

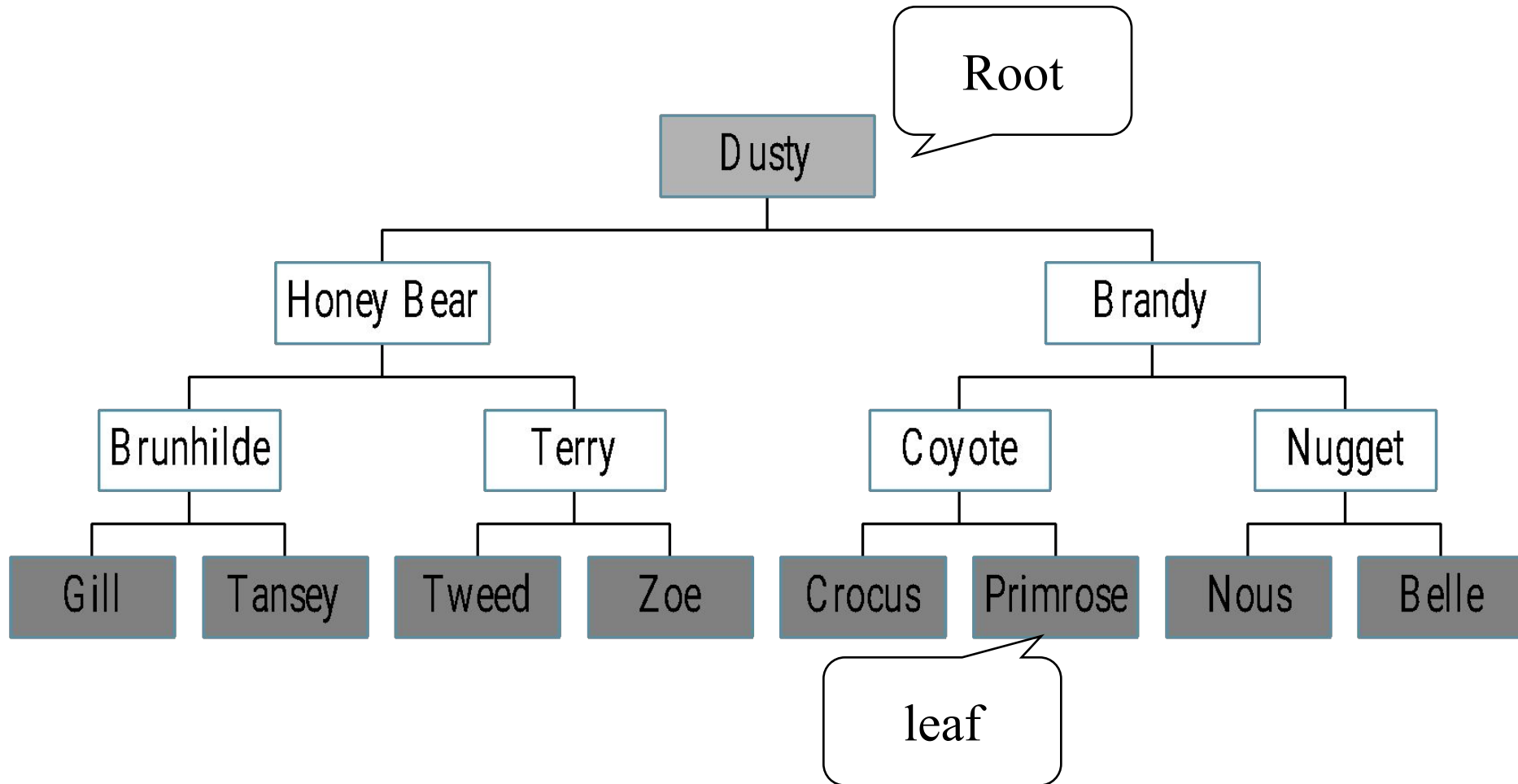
CHAPTER 5

Trees

All the programs in this file are selected from

Ellis Horowitz, Sartaj Sahni, and Susan Anderson-Freed
“Fundamentals of Data Structures in C”,
Computer Science Press, 1992.

Trees



Definition of Tree

- A tree is a finite set of one or more nodes such that:
- There is a specially designated node called the root.
- The remaining nodes are partitioned into $n \geq 0$ disjoint sets T_1, \dots, T_n , where each of these sets is a tree.
- We call T_1, \dots, T_n the subtrees of the root.

Level and Depth

node (13)

degree of a node

leaf (terminal)

nonterminal

parent

children

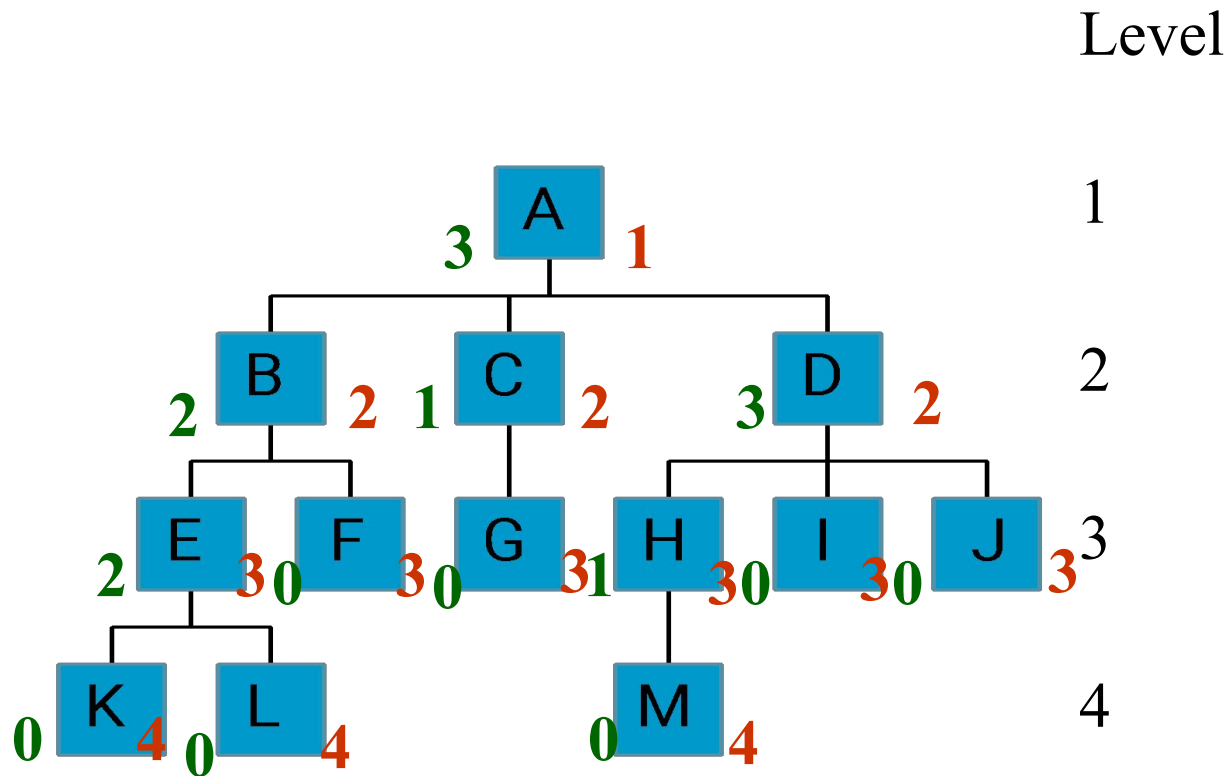
sibling

degree of a tree (3)

ancestor

level of a node

height of a tree (4)



Terminology

- The degree of a node is the number of subtrees of the node
 - The degree of A is 3; the degree of C is 1.
- The node with degree 0 is a leaf or terminal node.
- A node that has subtrees is the *parent* of the roots of the subtrees.
- The roots of these subtrees are the *children* of the node.
- Children of the same parent are *siblings*.
- The ancestors of a node are all the nodes along the path from the root to the node.

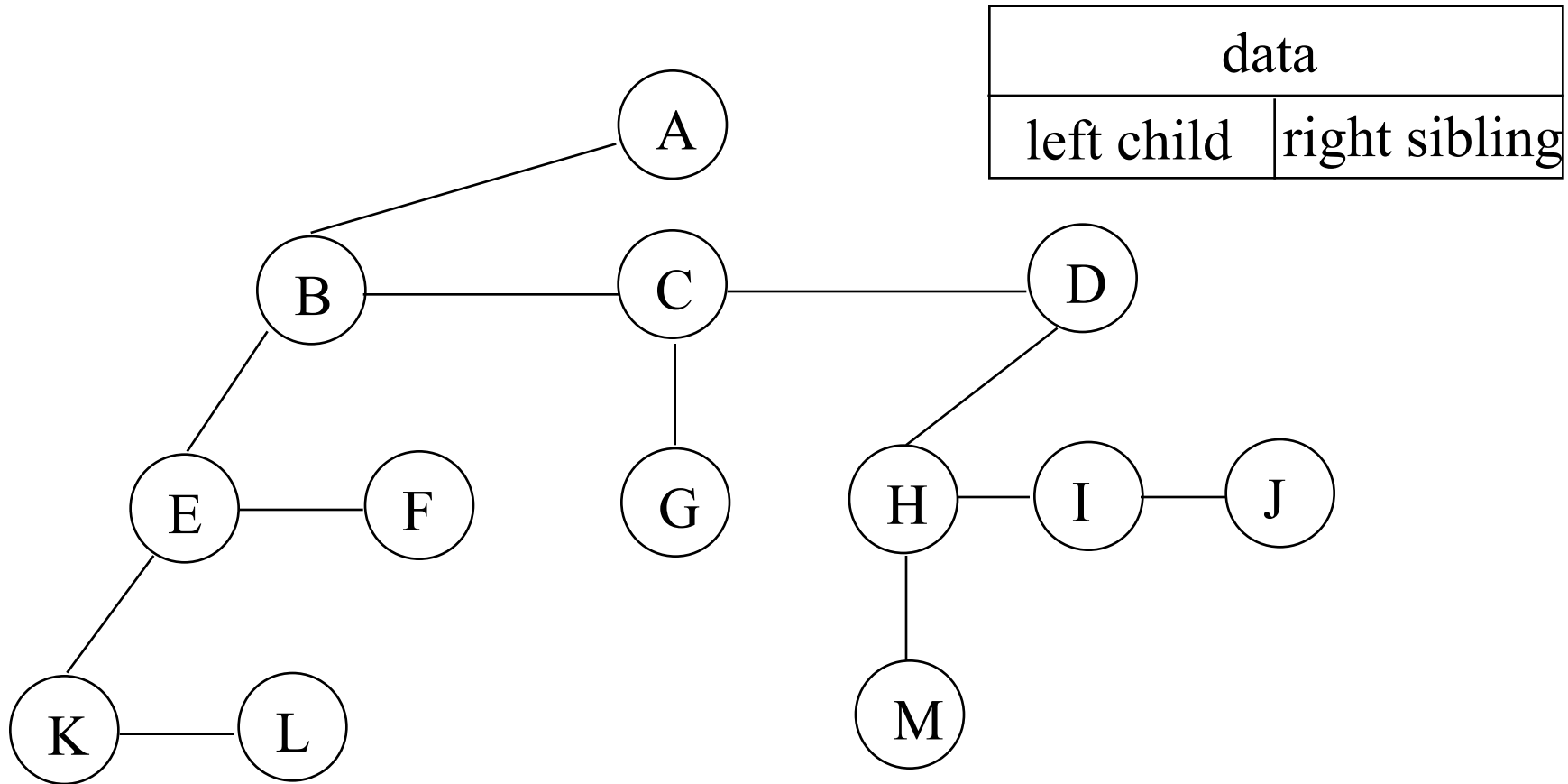
Representation of Trees

- List Representation
 - (A (B (E (K, L), F), C (G), D (H (M), I, J)))
 - The root comes first, followed by a list of sub-trees

data	link 1	link 2	...	link n
------	--------	--------	-----	--------

How many link fields are needed in such a representation?

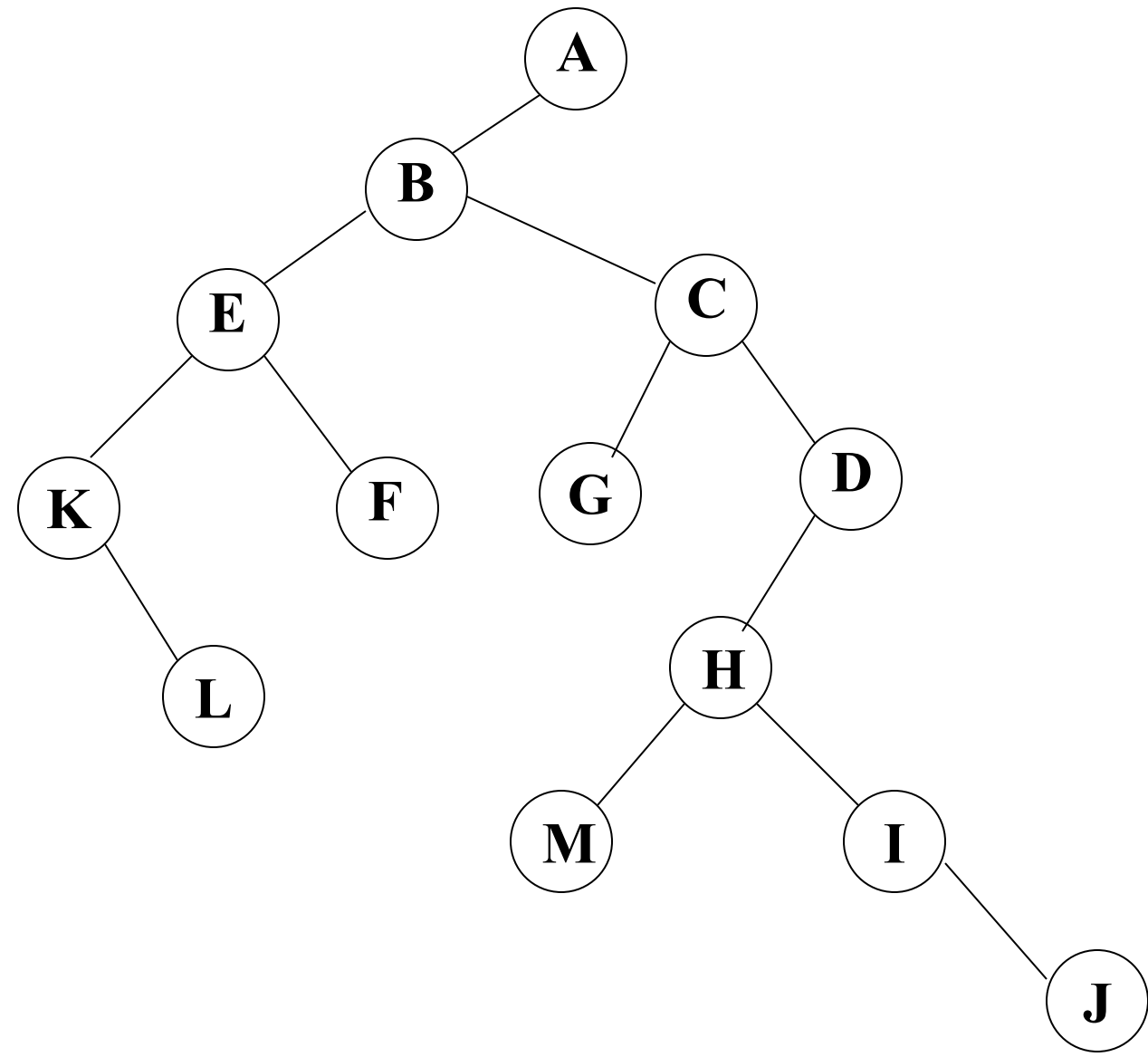
Left Child - Right Sibling



Binary Trees

- A binary tree is a finite set of nodes that is either empty or consists of a root and two disjoint binary trees called *the left subtree* and *the right subtree*.
- Any tree can be transformed into binary tree.
 - by left child-right sibling representation
- The left subtree and the right subtree are distinguished.

