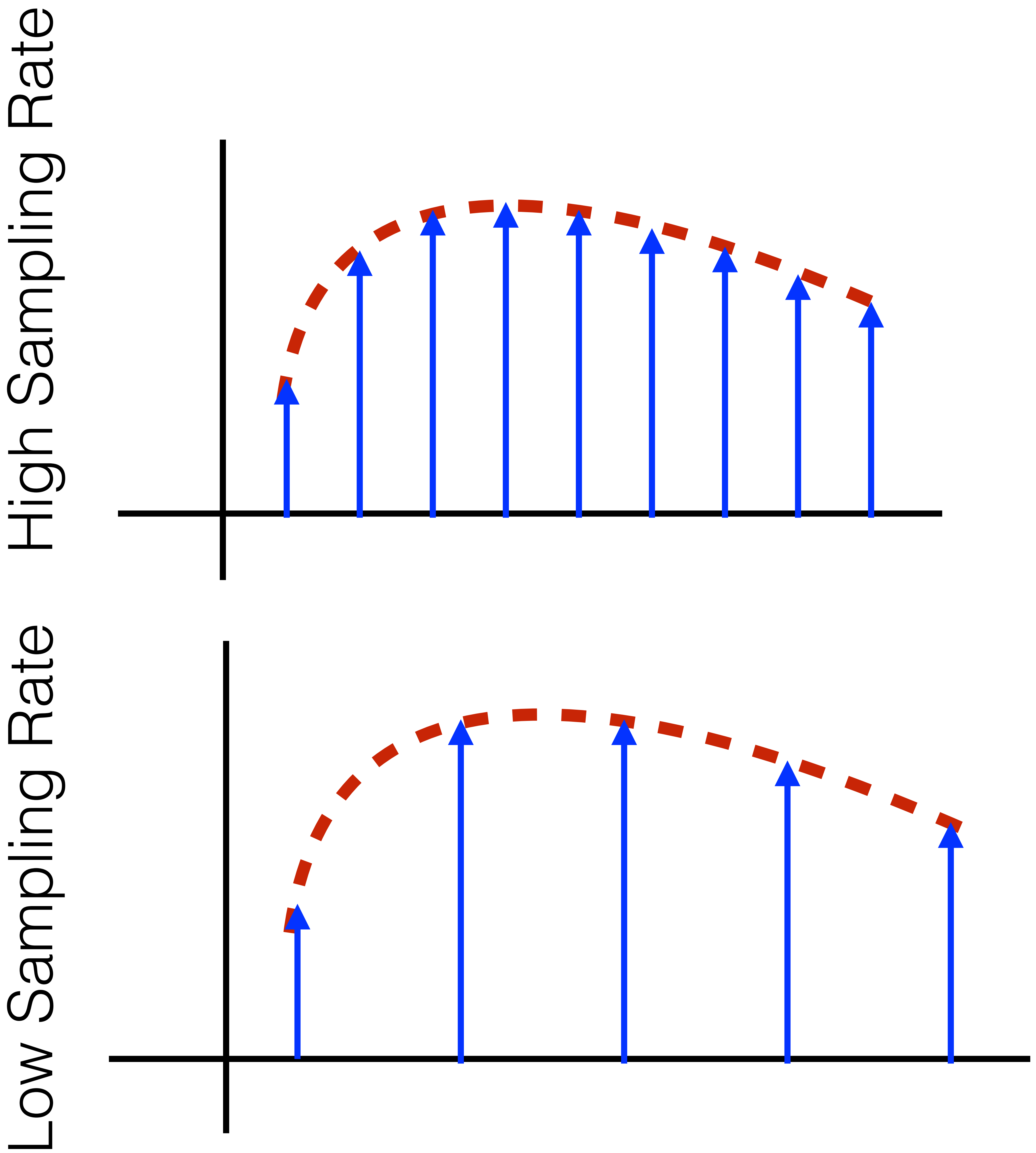


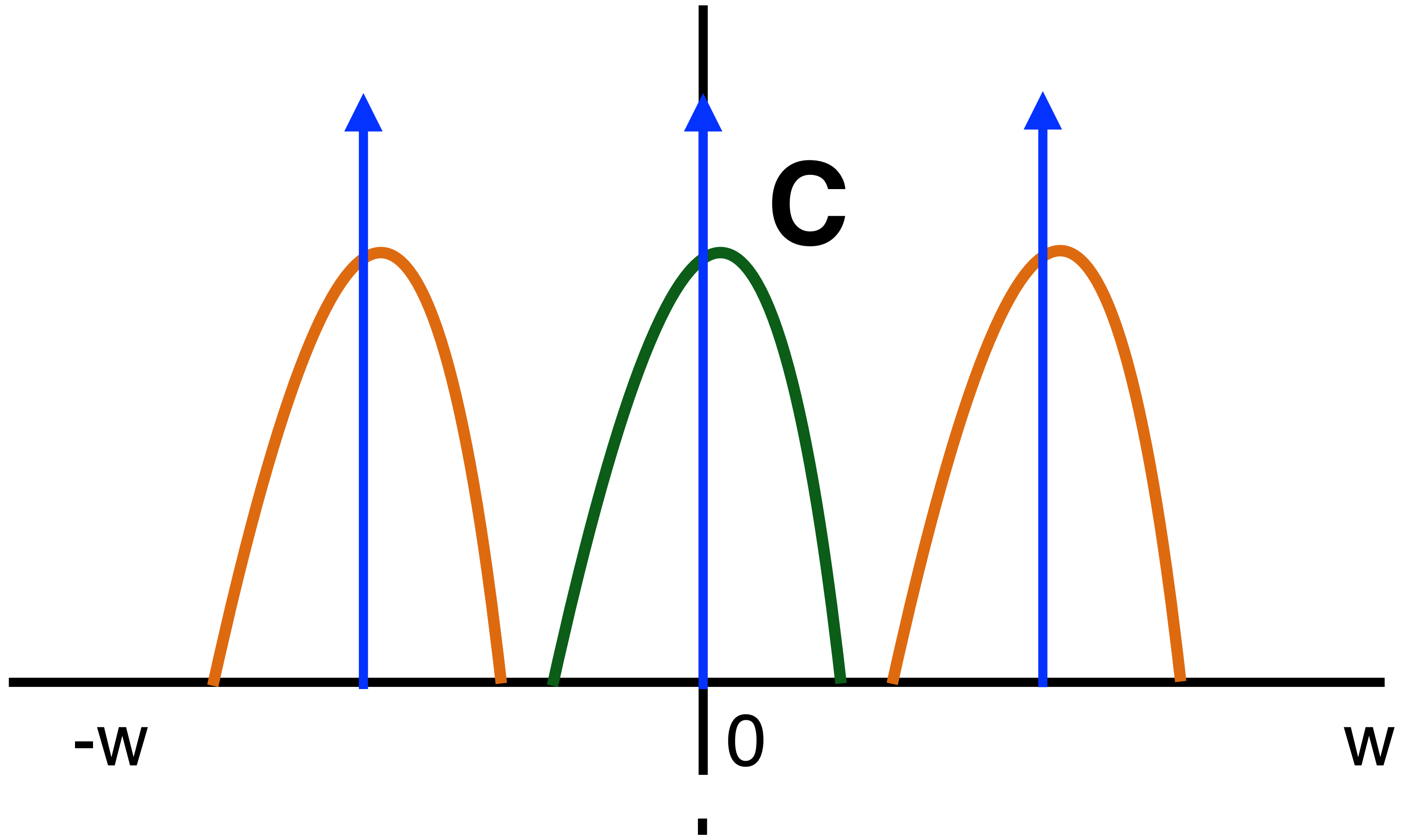

$f(x)\,\mathbf{S}(x)$

$\hat{f}(\omega) \otimes \hat{\mathbf{S}}(\omega)$

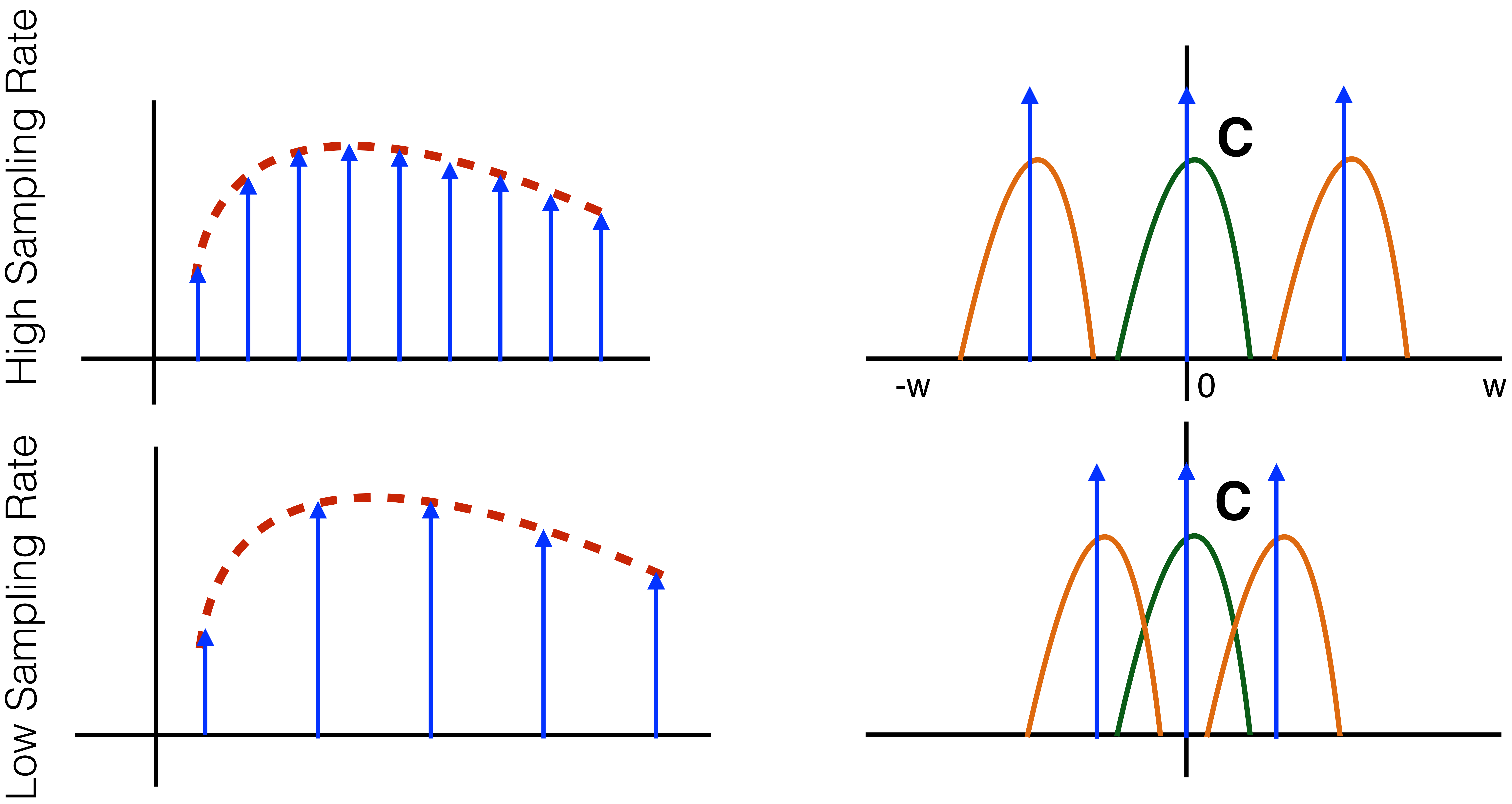
**Fredo Durand [2011]**

# Aliasing in Reconstruction



# Aliasing in Reconstruction

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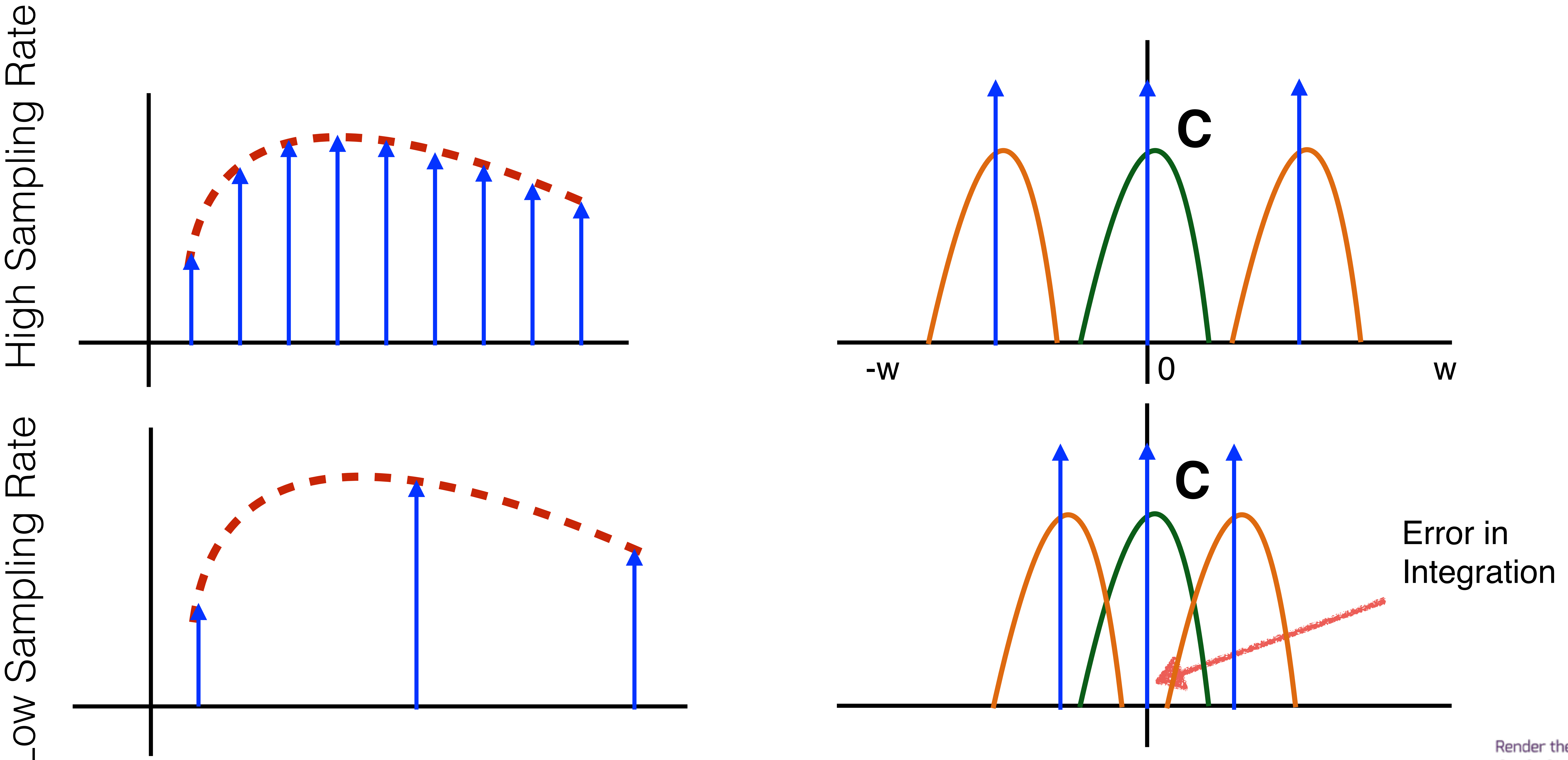
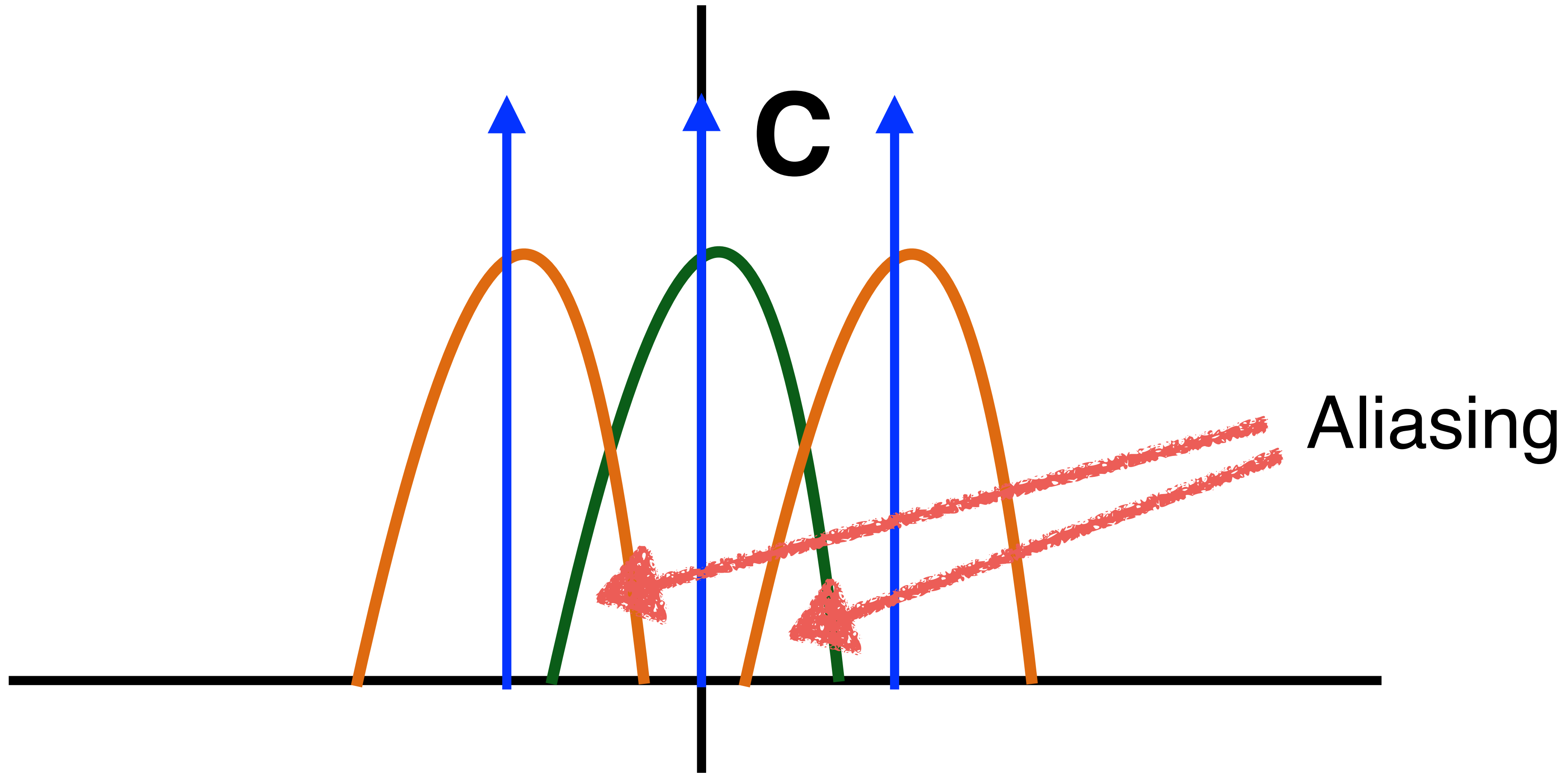
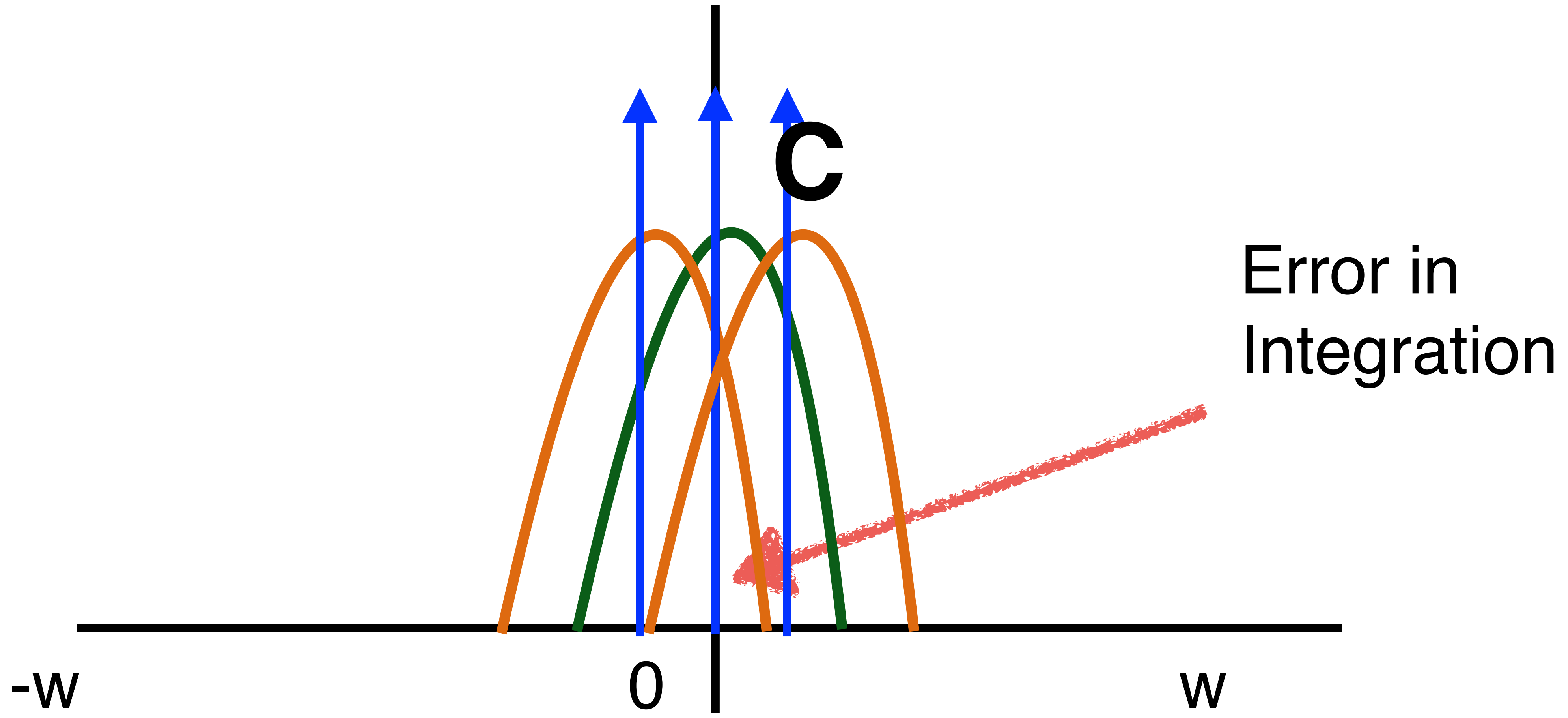
# Error in Monte Carlo Integration



