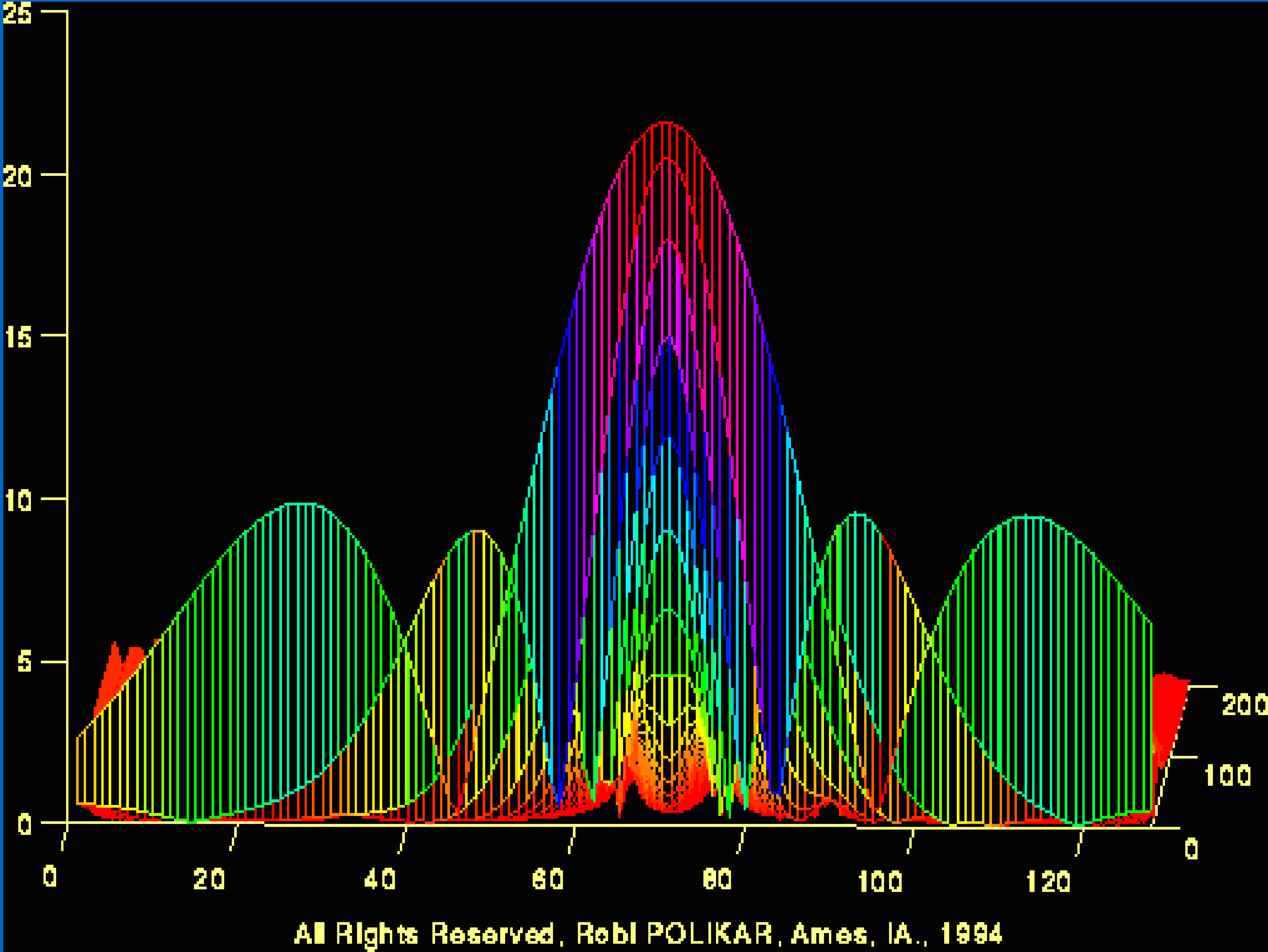
25
20
15
10
5
0
0
20
40
60
80
100
120
200
100
0
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First of all, why do we need a transform, or what is a transform anyway?

- Mathematical transformations are applied to signals to obtain a further information from that signal that is not readily available in the raw signal.
- In many cases, the most distinguished information is hidden in the frequency content of the signal. The **frequency SPECTRUM** of a signal is basically the frequency components of that signal. The frequency spectrum of a signal shows what frequencies exist in the signal.