***Figure 5.6: Left child-right child tree representation of a tree (p.191)**



Abstract Data Type *Binary_Tree*

structure *Binary_Tree*(abbreviated *BinTree*) is objects: a finite set of nodes either empty or consisting of a root node, left *Binary_Tree*, and right *Binary_Tree*.

functions:

for all $bt, bt1, bt2 \in BinTree, item \in element$
Bintree Create() ::= creates an empty binary tree
Boolean IsEmpty(bt) ::= if ($bt == \text{empty binary tree}$) return *TRUE* else return *FALSE*

BinTree MakeBT($bt1$, $item$, $bt2$) ::= return a binary tree
whose left subtree is $bt1$, whose right subtree is $bt2$,
and whose root node contains the data $item$

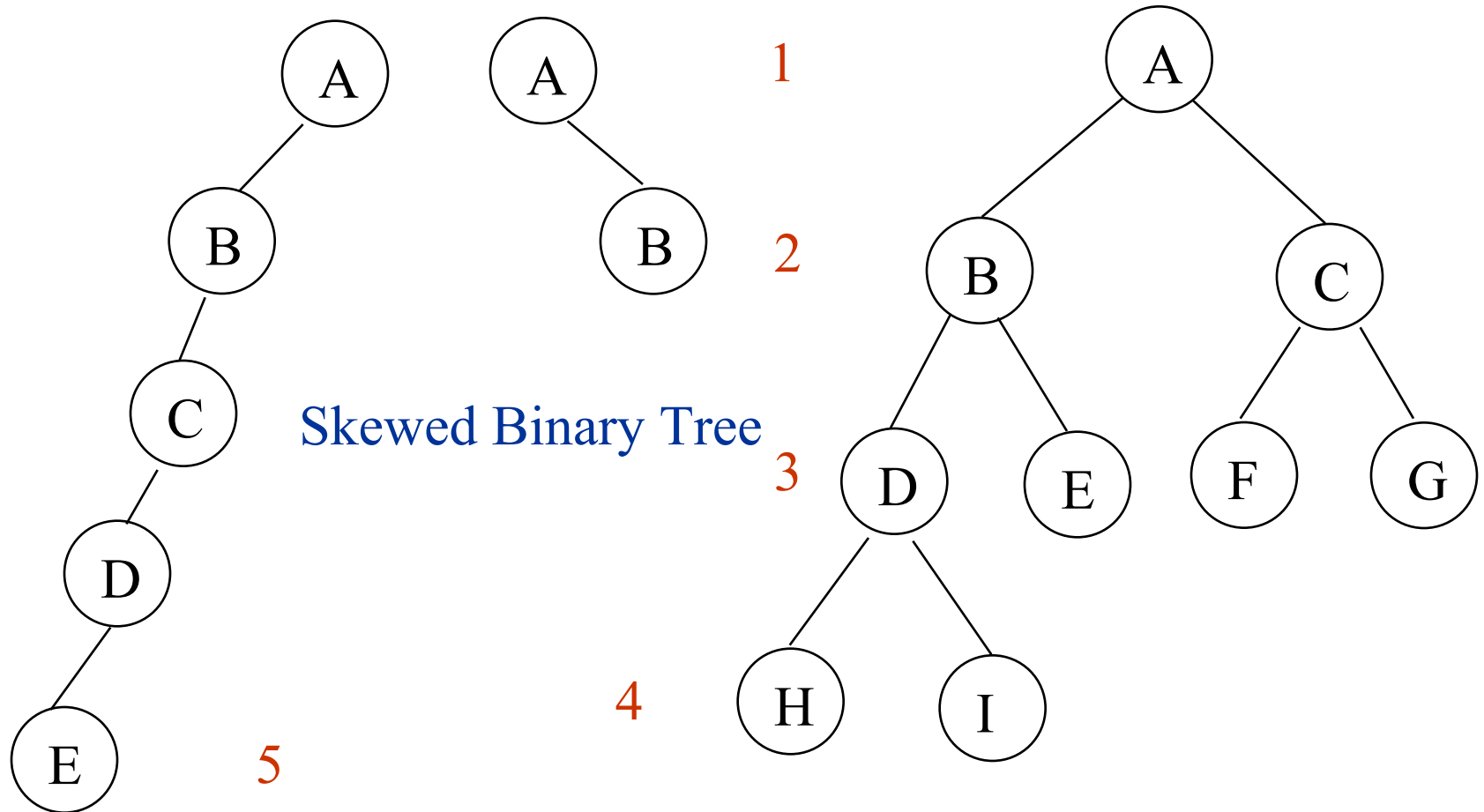
Bintree Lchild(bt) ::= if (IsEmpty(bt)) return error
else return the left subtree of bt

element Data(bt) ::= if (IsEmpty(bt)) return error
else return the data in the root node of bt

Bintree Rchild(bt) ::= if (IsEmpty(bt)) return error
else return the right subtree of bt

Samples of Trees

Complete Binary Tree



Maximum Number of Nodes in BT

- The maximum number of nodes on level i of a binary tree is 2^{i-1} , $i \geq 1$.
- The maximum number of nodes in a binary tree of depth k is $2^k - 1$, $k \geq 1$.

Prove by induction.

$$\sum_{i=1}^k 2^{i-1} = 2^k - 1$$

Relations between Number of Leaf Nodes and Nodes of Degree 2

For any nonempty binary tree, T , if n_0 is the number of leaf nodes and n_2 the number of nodes of degree 2, then $n_0 = n_2 + 1$

proof:

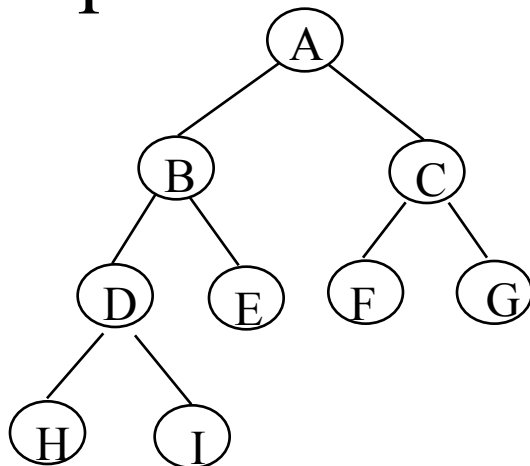
Let n and B denote the total number of nodes & branches in T .

Let n_0 , n_1 , n_2 represent the nodes with no children, single child, and two children respectively.

$$\begin{aligned} n &= n_0 + n_1 + n_2, \quad B + 1 = n, \quad B = n_1 + 2n_2 \implies n_1 + 2n_2 + 1 = n \\ n_1 + 2n_2 + 1 &= n_0 + n_1 + n_2 \implies n_0 = n_2 + 1 \end{aligned}$$

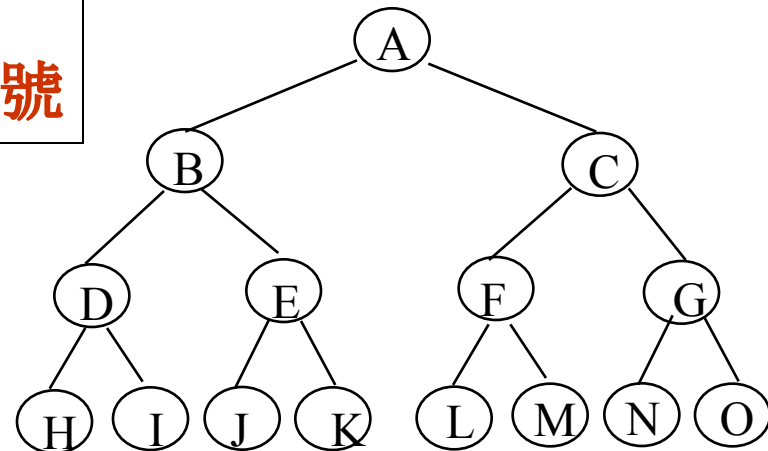
Full BT VS Complete BT

- A full binary tree of depth k is a binary tree of depth k having $2^{k+1}-1$ nodes, $k \geq 0$.
- A binary tree with n nodes and depth k is complete *iff* its nodes correspond to the nodes numbered from 1 to n in the full binary tree of depth k .



Complete binary tree

由上至下，
由左至右編號

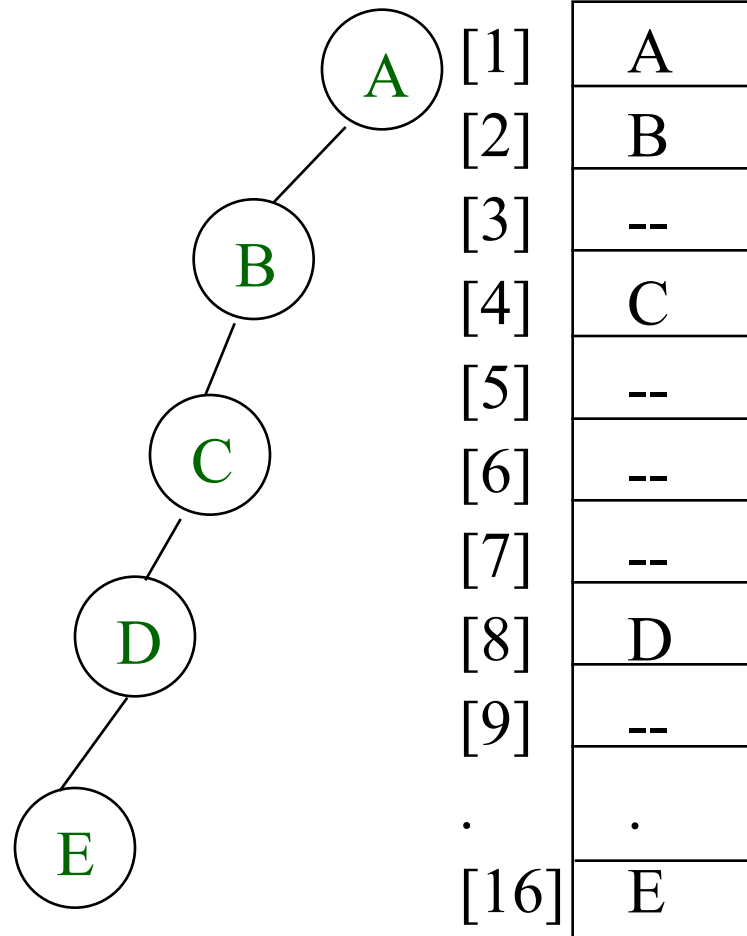


Full binary tree of depth 4

Binary Tree Representations

- If a complete binary tree with n nodes (depth = $\log n + 1$) is represented sequentially, then for any node with index i , $1 \leq i \leq n$, we have:
 - $parent(i)$ is at $i/2$ if $i \neq 1$. If $i=1$, i is at the root and has no parent.
 - $left_child(i)$ is at $2i$ if $2i \leq n$. If $2i > n$, then i has no left child.
 - $right_child(i)$ is at $2i+1$ if $2i+1 \leq n$. If $2i+1 > n$, then i has no right child.

Sequential Representation

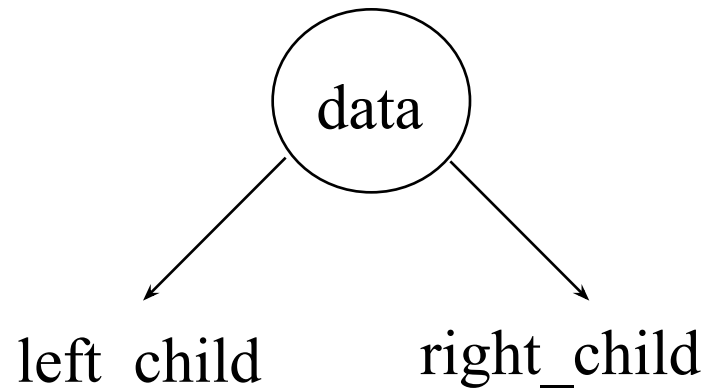
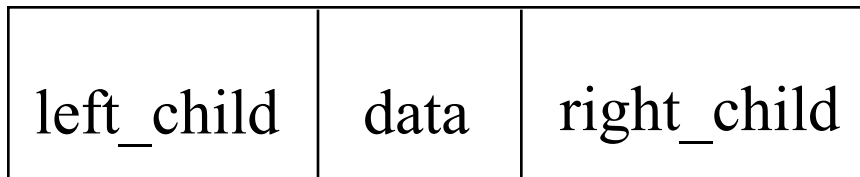


(1) waste space
(2) insertion/deletion problem

[1]	A
[2]	B
[3]	C
[4]	D
[5]	E
[6]	F
[7]	G
[8]	H
[9]	I

Linked Representation

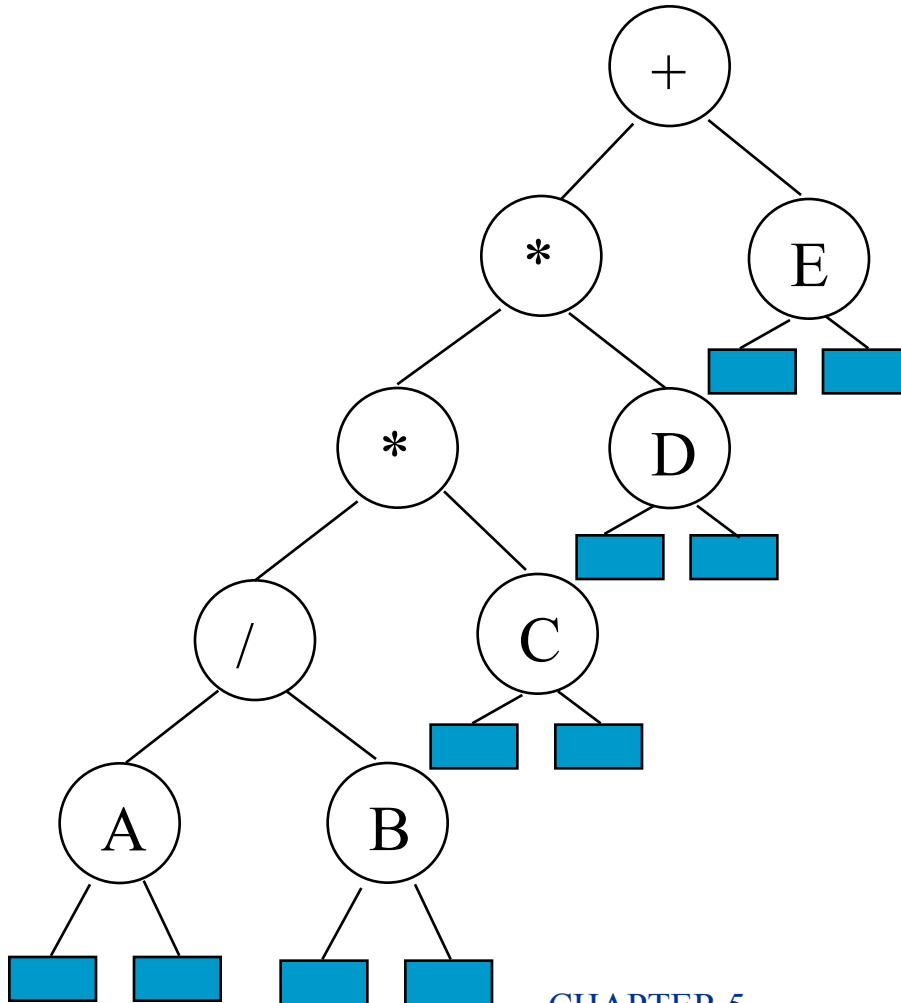
```
typedef struct node *tree_pointer;  
typedef struct node {  
    int data;  
    tree_pointer left_child, right_child;  
};
```



Binary Tree Traversals

- Let L, V, and R stand for moving left, visiting the node, and moving right.
- There are six possible combinations of traversal
 - LVR, LRV, VLR, VRL, RVL, RLV
- Adopt convention that we traverse left before right, only 3 traversals remain
 - LVR, LRV, VLR
 - inorder, postorder, preorder

Arithmetic Expression Using BT



inorder traversal

A / B * C * D + E

infix expression

preorder traversal

+ * * / A B C D E

prefix expression

postorder traversal

A B / C * D * E +

postfix expression

level order traversal

+ * E * D / C A B

Inorder Traversal (recursive version)

```
void inorder(tree_pointer ptr)
/* inorder tree traversal */
{
    if (ptr) {
        inorder(ptr->left_child);
        printf("%d", ptr->data);
        indorder(ptr->right_child);
    }
}
```

A / B * C * D + E

Preorder Traversal (recursive version)

```
void preorder(tree_pointer ptr)
/* preorder tree traversal */
{
    if (ptr) {
        printf("%d", ptr->data);
        preorder(ptr->left_child);
        predorder(ptr->right_child);
    }
}
```