# Aliasing (Reconstruction) vs. Error (Integration)

**Fredo Durand [2011]**
**Belcour et al. [2013]**



# Integration in the Fourier Domain

# Integration is the DC term in the Fourier Domain

Spatial Domain:

$$I = \int_D f(x)dx$$

Fourier Domain:

$$\hat{f}(0)$$

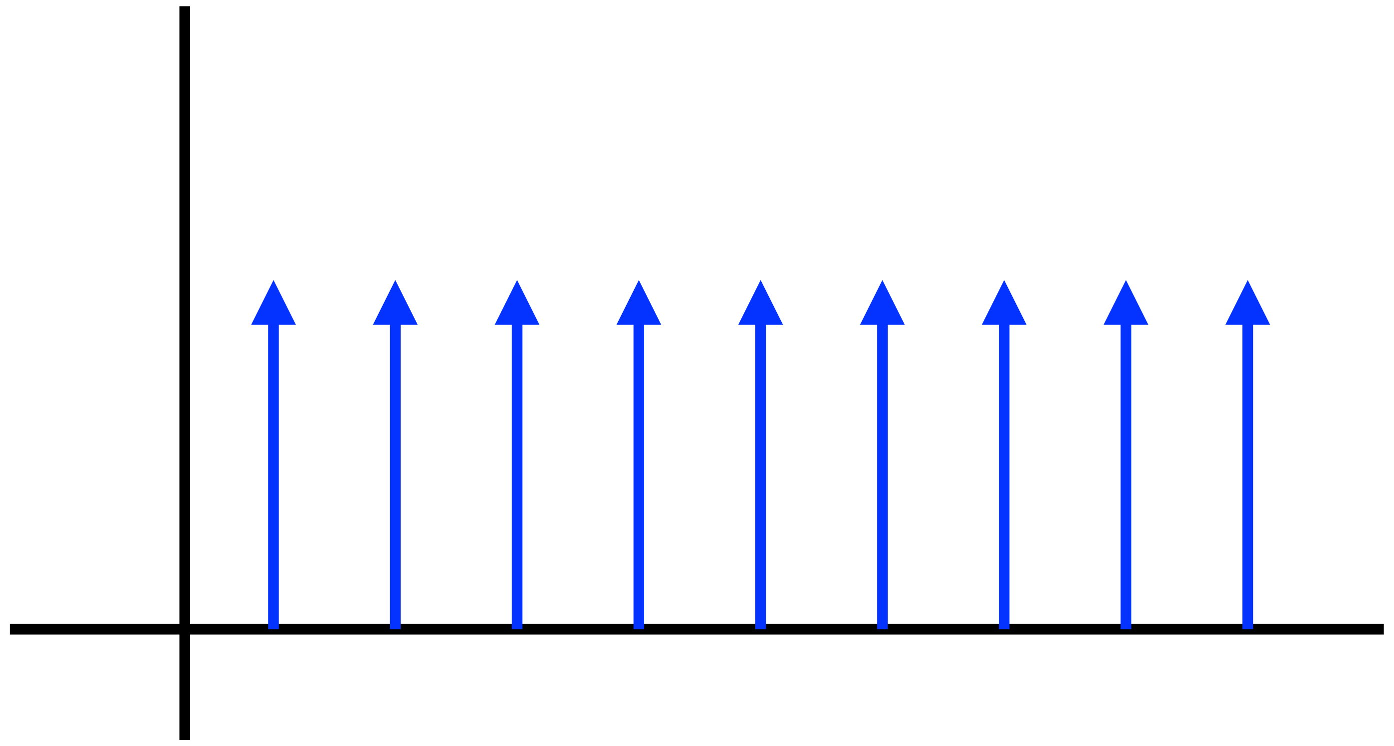
# Monte Carlo Estimator in Spatial Domain

$$\tilde{\mu}_N = \int_D f(x)\mathbf{S}(x)dx$$

# Monte Carlo Estimator in Spatial Domain

$$\boxed{\tilde{\mu}_N} = \int_D f(x)\mathbf{S}(x)dx$$

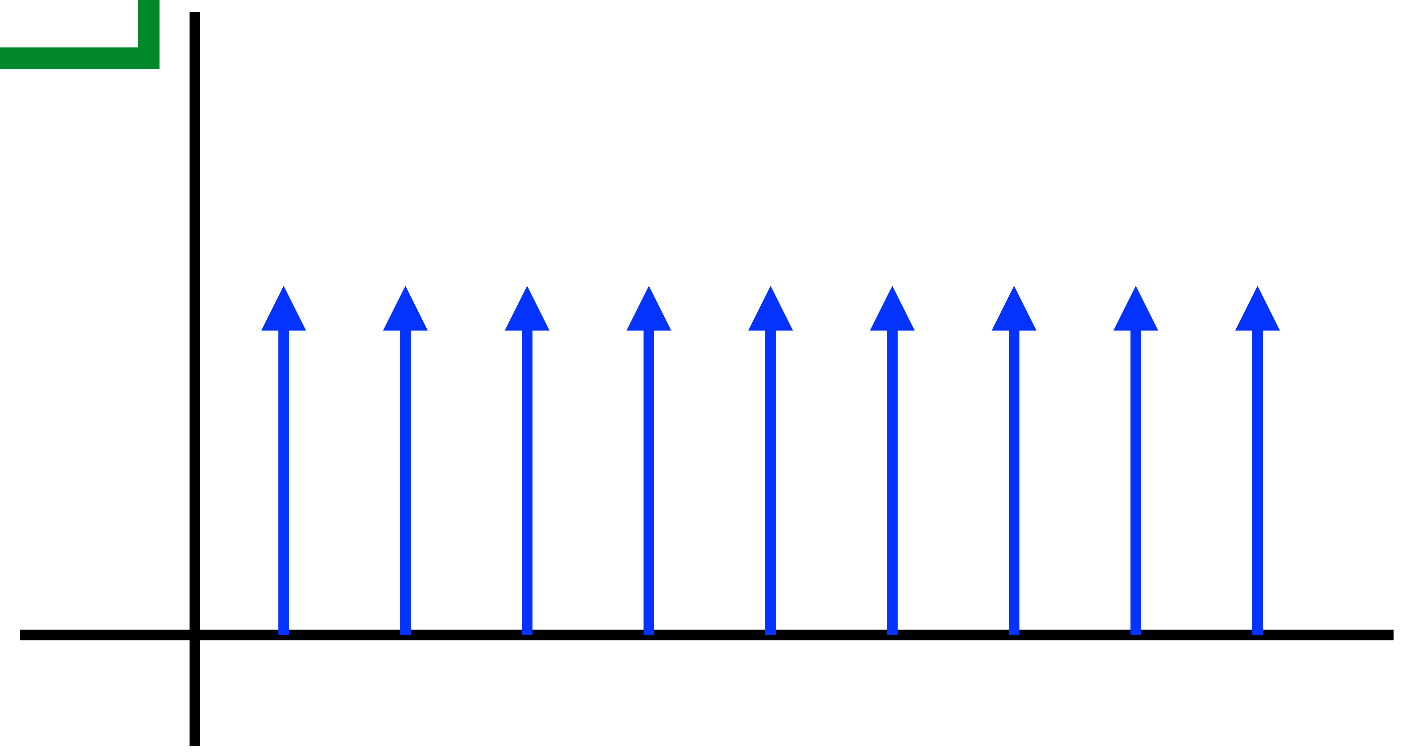
$$\boxed{\mathbf{S}(x)} = \frac{1}{N}\sum_{k=1}^{N}\delta(x - \boxed{x_k})$$



# Monte Carlo Estimator in Fourier Domain

$$\tilde{\mu}_N = \int_D f(x)\mathbf{S}(x)dx = \boxed{\int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega}$$

$$\mathbf{S}(x) = \frac{1}{N}\sum_{k=1}^{N}\delta(x - x_k)$$



# Monte Carlo Estimator in Fourier Domain

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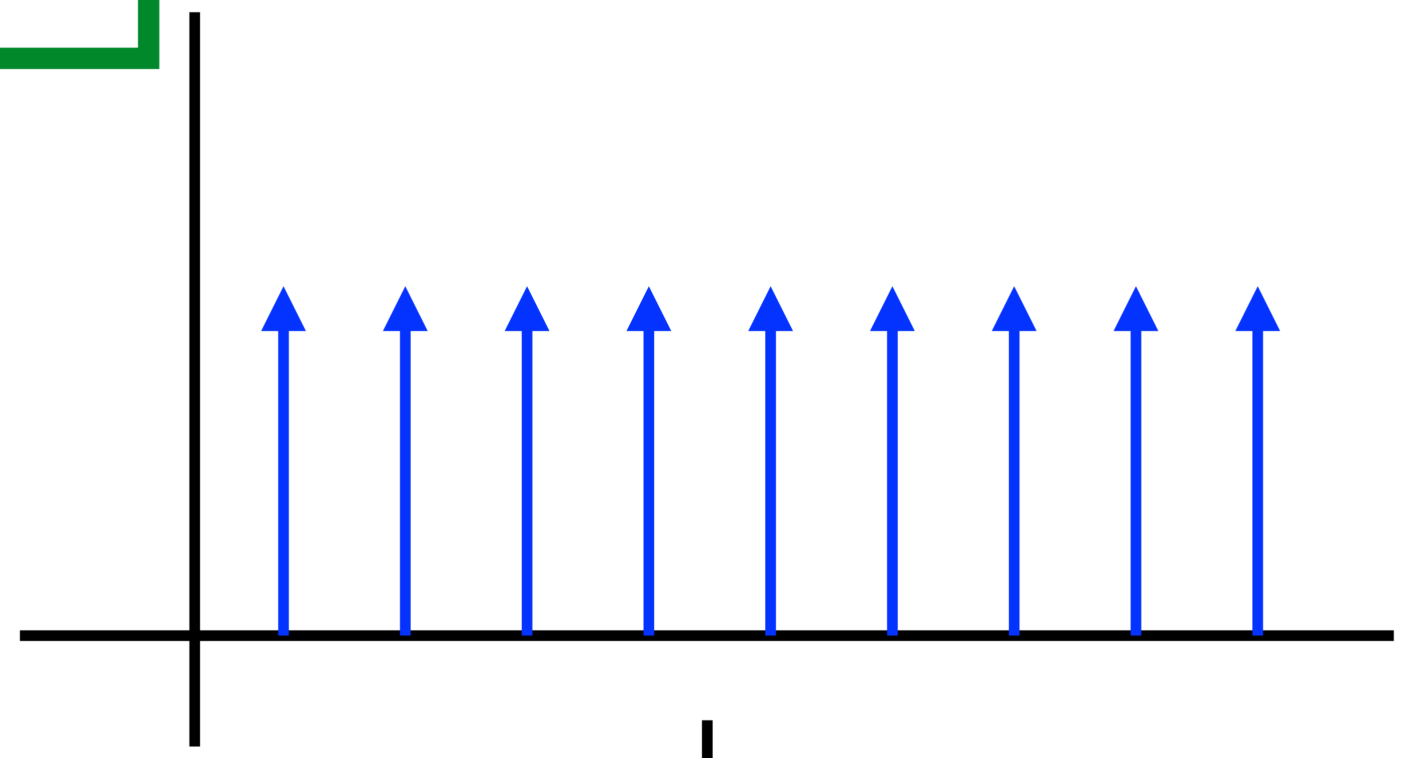
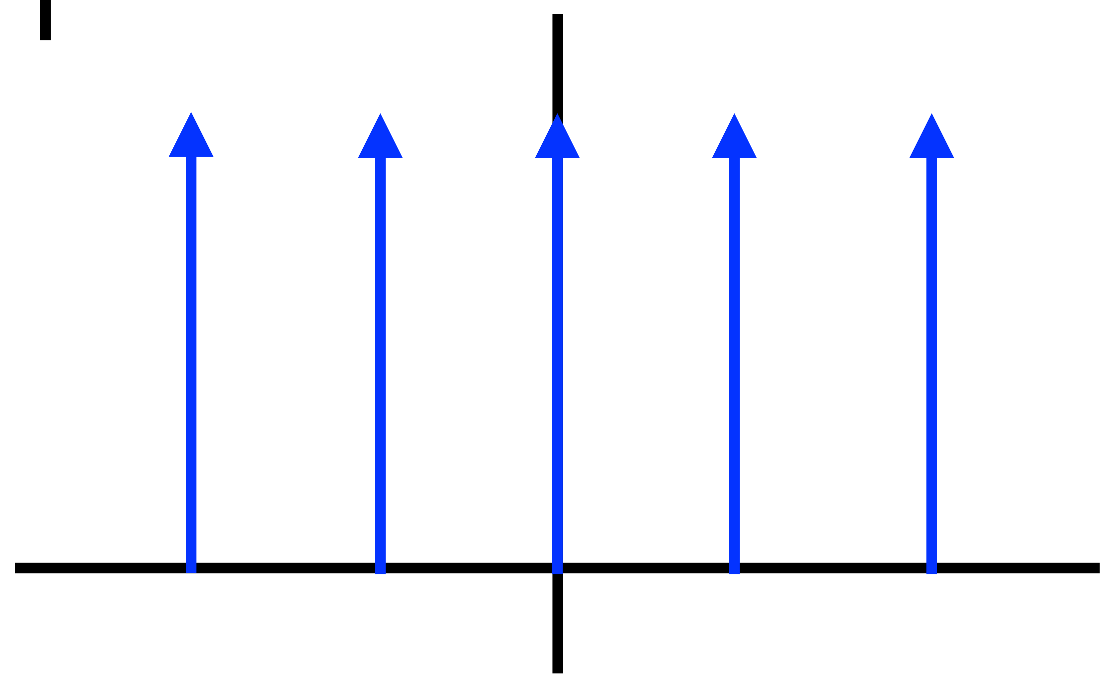


$$\hat{\mathbf{S}}(\omega) = \frac{1}{N}\sum_{k=1}^{N} e^{-i2\pi\omega x_k}$$





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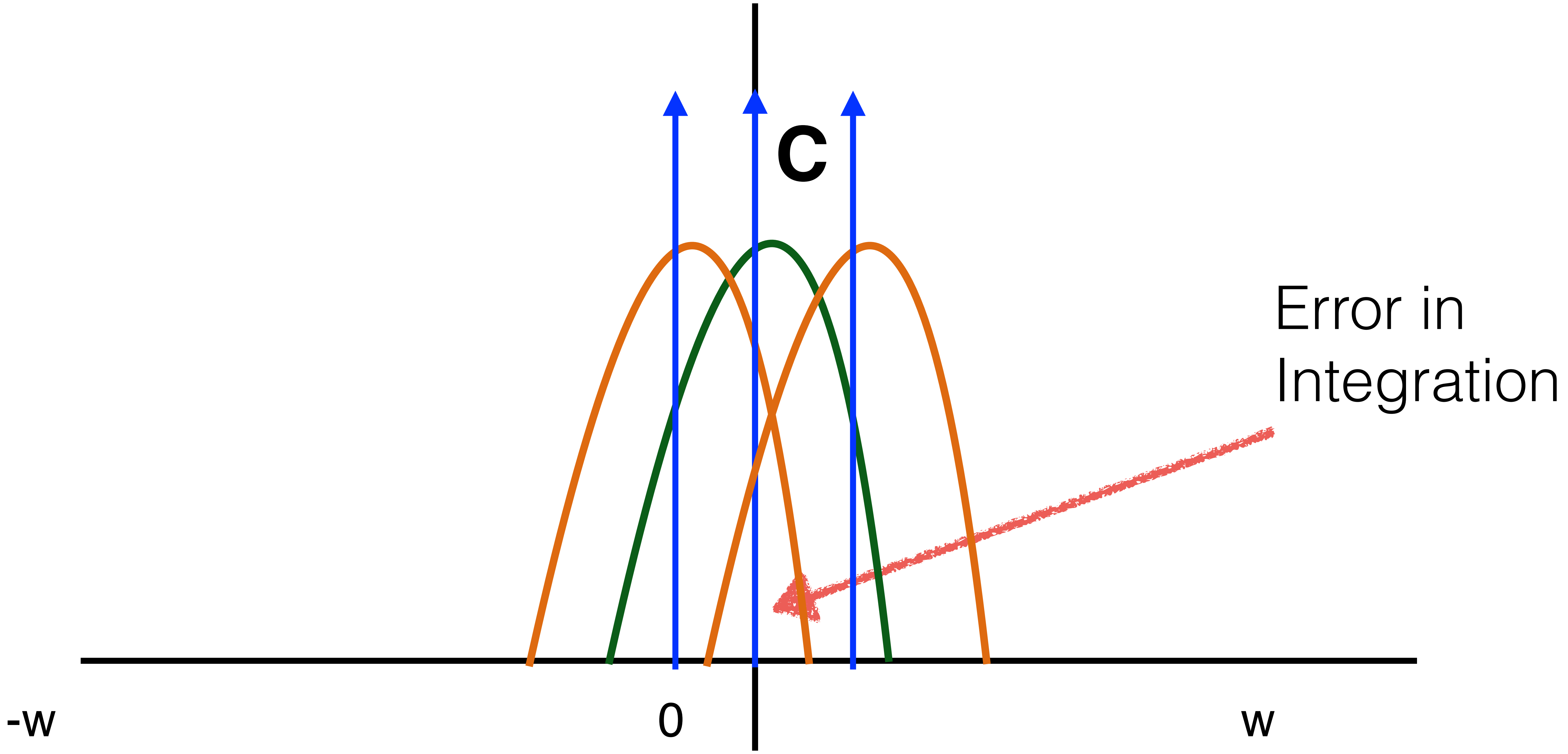
# How to Formulate Error in Fourier Domain ?

$$I = \hat{f}(0)$$

$$\tilde{\mu}_N = \int_{\Omega} \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$




**Fredo Durand [2011]**

# Error in Spatial Domain

$$I = \hat{f}(0)$$

$$\tilde{\mu}_N = \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

True Integral

$$I - \tilde{\mu}_N = \boxed{\int_D f(x)dx} - \boxed{\int_D f(x)\mathbf{S}(x)dx}$$

Monte Carlo Estimator

# Error in Spatial Domain

$$I = \hat{f}(0)$$

$$\tilde{\mu}_N = \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

$$I - \tilde{\mu}_N = \int_D f(x)dx - \int_D f(x)\mathbf{S}(x)dx$$

# Error in Fourier Domain

$$I = \hat{f}(0)$$

$$\tilde{\mu}_N = \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

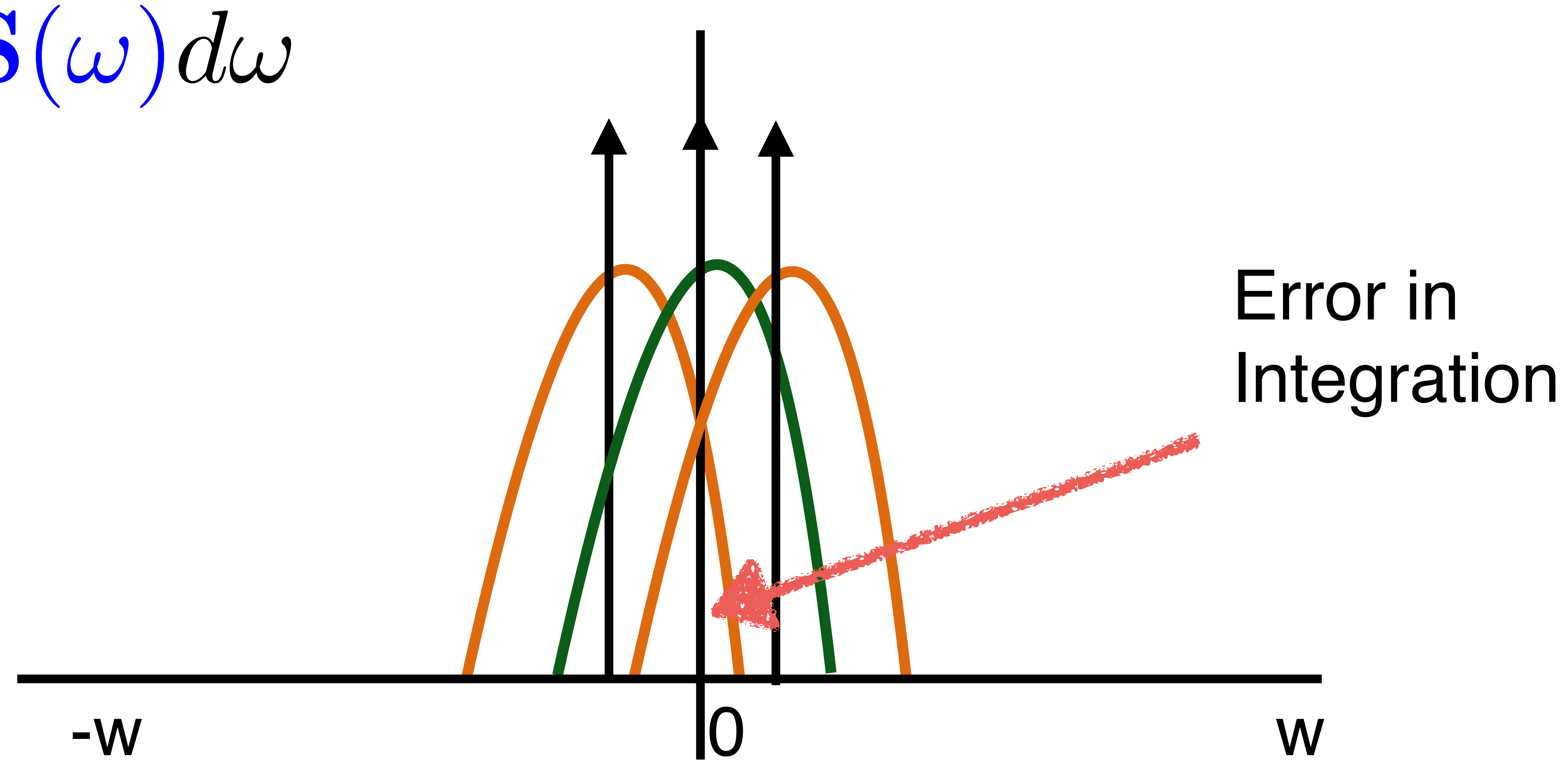
$$I - \tilde{\mu}_N = \int_D f(x)dx - \int_D f(x)\mathbf{S}(x)dx$$

$$I - \tilde{\mu}_N = \hat{f}(0) - \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

**Fredo Durand [2011]**

# Error in Fourier Domain

$$I - \tilde{\mu}_N = \hat{f}(0) - \int_{\Omega} \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$




**Fredo Durand [2011]**

$$\text{Error} = \text{Bias}^2 + \text{Variance}$$

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# Properties of Error

- Bias
- Variance

**Subr and Kautz [2013]**

# Bias in the Monte Carlo Estimator

# Bias in Fourier Domain

Error: $$I - \tilde{\mu}_N = \hat{f}(0) - \int_{\Omega} \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

# Bias in Fourier Domain

Error:

$$I - \tilde{\mu}_N = \hat{f}(0) - \int_{\Omega} \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

# Bias in Fourier Domain

$$\langle I - \tilde{\mu}_N \rangle = \hat{f}(0) - \left\langle \int_{\Omega} \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega \right\rangle$$

# Bias in Fourier Domain

$$\langle I - \tilde{\mu}_N \rangle = \hat{f}(0) - \left\langle \int_\Omega \hat{f}^*(\omega) \hat{\mathbf{S}}(\omega) d\omega \right\rangle$$

$$\langle I - \tilde{\mu}_N \rangle = \hat{f}(0) - \int_\Omega \hat{f}^*(\omega) \langle \hat{\mathbf{S}}(\omega) \rangle \, d\omega$$

**Subr and Kautz [2013]**

# Bias in Fourier Domain

$$\langle I - \tilde{\mu}_N \rangle = \hat{f}(0) - \boxed{\int_{\Omega} \hat{f}^*(\omega) \, \langle \hat{\mathbf{S}}(\omega) \rangle \, d\omega}$$

To obtain an unbiased estimator:

**Subr and Kautz [2013]**

$$\langle \hat{\mathbf{S}}(\omega) \rangle = 0$$

for frequencies other than zero

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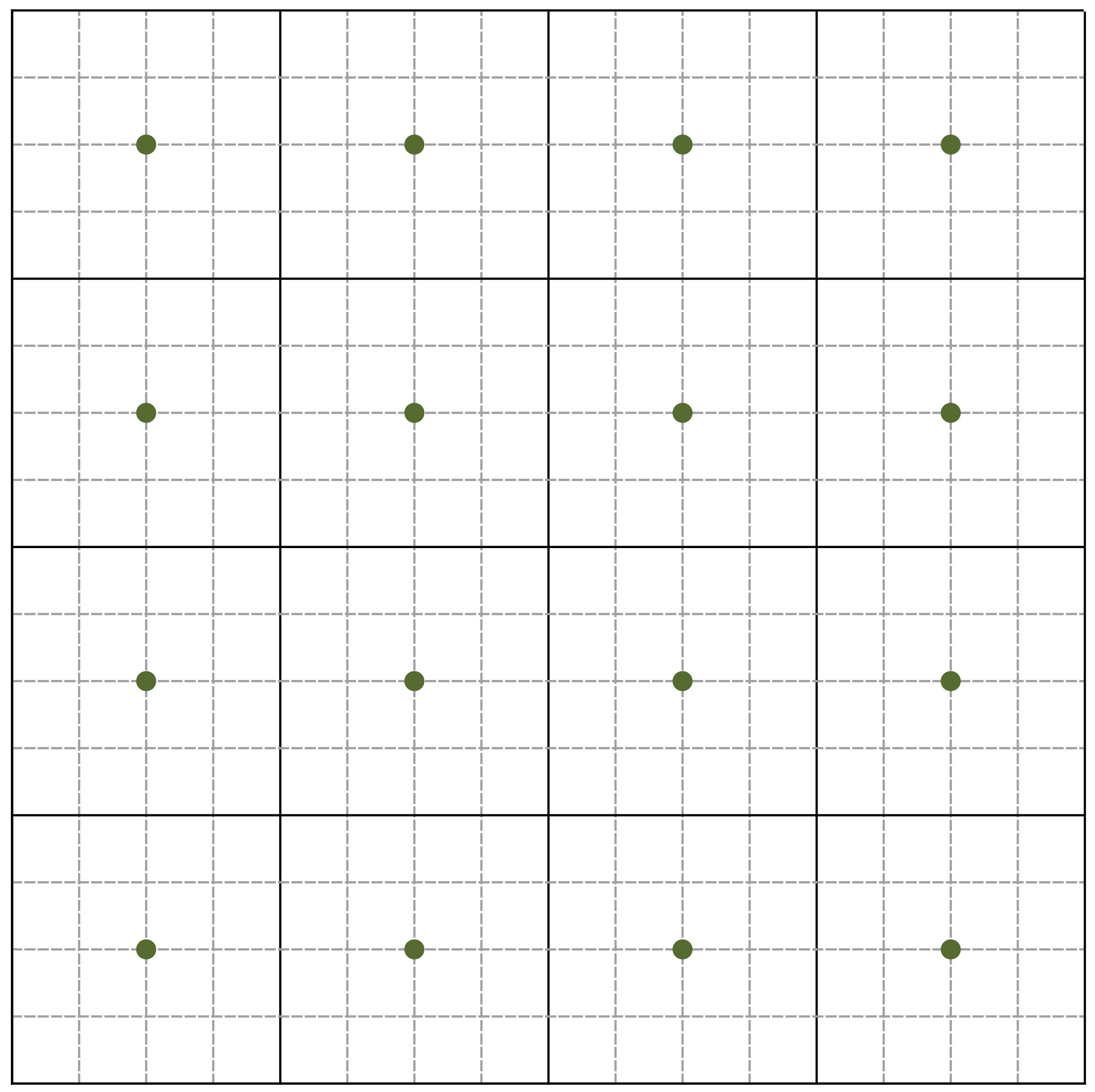
How to obtain $\langle \hat{\mathbf{S}}(\omega) \rangle = 0$ ?

