Introduction to algorithms

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Fundamentals of algorithmic problem solving

An algorithm is a sequence of unambiguous instructions for solving a problem, i.e., for obtaining a required output for any legitimate input in a finite amount of time.

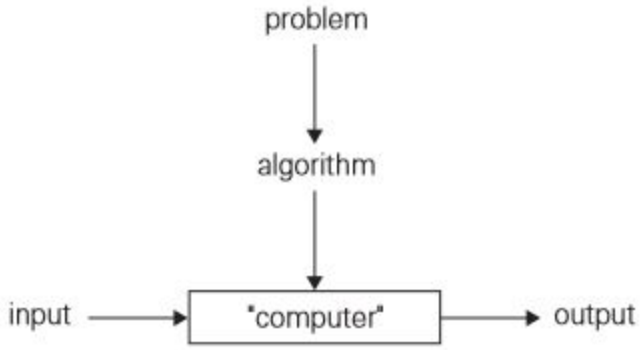


FIGURE 1.1 The notion of the algorithm.

- The non ambiguity requirement for each step of an algorithm cannot be compromised.
- The range of inputs for which an algorithm works has to be specified carefully.
- The same algorithm can be represented in several different ways.
- There may exist several algorithms for solving the same problem.
- Algorithms for the same problem can be based on very different ideas and can solve the problem with dramatically different speeds.

Fundamentals of algorithmic problem solving

Euclid's algorithm for computing gcd(m, n)

Step 1 If $n = 0$, return the value of m as the answer and stop; otherwise, proceed to Step 2.

Step 2 Divide m by n and assign the value of the remainder to r.

Step 3 Assign the value of n to m and the value of r to n. Go to Step 1.

Pseudocode:

ALGORITHM Euclid(m, n)

//Computes gcd(m, n) by Euclid's algorithm

//Input: Two nonnegative, not-both-zero integers m and n

//Output: Greatest common divisor of m and n

while (n!= 0) do

    r ←m mod n

    m←n

    n←r

End while

    return m

Algorithmic Design and Analysis Process

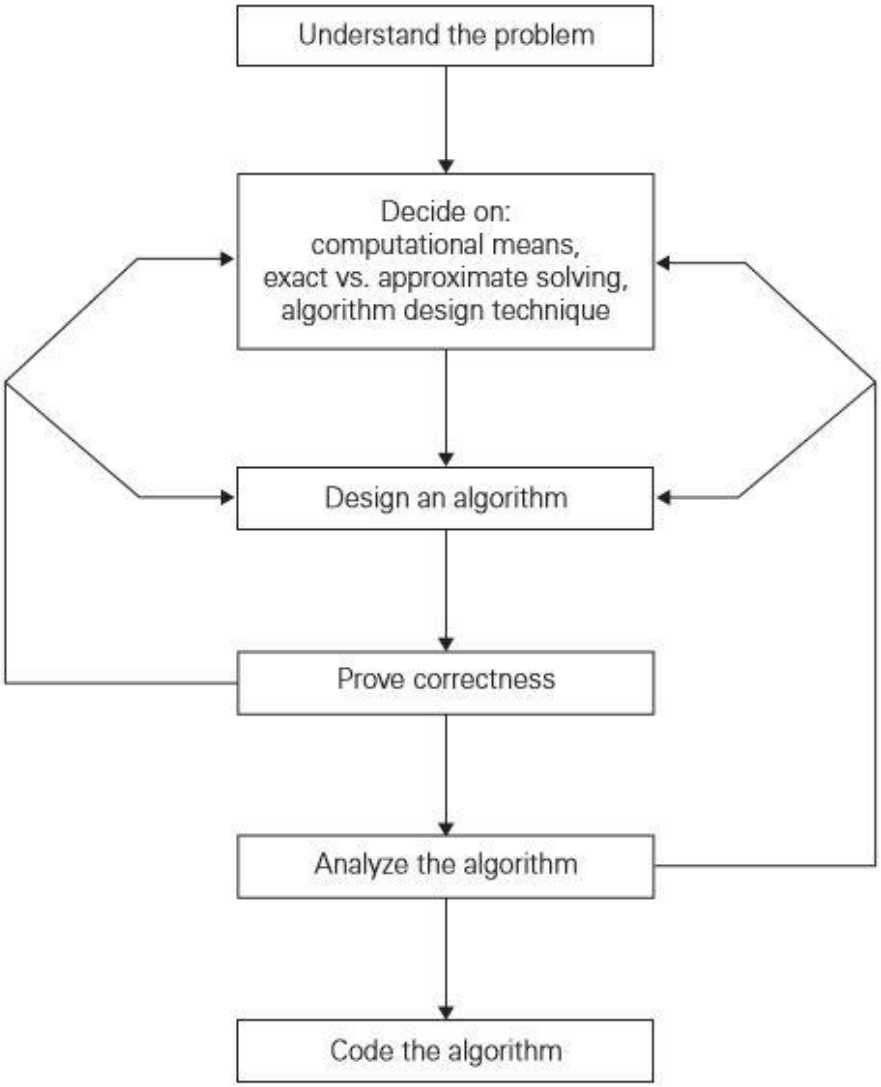


FIGURE 1.2 Algorithm design and analysis process.