+ * * / A B C D E

Postorder Traversal (recursive version)

```
void postorder(tree_pointer ptr)
/* postorder tree traversal */
{
    if (ptr) {
        postorder(ptr->left_child);
        postorder(ptr->right_child);
        printf("%d", ptr->data);
    }
}
```

A B / C * D * E +

Iterative Inorder Traversal

(using stack)

```
void iter_inorder(tree_pointer node)
{
    int top= -1; /* initialize stack */
    tree_pointer stack[MAX_STACK_SIZE];
    for (;;) {
        for (; node; node=node->left_child)
            add(&top, node); /* add to stack */
        node= delete(&top);
        /* delete from stack */
        if (!node) break; /* empty stack */
        printf("%D", node->data);
        node = node->right_child;
    }
}
```

O(n)

Trace Operations of Inorder Traversal

Call of inorder	Value in root	Action	Call of inorder	Value in root	Action
1	+		11	C	
2	*		12	NULL	
3	*		11	C	printf
4	/		13	NULL	
5	A		2	*	printf
6	NULL		14	D	
5	A	printf	15	NULL	
7	NULL		14	D	printf
4	/	printf	16	NULL	
8	B		1	+	printf
9	NULL		17	E	
8	B	printf	18	NULL	
10	NULL		17	E	printf
3	*	printf	19	NULL	

Level Order Traversal

(using queue)

```
void level_order(tree_pointer ptr)
/* level order tree traversal */
{
    int front = rear = 0;
    tree_pointer queue[MAX_QUEUE_SIZE];
    if (!ptr) return; /* empty queue */
    addq(front, &rear, ptr);
    for (;;) {
        ptr = deleteq(&front, rear);
```

```

if (ptr) {
    printf("%d", ptr->data);
    if (ptr->left_child)
        addq(front, &rear,
              ptr->left_child);
    if (ptr->right_child)
        addq(front, &rear,
              ptr->right_child);
}
else break;
}
}

```

$+ * E * D / C A B$

Copying Binary Trees

```
tree_pointer copy(tree_pointer original)
{
    tree_pointer temp;
    if (original) {
        temp=(tree_pointer) malloc(sizeof(node));
        if (IS_FULL(temp)) {
            fprintf(stderr, "the memory is full\n");
            exit(1);
        }
        temp->left_child=copy(original->left_child);
        temp->right_child=copy(original->right_child);
        temp->data=original->data;
        return temp;
    }
    return NULL;
}
```

postorder

Equality of Binary Trees

the same topology and data

```
int equal(tree_pointer first, tree_pointer second)
{
    /* function returns FALSE if the binary trees first and
       second are not equal, otherwise it returns TRUE */

    return ((!first && !second) || (first && second &&
        (first->data == second->data) &&
        equal(first->left_child, second->left_child) &&
        equal(first->right_child, second->right_child)))
}
```

Propositional Calculus Expression

- A variable is an expression.
- If x and y are expressions, then $\neg x$, $x \wedge y$, $x \vee y$ are expressions.
- Parentheses can be used to alter the normal order of evaluation ($\neg > \wedge > \vee$).
- Example: $x_1 \vee (x_2 \wedge \neg x_3)$
- satisfiability problem: Is there an assignment to make an expression true?

$$(x_1 \wedge \neg x_2) \vee (\neg x_1 \wedge x_3) \vee \neg x_3$$

(t,t,t)

(t,t,f)

(t,f,t)

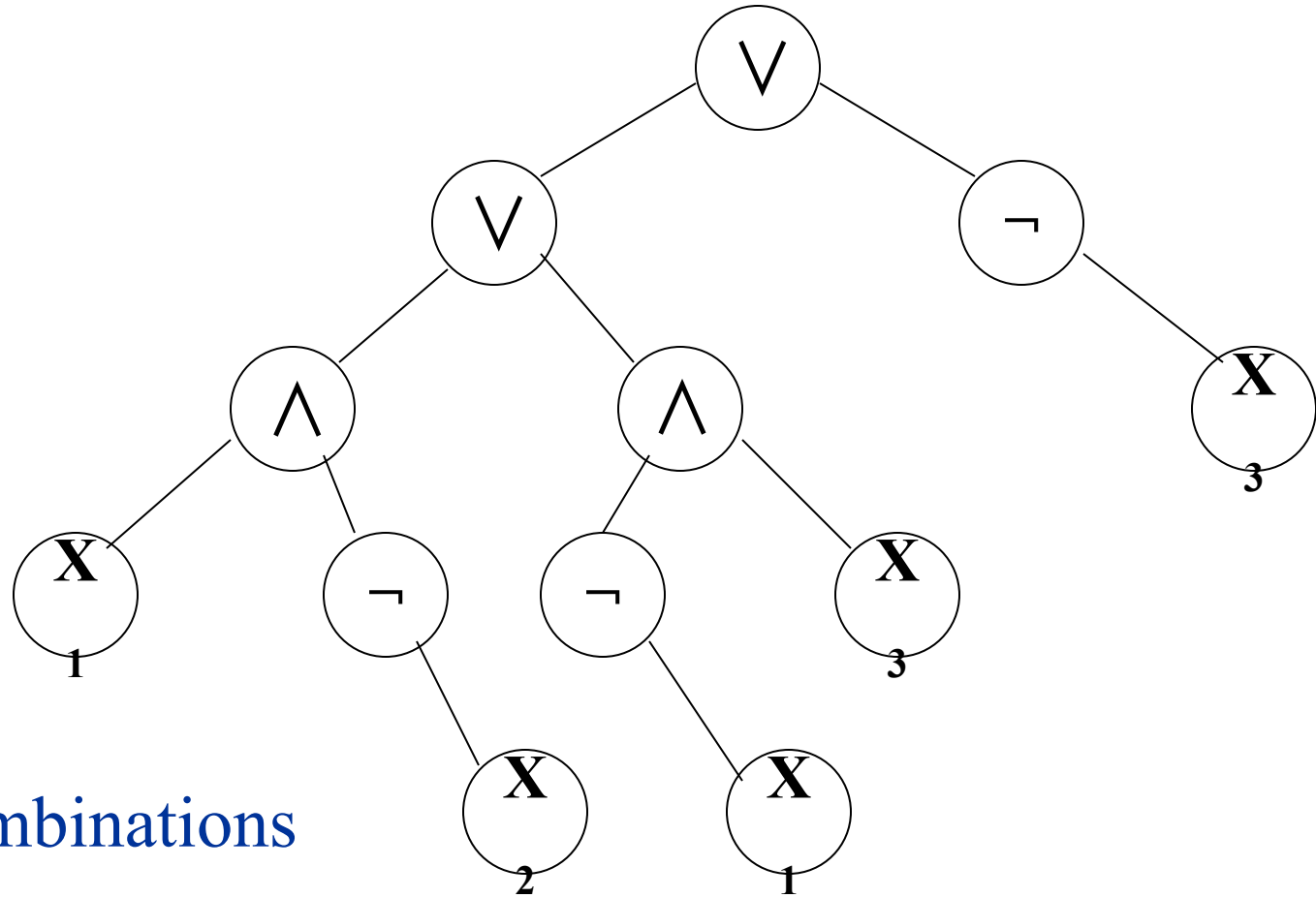
(t,f,f)

(f,t,t)

(f,t,f)

(f,f,t)

(f,f,f)



2^n possible combinations
for n variables

postorder traversal (postfix evaluation)

node structure

<i>left_child</i>	<i>data</i>	<i>value</i>	<i>right_child</i>
-------------------	-------------	--------------	--------------------

```
typedef enum {not, and, or, true, false } logical;  
typedef struct node *tree_pointer;  
typedef struct node {  
    tree_pointer list_child;  
    logical      data;  
    short int    value;  
    tree_pointer right_child;  
} ;
```


First version of satisfiability algorithm

```
for (all  $2^n$  possible combinations) {  
    generate the next combination;  
    replace the variables by their values;  
    evaluate root by traversing it in postorder;  
    if (root->value) {  
        printf(<combination>);  
        return;  
    }  
}  
printf("No satisfiable combination \n");
```

Post-order-eval function

```
void post_order_eval(tree_pointer node)
{
/* modified post order traversal to evaluate a propositional
calculus tree */
    if (node) {
        post_order_eval(node->left_child);
        post_order_eval(node->right_child);
        switch(node->data) {
            case not: node->value =
                        !node->right_child->value;
                        break;
```

```
case and:    node->value =  
            node->right_child->value &&  
            node->left_child->value;  
            break;
```

```
case or:     node->value =  
            node->right_child->value ||  
            node->left_child->value;  
            break;
```

```
case true:   node->value = TRUE;  
            break;
```

```
case false: node->value = FALSE;
```

```
}
```

```
}
```

```
}
```

Threaded Binary Trees

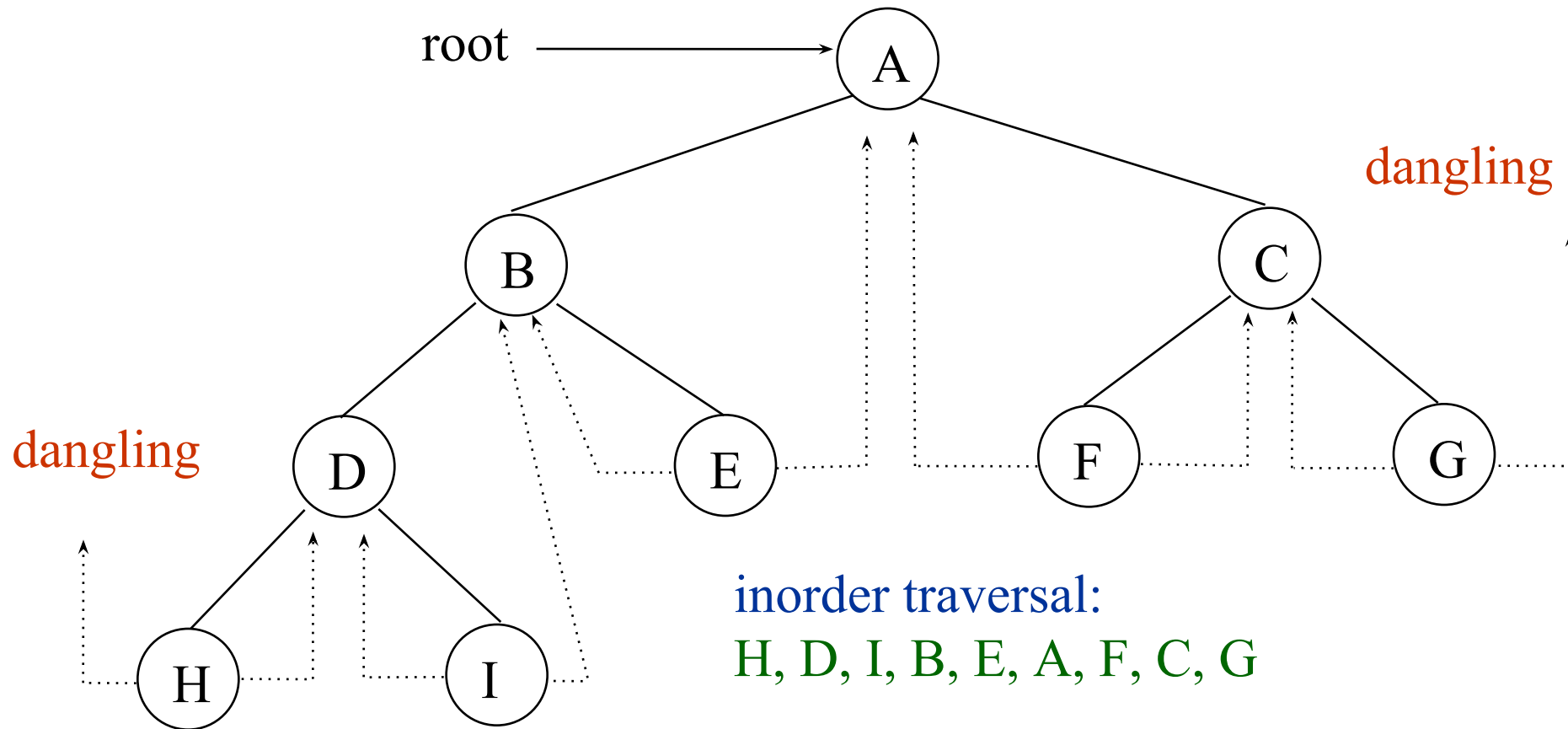
- Two many null pointers in current representation of binary trees
 - n: number of nodes
 - number of non-null links: $n-1$
 - total links: $2n$
 - null links: $2n-(n-1)=n+1$
- Replace these null pointers with some useful “threads”.

Threaded Binary Trees *(Continued)*

If `ptr->left_child` is null,
replace it with a pointer to the node that would be
visited *before* `ptr` in an *inorder traversal*

If `ptr->right_child` is null,
replace it with a pointer to the node that would be
visited *after* `ptr` in an *inorder traversal*

A Threaded Binary Tree



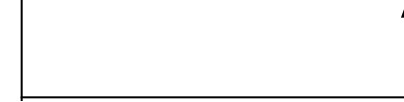
Data Structures for Threaded BT

left_thread left_child data right_child right_thread

TRUE	•	—	•	FALSE
------	---	---	---	-------



TRUE: thread



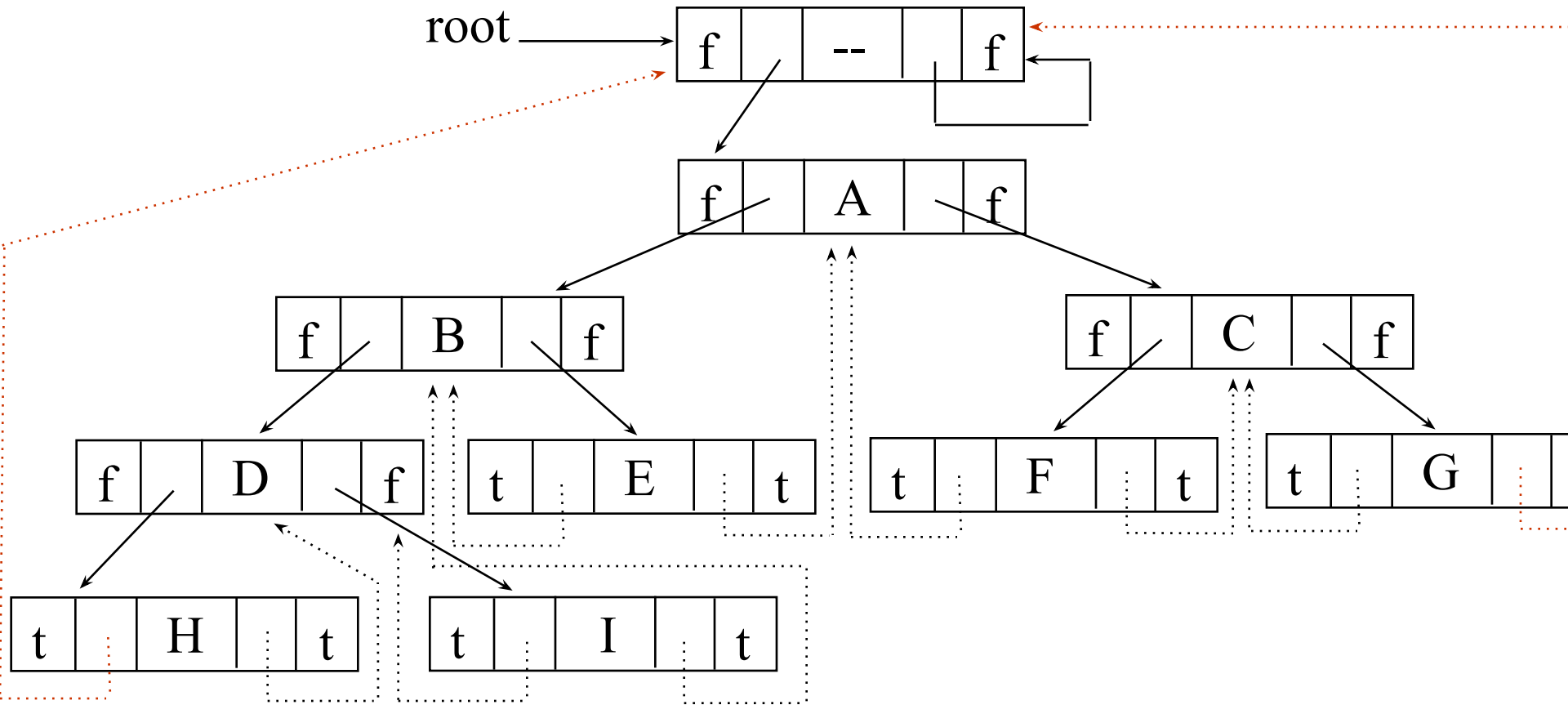
FALSE: child

```
typedef struct threaded_tree
    *threaded_pointer;

typedef struct threaded_tree {
    short int left_thread;
    threaded_pointer left_child;
    char data;
    threaded_pointer right_child;
    short int right_thread;  };

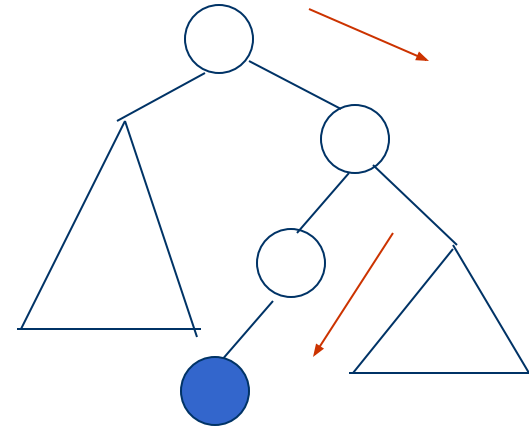
```