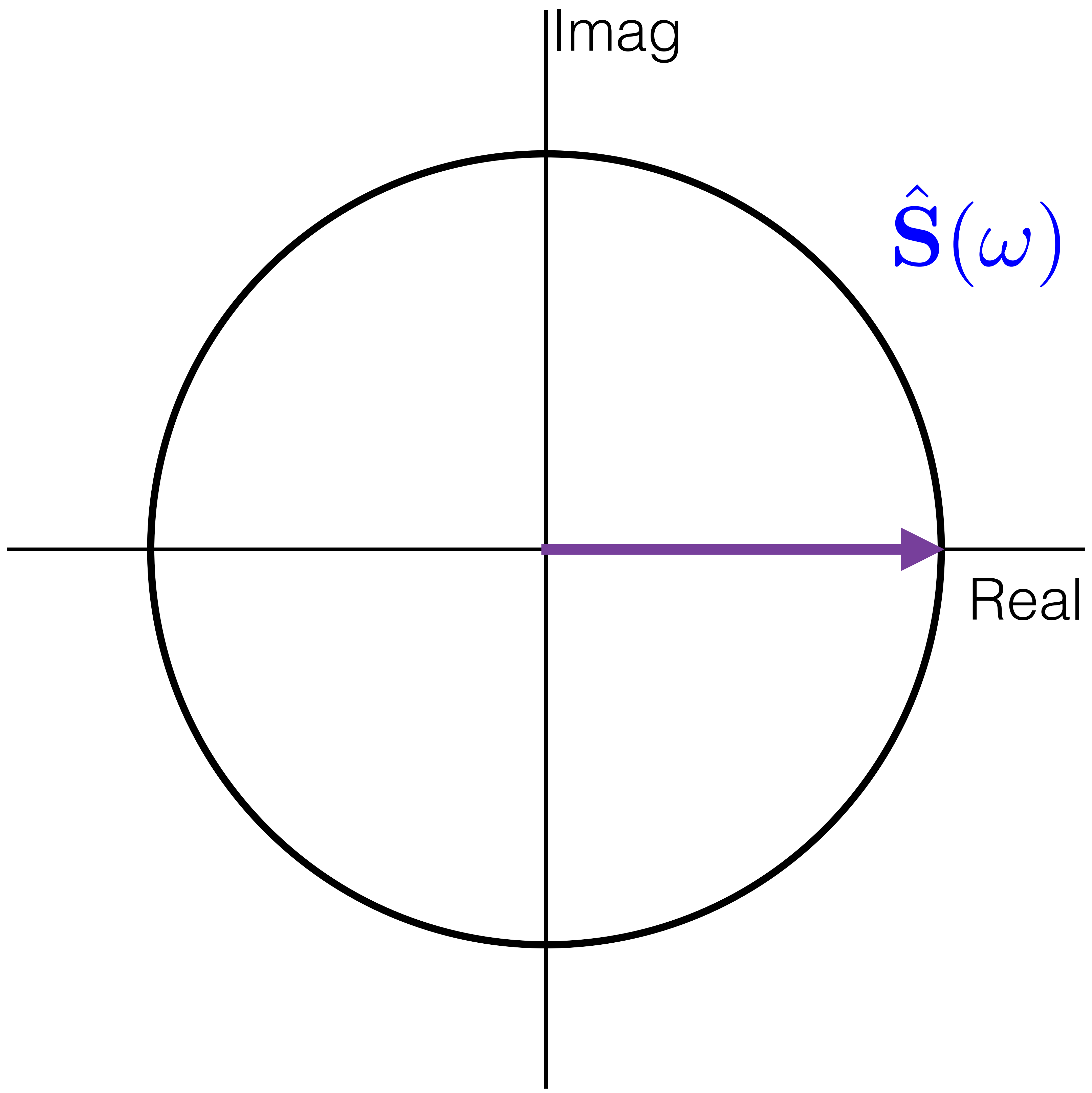
# Complex form in Amplitude and Phase

$$\langle \hat{\mathbf{S}}(\omega) \rangle = \underbrace{\boxed{|\langle \hat{\mathbf{S}}(\omega) \rangle|}}_{\text{Amplitude}} e^{-\overbrace{\boxed{\Phi(\langle \hat{\mathbf{S}}(\omega) \rangle)}}^{\text{Phase}}}$$

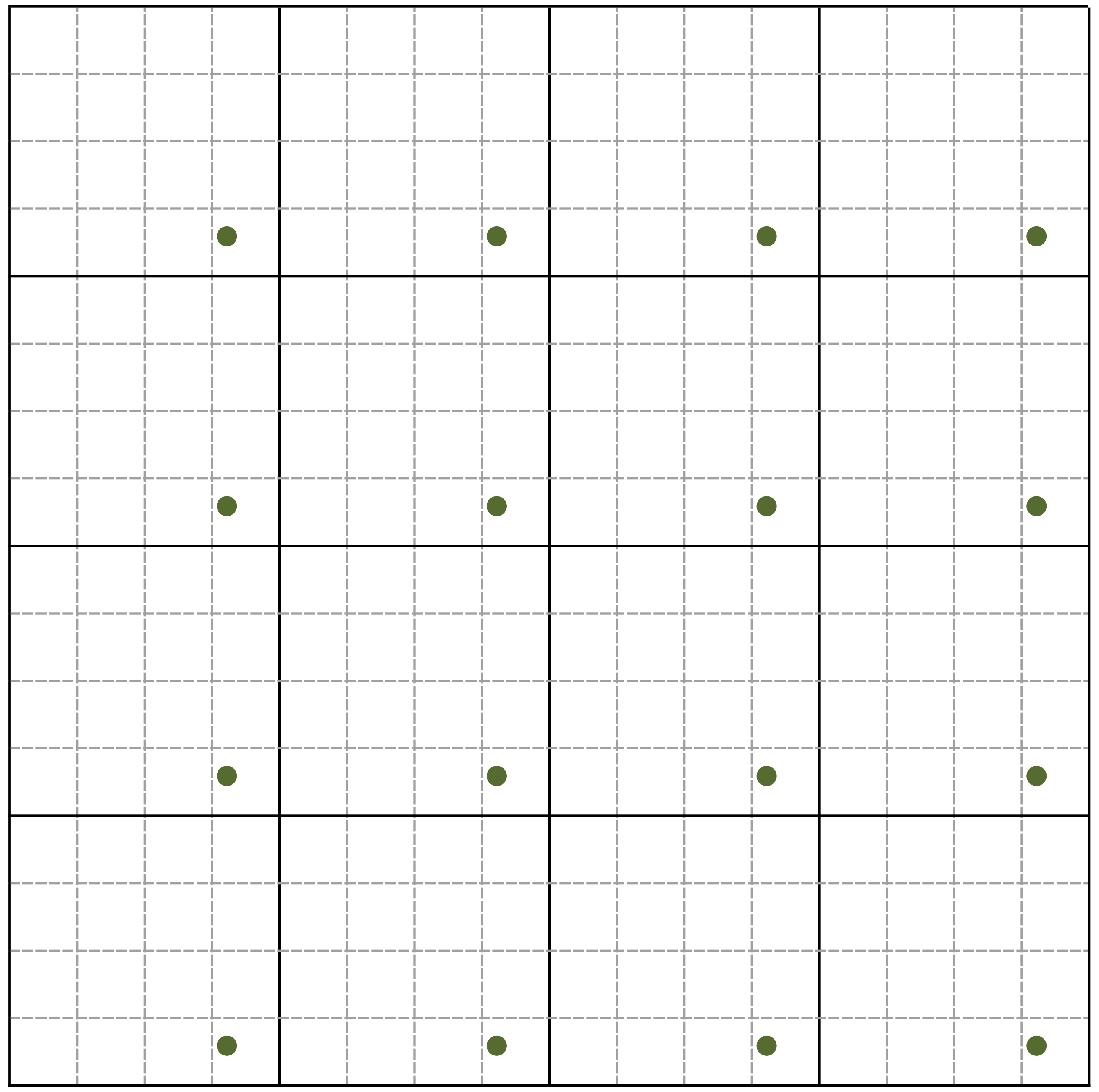
# Phase change due to Random Shift



For a given frequency $\omega$

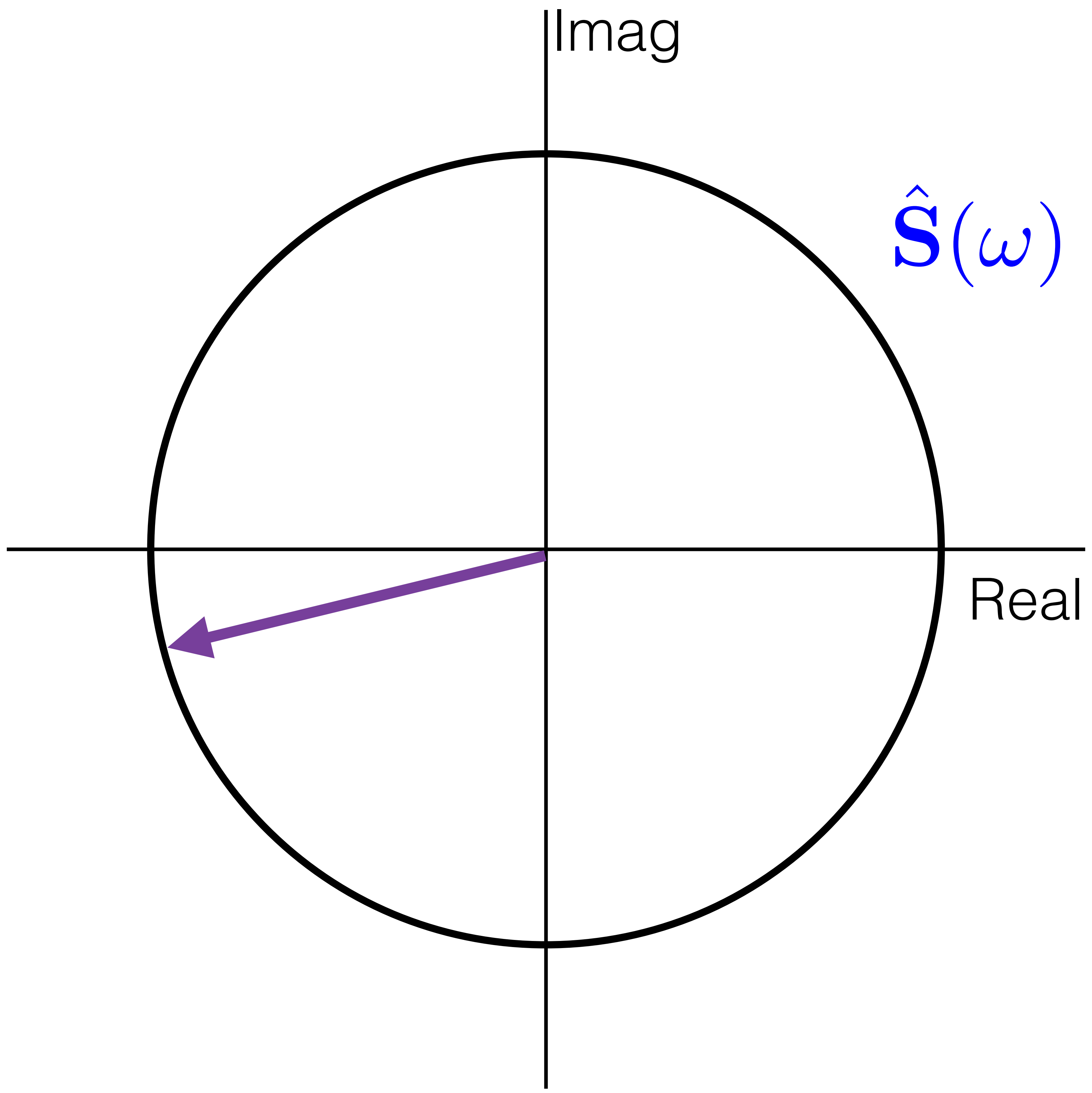


# Phase change due to Random Shift



**Pauly et al. [2000]**
**Ramamoorthi et al. [2012]**

For a given frequency $\omega$



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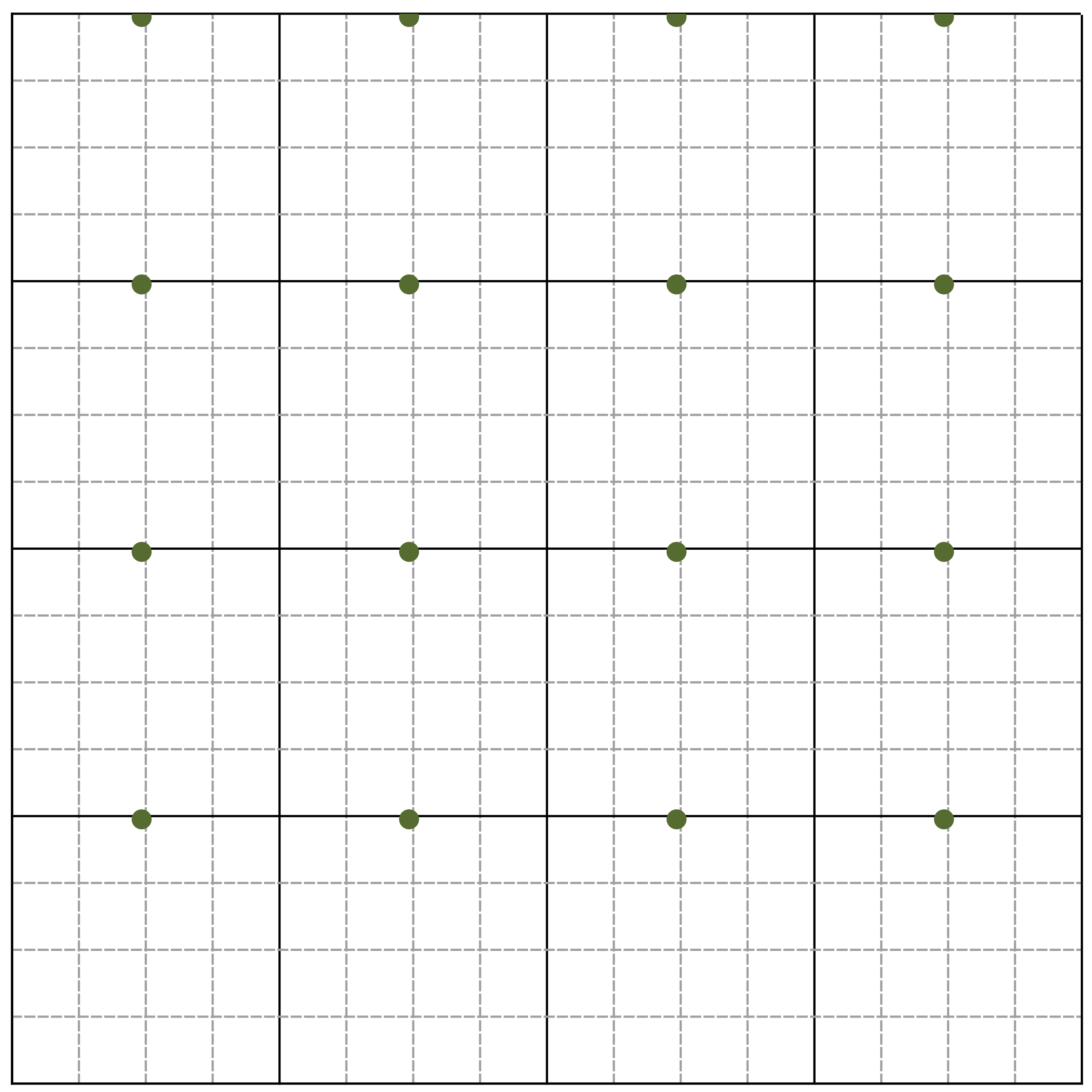
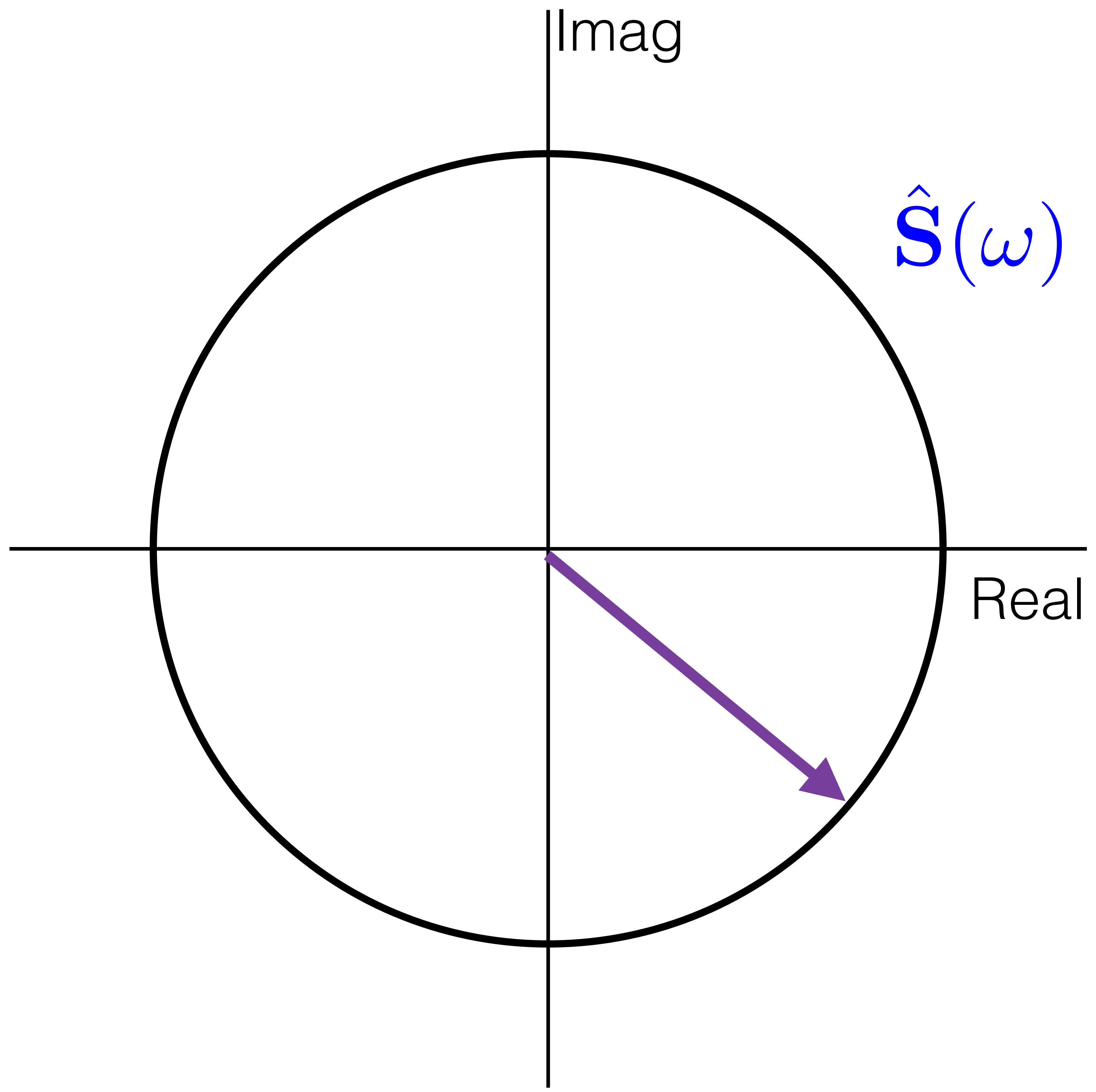
# Phase change due to Random Shift