Algorithmic Design and Analysis Process : Steps

- Understanding the Problem
- Ascertaining the Capabilities of the Computational Device: Sequential, Parallel
- Choosing between Exact and Approximate Problem Solving :

  Choose between solving the problem exactly (Exact algorithm) or solving it approximately (approximation algorithm)
- Algorithm Design Techniques

  An algorithm design technique (or "strategy" or "paradigm") is a general approach to solving problems algorithmically that is applicable to a variety of problems from different areas of computing.
- Designing an Algorithm and Data Structures: Arrays, Linked Lists, Queue, Stack
- Methods of Specifying an Algorithm : Flow Chart, Pseudo code, Algorithm
- Proving an Algorithm's Correctness : Tracing using paper and pen method
- Analyzing an Algorithm : Time Efficiency, Space Efficiency
- Coding an Algorithm: Using C++, Java, Python programming languages etc.

Important Problem Types

- Sorting
- Searching
- String processing
- Graph problems
- Combinatorial problems
- Geometric problems
- Numerical problems

The Analysis Framework

Time efficiency: Time Efficiency also called time complexity, indicates how fast an algorithm in question runs.

Space efficiency: Space Efficiency is also called space complexity, refers to the amount of memory units required by the algorithm in addition to the space needed for its input and output.

$$T(n) \approx Cop C(n)$$

Where

Let Cop be the execution time of an algorithm's basic operation on a particular computer,

let C(n) be the number of times this operation needs to be executed for this algorithm.

Order of Growth

TABLE 2.1 Values (some approximate) of several functions important for analysis of algorithms

n	$\log_2 n$	n	$n \log_2 n$	n^2	n^3	2^n	$n!$
10	3.3	10^1	$3.3 \cdot 10^1$	10^2	10^3	10^3	$3.6 \cdot 10^6$
10^2	6.6	10^2	$6.6 \cdot 10^2$	10^4	10^6	$1.3 \cdot 10^{30}$	$9.3 \cdot 10^{157}$
10^3	10	10^3	$1.0 \cdot 10^4$	10^6	10^9		
10^4	13	10^4	$1.3 \cdot 10^5$	10^8	10^{12}		
10^5	17	10^5	$1.7 \cdot 10^6$	10^{10}	10^{15}		
10^6	20	10^6	$2.0 \cdot 10^7$	10^{12}	10^{18}		

ALGORITHM *SequentialSearch*$(A[0..n-1], K)$

```
//Searches for a given value in a given array by sequential search
//Input: An array A[0..n - 1] and a search key K
//Output: The index of the first element in A that matches K or -1 if there are no
//          matching elements
i ←0
while i < n and A[i] ≠ K do
        i ←i + 1
if i < n return i
else return -1
```

Worst-case efficiency

- The ***worst-case efficiency of an algorithm is its efficiency for the worst case input of size n.***
- The algorithm runs the longest among all possible inputs of that size.
- For the input of size *n*, *the running time is Cworst(n) = n*.

Best case efficiency

- The ***best-case efficiency of an algorithm is its efficiency for the best case input of size n.***
- The algorithm runs the fastest among all possible inputs of that size n.
- In sequential search, If we search a first element in list of size *n. (i.e. first element equal to* a search key), then the running time is *Cbest(n) = 1*

Average case efficiency

- The Average case efficiency lies between best case and worst case.
- To analyze the algorithm's average case efficiency, we must make some assumptions about possible inputs of size n.
- The standard assumptions are that
  - The probability of a successful search is equal to p $(0 \le p \le 1)$ and
  - The probability of the first match occurring in the ith position of the list is the same for every i.