Memory Representation of A Threaded BT



Next Node in Threaded BT

```
threaded_pointer insucc(threaded_pointer
    tree)
{
    threaded_pointer temp;
    temp = tree->right_child;
    if (!tree->right_thread)
        while (!temp->left_thread)
            temp = temp->left_child;
    return temp;
}
```



Inorder Traversal of Threaded BT

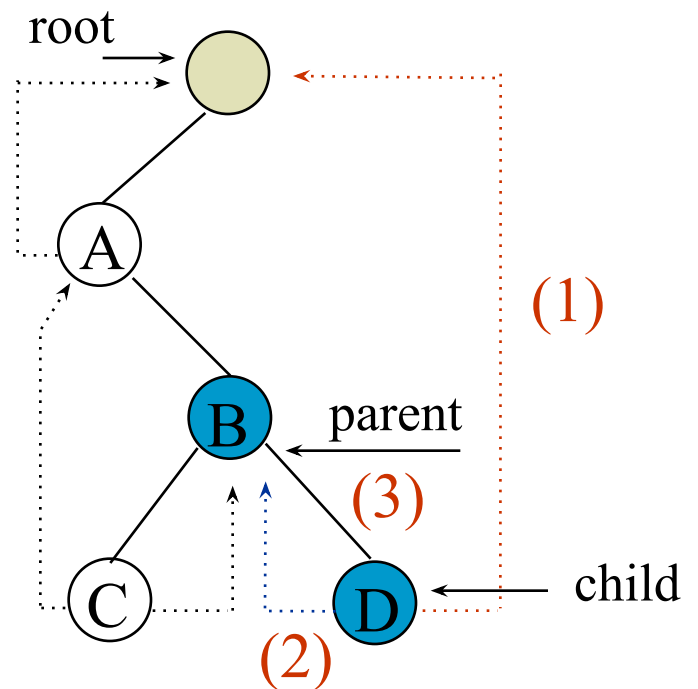
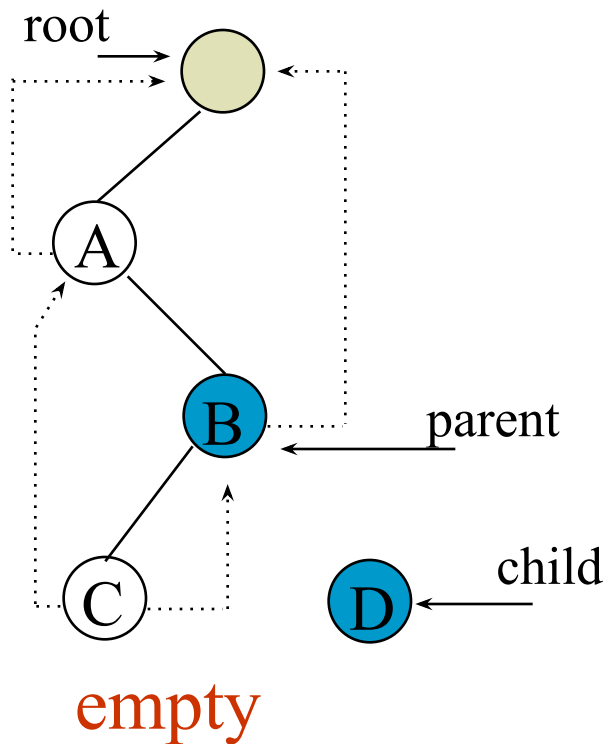
```
void tinorder(threaded_pointer tree)
{
    /* traverse the threaded binary tree
       inorder */
    threaded_pointer temp = tree;
    for (;;) {
        temp = insucc(temp);
        O(n) if (temp==tree) break;
        printf("%3c", temp->data);
    }
}
```

Inserting Nodes into Threaded BTs

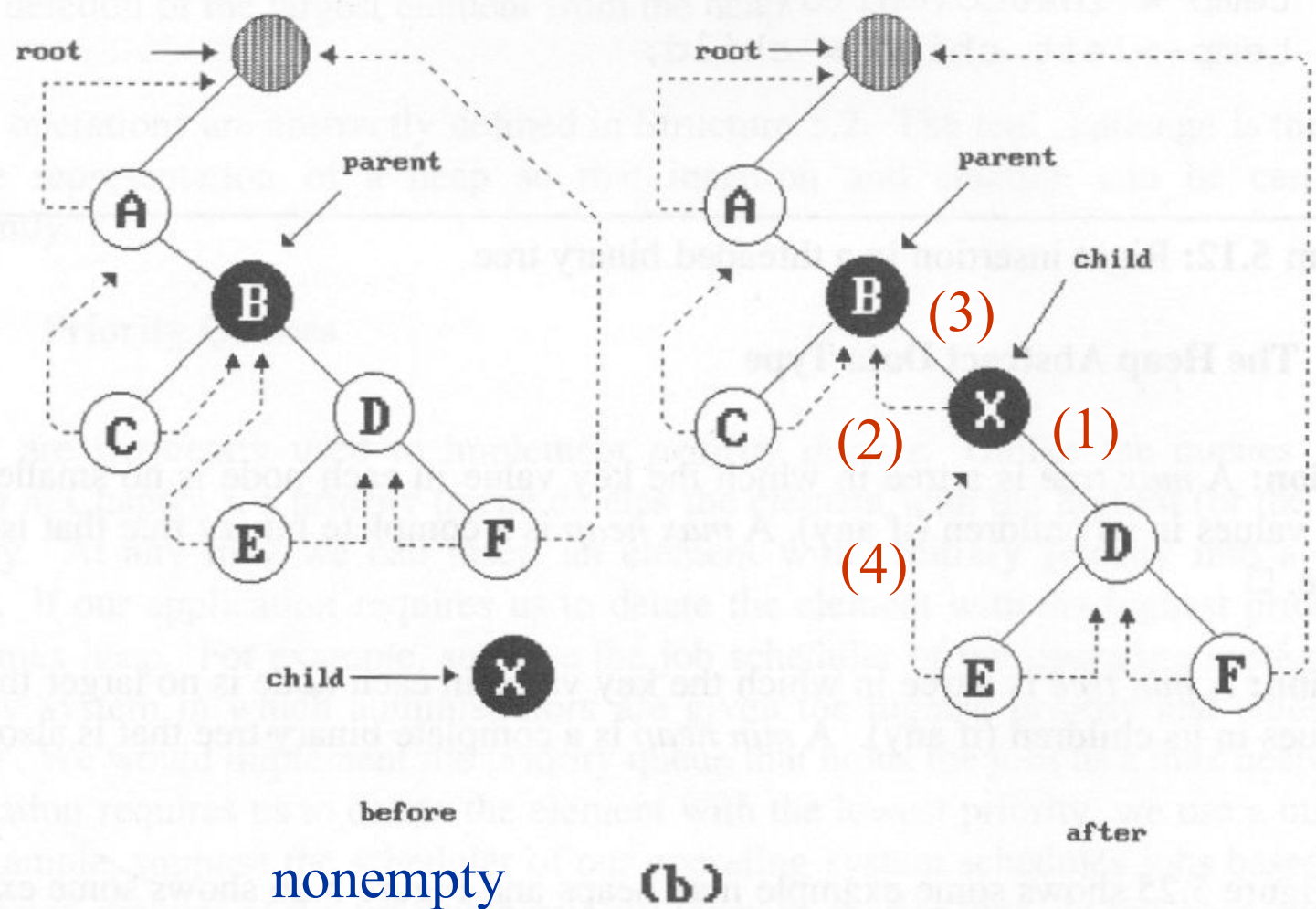
- Insert `child` as the right child of node `parent`
 - **change** `parent->right_thread` to **FALSE**
 - **set** `child->left_thread` **and** `child->right_thread` to **TRUE**
 - **set** `child->left_child` to **point to** `parent`
 - **set** `child->right_child` to `parent->right_child`
 - **change** `parent->right_child` to **point to** `child`

Examples

Insert a node D as a right child of B.



***Figure 5.24:** Insertion of child as a right child of parent in a threaded binary tree (p.217)



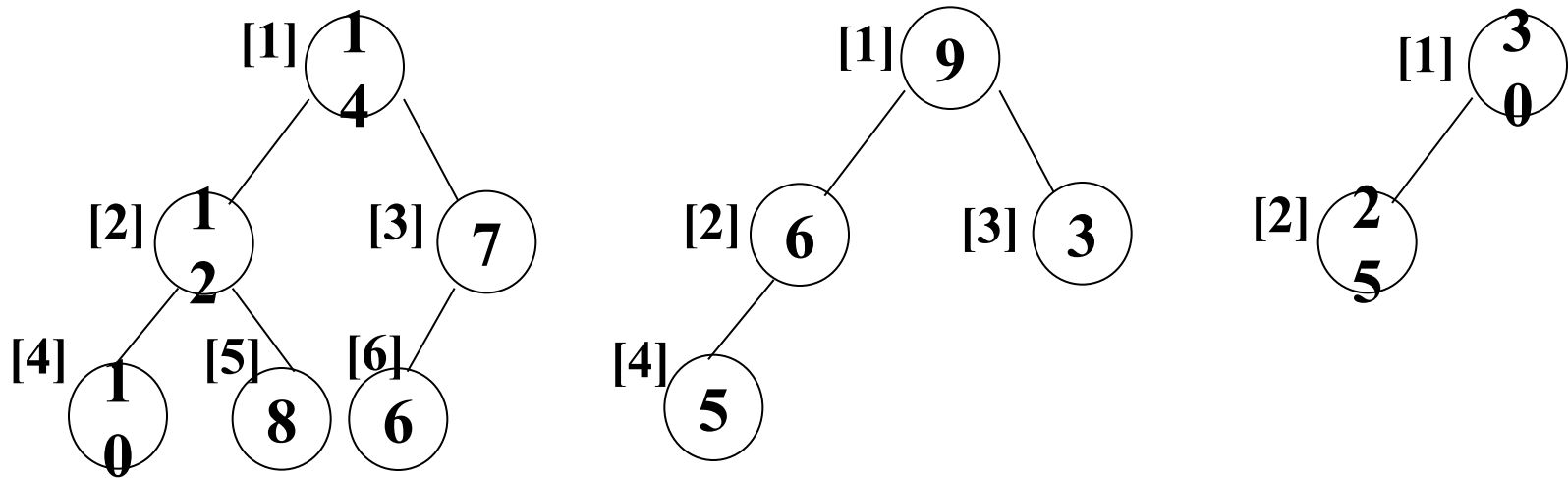
Right Insertion in Threaded BTs

```
void insert_right(threaded_pointer parent,
                  threaded_pointer child)
{
    threaded_pointer temp;
    (1) child->right_child = parent->right_child;
    child->right_thread = parent->right_thread;
    child->left_child = parent;    case (a)
    (2) child->left_thread = TRUE;
    parent->right_child = child;
    (3) parent->right_thread = FALSE;
    if (!child->right_thread) { case (b)
        temp = insucc(child);
    (4) temp->left_child = child;
    }
}
```

Heap

- A *max tree* is a tree in which the key value in each node is **no smaller than** the key values in its children. A *max heap* is a **complete binary tree** that is also a max tree.
- A *min tree* is a tree in which the key value in each node is **no larger than** the key values in its children. A *min heap* is a **complete binary tree** that is also a min tree.
- Operations on heaps
 - creation of an empty heap
 - insertion of a new element into the heap;
 - deletion of the largest element from the heap

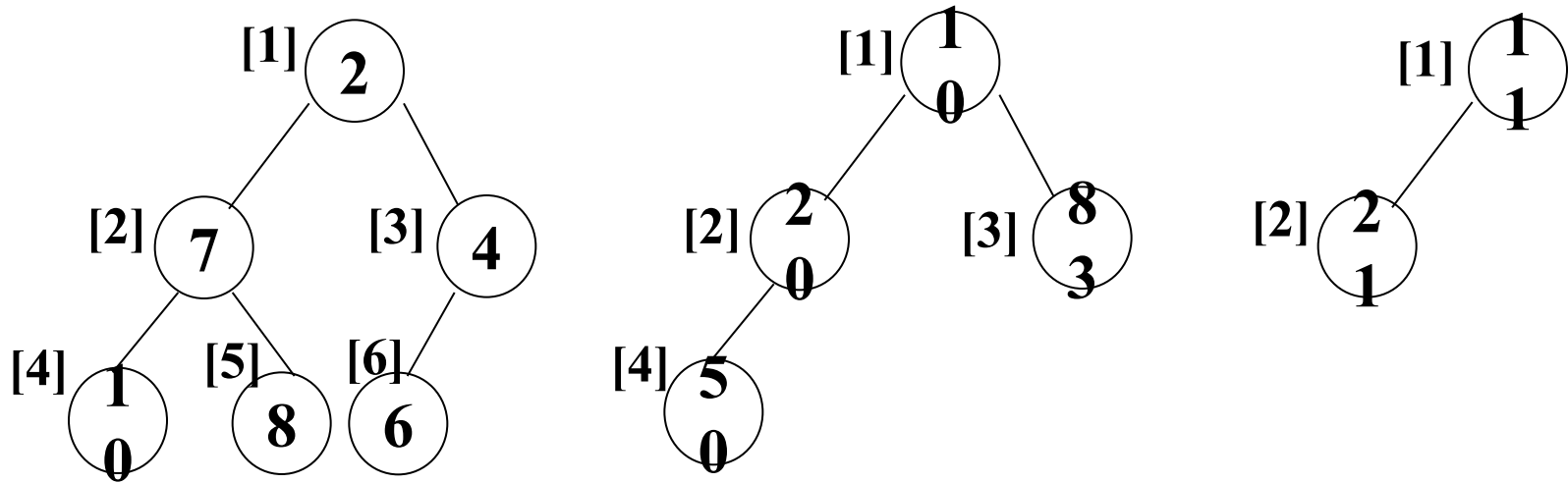
***Figure 5.25: Sample max heaps (p.219)**



Property:

The root of **max heap** (**min heap**) contains the **largest** (**smallest**).