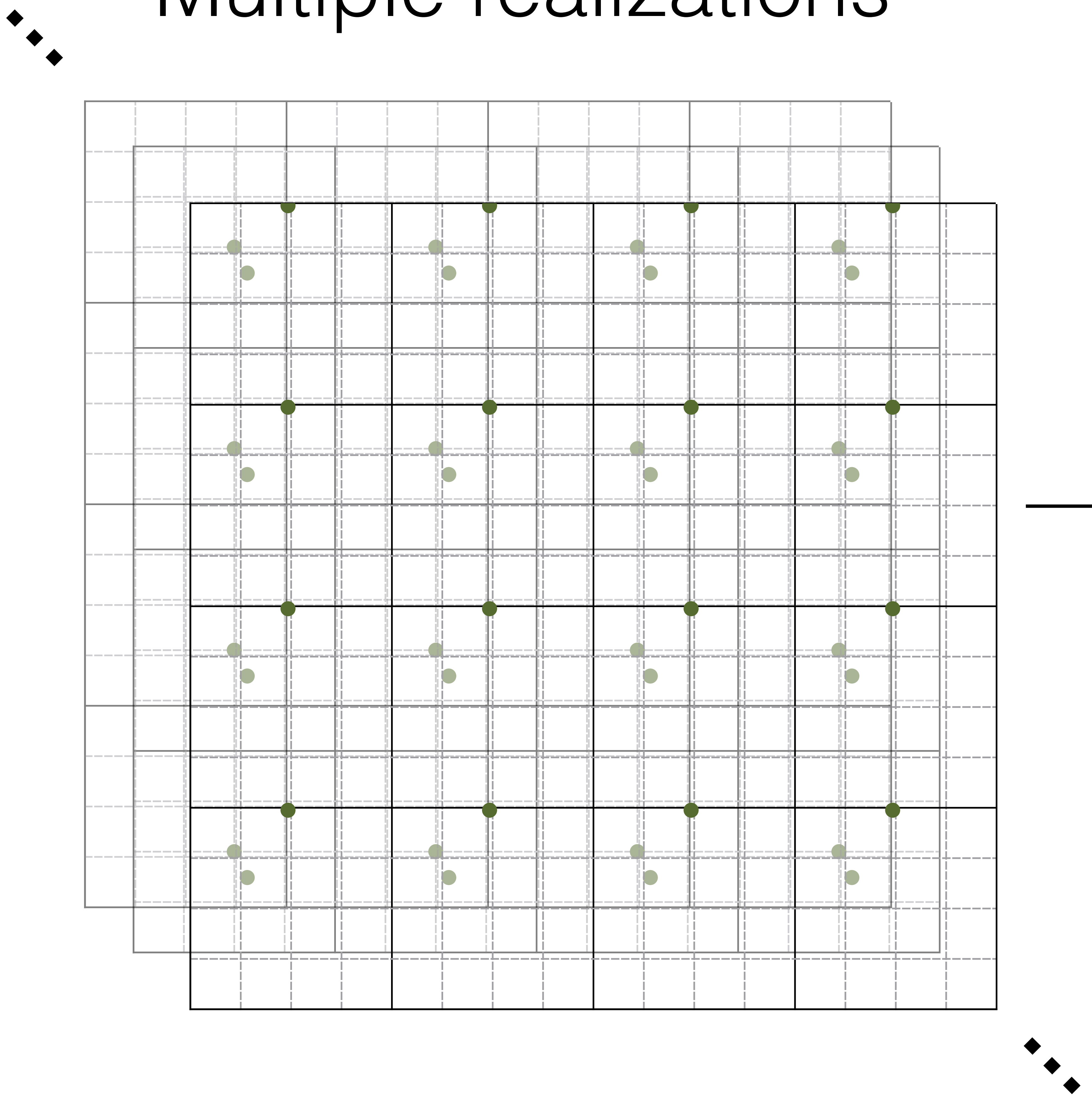


For a given frequency $\omega$

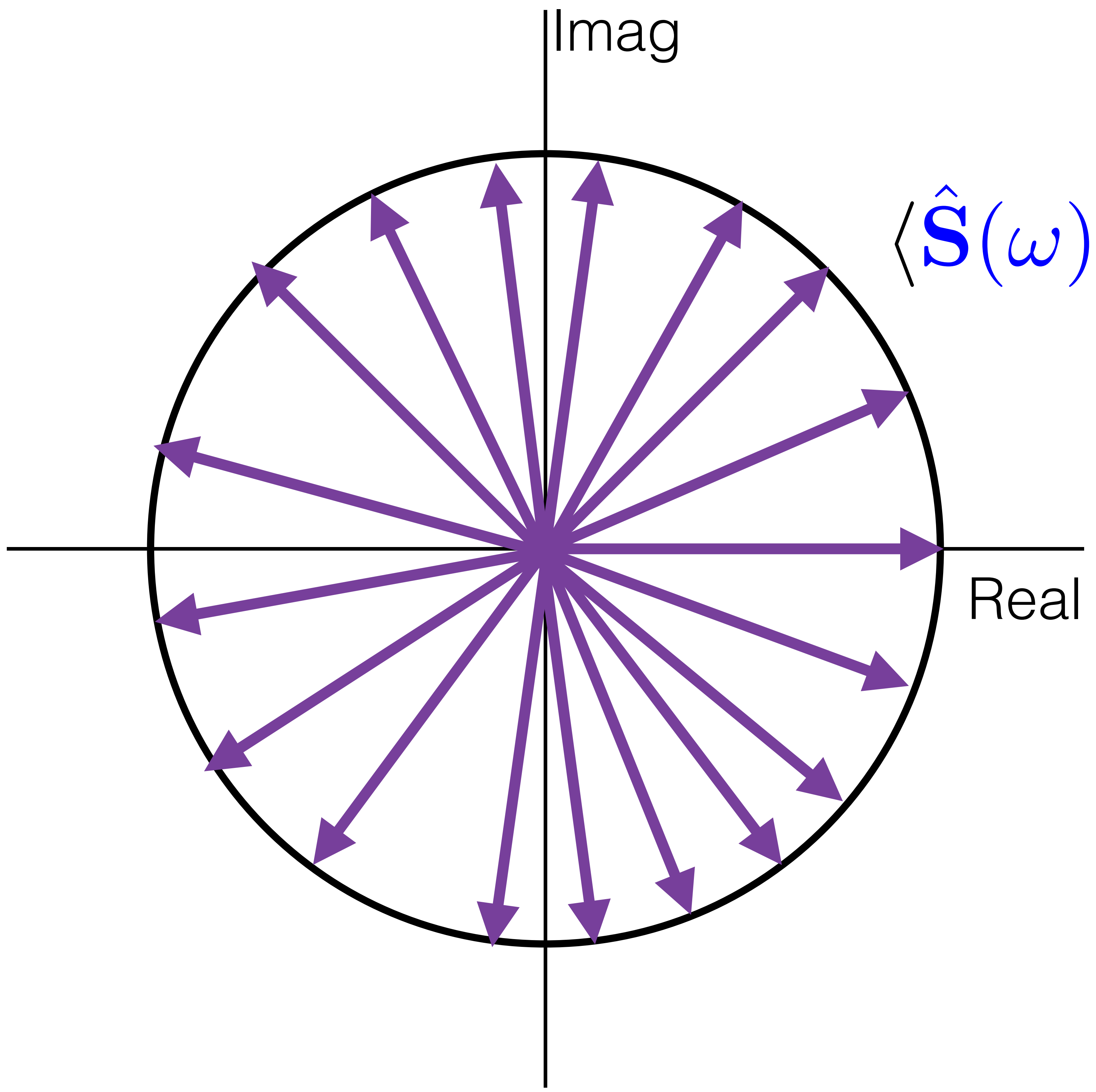


# Phase change due to Random Shift

Multiple realizations



For a given frequency $\omega$




$$\langle \hat{\mathbf{S}}(\omega) \rangle = 0 \;\; \forall \omega \neq 0$$

$$\text{Error} = \cancel{\text{Bias}^2} + \text{Variance}$$

- Homogenization allows representation of error only in terms of variance
- We can take any sampling pattern and homogenize it to make the Monte Carlo estimator unbiased.

# Variance in the Fourier domain

# Variance in the Fourier domain

Error: $$I - \tilde{\mu}_N = \hat{f}(0) - \int_{\Omega} \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

# Variance in the Fourier domain

$$\mathrm{Var}(I - \tilde{\mu}_N) = \mathrm{Var}\left( \hat{f}(0) - \int_\Omega \hat{f}^*(\omega)\, \hat{\mathbf{S}}(\omega)\, d\omega \right)$$

# Variance in the Fourier domain

$$\mathrm{Var}(I - \tilde{\mu}_N) = \mathrm{Var}\left(\hat{f}(0) - \int_{\Omega} \hat{f}^*(\omega)\,\hat{\mathbf{S}}(\omega)\,d\omega\right)$$

$$\mathrm{Var}(\tilde{\mu}_N) = \mathrm{Var}\left(\int_{\Omega} \hat{f}^*(\omega)\,\hat{\mathbf{S}}(\omega) d\omega\right)$$

# Variance in the Fourier domain

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# Variance in the Fourier domain

$$\mathrm{Var}(\tilde{\mu}_N) = \mathrm{Var}\left(\int_\Omega \hat{f}^*(\omega)\,\hat{\mathbf{S}}(\omega) d\omega\right)$$

$$\mathrm{Var}(\tilde{\mu}_N) = \int_\Omega P_f(\omega)\,\mathrm{Var}\left(\hat{\mathbf{S}}(\omega)\right) d\omega$$

where,

$$P_f(\omega) = |\hat{f}^*(\omega)|^2$$ Power Spectrum

# Variance in the Fourier domain

$$\boxed{\mathrm{Var}(\tilde{\mu}_N) = \int_{\Omega} P_f(\omega)\,\mathrm{Var}\left(\hat{\mathbf{S}}(\omega)\right) d\omega}$$

**Subr and Kautz [2013]**

This is a general form, both for homogenised as well as non-homogenised sampling patterns

# Variance in the Fourier domain

$$\boxed{\mathrm{Var}(\tilde{\mu}_N) = \int_{\Omega} P_f(\omega)\,\mathrm{Var}\left(\hat{\mathbf{S}}(\omega)\right) d\omega}$$

# Variance in the Fourier domain

$$\boxed{\mathrm{Var}(\tilde{\mu}_N) = \int_\Omega P_f(\omega)\, \mathrm{Var}\left(\hat{\mathbf{S}}(\omega)\right) d\omega}$$

For purely random samples: $\langle \hat{\mathbf{S}}(\omega) \rangle = 0$

$$\boxed{\mathrm{Var}(\tilde{\mu}_N) = \int_\Omega P_f(\omega)\, \langle P_S(\omega) \rangle\, d\omega}$$

**Fredo Durand [2011]**

where,

$$P_S(\omega) = |\hat{\mathbf{S}}(\omega)|^2$$

# Variance using Homogenized Samples

Homogenizing any sampling pattern makes $\langle \hat{\mathbf{S}}(\omega) \rangle = 0$

$$\boxed{\mathrm{Var}(\tilde{\mu}_N) = \int_{\Omega} P_f(\omega)\ \langle P_S(\omega) \rangle\, d\omega}$$

**Pilleboue et al. [2015]**

where,

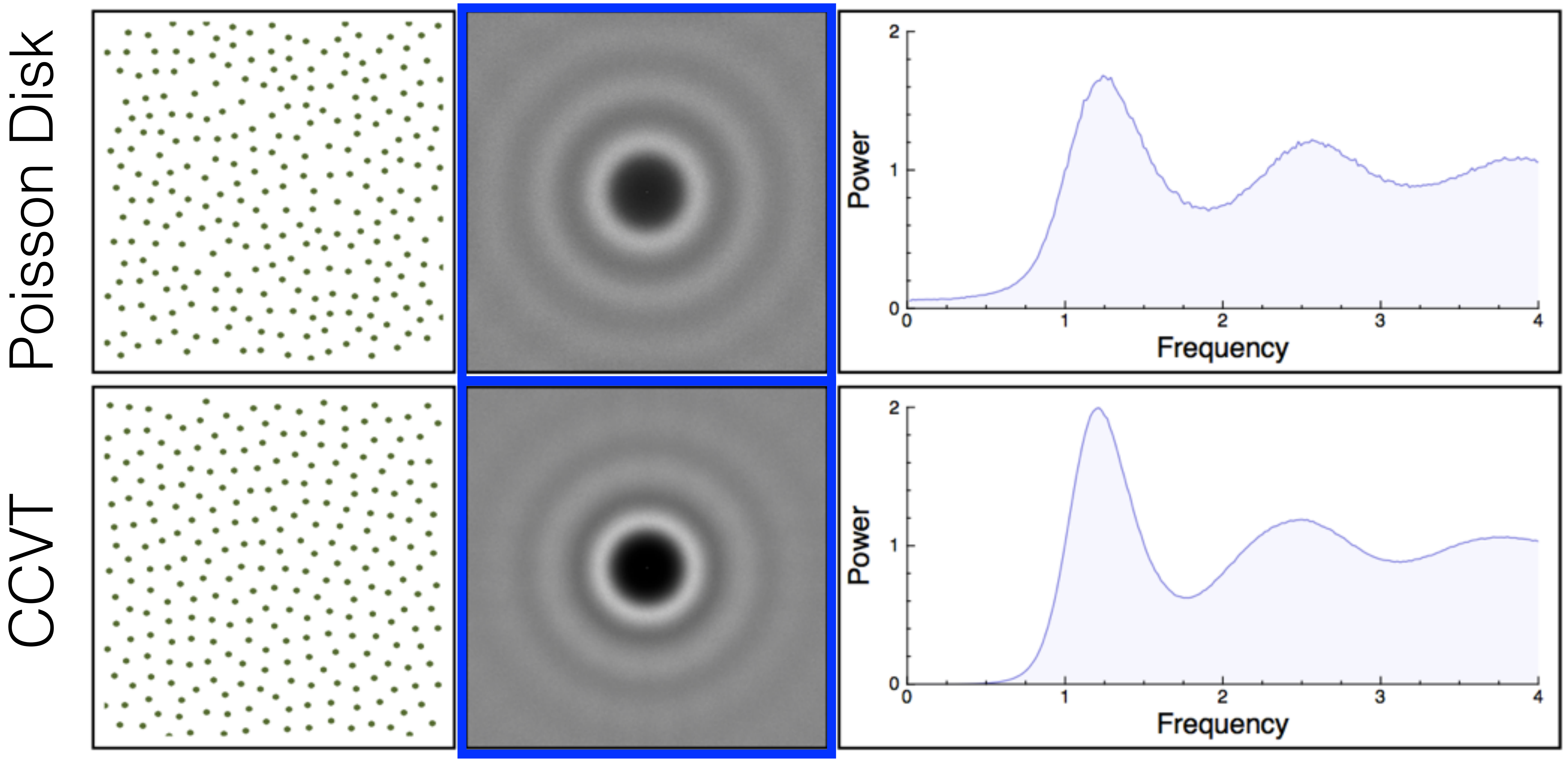
$$P_S(\omega) = |\hat{\mathbf{S}}(\omega)|^2$$

# Variance using Homogenized Samples

$$\boxed{\mathrm{Var}(\tilde{\mu}_N) = \int_{\Omega} P_f(\omega)\ \langle P_S(\omega)\rangle\, d\omega}$$

# Variance in terms of n-dimensional Power Spectra

$$\mathrm{Var}(\tilde{\mu}_N) = \int_\Omega P_f(\omega)\, \langle P_S(\omega) \rangle \, d\omega$$



# Variance in the Polar Coordinates

$$\boxed{\mathrm{Var}(\tilde{\mu}_N) = \int_{\Omega} P_f(\omega)\ \langle P_S(\omega) \rangle \, d\omega}$$

In polar coordinates:

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^{\infty} \int_{\mathcal{S}^{d-1}} P_f(\rho\mathbf{n}) \, \langle P_{\mathbf{S}}(\rho\mathbf{n}) \rangle \, d\mathbf{n} \, d\rho$$

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In polar coordinates:

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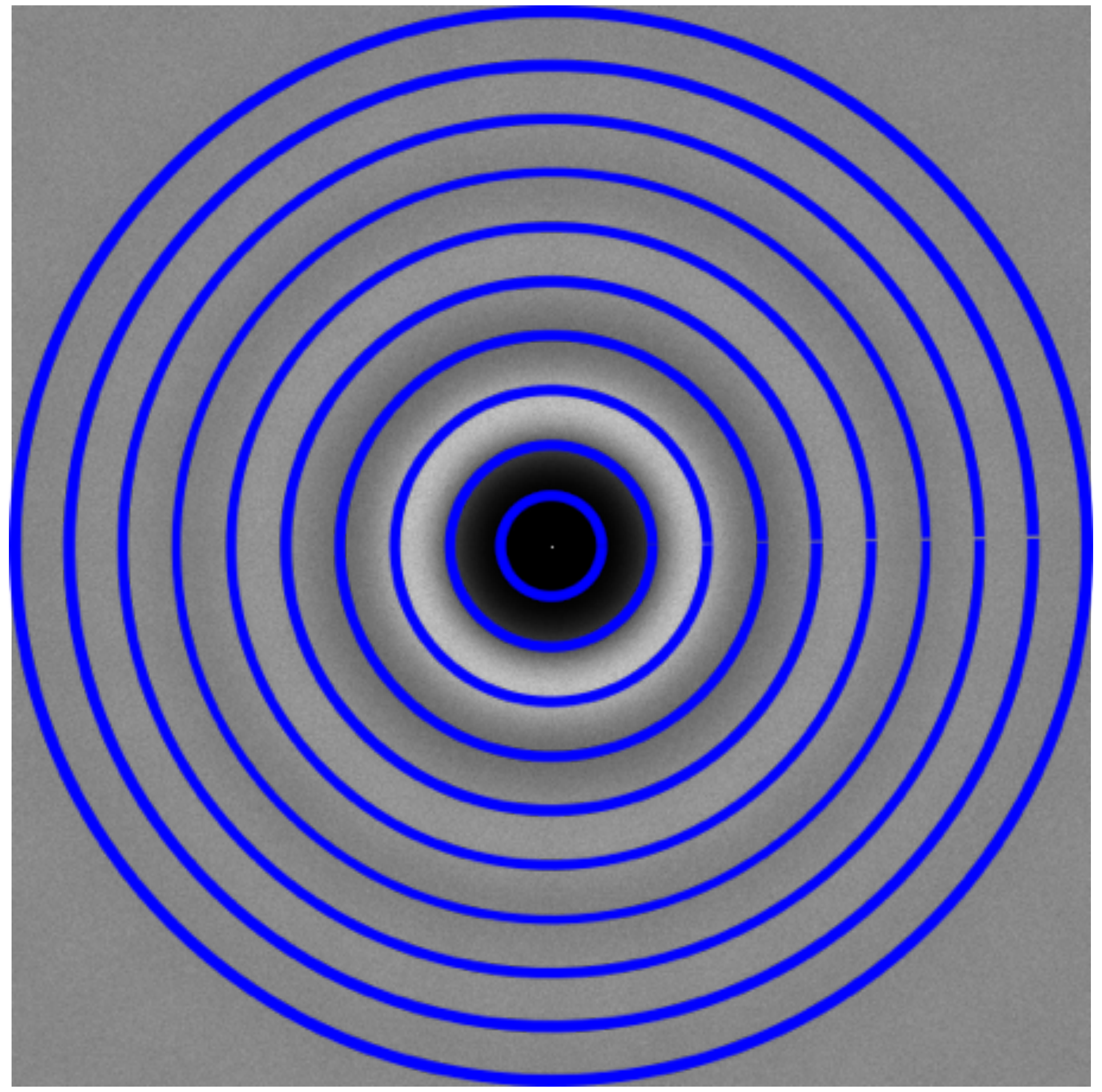
# Variance for Isotropic Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^{\infty} \int_{\mathcal{S}^{d-1}} P_f(\rho\mathbf{n}) \left\langle P_{\mathbf{S}}(\rho\mathbf{n}) \right\rangle d\mathbf{n}\, d\rho$$

For isotropic power spectra:

$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^{\infty} \tilde{P}_f(\rho) \left\langle \tilde{P}_{\mathbf{S}}(\rho) \right\rangle d\rho}$$



# Variance for Isotropic Power Spectra

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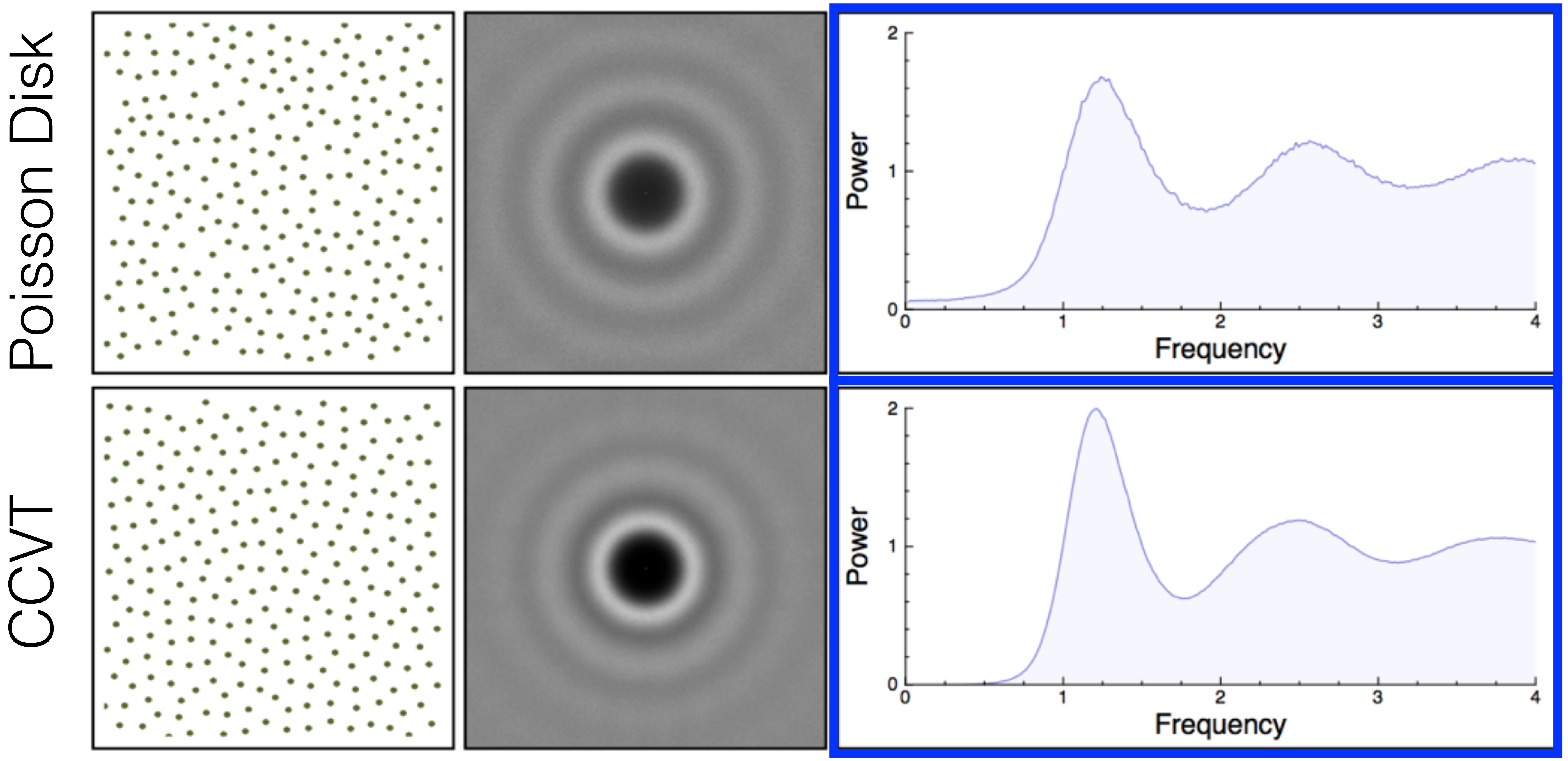
For isotropic power spectra:

$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \left\langle \tilde{P}_{\mathbf{S}}(\rho) \right\rangle d\rho}$$