$$C_{avg}(n) = [1 \cdot \frac{p}{n} + 2 \cdot \frac{p}{n} + \cdots + i \cdot \frac{p}{n} + \cdots + n \cdot \frac{p}{n}] + n \cdot (1 - p)$$

$$= \frac{p}{n}[1 + 2 + \cdots + i + \cdots + n] + n(1 - p)$$

$$= \frac{p}{n}\frac{n(n+1)}{2} + n(1 - p) = \frac{p(n+1)}{2} + n(1 - p).$$

Basic Efficiency Classes

TABLE 2.2 Basic asymptotic efficiency classes

Class	Name	Comments
1	*constant*	Short of best-case efficiencies, very few reasonable examples can be given since an algorithm's running time typically goes to infinity when its input size grows infinitely large.
$\log n$	*logarithmic*	Typically, a result of cutting a problem's size by a constant factor on each iteration of the algorithm (see Section 4.4). Note that a logarithmic algorithm cannot take into account all its input or even a fixed fraction of it: any algorithm that does so will have at least linear running time.
n	*linear*	Algorithms that scan a list of size n (e.g., sequential search) belong to this class.
$n \log n$	*linearithmic*	Many divide-and-conquer algorithms (see Chapter 5), including mergesort and quicksort in the average case, fall into this category.
n^2	*quadratic*	Typically, characterizes efficiency of algorithms with two embedded loops (see the next section). Elementary sorting algorithms and certain operations on $n \times n$ matrices are standard examples.
n^3	*cubic*	Typically, characterizes efficiency of algorithms with three embedded loops (see the next section). Several nontrivial algorithms from linear algebra fall into this class.
2^n	*exponential*	Typical for algorithms that generate all subsets of an n-element set. Often, the term "exponential" is used in a broader sense to include this and larger orders of growth as well.
$n!$	*factorial*	Typical for algorithms that generate all permutations of an n-element set.

Asymptotic Notations

O-notation DEFINITION: A function t(n) is said to be in O(g(n)), denoted t(n)∈O(g(n)), if t(n) is bounded above by some constant multiple of g(n) for all large n, i.e., if there exist some positive constant c and some nonnegative integer n0 such that t(n)≤cg(n) for all n≥n0.

Ex 1:100n+5≤105n (for all n≥n0)

Ex2: 100n+5≤101n (for all n≥n0

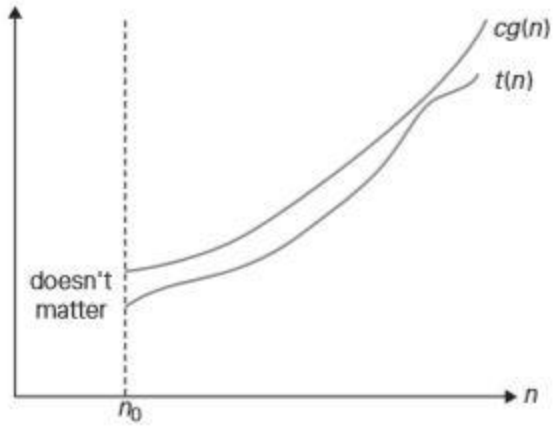


FIGURE 2.1 Big-oh notation: $t(n) \in O(g(n))$.

Asymptotic Notations

Ω-notation DEFINITION : A function t(n) is said to be in (g(n)), denoted t(n)∈ Ω(g(n)), if t(n) is bounded below by some positive constant multiple of g(n) for all large n, i.e., if there exist some positive constant c and some nonnegative integer n0 such that t(n)≥cg(n) for all n≥n0.

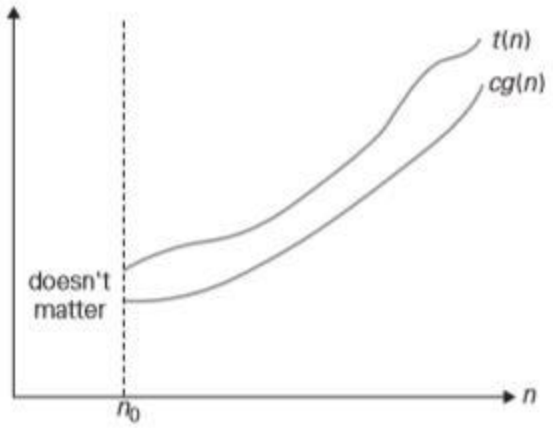


FIGURE 2.2 Big-omega notation: $t(n) \in \Omega(g(n))$.