***Figure 5.26: Sample min heaps (p.220)**



structure MaxHeap ADT for Max Heap

objects: a complete binary tree of $n > 0$ elements organized so that the value in each node is at least as large as those in its children

functions:

for all *heap* belong to *MaxHeap*, *item* belong to *Element*, n ,
max_size belong to integer

MaxHeap Create(*max_size*) ::= create an empty heap that can
hold a maximum of *max_size* elements

Boolean HeapFull(*heap*, n) ::= if ($n == \text{max_size}$) return TRUE
else return FALSE

MaxHeap Insert(*heap*, *item*, n) ::= if (!HeapFull(*heap*, n)) insert
item into heap and return the resulting heap
else return error

Boolean HeapEmpty(*heap*, n) ::= if ($n > 0$) return FALSE
else return TRUE

Element Delete(*heap*, n) ::= if (!HeapEmpty(*heap*, n)) return one
instance of the **largest** element in the heap
and remove it from the heap

else return error

Application: priority queue

- machine service
 - amount of time (min heap)
 - amount of payment (max heap)
- factory
 - time tag

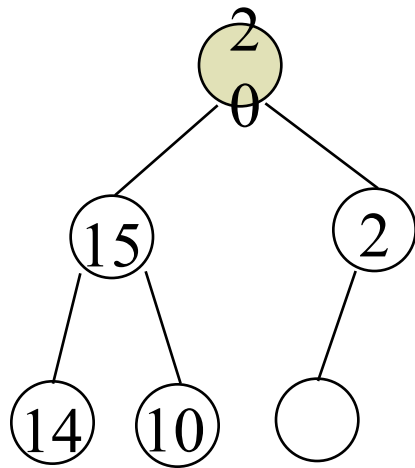
Data Structures

- unordered linked list
- unordered array
- sorted linked list
- sorted array
- heap

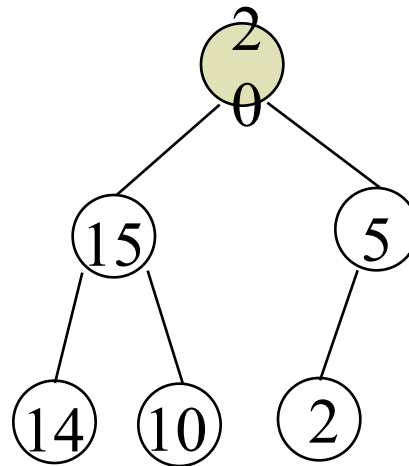
***Figure 5.27: Priority queue representations (p.221)**

Representation	Insertion	Deletion
Unordered array	$\Theta(1)$	$\Theta(n)$
Unordered linked list	$\Theta(1)$	$\Theta(n)$
Sorted array	$O(n)$	$\Theta(1)$
Sorted linked list	$O(n)$	$\Theta(1)$
Max heap	$O(\log_2 n)$	$O(\log_2 n)$

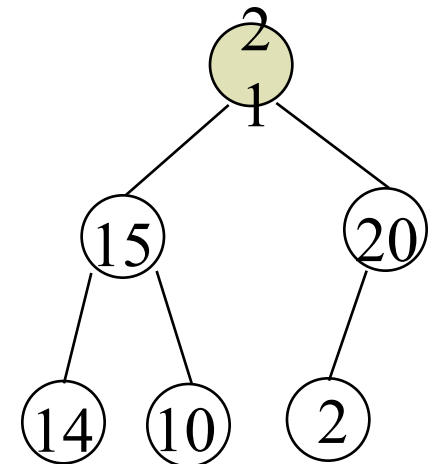
Example of Insertion to Max Heap



initial location of new node



insert 5 into heap



insert 21 into heap

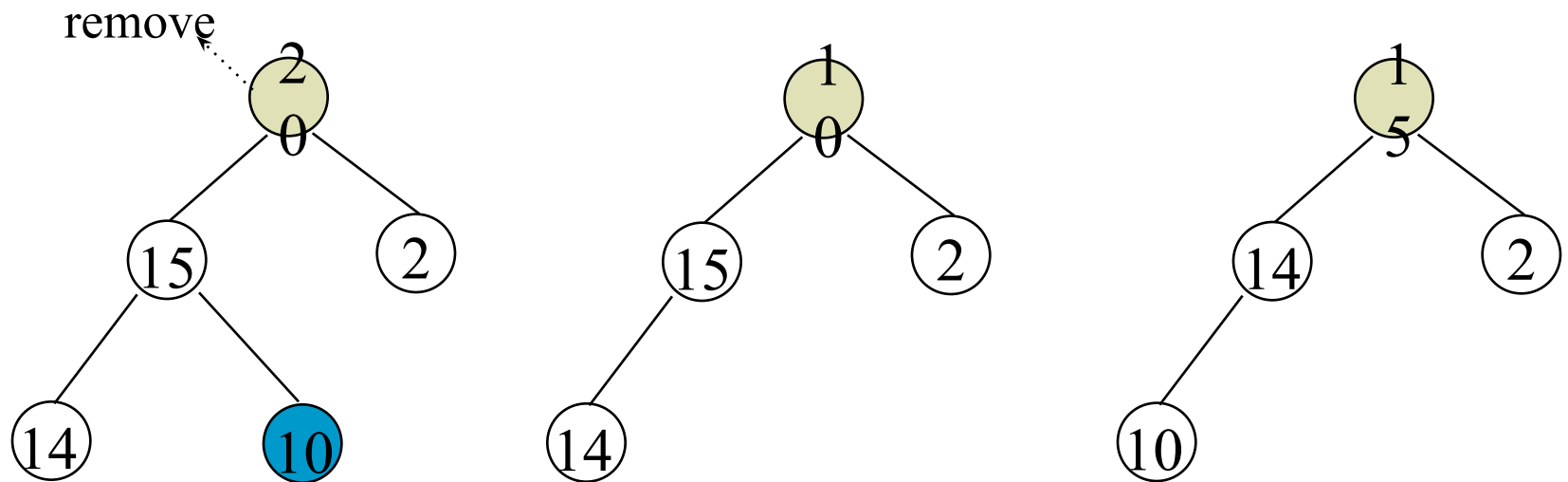
Insertion into a Max Heap

```
void insert_max_heap(element item, int *n)
{
    int i;
    if (HEAP_FULL(*n)) {
        fprintf(stderr, "the heap is full.\n");
        exit(1);
    }
    i = ++(*n);
    while ((i!=1) && (item.key>heap[i/2].key)) {
        heap[i] = heap[i/2];
        i /= 2;
    }
    heap[i] = item;
}
```

$$2^k - 1 = n \implies k = \lceil \log_2(n+1) \rceil$$

$O(\log_2 n)$

Example of Deletion from Max Heap



Deletion from a Max Heap

```
element delete_max_heap(int *n)
{
    int parent, child;
    element item, temp;
    if (HEAP_EMPTY(*n)) {
        fprintf(stderr, "The heap is empty\n");
        exit(1);
    }
    /* save value of the element with the
       highest key */
    item = heap[1];
    /* use last element in heap to adjust heap
    temp = heap[(*n)--];
    parent = 1;
    child = 2;
```

```

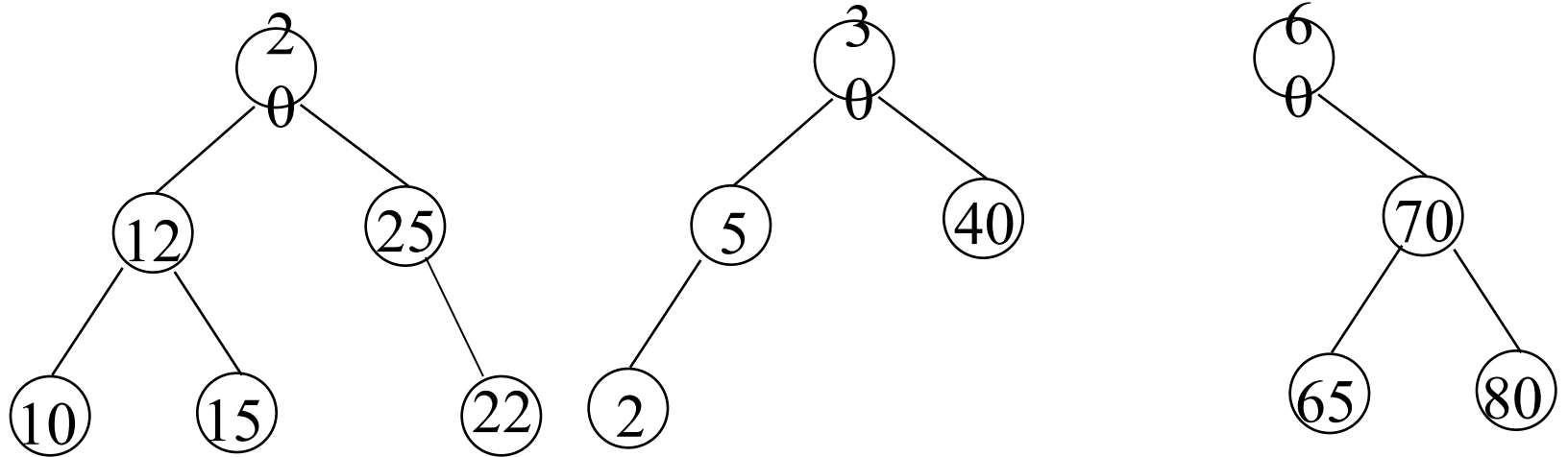
while (child <= *n) {
    /* find the larger child of the current
       parent */
    if ((child < *n) &&
        (heap[child].key < heap[child+1].key))
        child++;
    if (temp.key >= heap[child].key) break;
    /* move to the next lower level */
    heap[parent] = heap[child];
    child *= 2;
}
heap[parent] = temp;
return item;
}

```

Binary Search Tree

- Heap
 - a min (max) element is deleted. $O(\log_2 n)$
 - deletion of an arbitrary element $O(n)$
 - search for an arbitrary element $O(n)$
- Binary search tree
 - Every element has a unique key.
 - The keys in a nonempty left subtree (right subtree) are smaller (larger) than the key in the root of subtree.
 - The left and right subtrees are also binary search trees.

Examples of Binary Search Trees



Searching a Binary Search Tree

```
tree_pointer search(tree_pointer root,
                    int key)
{
    /* return a pointer to the node that
       contains key. If there is no such
       node, return NULL */

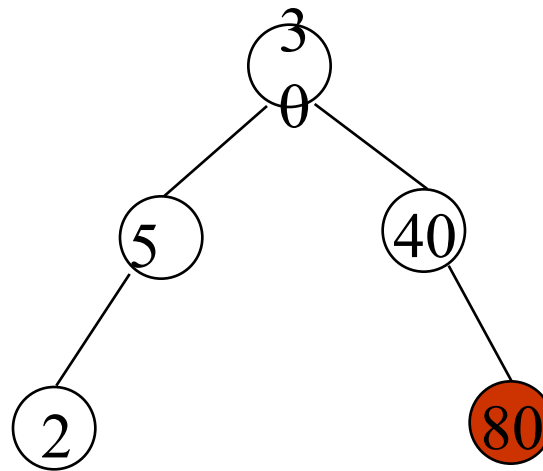
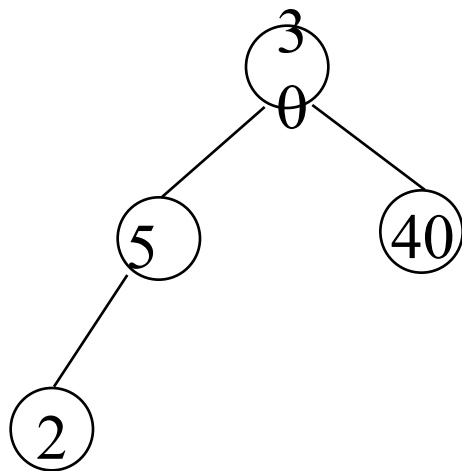
    if (!root) return NULL;
    if (key == root->data) return root;
    if (key < root->data)
        return search(root->left_child,
                      key);
    return search(root->right_child, key);
}
```

Another Searching Algorithm

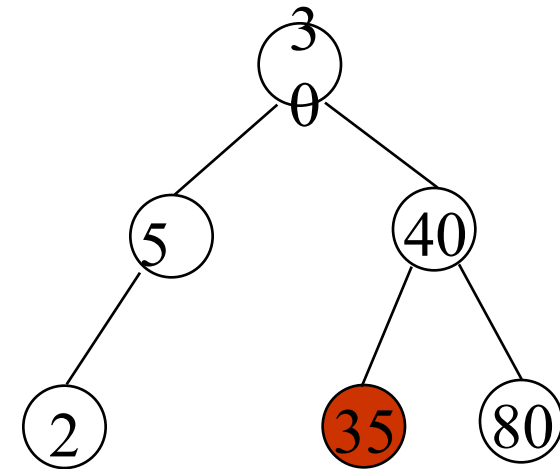
```
tree_pointer search2 (tree_pointer tree,  
    int key)  
{  
    while (tree) {  
        if (key == tree->data) return tree;  
        if (key < tree->data)  
            tree = tree->left_child;  
        else tree = tree->right_child;  
    }  
    return NULL;  
}
```

$O(h)$

Insert Node in Binary Search Tree



Insert 80

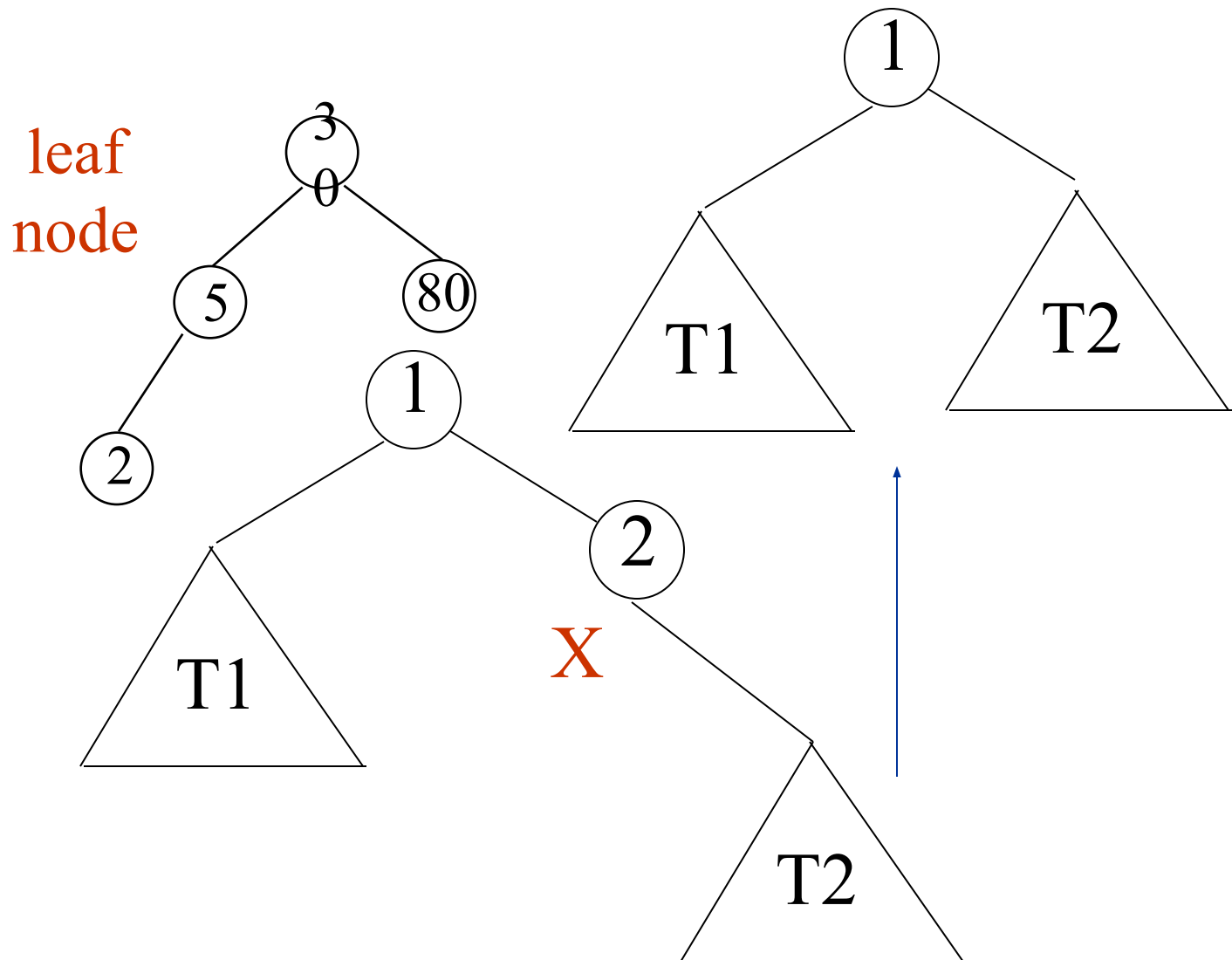


Insert 35

Insertion into A Binary Search Tree

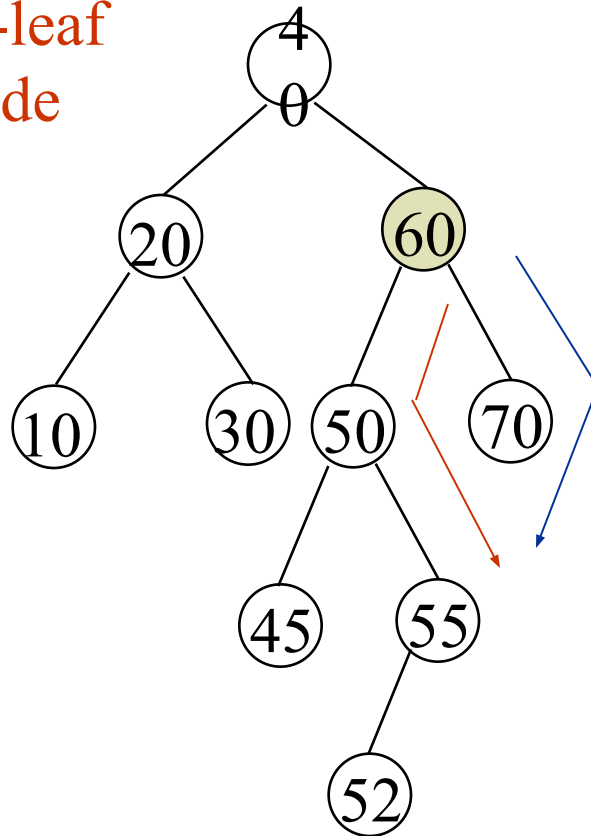
```
void insert_node(tree_pointer *node, int num)
{
    tree_pointer ptr,
        temp = modified_search(*node, num);
    if (temp || !(*node)) {
        ptr = (tree_pointer) malloc(sizeof(node));
        if (IS_FULL(ptr)) {
            fprintf(stderr, "The memory is full\n");
            exit(1);
        }
        ptr->data = num;
        ptr->left_child = ptr->right_child = NULL;
        if (*node)
            if (num < temp->data) temp->left_child = ptr;
            else temp->right_child = ptr;
        else *node = ptr;
    }
}
```