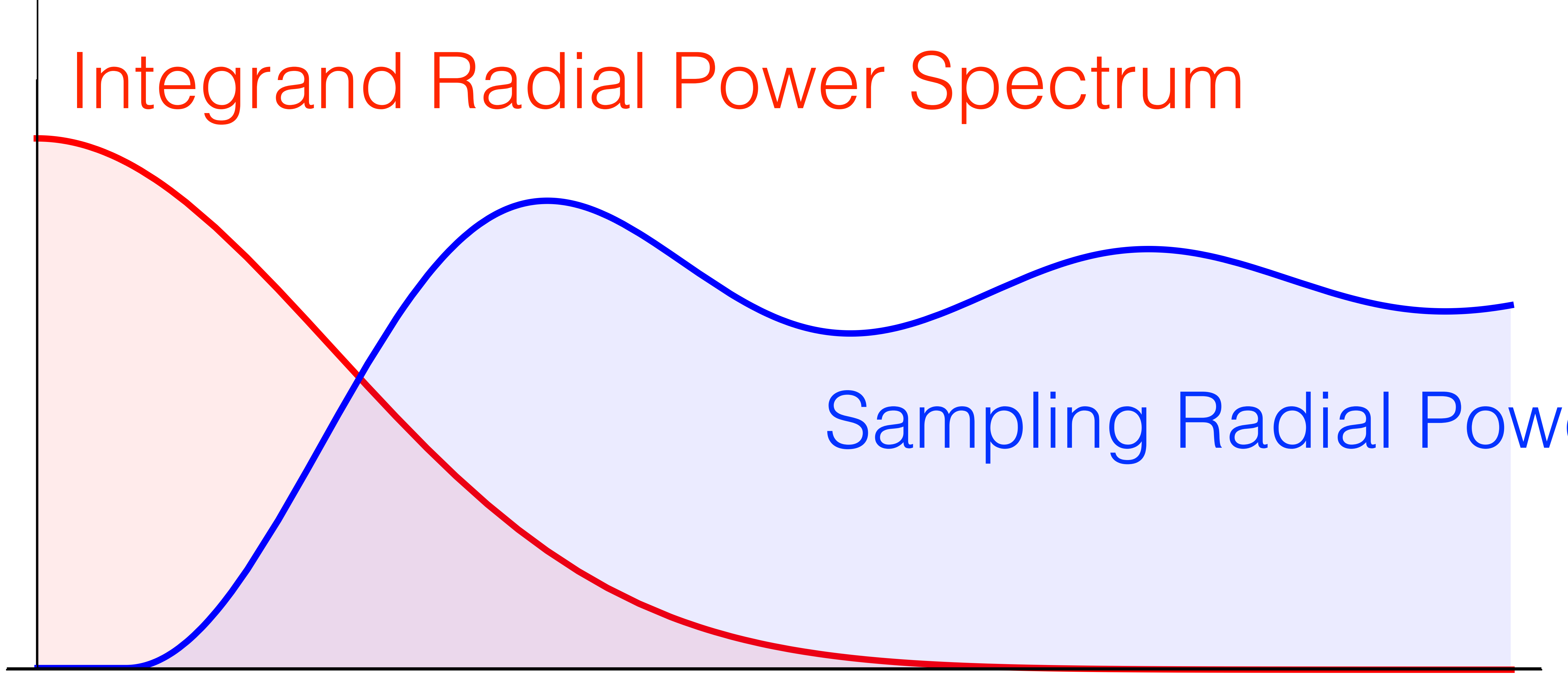
# Variance in terms of 1-dimensional Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \, \langle \tilde{P}_{\mathbf{S}}(\rho) \rangle \, d\rho$$



# Variance: Integral over Product of Power Spectra

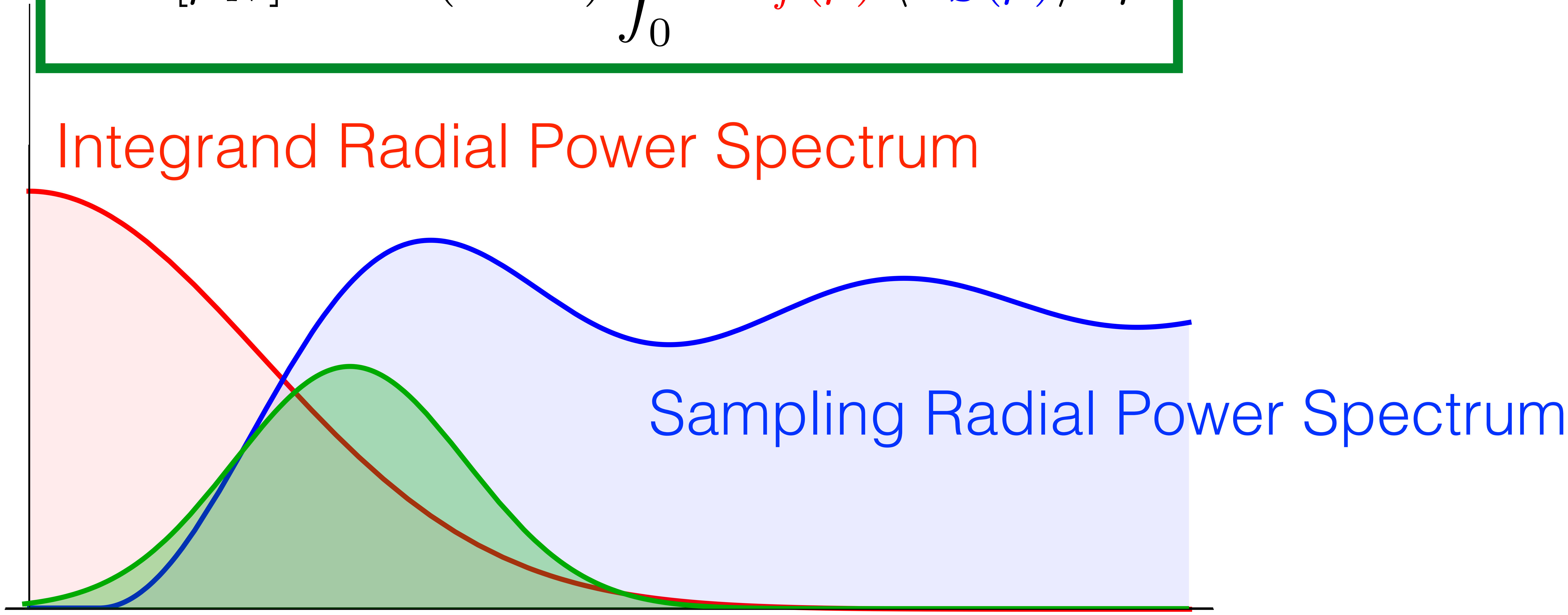
$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \color{red}{\tilde{P}_f(\rho)} \, \langle \color{blue}{\tilde{P}_{\mathbf{S}}(\rho)} \rangle \, d\rho}$$




For given number of Samples

# Variance: Integral over Product of Power Spectra

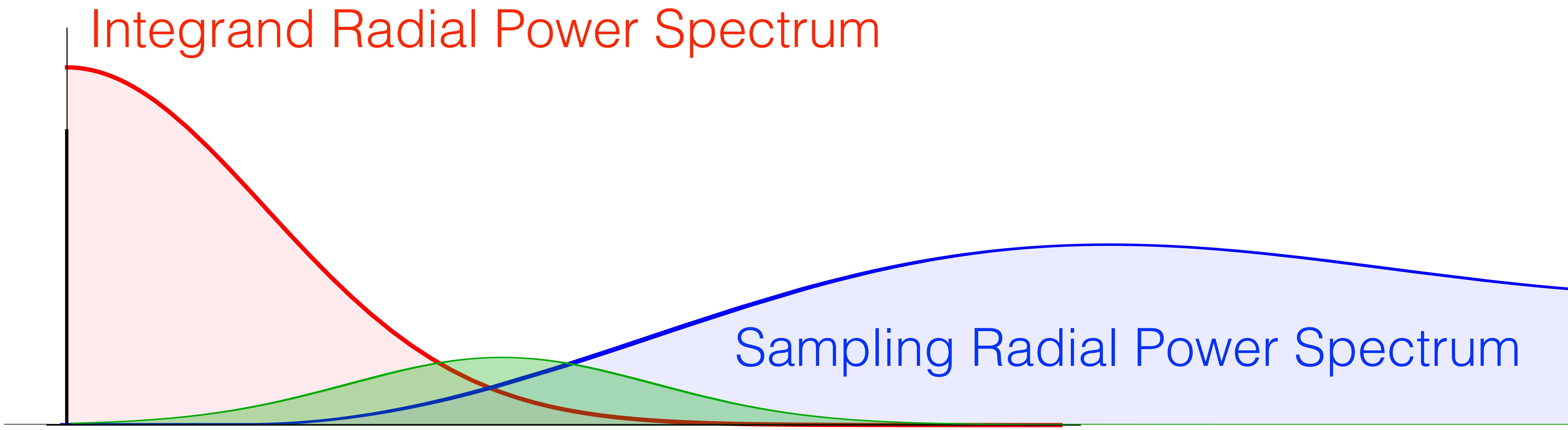
$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^{\infty} \tilde{P}_f(\rho) \left\langle \tilde{P}_{\mathbf{S}}(\rho) \right\rangle d\rho}$$




For given number of Samples

# Variance: Integral over Product of Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \left\langle \tilde{P}_{\mathbf{S}}(\rho) \right\rangle d\rho$$




For given number of Samples

# Variance: Integral over Product of Power Spectra

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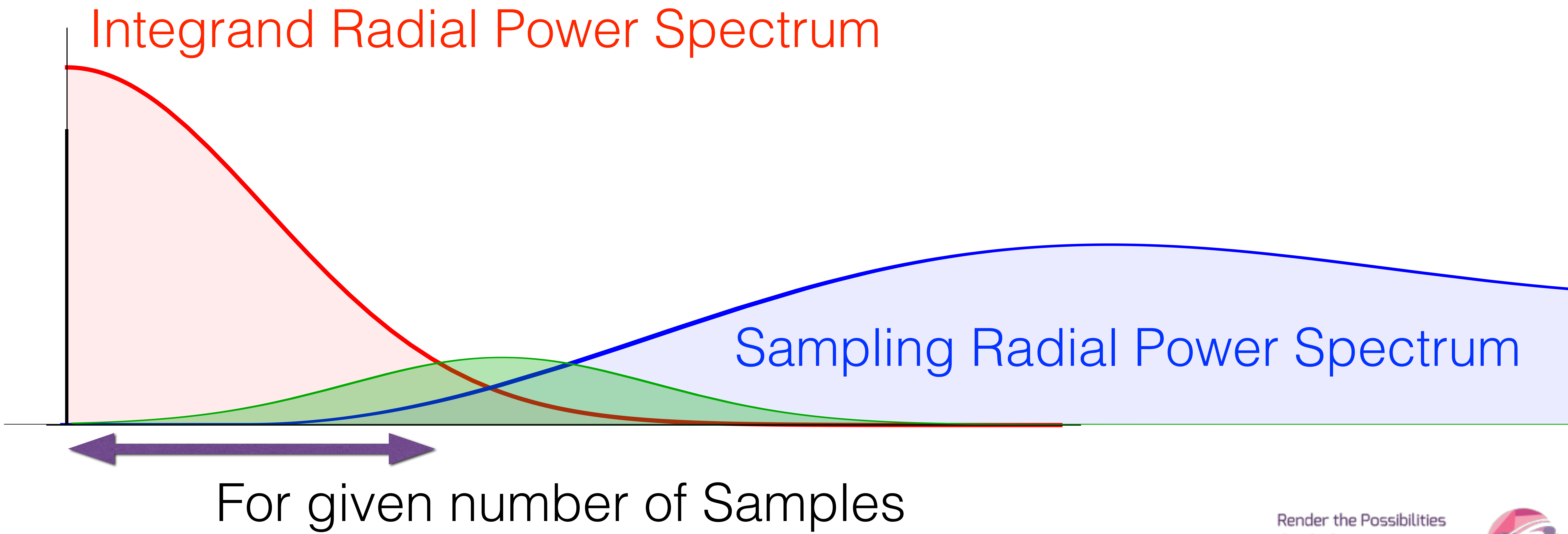
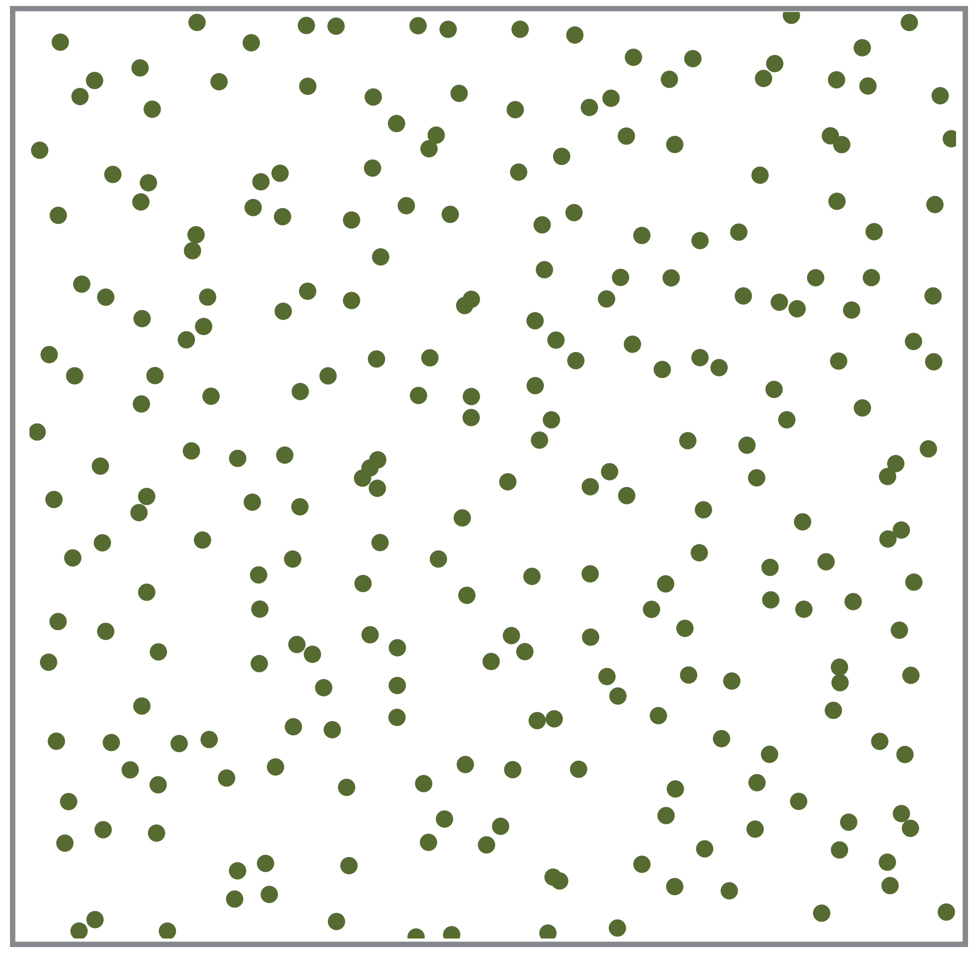
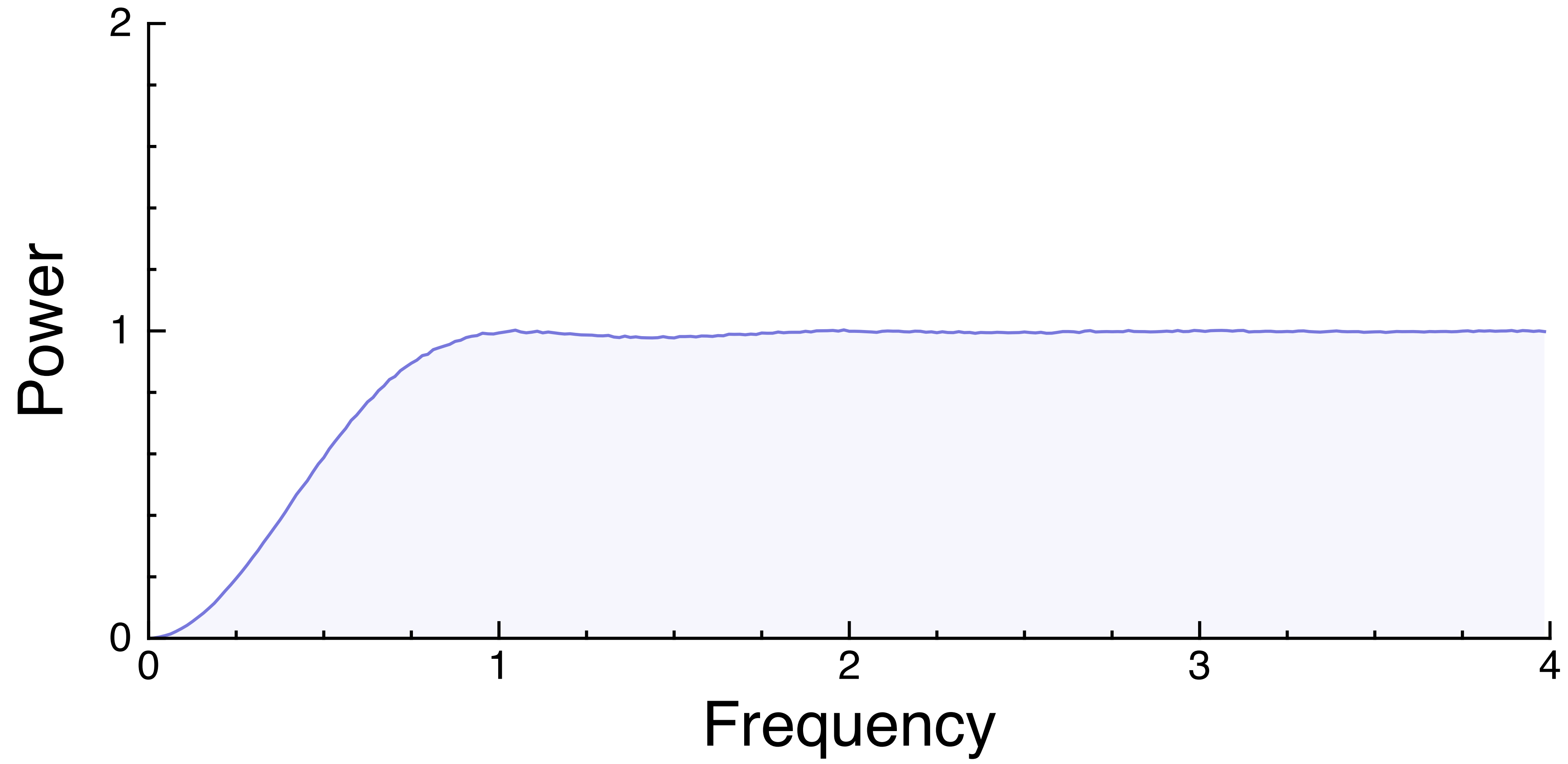


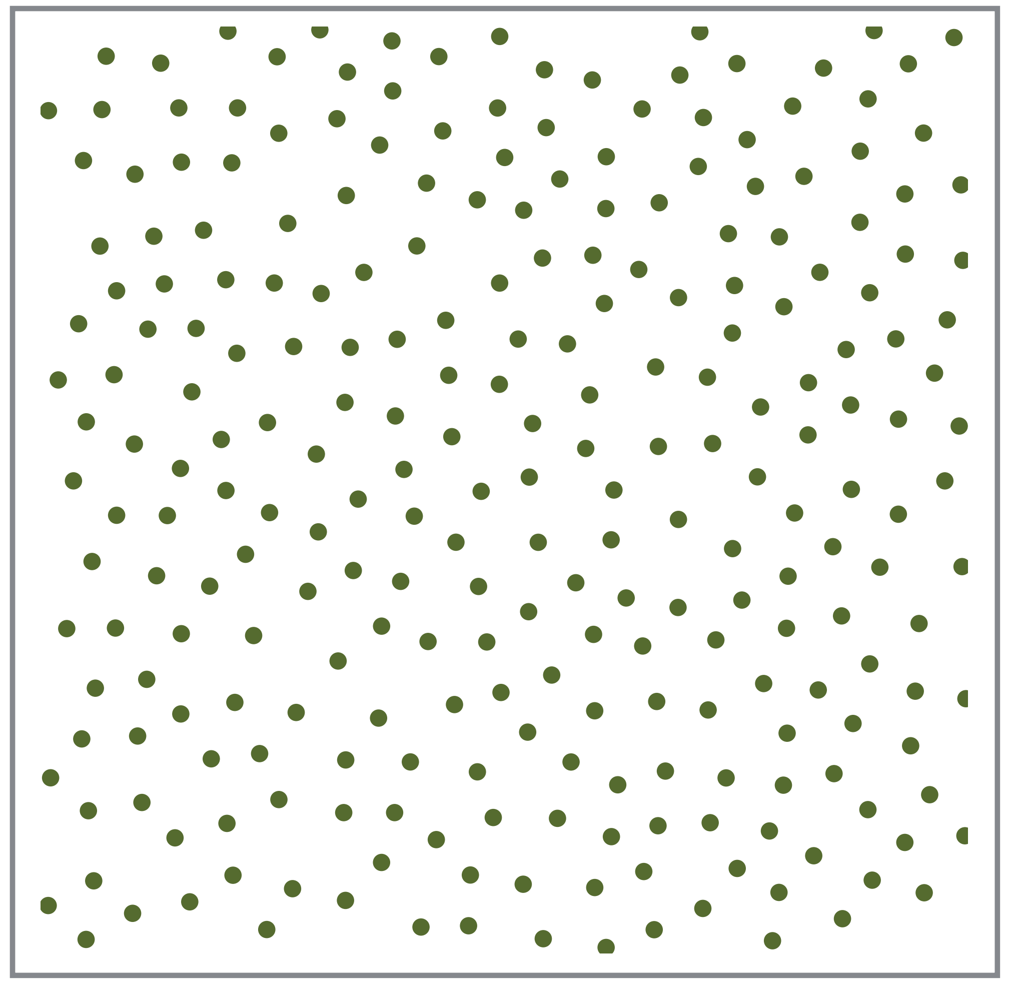
# Spatial Distribution vs Radial Mean Power Spectra