Asymptotic Notations

Θ-notation DEFINITION :A function t(n) is said to be in (g(n)), denoted t(n)∈Θ (g(n)), if t(n) is bounded both above and below by some positive constant multiples of g(n) for all large n, i.e., if there exist some positive constants c1 and c2 and some nonnegative integer n0 such that c2g(n)≤t(n)≤c1g(n) for all n≥n0

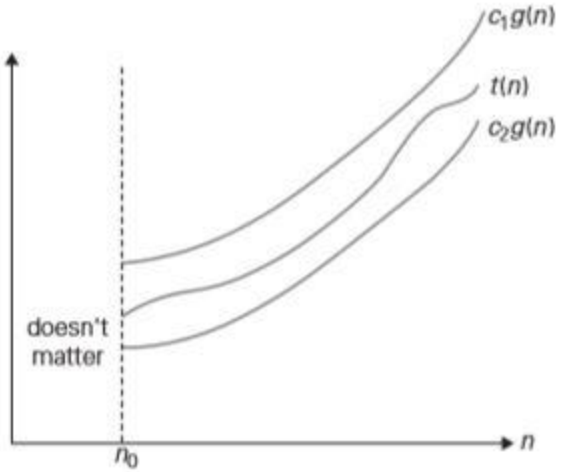


FIGURE 2.3 Big-theta notation: $t(n) \in \Theta(g(n))$.

THEOREM:

THEOREM:

If $t1(n) \in O(g1(n))$ and $t2(n) \in O(g2(n))$, then

$t1(n)+t2(n) \in O(\max\{g1(n), g2(n)\})$.

Proof: $t1(n) \leq c1g1(n)$ for all $n \geq n1$.

$t2(n) \leq c2g2(n)$ for all $n \geq n2$.

$t1(n)+t2(n) \leq c1g1(n)+c2g2(n)$

Let $c3=\max\{c1,c\ 2\}$ and consider $n \geq \max\{n1,n\ 2\}$, we can write

$\leq c3g1(n)+c3g2(n)$

$=c3[g1(n)+g2(n)]$

$\leq c3\max\{g1(n), g2(n)\}$.

Comparing Orders of Growth using limits

$$\lim_{n\to\infty}\frac{t(n)}{g(n)}=\begin{cases}0 & \text{implies that } t(n) \text{ has a smaller order of growth than } g(n),\\ c & \text{implies that } t(n) \text{ has the same order of growth as } g(n),\\ \infty & \text{implies that } t(n) \text{ has a larger order of growth than } g(n).^{3}\end{cases}$$

EXAMPLE 1 Compare the orders of growth of $\frac{1}{2}n(n-1)$ and n^2. (This is one of the examples we used at the beginning of this section to illustrate the definitions.)

$$\lim_{n\to\infty}\frac{\frac{1}{2}n(n-1)}{n^2}=\frac{1}{2}\lim_{n\to\infty}\frac{n^2-n}{n^2}=\frac{1}{2}\lim_{n\to\infty}(1-\frac{1}{n})=\frac{1}{2}.$$

Since the limit is equal to a positive constant, the functions have the same order of growth or, symbolically, $\frac{1}{2}n(n-1)\in\Theta(n^2)$. ■

EXAMPLE 2 Compare the orders of growth of $\log_2 n$ and $\sqrt{n}$. (Unlike Example 1, the answer here is not immediately obvious.)

$$\lim_{n\to\infty}\frac{\log_2 n}{\sqrt{n}}=\lim_{n\to\infty}\frac{(\log_2 n)'}{(\sqrt{n})'}=\lim_{n\to\infty}\frac{(\log_2 e)\frac{1}{n}}{\frac{1}{2\sqrt{n}}}=2\log_2 e\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0.$$

EXAMPLE 3 Compare the orders of growth of $n!$ and 2^n. (We discussed this informally in Section 2.1.) Taking advantage of Stirling's formula, we get

$$\lim_{n\to\infty}\frac{n!}{2^n}=\lim_{n\to\infty}\frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{2^n}=\lim_{n\to\infty}\sqrt{2\pi n}\frac{n^n}{2^n e^n}=\lim_{n\to\infty}\sqrt{2\pi n}\left(\frac{n}{2e}\right)^n=\infty.$$

General Plan for Analyzing the Time Efficiency of Nonrecursive Algorithms

1. Decide on a parameter (or parameters) indicating an input's size.
2. Identify the algorithm's basic operation. (As a rule, it is located in the innermost loop.)
3. Check whether the number of times the basic operation is executed depends only on the size of an input. If it also depends on some additional property, the worst-case, average-case, and, if necessary, best-case efficiencies have to be investigated separately.
4. Set up a sum expressing the number of times the algorithm's basic operation is executed.[4]
5. Using standard formulas and rules of sum manipulation, either find a closed-form formula for the count or, at the very least, establish its order of growth.

EXAMPLE 1 Consider the problem of finding the value of the largest element in a list of n numbers. For simplicity, we assume that the list is implemented as an array. The following is pseudocode of a standard algorithm for solving the problem.