Deletion for A Binary Search Tree

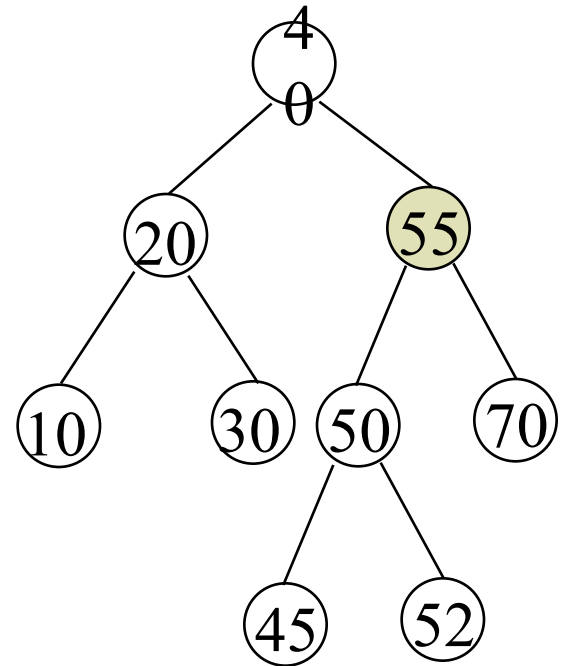


Deletion for A Binary Search Tree

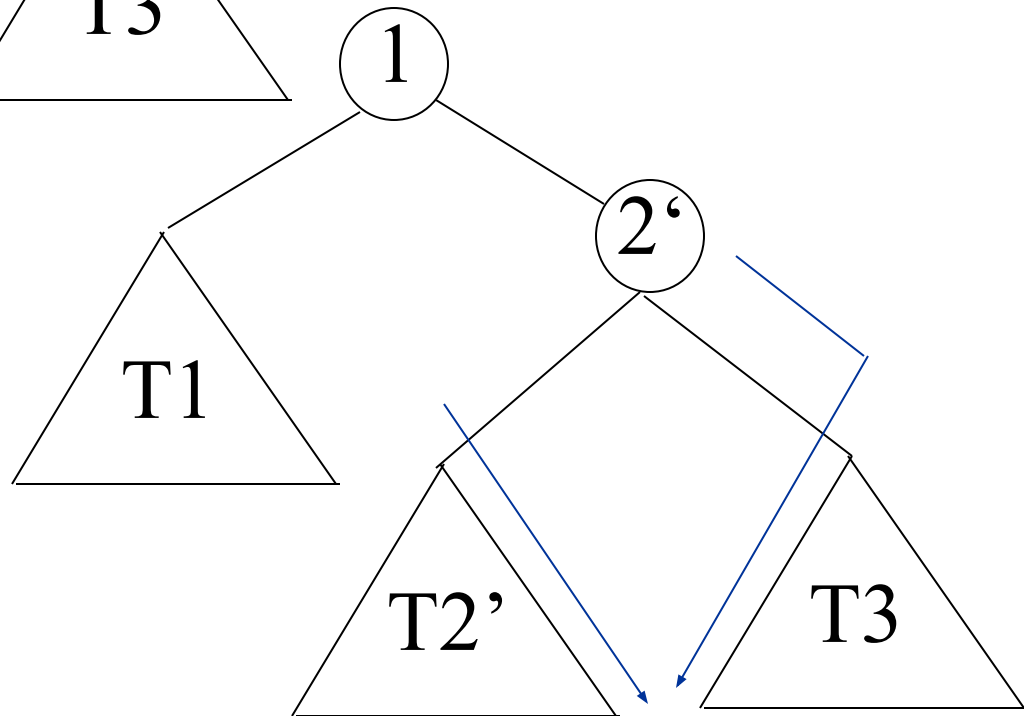
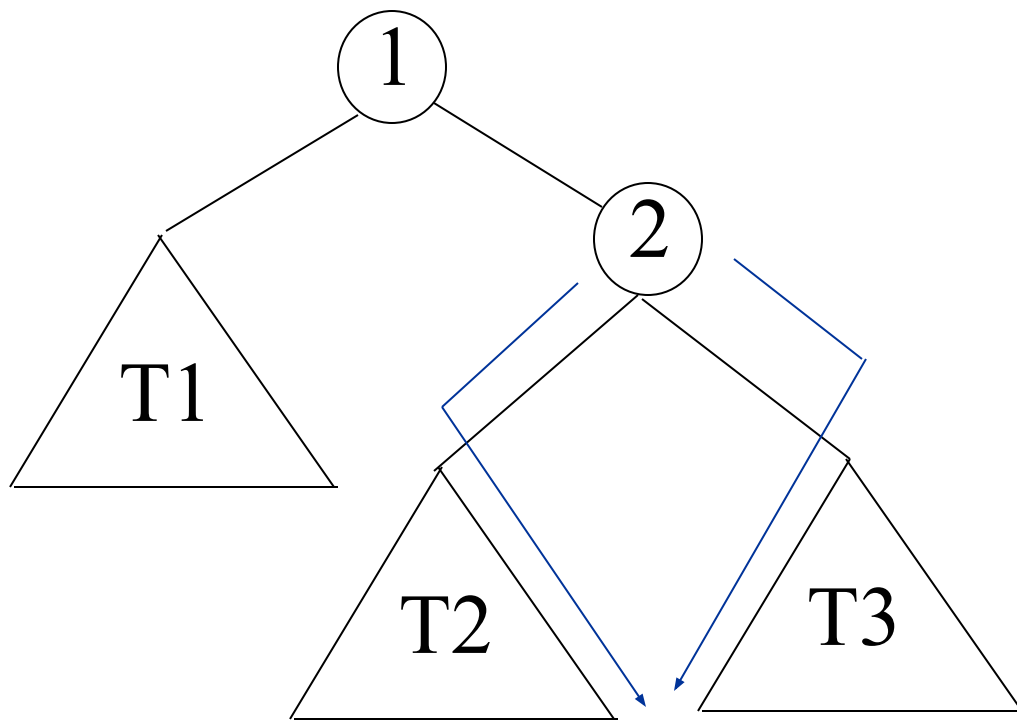
non-leaf
node



Before deleting 60



After deleting 60



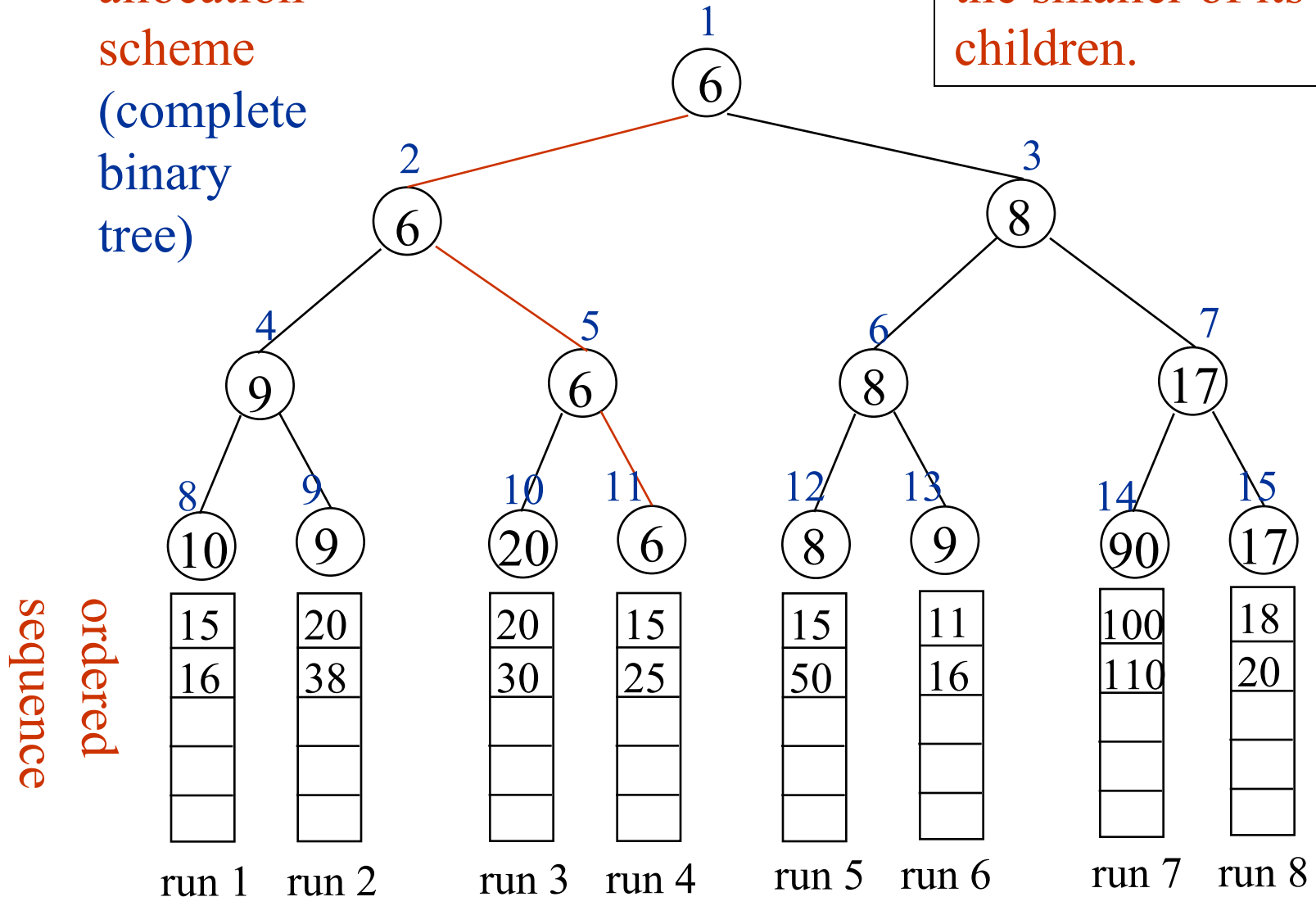
Selection Trees

- (1) winner tree
- (2) loser tree

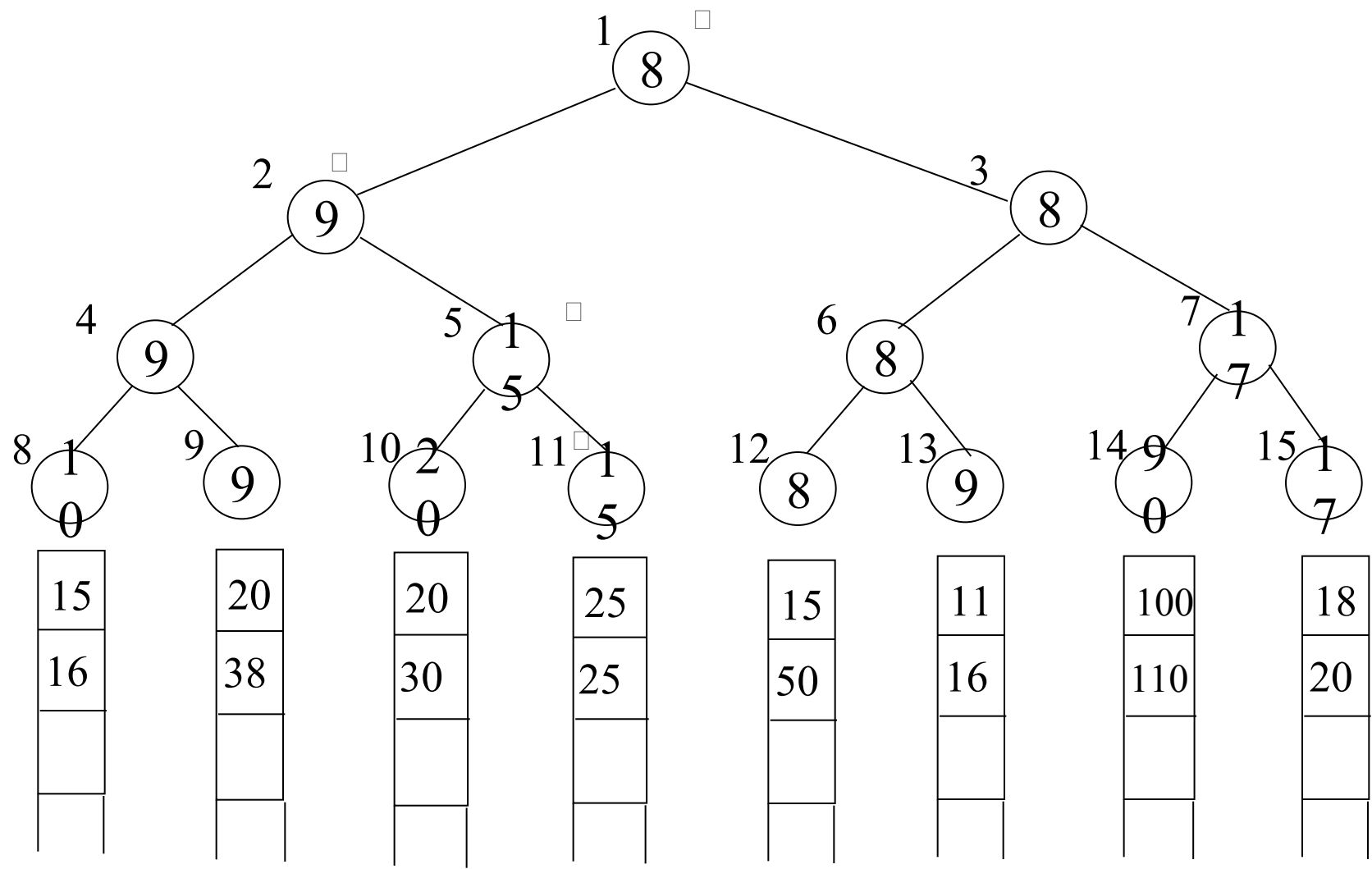
sequential
allocation
scheme
(complete
binary
tree)

winner tree

Each node represents
the smaller of its two
children.