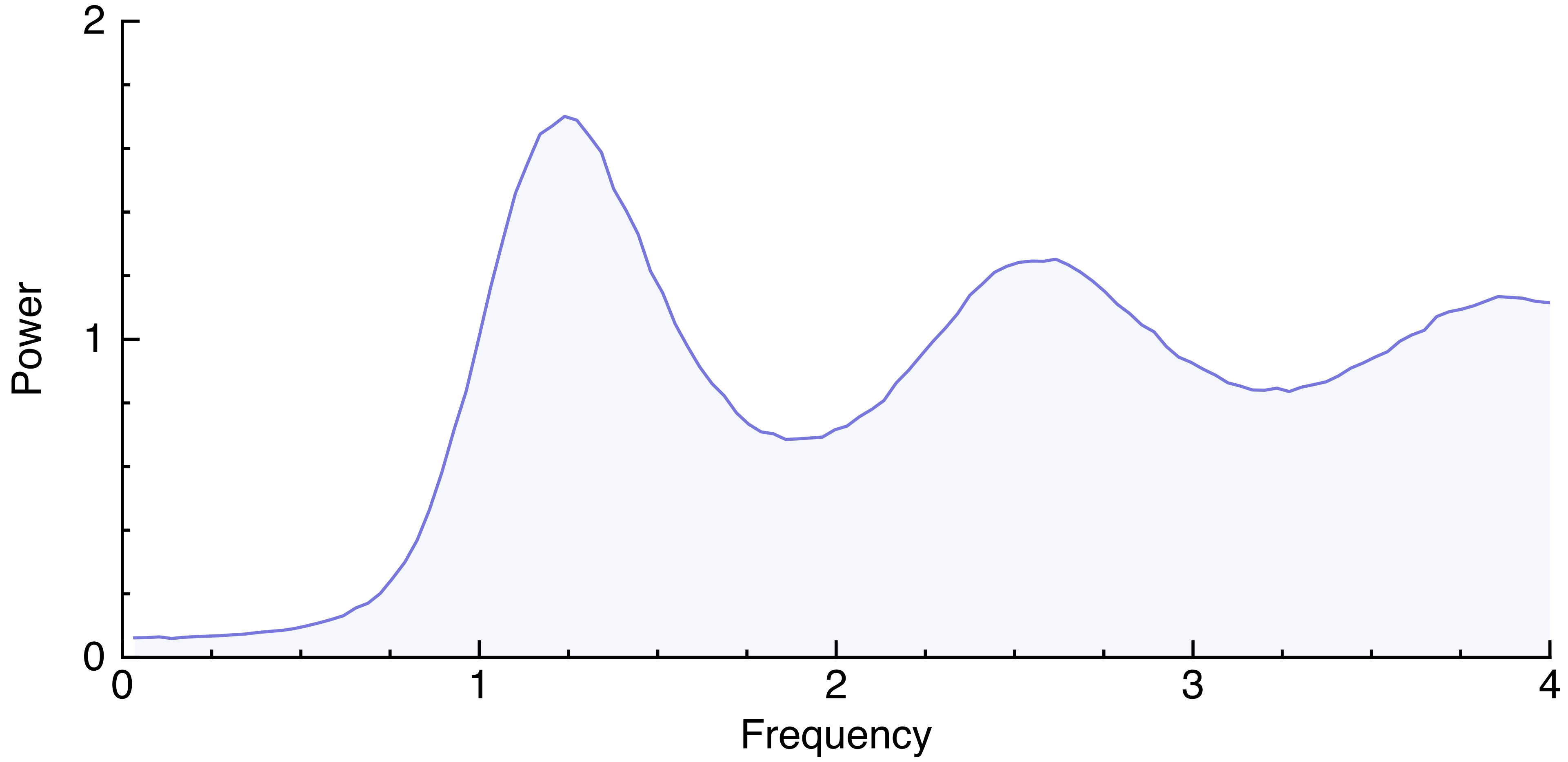
Jitter






Poisson Disk





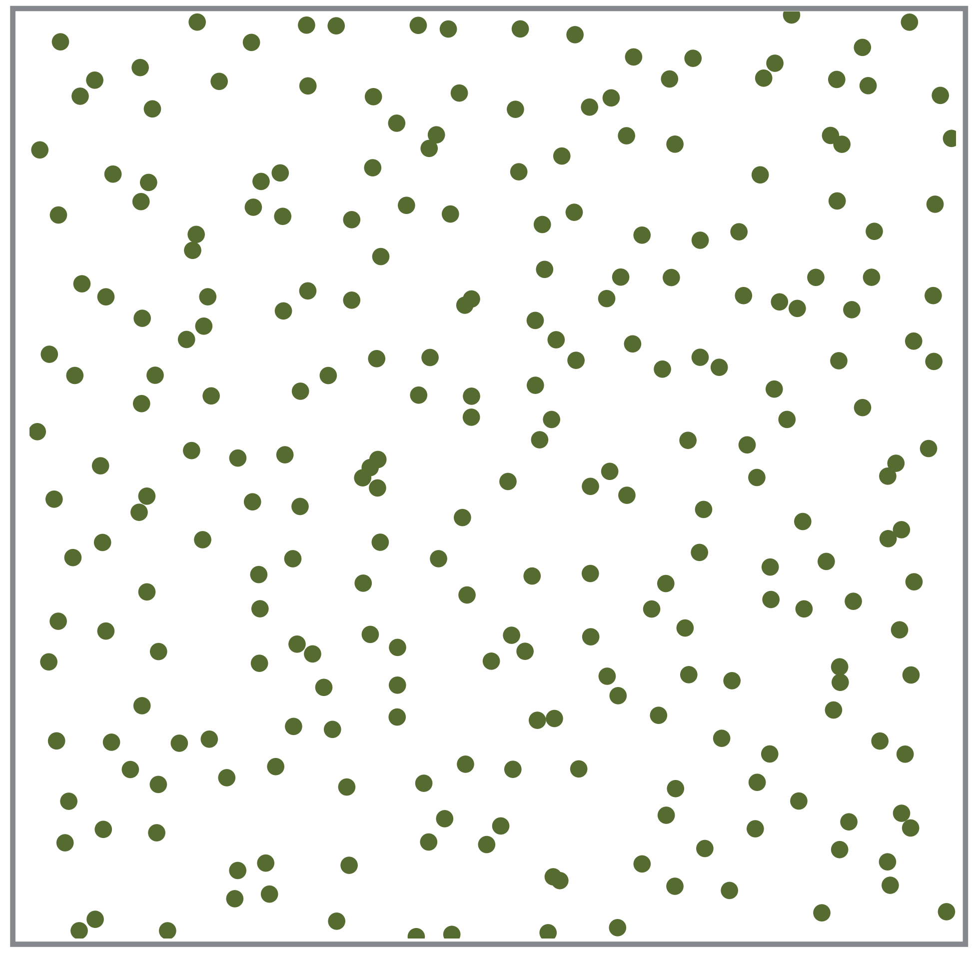
# For 2-dimensions

| Samplers | Worst Case | Best Case |
|---|---|---|
| Random | $\mathcal{O}(N^{-1})$ | $\mathcal{O}(N^{-1})$ |
| Jitter | $\mathcal{O}(N^{-1.5})$ | $\mathcal{O}(N^{-2})$ |
| Poisson Disk | $\mathcal{O}(N^{-1})$ | $\mathcal{O}(N^{-1})$ |
| CCVT | $\mathcal{O}(N^{-1.5})$ | $\mathcal{O}(N^{-3})$ |

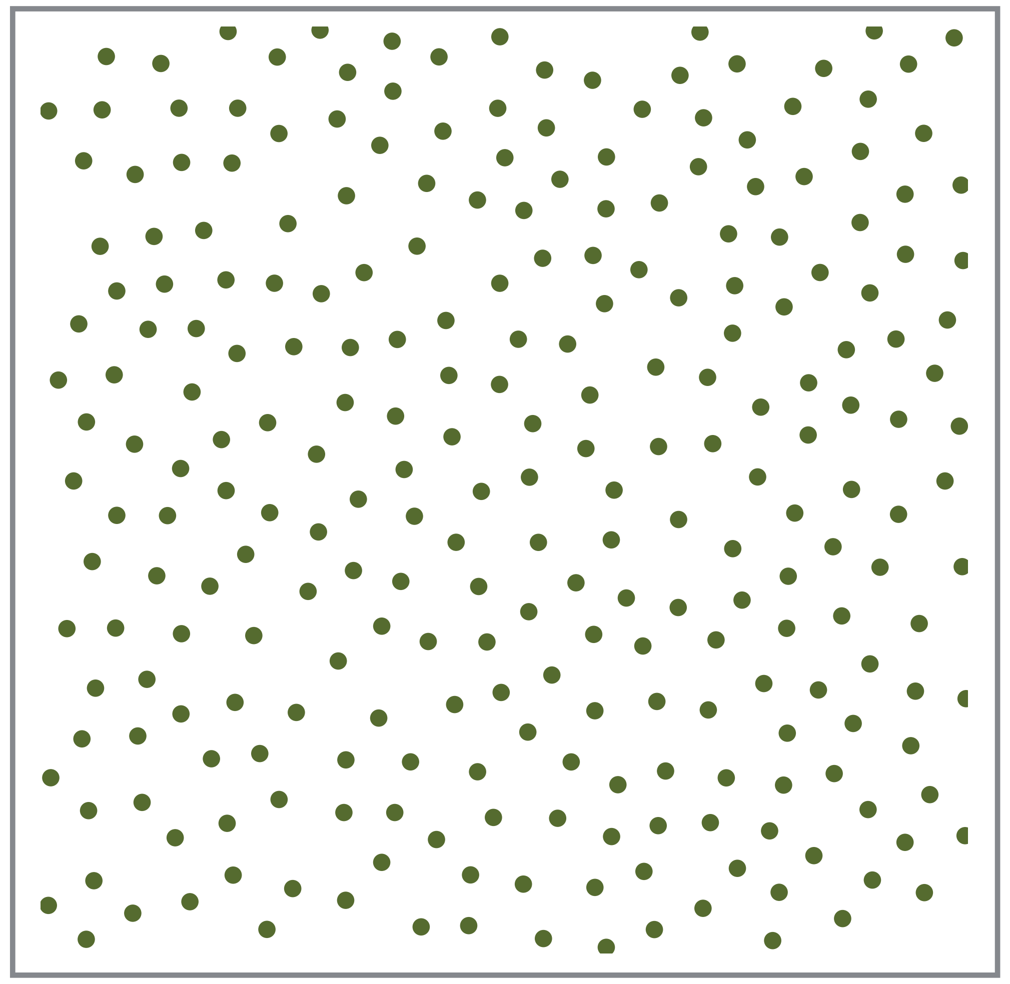
**Pilleboue et al. [2015]**

# For 2-dimensions

Jitter



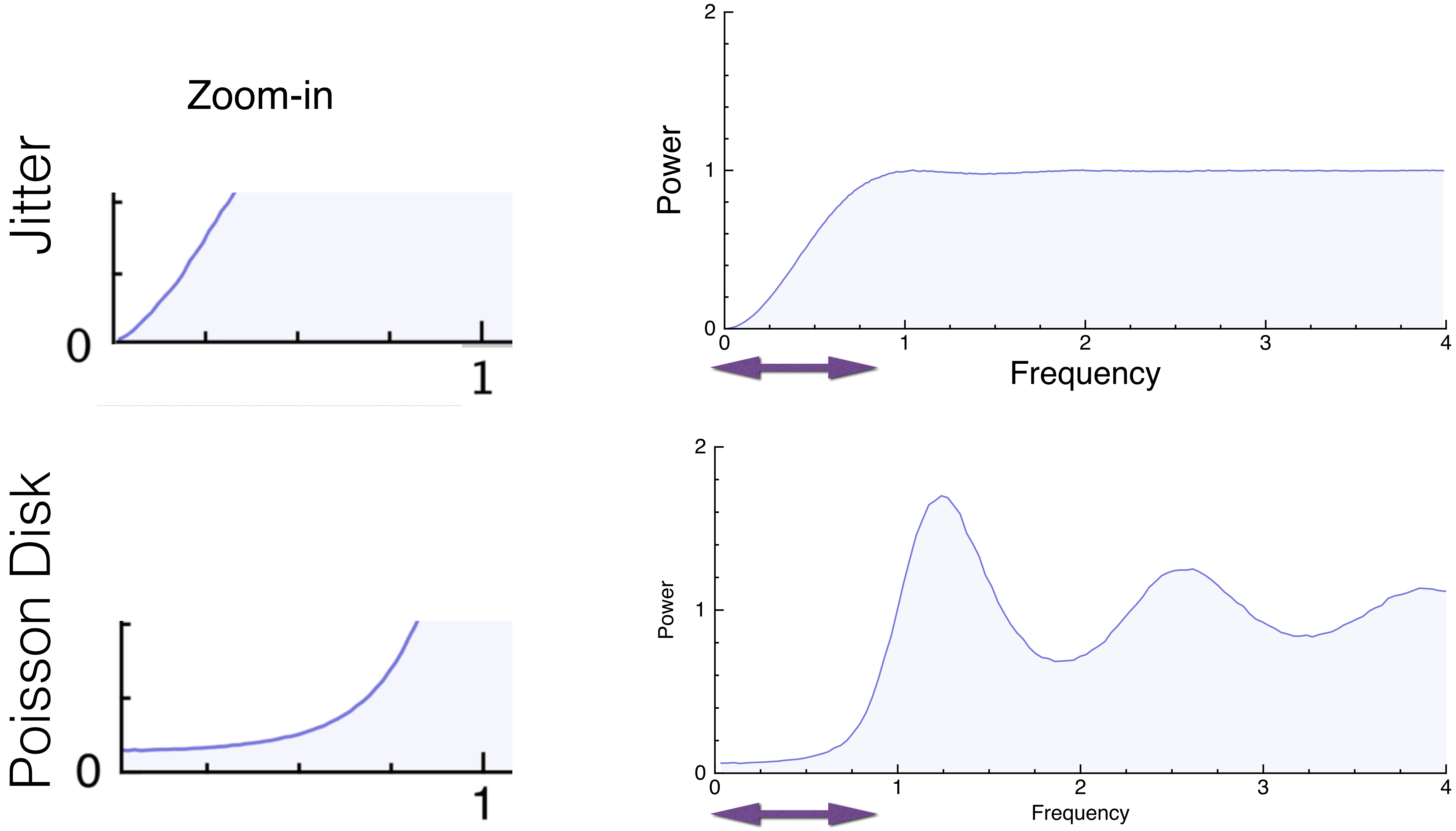
Poisson Disk



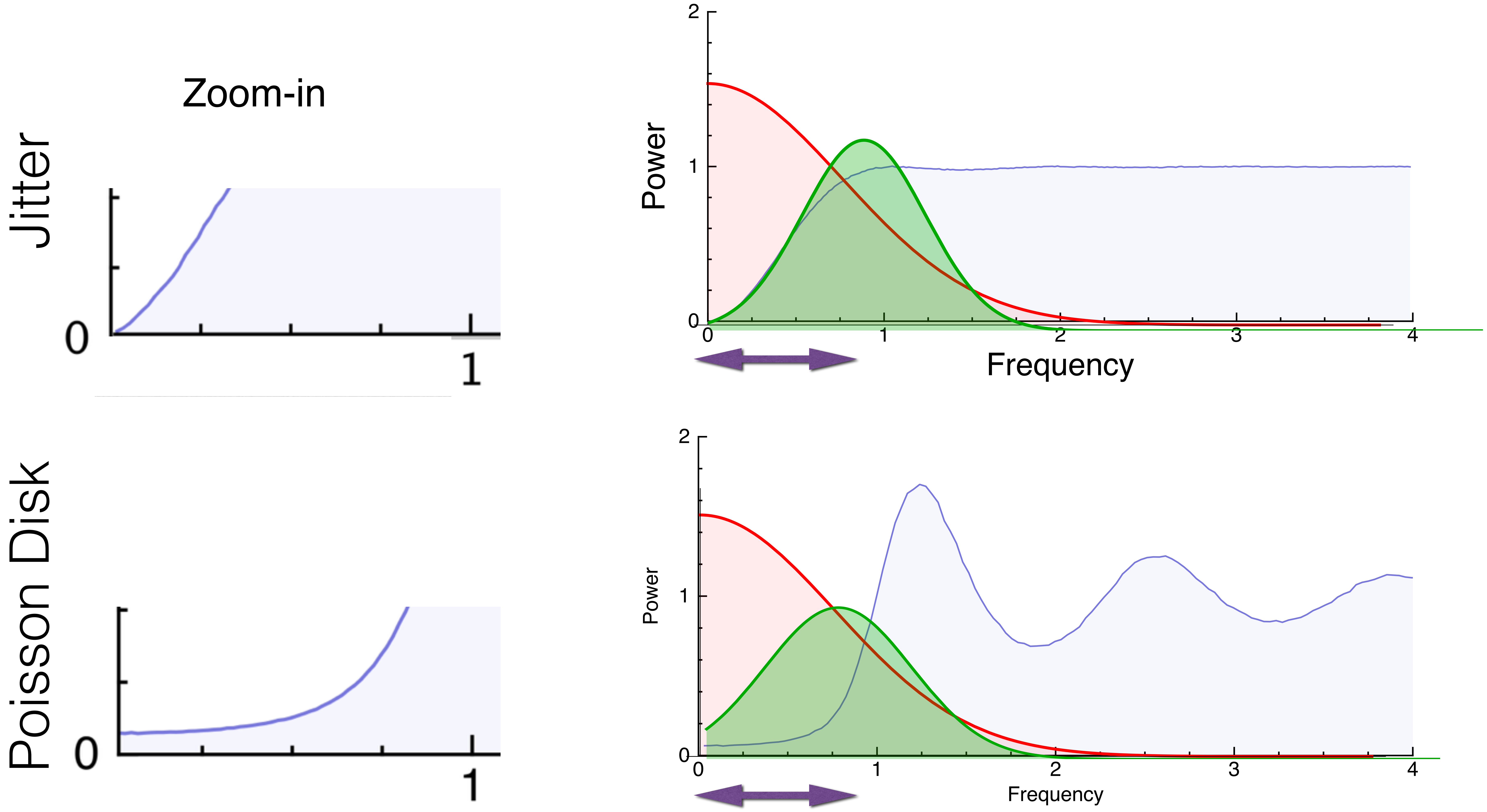
| Samplers | Worst Case | Best Case |
| --- | --- | --- |
| Random | $\mathcal{O}(N^{-1})$ | $\mathcal{O}(N^{-1})$ |
| Jitter | $\mathcal{O}(N^{-1.5})$ | $\mathcal{O}(N^{-2})$ |
| Poisson Disk | $\mathcal{O}(N^{-1})$ | $\mathcal{O}(N^{-1})$ |
| CCVT | $\mathcal{O}(N^{-1.5})$ | $\mathcal{O}(N^{-3})$ |