ALGORITHM *MaxElement*$(A[0..n-1])$

```
//Determines the value of the largest element in a given array
//Input: An array A[0..n − 1] of real numbers
//Output: The value of the largest element in A
maxval ← A[0]
for i ← 1 to n − 1 do
    if A[i] > maxval
        maxval ← A[i]
return maxval
```

EXAMPLE 2 Consider the ***element uniqueness problem***: check whether all the elements in a given array of n elements are distinct. This problem can be solved by the following straightforward algorithm.

ALGORITHM *UniqueElements*$(A[0..n-1])$

```
//Determines whether all the elements in a given array are distinct
//Input: An array A[0..n − 1]
//Output: Returns "true" if all the elements in A are distinct
//                and "false" otherwise
for i ← 0 to n − 2 do
    for j ← i + 1 to n − 1 do
        if A[i] = A[j] return false
return true
```

$$C_{worst}(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} [(n-1) - (i+1) + 1] = \sum_{i=0}^{n-2} (n-1-i)$$

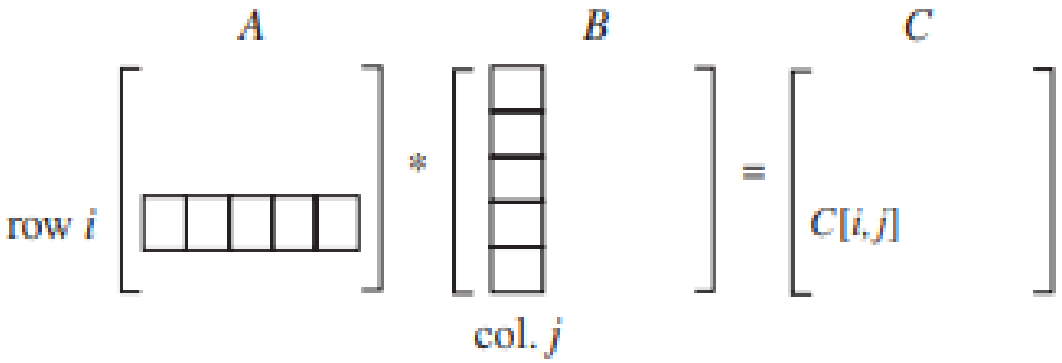
$$= \sum_{i=0}^{n-2} (n-1) - \sum_{i=0}^{n-2} i = (n-1) \sum_{i=0}^{n-2} 1 - \frac{(n-2)(n-1)}{2}$$

$$= (n-1)^2 - \frac{(n-2)(n-1)}{2} = \frac{(n-1)n}{2} \approx \frac{1}{2} n^2 \in \Theta(n^2).$$

We also could have computed the sum $\sum_{i=0}^{n-2} (n-1-i)$ faster as follows:

$$\sum_{i=0}^{n-2} (n-1-i) = (n-1) + (n-2) + \cdots + 1 = \frac{(n-1)n}{2},$$

EXAMPLE 3 Given two $n \times n$ matrices A and B, find the time efficiency of the definition-based algorithm for computing their product $C = AB$. By definition, C is an $n \times n$ matrix whose elements are computed as the scalar (dot) products of the rows of matrix A and the columns of matrix B:



where $C[i, j] = A[i, 0]B[0, j] + \cdots + A[i, k]B[k, j] + \cdots + A[i, n-1]B[n-1, j]$ for every pair of indices $0 \le i, j \le n-1$.

ALGORITHM *MatrixMultiplication*$(A[0..n-1, 0..n-1], B[0..n-1, 0..n-1])$

```
//Multiplies two square matrices of order n by the definition-based algorithm
//Input: Two n × n matrices A and B
//Output: Matrix C = AB
for i ← 0 to n − 1 do
    for j ← 0 to n − 1 do
        C[i, j] ← 0.0
        for k ← 0 to n − 1 do
            C[i, j] ← C[i, j] + A[i, k] * B[k, j]
return C
```

$$\sum_{k=0}^{n-1} 1,$$

and the total number of multiplications $M(n)$ is expressed by the following triple sum:

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1.$$

Now, we can compute this sum by using formula (S1) and rule (R1) given above. Starting with the innermost sum $\sum_{k=0}^{n-1} 1$, which is equal to n (why?), we get

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1 = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} n = \sum_{i=0}^{n-1} n^2 = n^3.$$

EXAMPLE 4 The following algorithm finds the number of binary digits in the binary representation of a positive decimal integer.