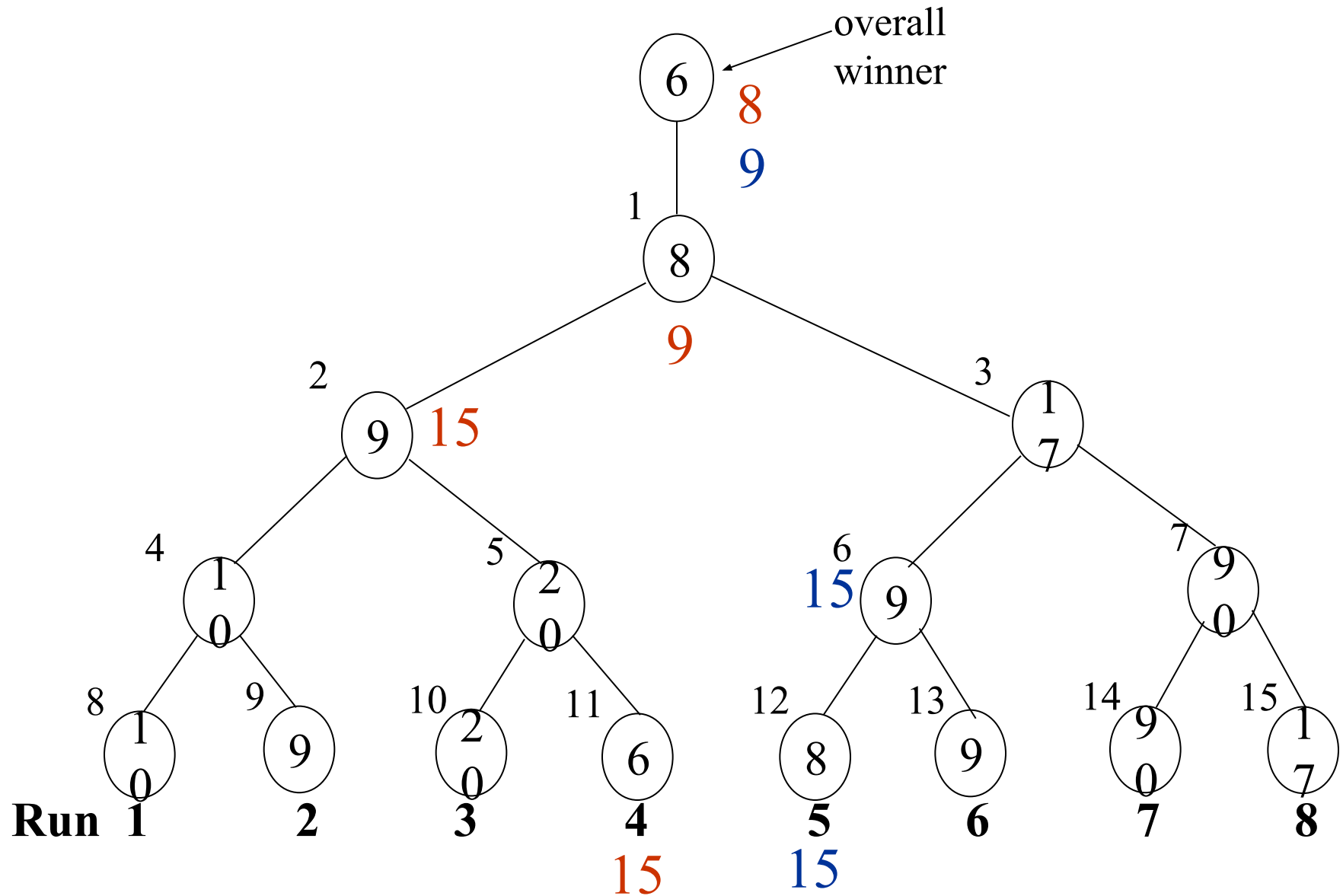
***Figure 5.35:** Selection tree of Figure 5.34 after one record has been output and the tree restructured(nodes that were changed are ticked)



Analysis

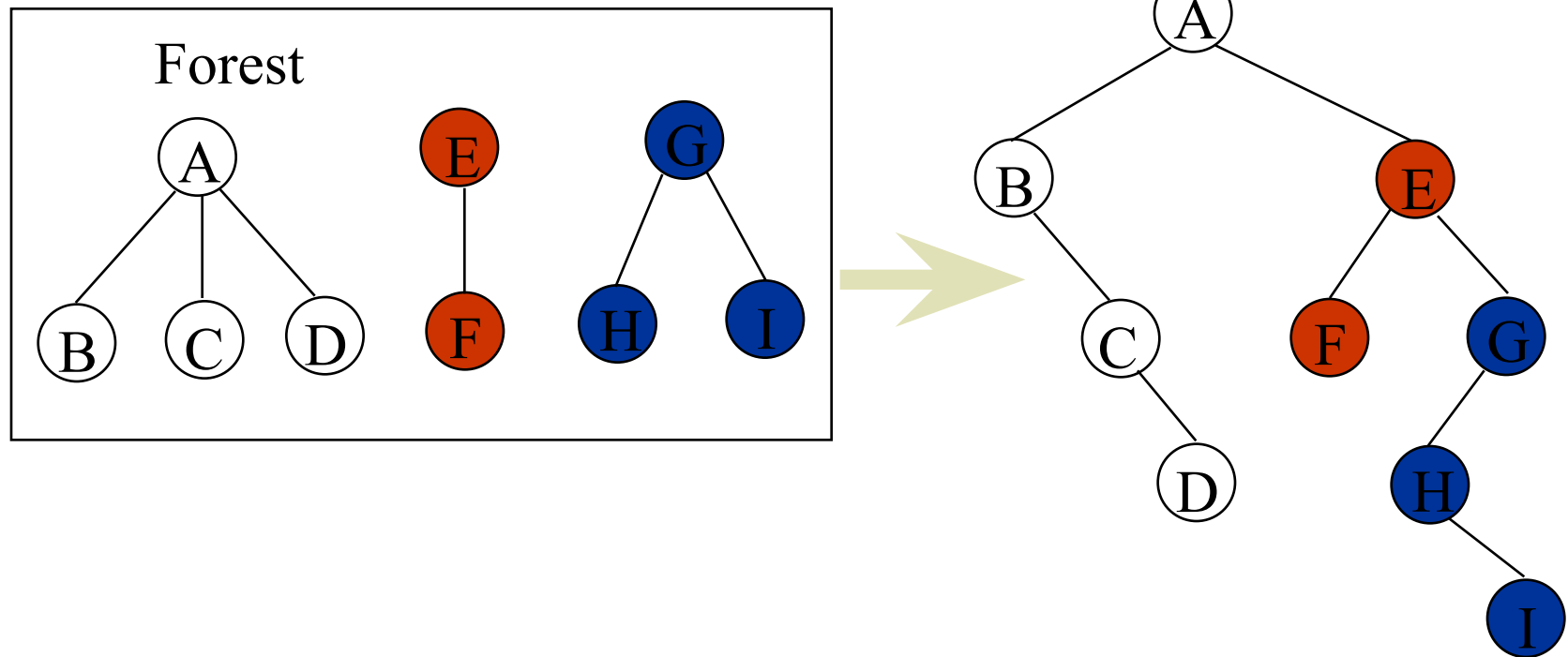
- K : # of runs
- n : # of records
- setup time: $O(K)$ $(K-1)$
- restructure time: $O(\log_2 K)$ $\lceil \log_2(K+1) \rceil$
- merge time: $O(n \log_2 K)$
- slight modification: **tree of loser**
 - consider the parent node only (vs. sibling nodes)

***Figure 5.36: Tree of losers corresponding to Figure 5.34 (p.235)**



Forest

- A forest is a set of $n \geq 0$ disjoint trees



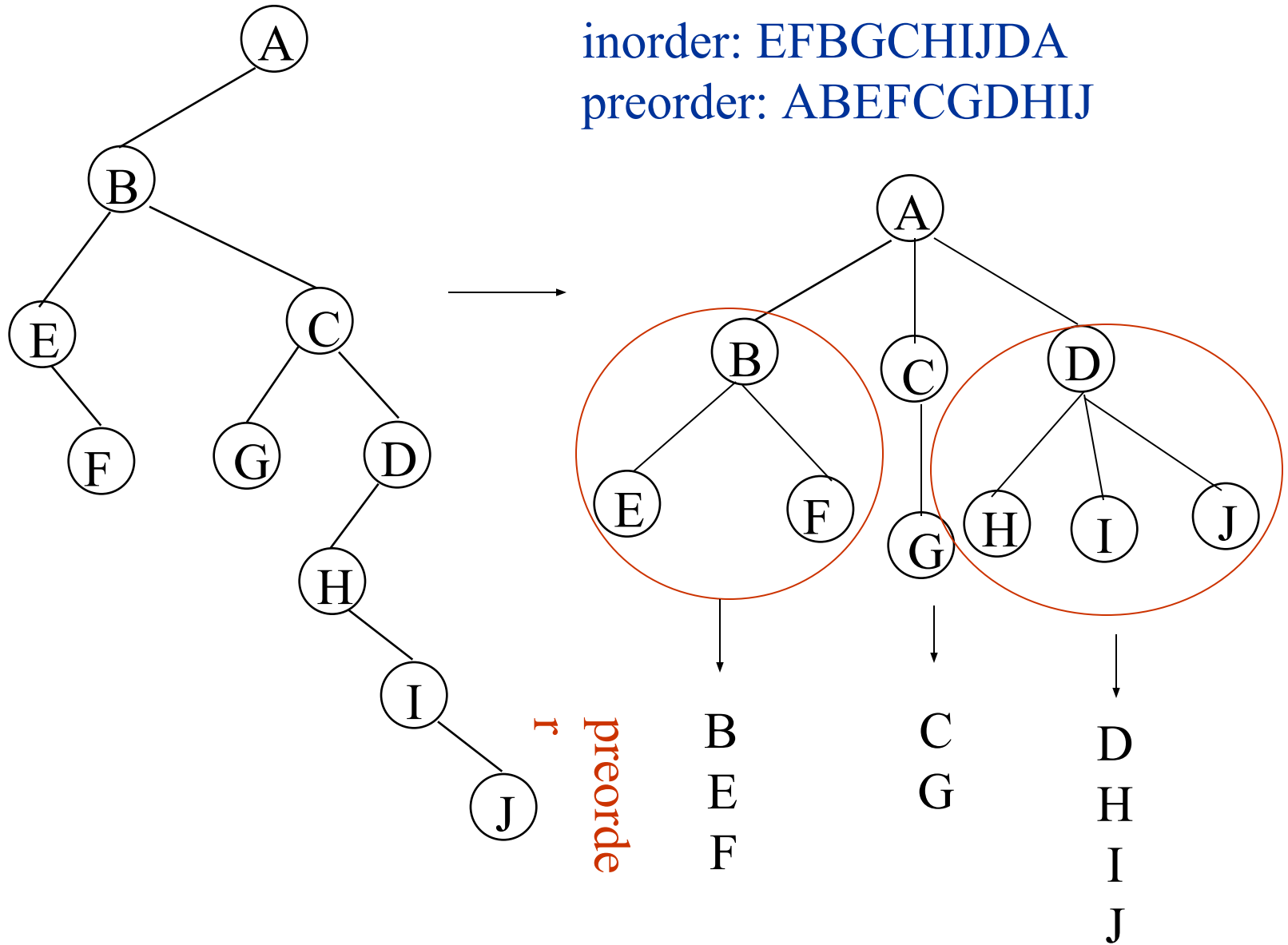
Transform a forest into a binary tree

- T_1, T_2, \dots, T_n : a forest of trees
 $B(T_1, T_2, \dots, T_n)$: a binary tree corresponding to this forest
- algorithm
 - (1) empty, if $n = 0$
 - (2) has root equal to $\text{root}(T_1)$
 - has left subtree equal to $B(T_{11}, T_{12}, \dots, T_{1m})$
 - has right subtree equal to $B(T_2, T_3, \dots, T_n)$

Forest Traversals

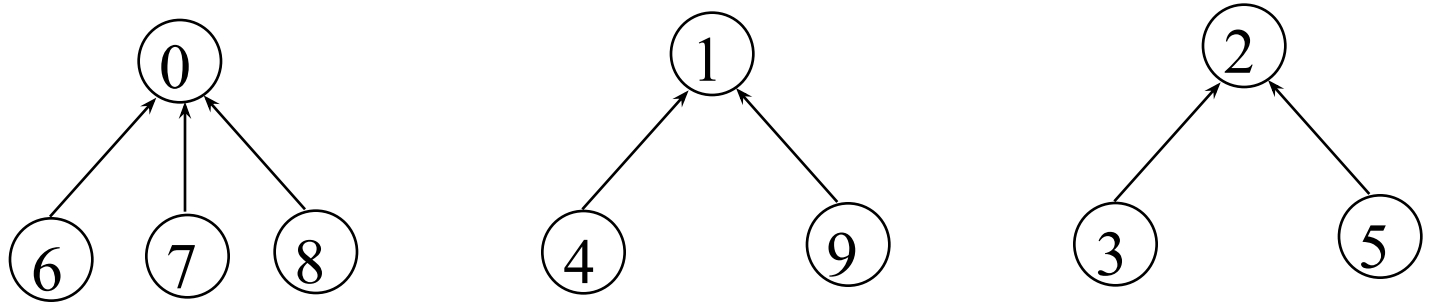
- Preorder
 - If F is empty, then return
 - Visit the root of the first tree of F
 - Traverse the subtrees of the first tree in tree preorder
 - Traverse the remaining trees of F in preorder
- Inorder
 - If F is empty, then return
 - Traverse the subtrees of the first tree in tree inorder
 - Visit the root of the first tree
 - Traverse the remaining trees of F in inorder

inorder: EFBGCHIJDA
preorder: ABEFCGDHIJ



Set Representation

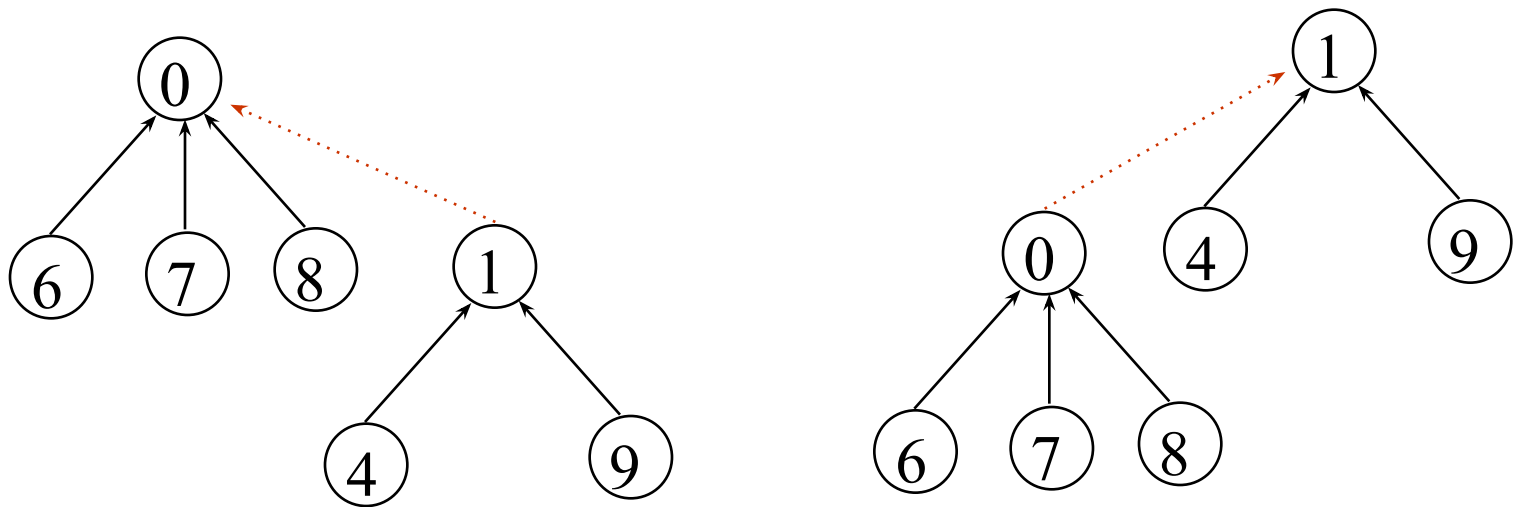
- $S_1 = \{0, 6, 7, 8\}$, $S_2 = \{1, 4, 9\}$, $S_3 = \{2, 3, 5\}$



- $S_i \cap S_j = \varnothing$
Two operations considered here
 - Disjoint set union $S_1 \cup S_2 = \{0, 6, 7, 8, 1, 4, 9\}$
 - Find(i): Find the set containing the element i .
 $3 \in S_3, 8 \in S_1$