**Pilleboue et al. [2015]**

# Low Frequency Region



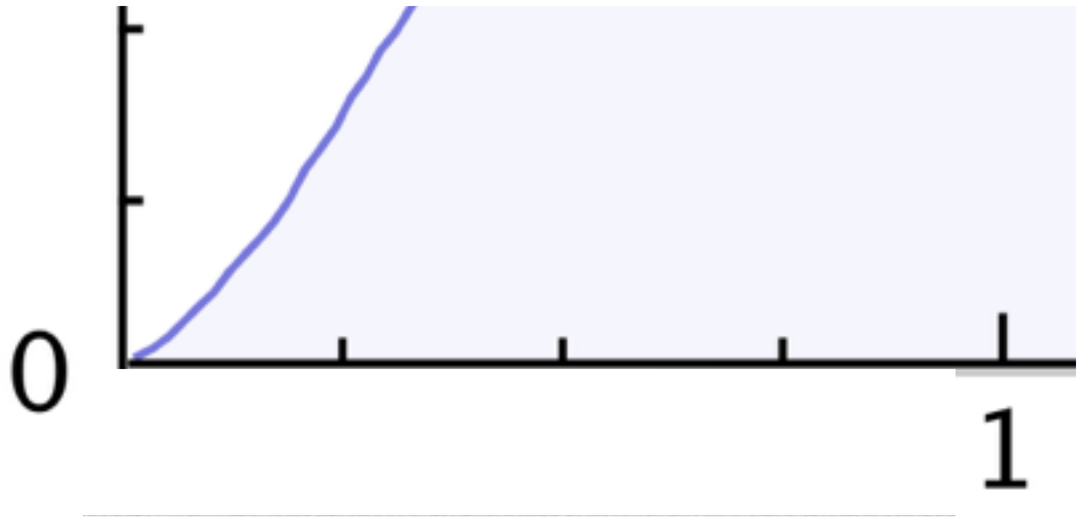
# Variance for Low Sample Count

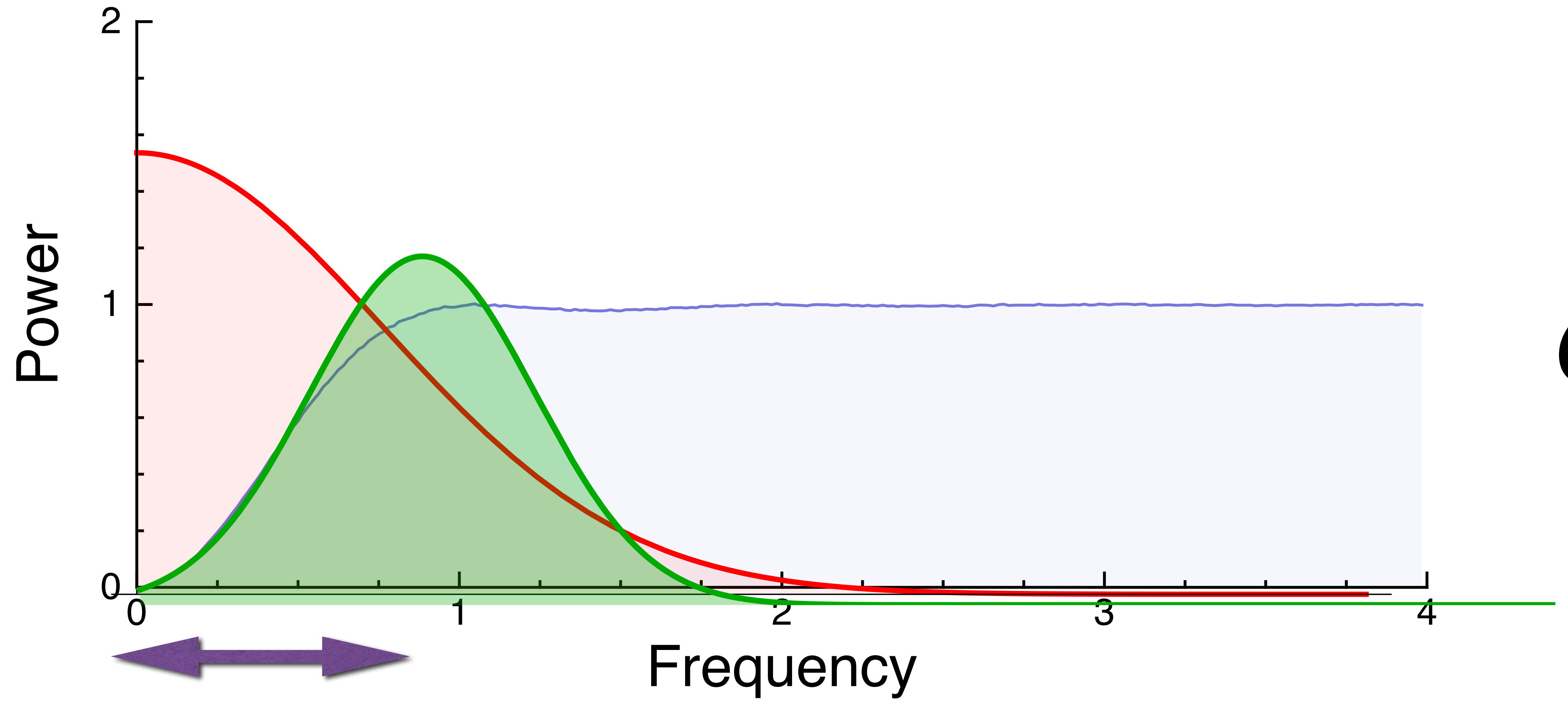


# Variance for Increasing Sample Count

Zoom-in

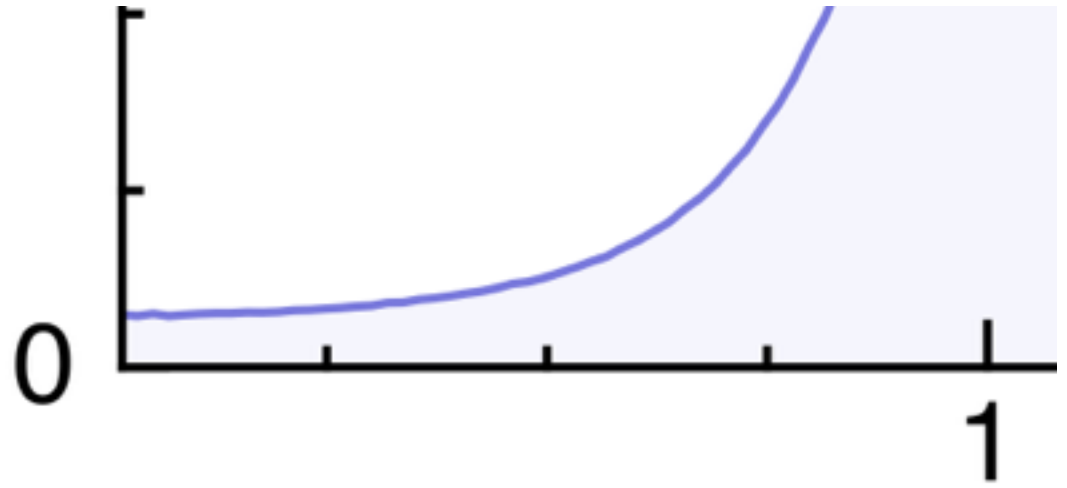
Jitter

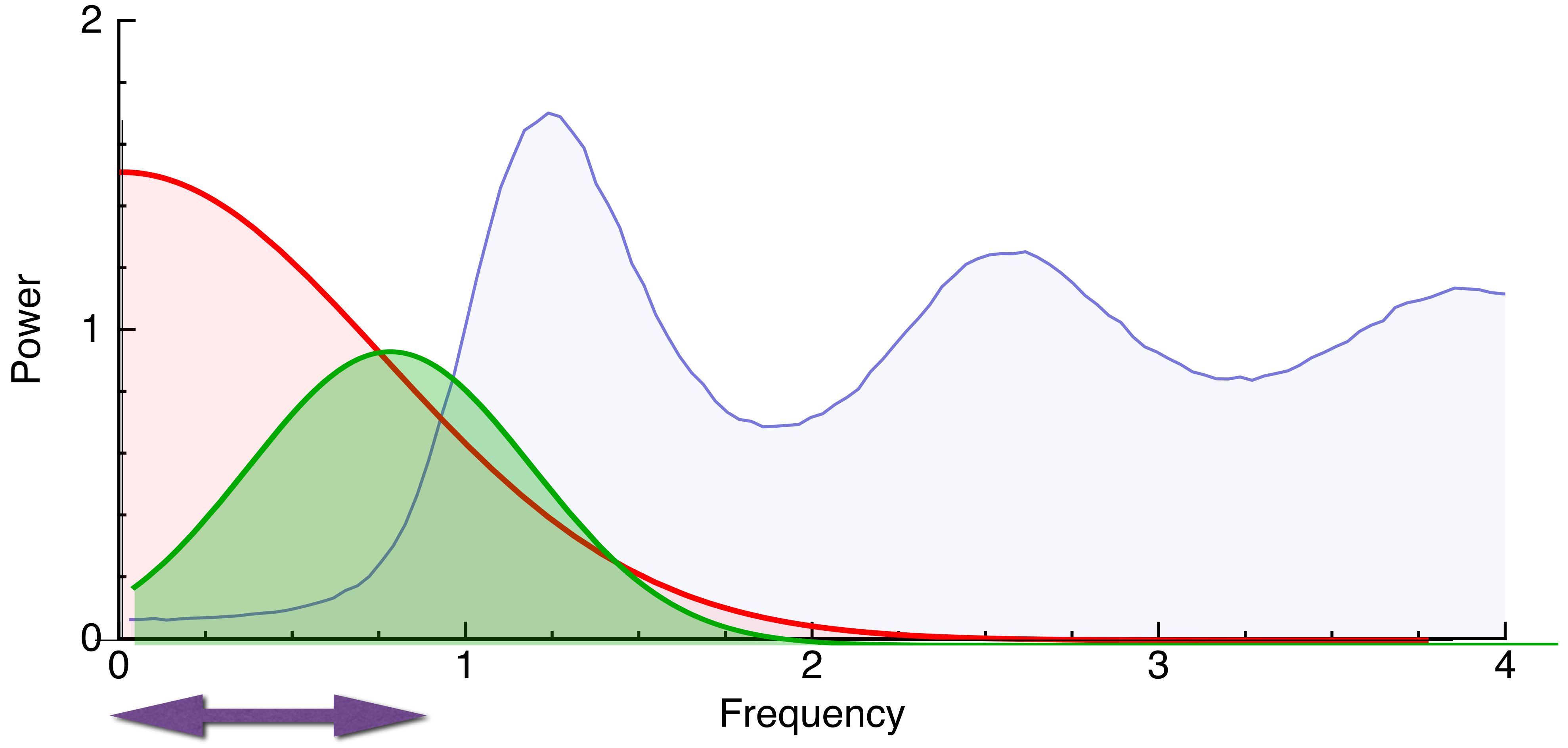



$\mathcal{O}(N^{-2})$

Poisson Disk

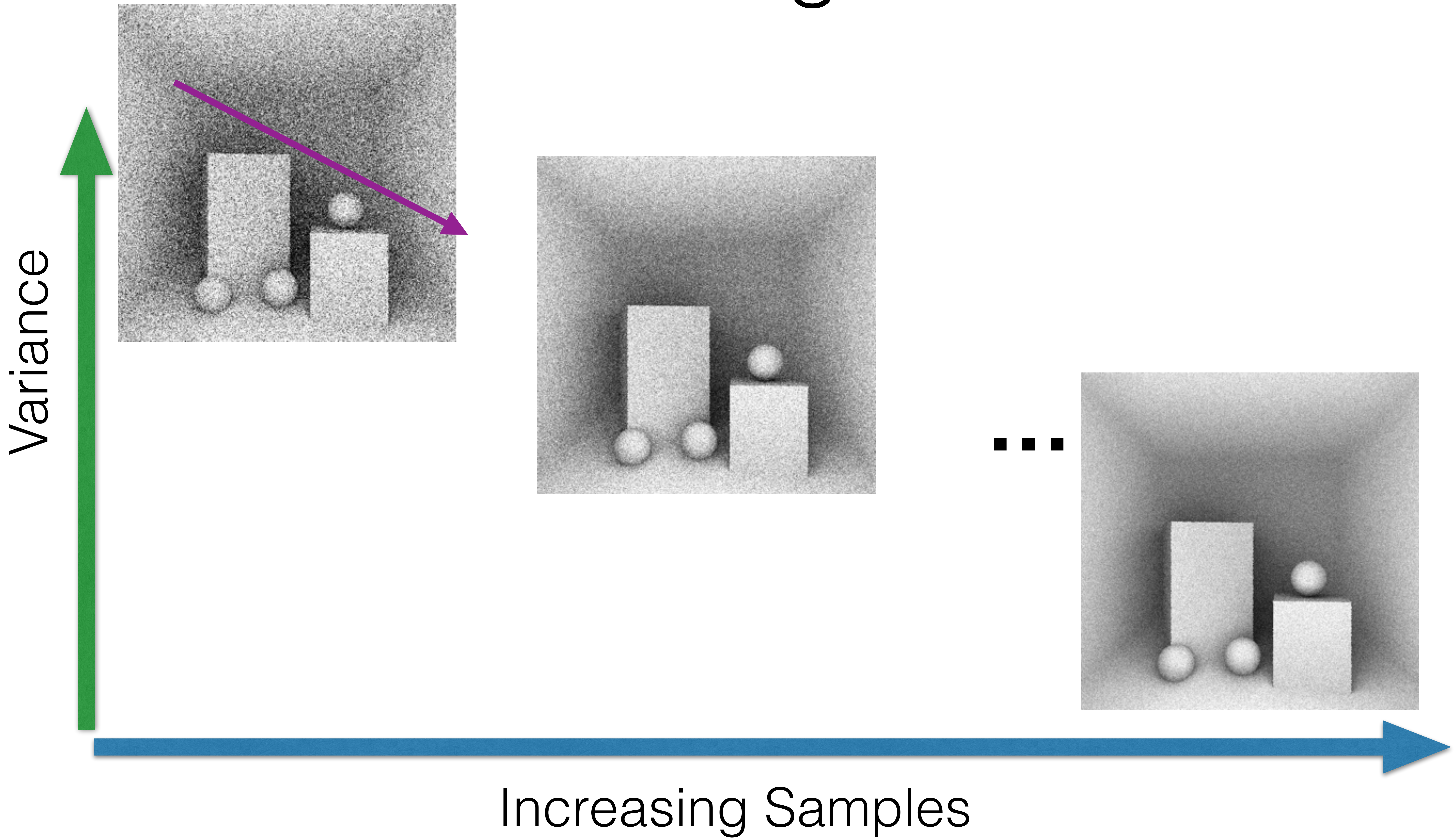


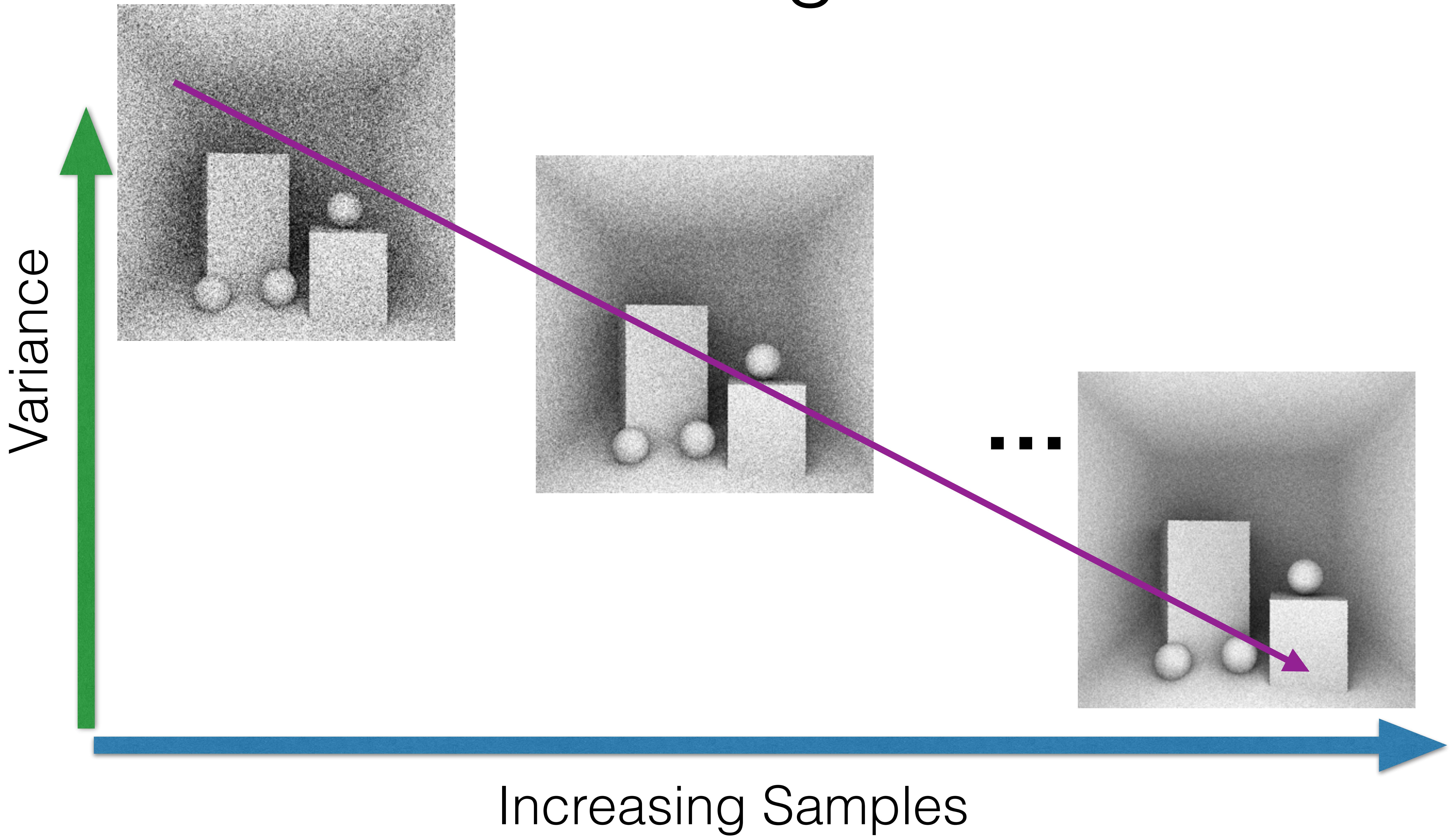

$\mathcal{O}(N^{-1})$

# Experimental Verification

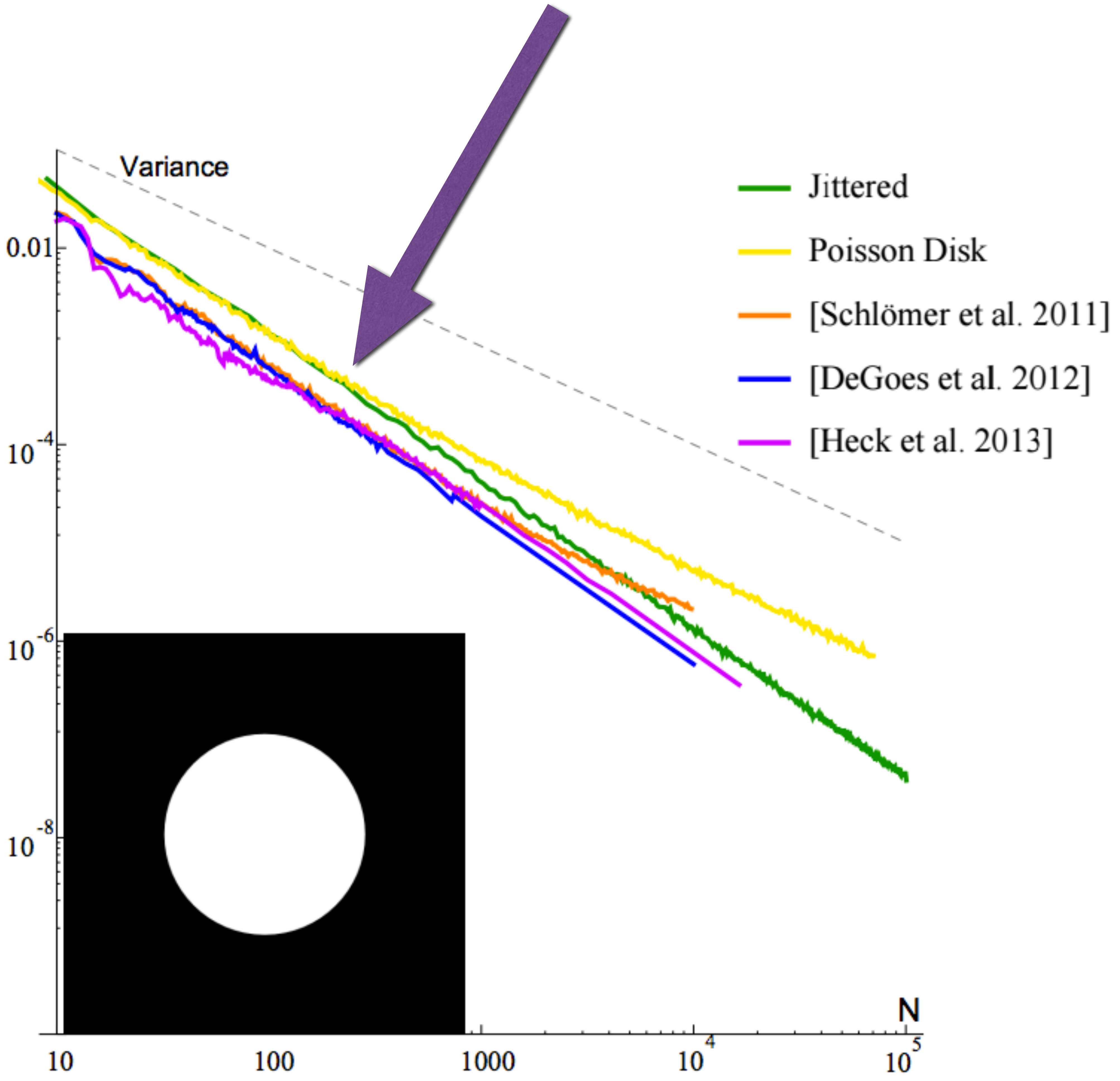
# Convergence rate



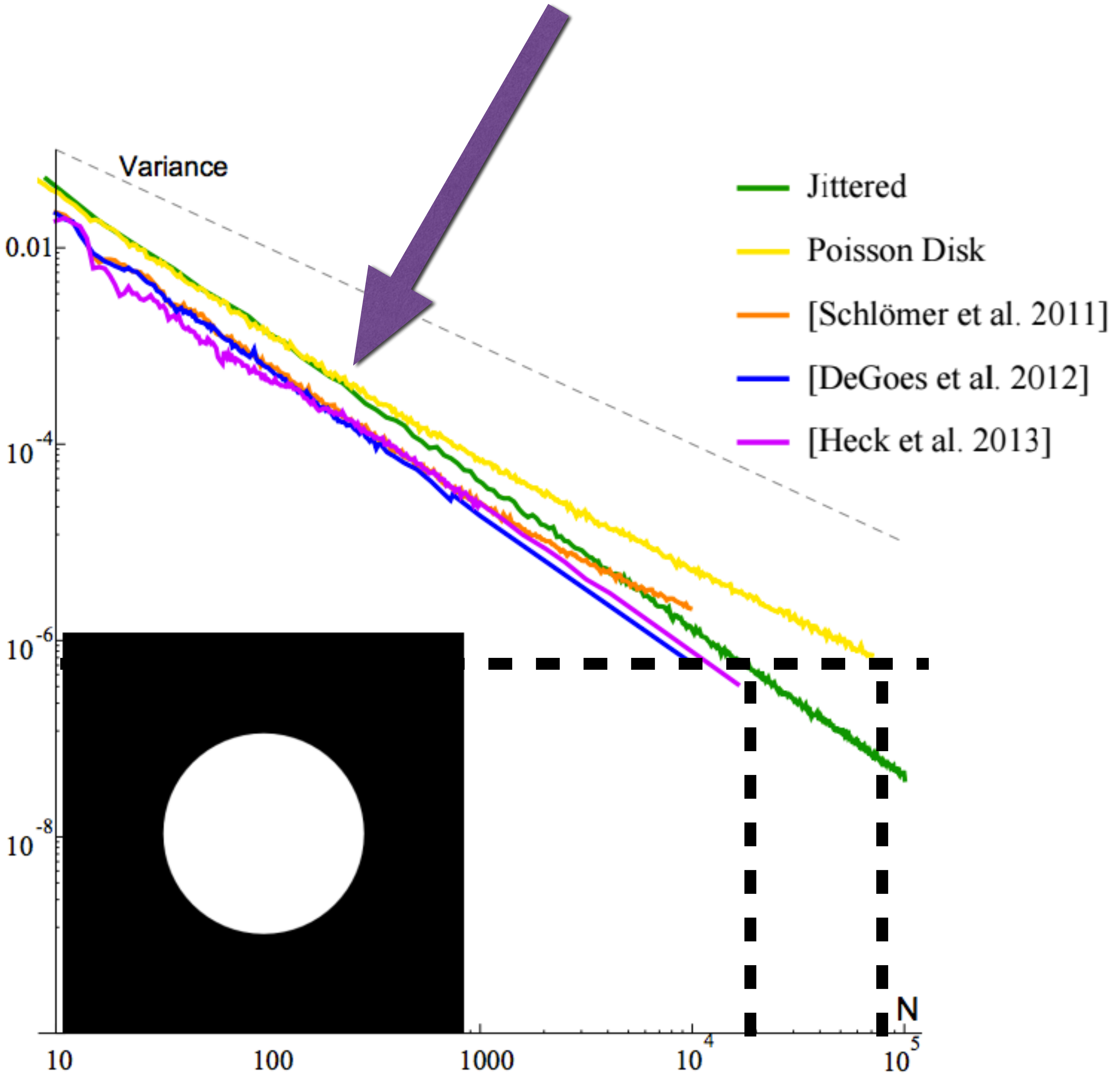
# Convergence rate



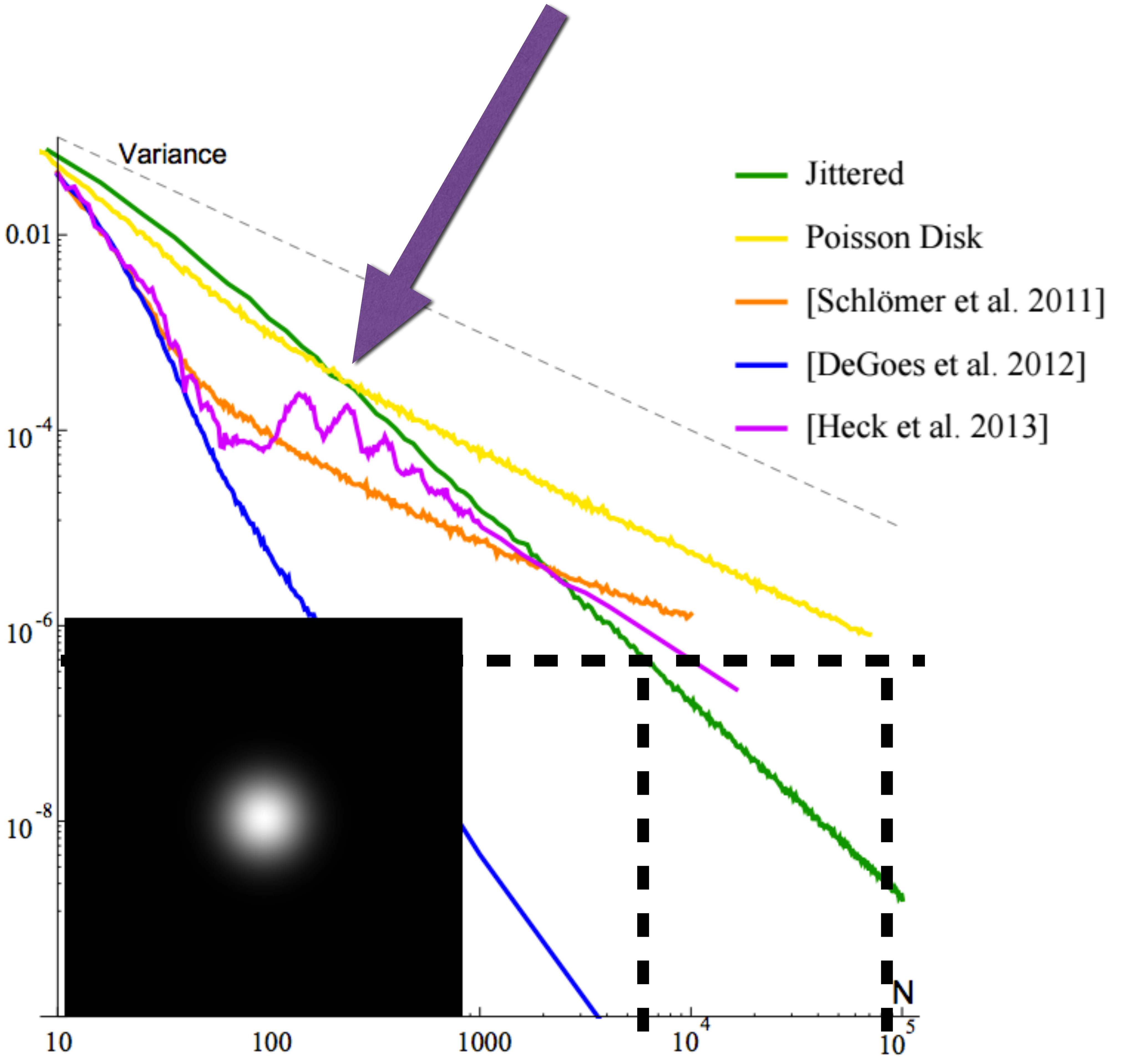
# Disk Function as Worst Case



# Disk Function as Worst Case



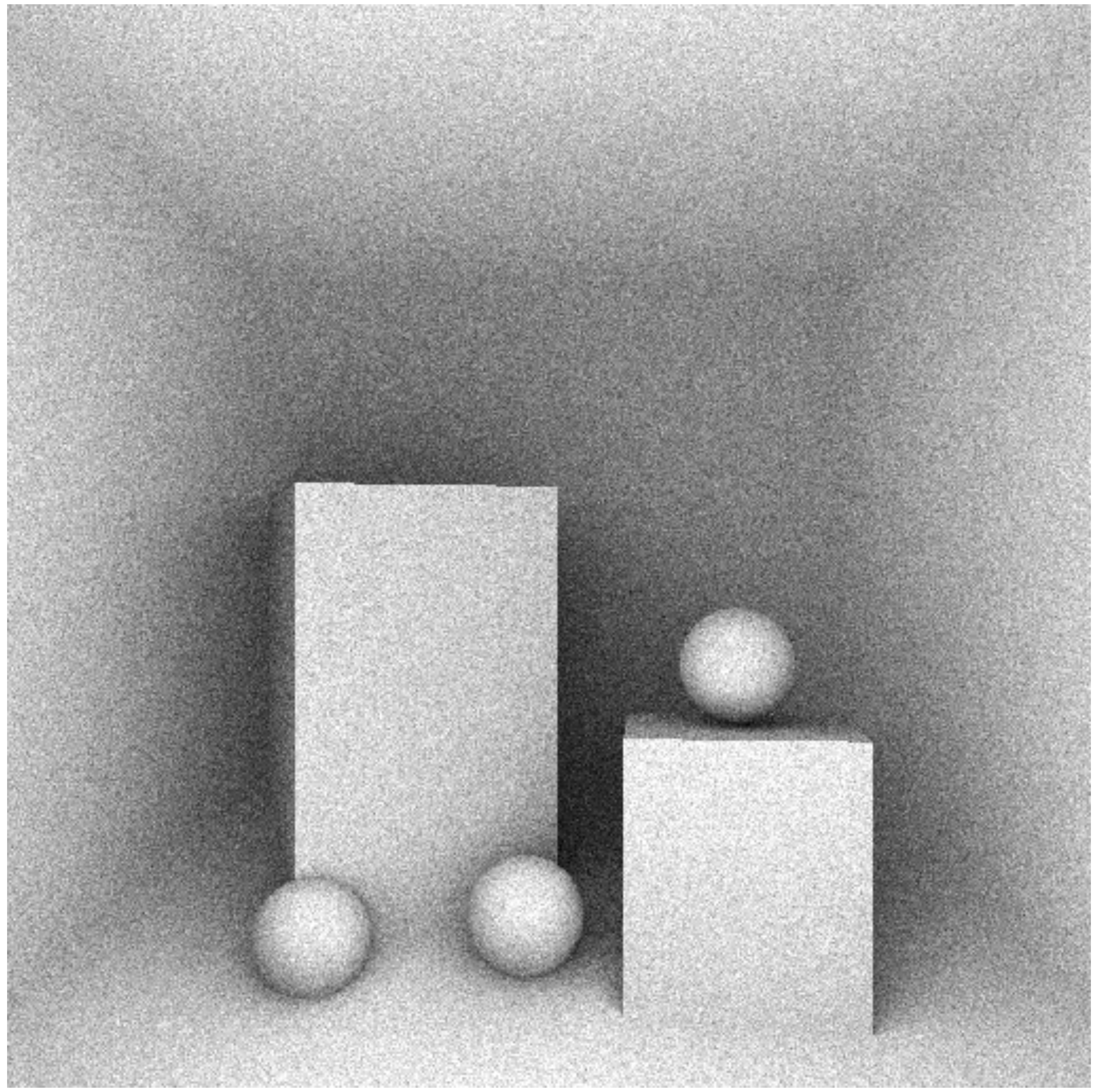