ALGORITHM *Binary*(n)

```
//Input: A positive decimal integer n
//Output: The number of binary digits in n's binary representation
count ← 1
while n > 1 do
    count ← count + 1
    n ← ⌊n/2⌋
return count
```

4. Consider the following algorithm.

ALGORITHM *Mystery*(n)

```
//Input: A nonnegative integer n
S ← 0
for i ← 1 to n do
    S ← S + i * i
return S
```

a. What does this algorithm compute?

b. What is its basic operation?

c. How many times is the basic operation executed?

d. What is the efficiency class of this algorithm?

Mathematical Analysis of Recursive Algorithms

General Plan for Analyzing the Time Efficiency of Recursive Algorithms

1. Decide on a parameter (or parameters) indicating an input's size.

2. Identify the algorithm's basic operation.

3. Check whether the number of times the basic operation is executed can vary on different inputs of the same size; if it can, the worst-case, average-case, and best-case efficiencies must be investigated separately.

4. Set up a recurrence relation, with an appropriate initial condition, for the number of times the basic operation is executed.

5. Solve the recurrence or, at least, ascertain the order of growth of its solution.

EXAMPLE 1 Compute the factorial function $F(n) = n!$ for an arbitrary nonnegative integer n. Since

$$n! = 1 \cdot \ldots \cdot (n-1) \cdot n = (n-1)! \cdot n \quad \text{for } n \geq 1$$

and $0! = 1$ by definition, we can compute $F(n) = F(n-1) \cdot n$ with the following recursive algorithm.

ALGORITHM $F(n)$

//Computes $n!$ recursively
//Input: A nonnegative integer n
//Output: The value of $n!$
if $n = 0$ **return** 1
else return $F(n-1) * n$

$$F(n) = F(n-1) \cdot n \quad \text{for } n > 0,$$

$$M(n) = \underbrace{M(n-1)}_{\substack{\text{to compute} \\ F(n-1)}} + \underbrace{1}_{\substack{\text{to multiply} \\ F(n-1) \text{ by } n}} \quad \text{for } n > 0.$$

if $n = 0$ return 1.

$M(0) = 0$.

the calls stop when $n = 0$ ← no multiplications when $n = 0$

$$M(n) = M(n-1) + 1 \quad \text{for } n > 0,$$
$$M(0) = 0.$$

$$M(n) = M(n-1) + 1 \qquad \text{substitute } M(n-1) = M(n-2) + 1$$

$$= [M(n-2) + 1] + 1 = M(n-2) + 2 \qquad \text{substitute } M(n-2) = M(n-3) + 1$$

$$= [M(n-3) + 1] + 2 = M(n-3) + 3.$$

$$M(n) = M(n-i) + i.$$

$$M(n) = M(n-1) + 1 = \cdots = M(n-i) + i = \cdots = M(n-n) + n = n.$$

Tower of Hanoi puzzle.