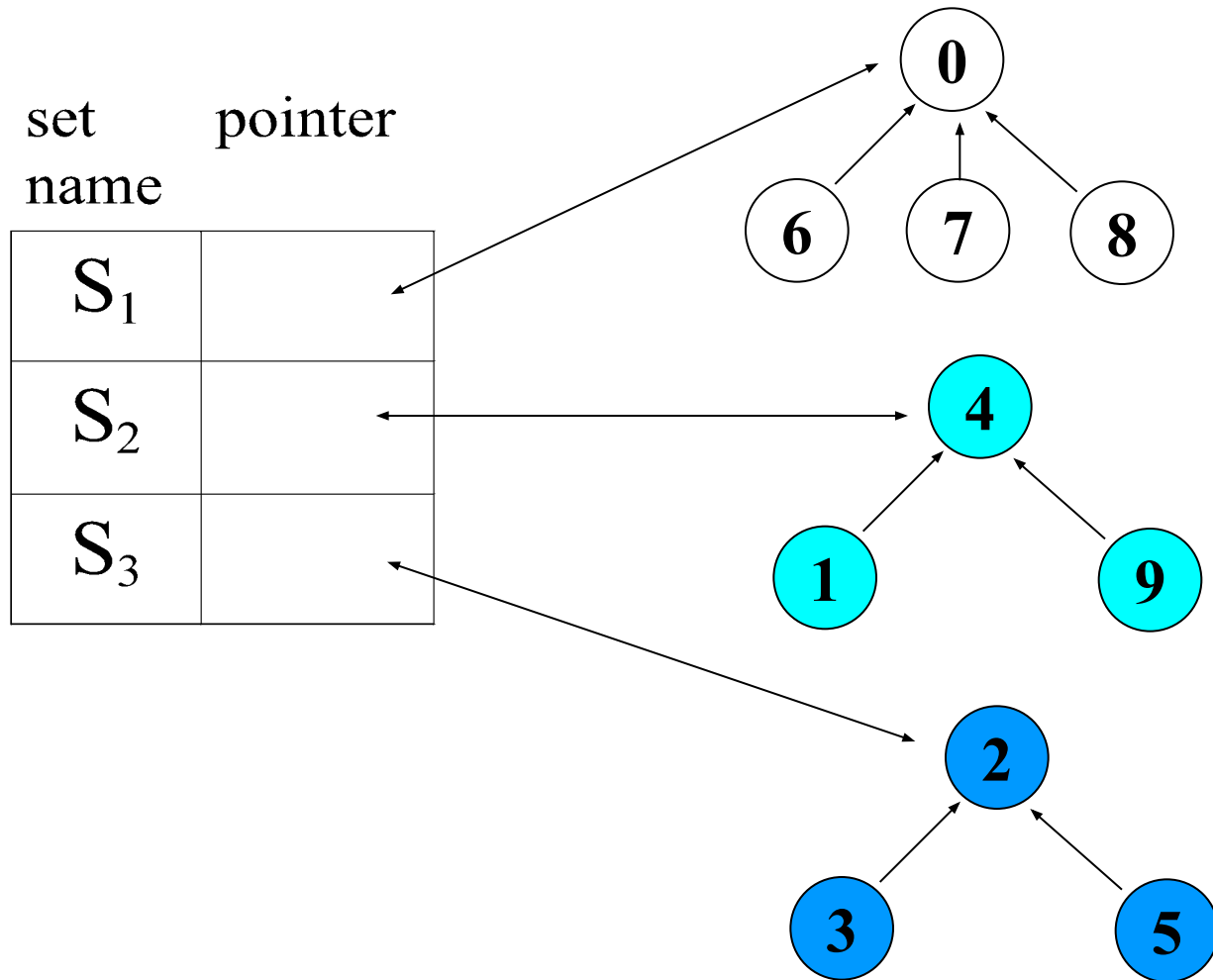
Disjoint Set Union

Make one of trees a subtree of the other



Possible representation for $S_1 \cup S_2$

***Figure 5.41: Data Representation of S_1 , S_2 and S_3 (p.240)**



Array Representation for Set

i	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
parent	-1	4	-1	2	-1	2	0	0	0	4

```
int find1(int i)
{
    for (; parent[i]>=0; i=parent[i])
        return i;
}
```

```
void union1(int i, int j)
{
    parent[i]= j;
}
```


*Figure 5.43: Degenerate tree (p.242)

union operation

$O(n)$ $n-1$

find operation

$O(n^2)$ $\sum_{i=2}^n i$



union(0,1), find(0)

union(1,2), find(0)

.

.

.

union(n-2,n-1), find(0)

degenerate tree

***Figure 5.44: Trees obtained using the weighting rule(p.243)**

weighting rule for union(i,j): if # of nodes in $i < \#$ in j then j the parent of i

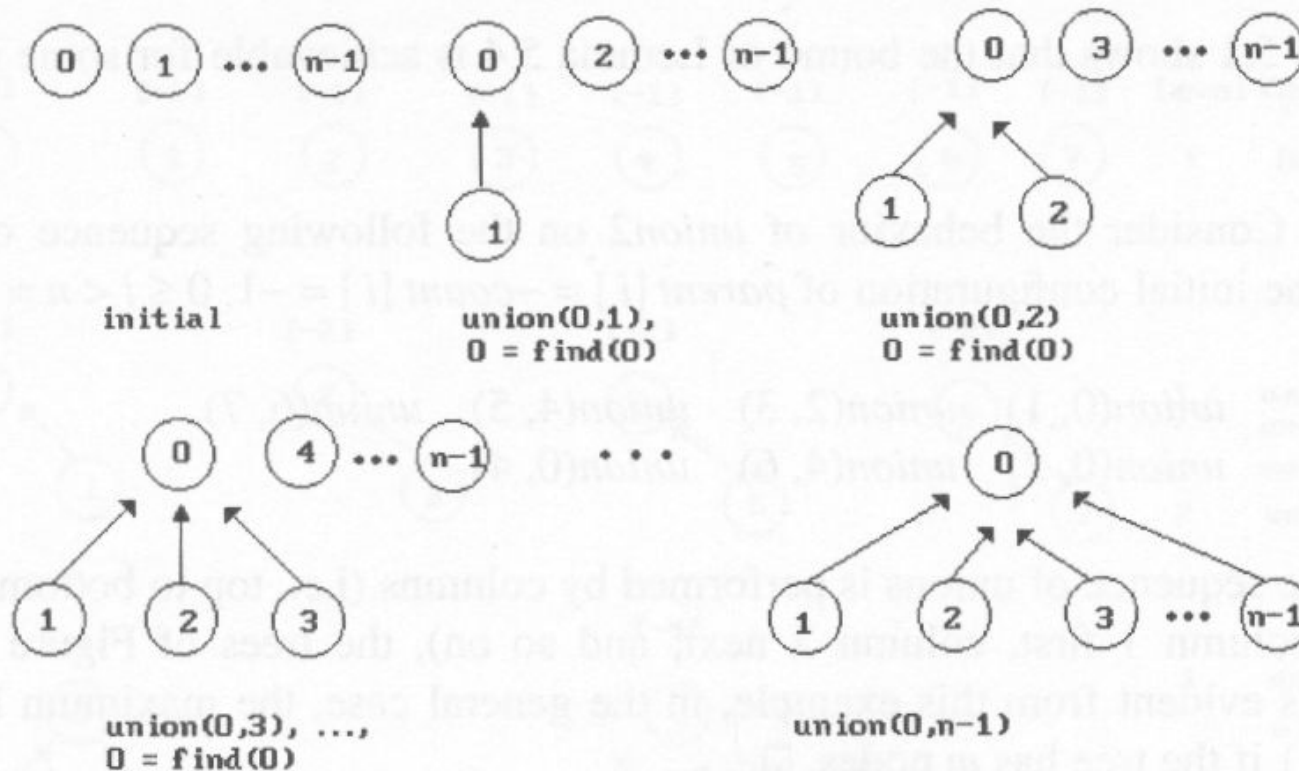


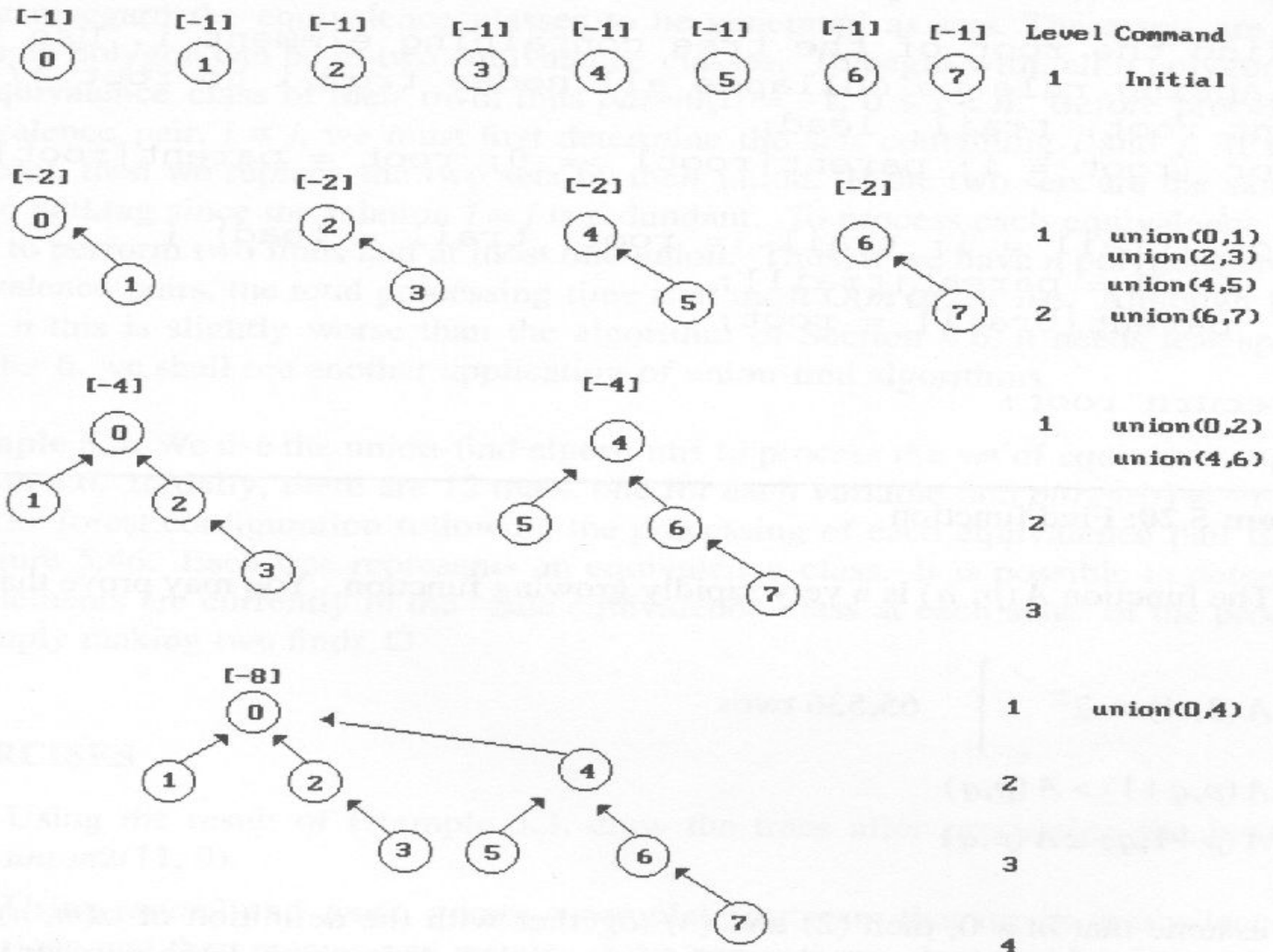
Figure 5.44: Trees obtained using the weighting rule

Modified Union Operation

```
void union2(int i, int j)
{
    Keep a count in the root of tree
    int temp = parent[i]+parent[j];
    if (parent[i]>parent[j]) {
        parent[i]=j;      i has fewer nodes.
        parent[j]=temp;
    }
    else { j has fewer nodes
        parent[j]=i;
        parent[i]=temp;
    }
}
```

If the number of nodes in tree i is less than the number in tree j, then make j the parent of i; otherwise make i the parent of j.

Figure 5.45: Trees achieving worst case bound (p.245)

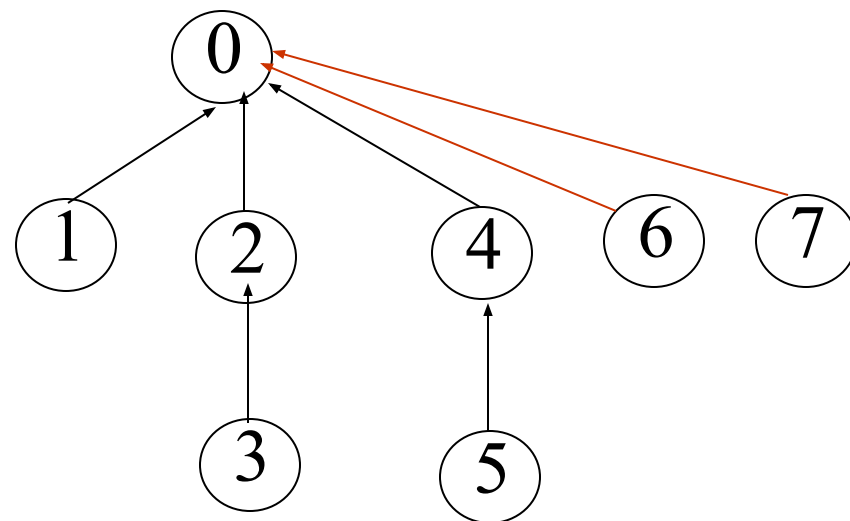
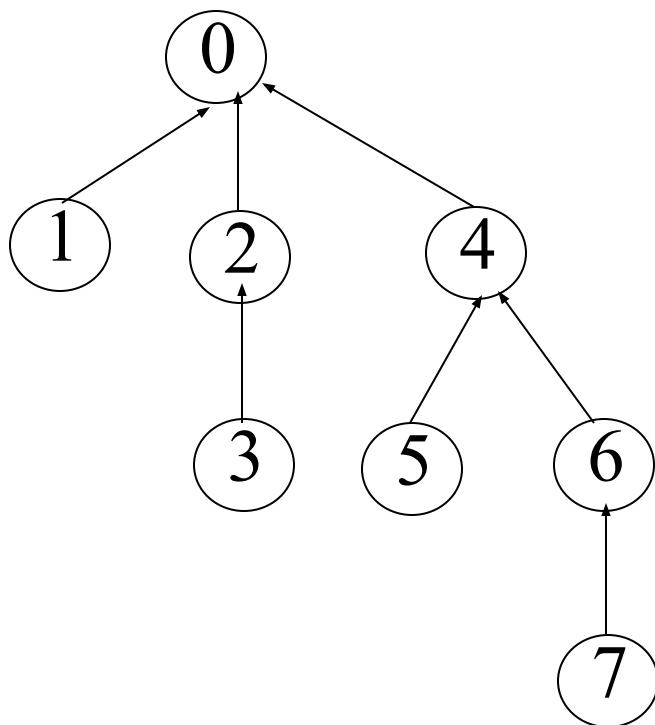


$$\lfloor \log_2 8 \rfloor + 1$$

Modified *Find(i)* Operation

```
int find2(int i)
{
    int root, trail, lead;
    for (root=i; parent[root]>=0;
        root=parent[root]) ;
    for (trail=i; trail!=root;
        trail=lead) {
        lead = parent[trail];
        parent[trail]= root;
    }
    return root;
}
```

If j is a node on the path from i to its root then make j a child of the root



find(7) find(7) find(7) find(7) find(7) find(7) find(7) find(7)

go up	3	1	1	1	1	1	1	1
reset	2							

12 moves (vs. 24 moves)

Applications

- Find equivalence class $i \equiv j$
- Find S_i and S_j such that $i \in S_i$ and $j \in S_j$
(two finds)
 - $S_i = S_j$ do nothing
 - $S_i \neq S_j$ union(S_i, S_j)
- example
 $0 \equiv 4, 3 \equiv 1, 6 \equiv 10, 8 \equiv 9, 7 \equiv 4, 6 \equiv 8,$
 $3 \equiv 5, 2 \equiv 11, 11 \equiv 0$
 $\{0, 2, 4, 7, 11\}, \{1, 3, 5\}, \{6, 8, 9, 10\}$

preorder: A B C D E F G H I

inorder: B C A E D G H F I