# Gaussian as Best Case



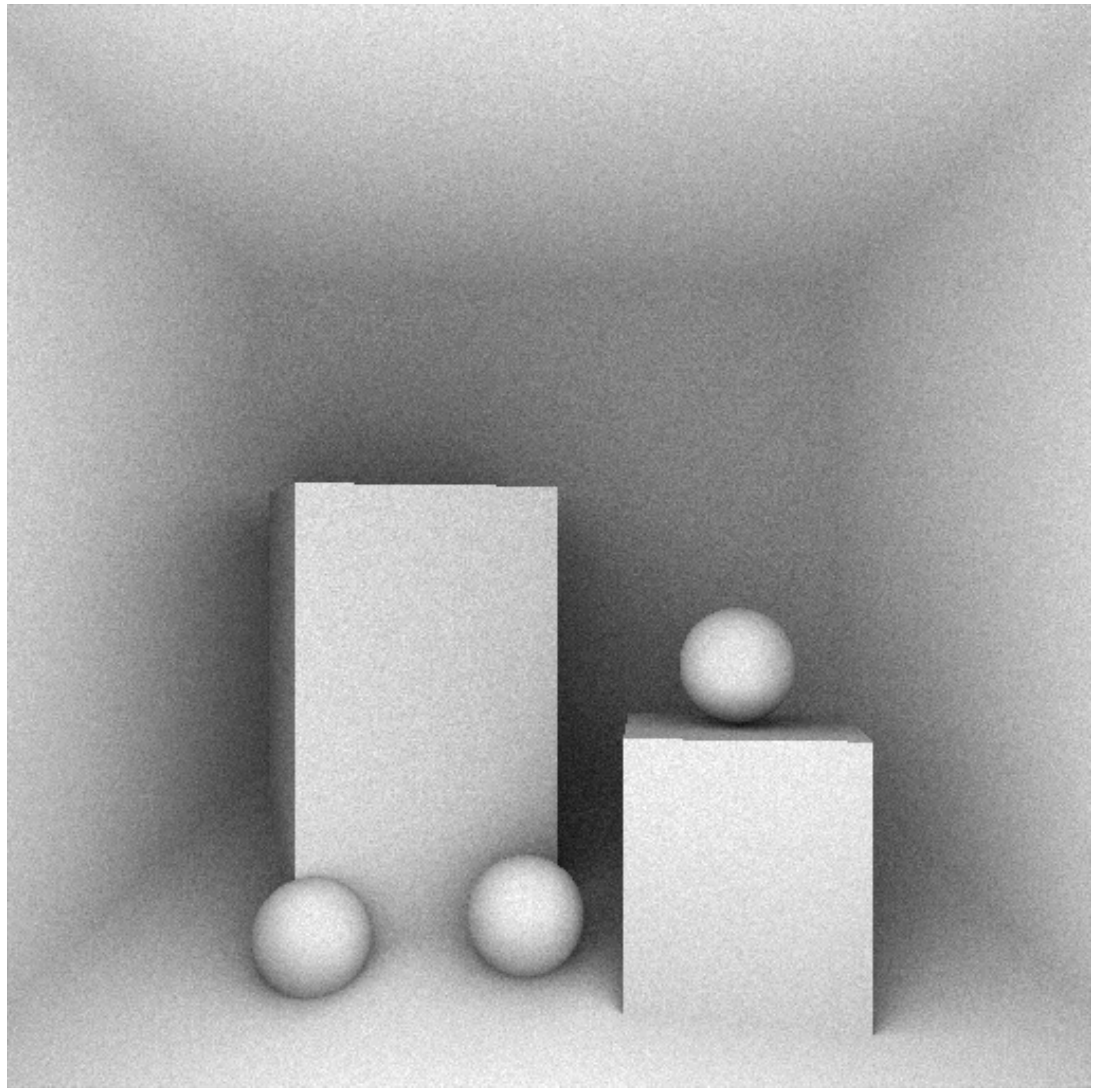
# Ambient Occlusion Examples

# Random vs Jittered

96 Secondary Rays



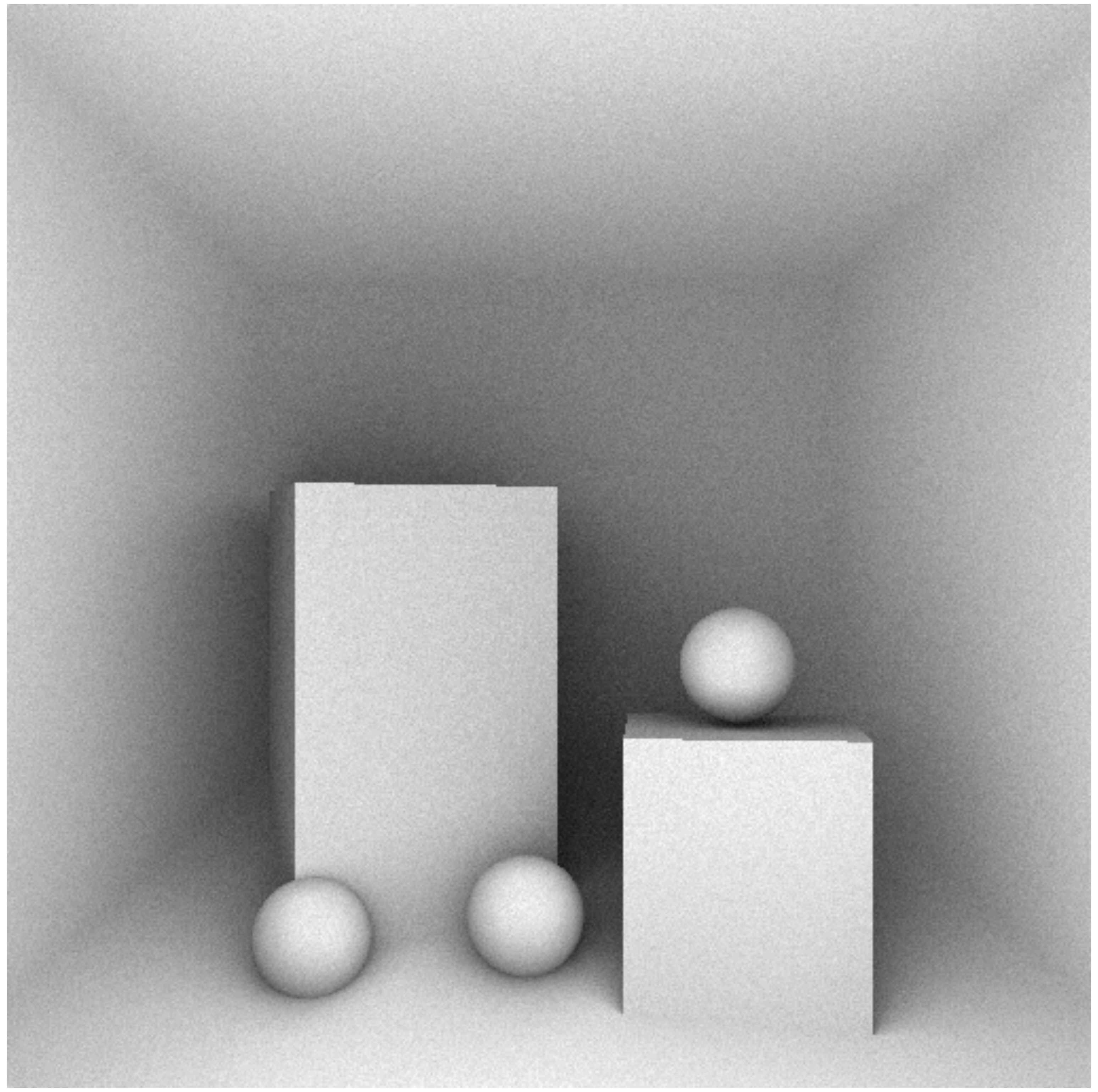
MSE: 4.74 x 10e-3



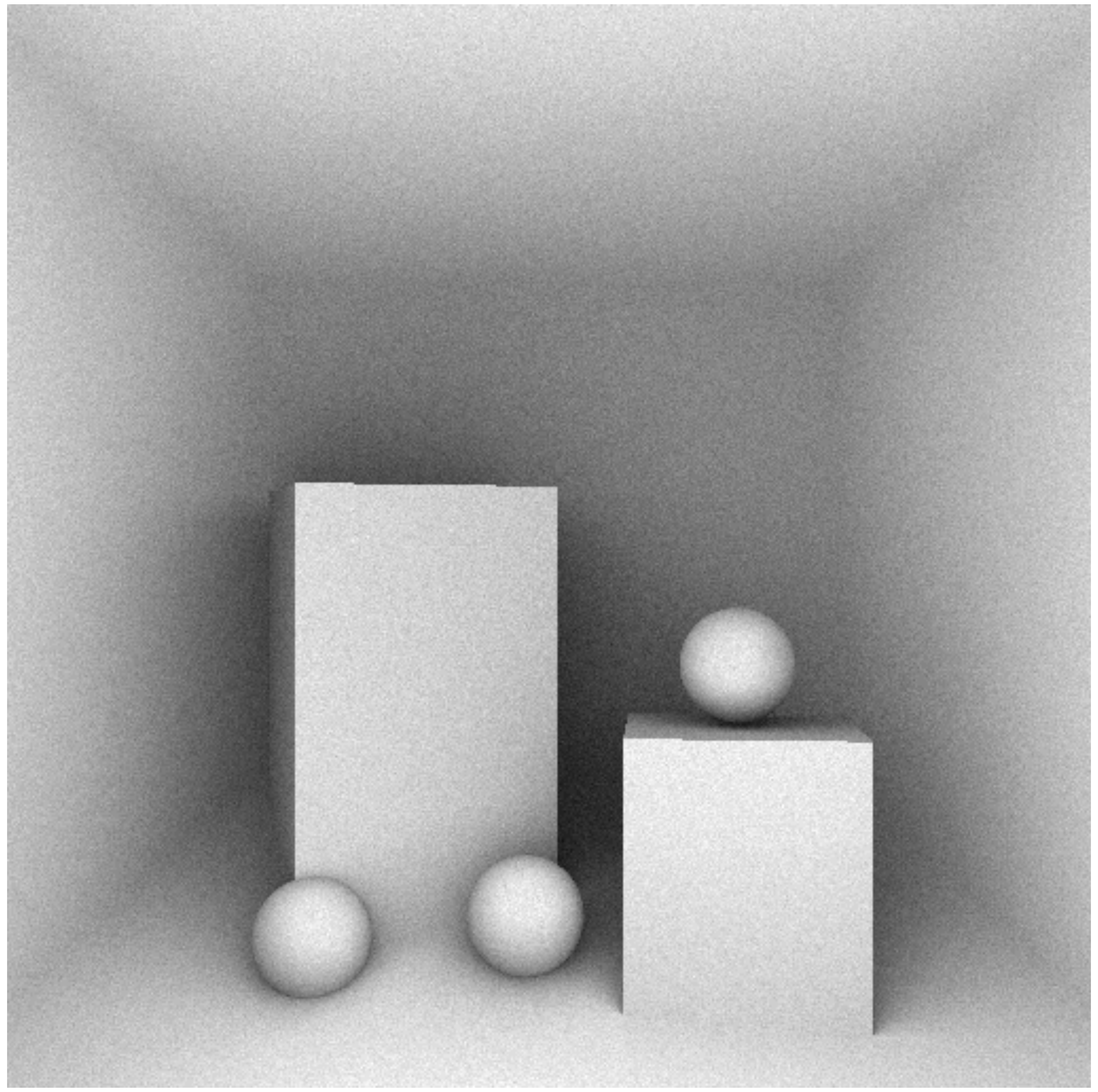
MSE: 8.56 x 10e-4

# CCVT vs. Poisson Disk

96 Secondary Rays

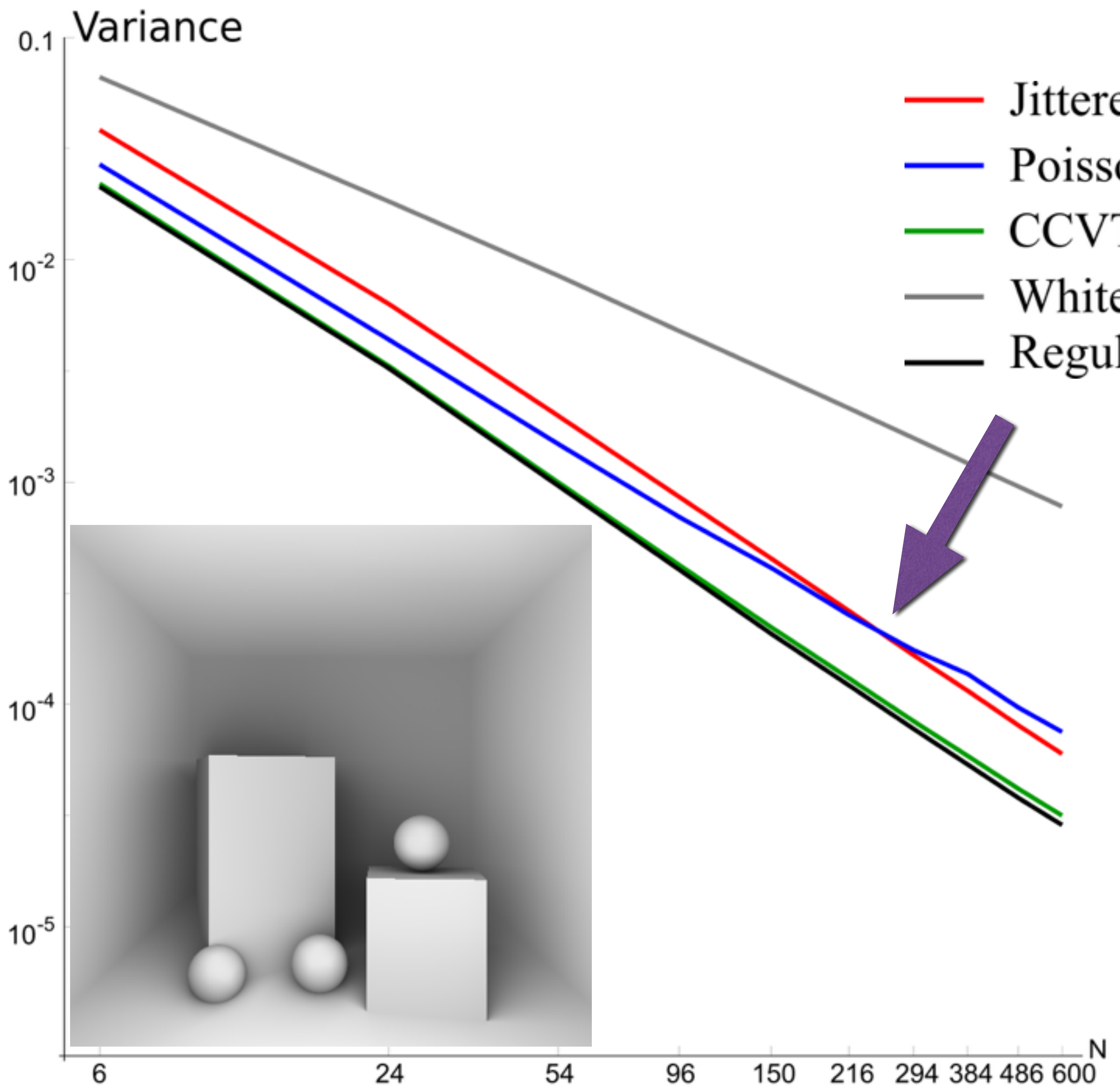


MSE: 4.24 x 10e-4

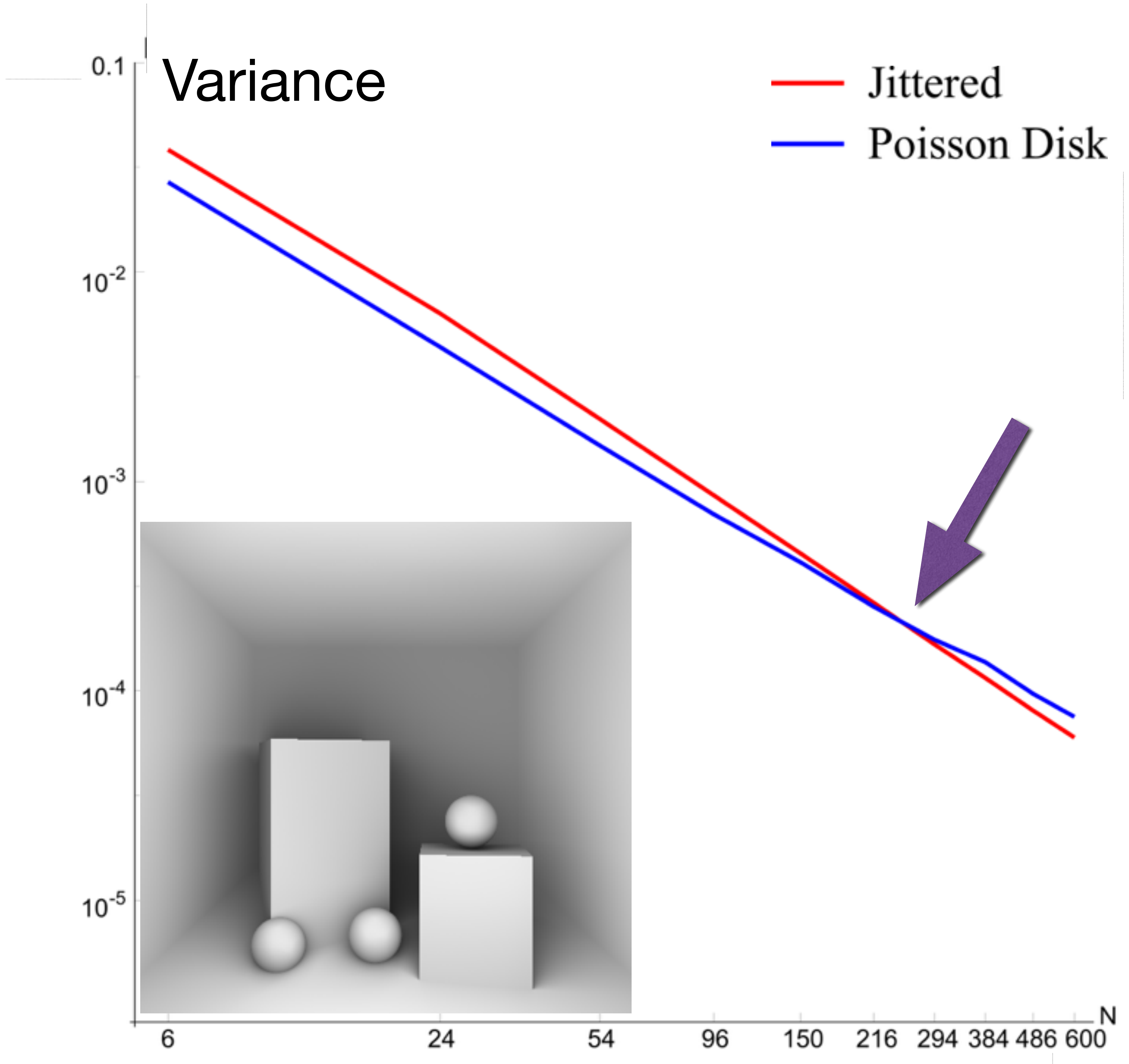


MSE: 6.95 x 10e-4

# Convergence rates



# Jittered vs Poisson Disk



# What are the benefits of this analysis ?

- For offline rendering, analysis tells which samplers would converge faster.
- For real time rendering, blue noise samples are more effective in reducing variance for a given number of samples