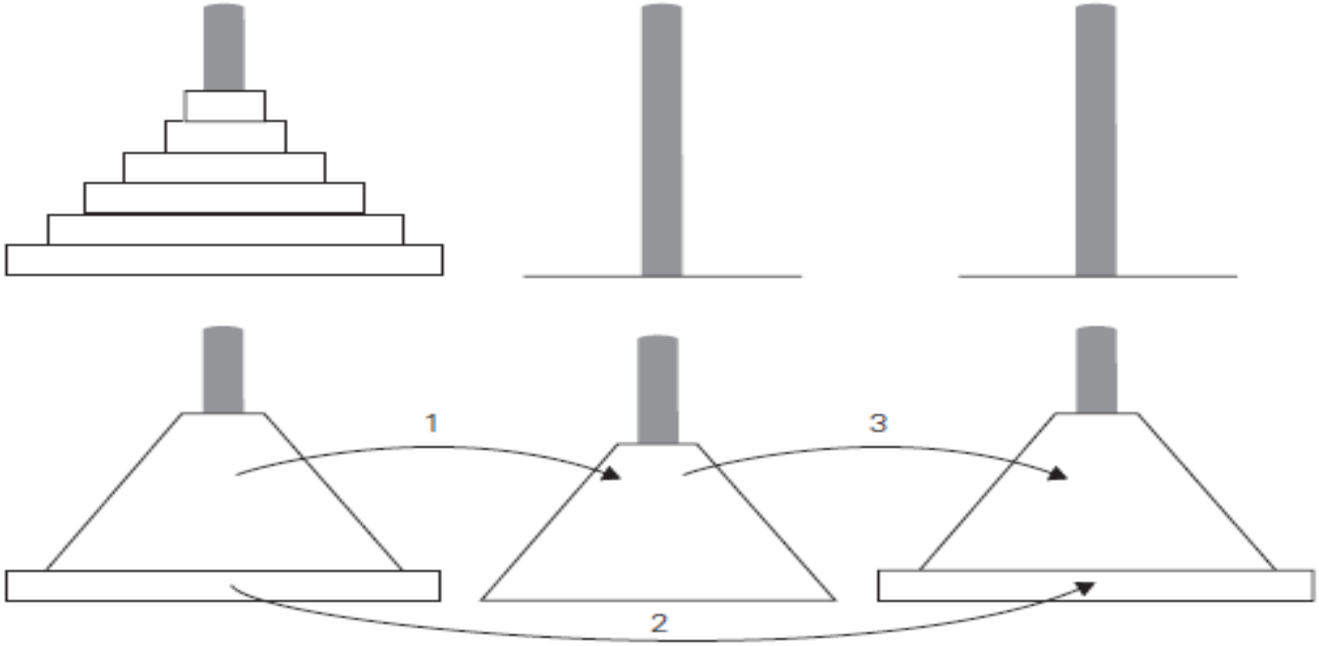


FIGURE 2.4 Recursive solution to the Tower of Hanoi puzzle.

$$M(n) = M(n-1) + 1 + M(n-1) \quad \text{for } n > 1.$$

With the obvious initial condition $M(1) = 1$, we have the following recurrence relation for the number of moves $M(n)$:

$$\begin{aligned} M(n) &= 2M(n-1) + 1 \quad \text{for } n > 1, \\ M(1) &= 1. \end{aligned} \tag{2.3}$$

We solve this recurrence by the same method of backward substitutions:

$$\begin{aligned} M(n) &= 2M(n-1) + 1 && \text{sub. } M(n-1) = 2M(n-2) + 1 \\ &= 2[2M(n-2) + 1] + 1 = 2^2 M(n-2) + 2 + 1 && \text{sub. } M(n-2) = 2M(n-3) + 1 \\ &= 2^2[2M(n-3) + 1] + 2 + 1 = 2^3 M(n-3) + 2^2 + 2 + 1. \end{aligned}$$

The pattern of the first three sums on the left suggests that the next one will be $2^4 M(n-4) + 2^3 + 2^2 + 2 + 1$, and generally, after i substitutions, we get

$$M(n) = 2^i M(n-i) + 2^{i-1} + 2^{i-2} + \cdots + 2 + 1 = 2^i M(n-i) + 2^i - 1.$$

Since the initial condition is specified for $n = 1$, which is achieved for $i = n - 1$, we get the following formula for the solution to recurrence (2.3):

$$\begin{aligned} M(n) &= 2^{n-1} M(n-(n-1)) + 2^{n-1} - 1 \\ &= 2^{n-1} M(1) + 2^{n-1} - 1 = 2^{n-1} + 2^{n-1} - 1 = 2^n - 1. \end{aligned}$$

EXAMPLE 3 As our next example, we investigate a recursive version of the algorithm discussed at the end of Section 2.3.

ALGORITHM *BinRec*(n)

//Input: A positive decimal integer n
//Output: The number of binary digits in n's binary representation
if $n = 1$ **return** 1
else return $BinRec(\lfloor n/2 \rfloor) + 1$

Let us set up a recurrence and an initial condition for the number of additions $A(n)$ made by the algorithm. The number of additions made in computing $BinRec(\lfloor n/2 \rfloor)$ is $A(\lfloor n/2 \rfloor)$, plus one more addition is made by the algorithm to increase the returned value by 1. This leads to the recurrence

$$A(n) = A(\lfloor n/2 \rfloor) + 1 \quad \text{for } n > 1. \tag{2.4}$$

A(1) = 0.

$$A(2^k) = A(2^{k-1}) + 1 \quad \text{for } k > 0,$$
$$A(2^0) = 0.$$

Now backward substitutions encounter no problems:

$$\begin{aligned}
A(2^k) &= A(2^{k-1}) + 1 && \text{substitute } A(2^{k-1}) = A(2^{k-2}) + 1\\
&= [A(2^{k-2}) + 1] + 1 = A(2^{k-2}) + 2 && \text{substitute } A(2^{k-2}) = A(2^{k-3}) + 1\\
&= [A(2^{k-3}) + 1] + 2 = A(2^{k-3}) + 3 && \dots\\
&\dots\\
&= A(2^{k-i}) + i\\
&\dots\\
&= A(2^{k-k}) + k.
\end{aligned}$$

Thus, we end up with

$$A(2^k) = A(1) + k = k,$$

or, after returning to the original variable $n = 2^k$ and hence $k = \log_2 n$,

$$A(n) = \log_2 n \in \Theta(\log n).$$

1. Solve the following recurrence relations.

   a. $x(n) = x(n-1) + 5 \quad \text{for } n > 1, \quad x(1) = 0$

   b. $x(n) = 3x(n-1) \quad \text{for } n > 1, \quad x(1) = 4$

   c. $x(n) = x(n-1) + n \quad \text{for } n > 0, \quad x(0) = 0$

   d. $x(n) = x(n/2) + n \quad \text{for } n > 1, \quad x(1) = 1 \quad (\text{solve for } n = 2^k)$

   e. $x(n) = x(n/3) + 1 \quad \text{for } n > 1, \quad x(1) = 1 \quad (\text{solve for } n = 3^k)$

1. a. $x(n) = x(n-1) + 5 \quad \text{for } n > 1, \quad x(1) = 0$

$$\begin{aligned} x(n) &= x(n-1) + 5 \\ &= [x(n-2) + 5] + 5 = x(n-2) + 5 \cdot 2 \\ &= [x(n-3) + 5] + 5 \cdot 2 = x(n-3) + 5 \cdot 3 \\ &= \ldots \\ &= x(n-i) + 5 \cdot i \\ &= \ldots \\ &= x(1) + 5 \cdot (n-1) = 5(n-1). \end{aligned}$$

Note: The solution can also be obtained by using the formula for the n term of the arithmetical progression:

$$x(n) = x(1) + d(n-1) = 0 + 5(n-1) = 5(n-1).$$

b. $x(n) = 3x(n-1) \quad \text{for } n > 1, \quad x(1) = 4$

$$\begin{aligned} x(n) &= 3x(n-1) \\ &= 3[3x(n-2)] = 3^2 x(n-2) \\ &= 3^2[3x(n-3)] = 3^3 x(n-3) \\ &= \ldots \\ &= 3^i x(n-i) \\ &= \ldots \\ &= 3^{n-1} x(1) = 4 \cdot 3^{n-1}. \end{aligned}$$

Note: The solution can also be obtained by using the formula for the n term of the geometric progression:

$$x(n) = x(1)q^{n-1} = 4 \cdot 3^{n-1}.$$

c. $x(n) = x(n-1) + n \quad \text{for } n > 0, \quad x(0) = 0$

$$
\begin{aligned}
x(n) &= x(n-1) + n \\
&= [x(n-2) + (n-1)] + n = x(n-2) + (n-1) + n \\
&= [x(n-3) + (n-2)] + (n-1) + n = x(n-3) + (n-2) + (n-1) + n \\
&= \ldots \\
&= x(n-i) + (n-i+1) + (n-i+2) + \ldots + n \\
&= \ldots \\
&= x(0) + 1 + 2 + \ldots + n = \frac{n(n+1)}{2}.
\end{aligned}
$$

d. $x(n) = x(n/2) + n \quad \text{for } n > 1, \quad x(1) = 1 \quad (\text{solve for } n = 2^k)$

$$
\begin{aligned}
x(2^k) &= x(2^{k-1}) + 2^k \\
&= [x(2^{k-2}) + 2^{k-1}] + 2^k = x(2^{k-2}) + 2^{k-1} + 2^k \\
&= [x(2^{k-3}) + 2^{k-2}] + 2^{k-1} + 2^k = x(2^{k-3}) + 2^{k-2} + 2^{k-1} + 2^k \\
&= \ldots \\
&= x(2^{k-i}) + 2^{k-i+1} + 2^{k-i+2} + \ldots + 2^k \\
&= \ldots \\
&= x(2^{k-k}) + 2^1 + 2^2 + \ldots + 2^k = 1 + 2^1 + 2^2 + \ldots + 2^k \\
&= 2^{k+1} - 1 = 2 \cdot 2^k - 1 = 2n - 1.
\end{aligned}
$$

e. $x(n) = x(n/3) + 1 \quad \text{for } n > 1, \quad x(1) = 1 \quad (\text{solve for } n = 3^k)$

$$
\begin{aligned}
x(3^k) &= x(3^{k-1}) + 1 \\
&= [x(3^{k-2}) + 1] + 1 = x(3^{k-2}) + 2 \\
&= [x(3^{k-3}) + 1] + 2 = x(3^{k-3}) + 3 \\
&= \ldots \\
&= x(3^{k-i}) + i \\
&= \ldots \\
&= x(3^{k-k}) + k = x(1) + k = 1 + \log_3 n.
\end{aligned}
$$