- How do we find the frequency content of a signal?
- The answer is **FOURIER TRANSFORM (FT)**.
- If the FT of a signal in time domain is taken, the frequency-amplitude representation of that signal is obtained. In other words, we now have a plot with one axis being the frequency and the other being the amplitude. This plot tells us how much of each frequency exists in our signal.

- FT and WT both are reversible transform, that is, it allows to go back and forward between the raw and processed signals. However, only either of them is available at any given time.
- That is, no frequency information is available in the time-domain signal, and no time information is available in the Fourier transformed signal. The natural question that comes to mind is that is it necessary to have both the time and the frequency information at the same time?
- How much of each frequency exists in the signal, but it does not tell us **when in time** these frequency components exist.

Image Enhancement in the Frequency Domain

Background

- Any function that periodically repeats itself can be expressed as the sum of sines and/or cosines of different frequencies, each multiplied by a different coefficient (Fourier series).
- Even functions that are not periodic (but whose area under the curve is finite) can be expressed as the integral of sines and/or cosines multiplied by a weighting function (Fourier transform).

Background

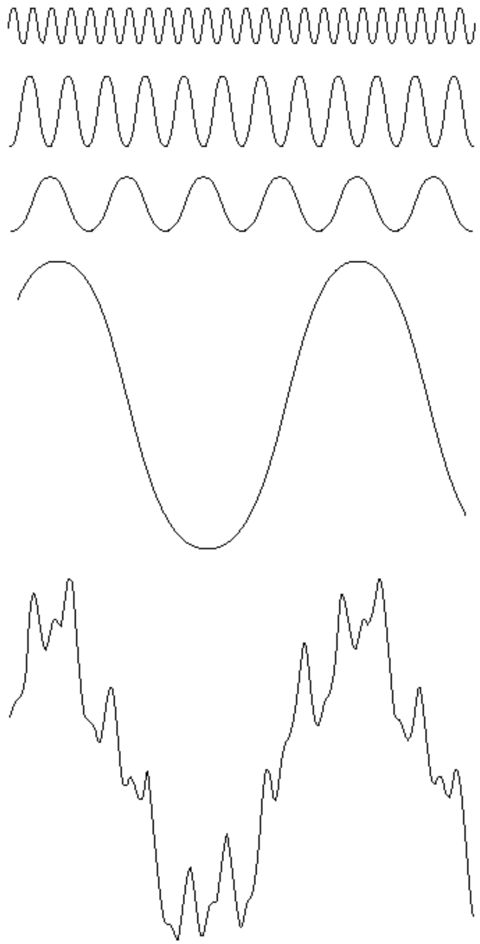


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

- The **frequency domain** refers to the plane of the two dimensional discrete Fourier transform of an image.
- The purpose of the Fourier transform is to represent a signal as a linear combination of sinusoidal signals of various frequencies.

Introduction to the Fourier Transform and the Frequency Domain

- The one-dimensional Fourier transform and its inverse
  - Fourier transform (continuous case)

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx \quad \text{where } j = \sqrt{-1}$$

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$$

  - Inverse Fourier transform:

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$$

- The two-dimensional Fourier transform and its inverse
  - Fourier transform (continuous case)

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dxdy$$

  - Inverse Fourier transform:

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} dudv$$

Continuous One-Dimensional Fourier Transform and Its Inverse

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx$$

Where $j = \sqrt{-1}$

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$$

- $(\boldsymbol{u})$ is the frequency variable.
- $\boldsymbol{F(u)}$ is composed of an infinite sum of sine and cosine terms and…
- Each value of u determines the frequency of its corresponding sine-cosine pair.

Discrete One-Dimensional Fourier Transform and Its Inverse

$$F(u) = \frac{1}{M} \sum_{x=0}^{M-1} f(x) e^{-j2\pi u \frac{x}{M}} \qquad u = [0, 1, 2, \ldots, M\text{-}1]$$

$$F(u) = \frac{1}{M} \sum_{x=0}^{M-1} f(x) \left[\cos 2\pi u \frac{x}{M} - j \sin 2\pi u \frac{x}{M} \right]$$

$$f(x) = \sum_{u=0}^{M-1} F(u) e^{j2\pi \frac{u}{M} x} \qquad x = [0, 1, 2, \ldots, M\text{-}1]$$

Introduction to the Fourier Transform and the Frequency Domain

- The two-dimensional Fourier transform and its inverse
  - Fourier transform (discrete case) DTC

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

$$\text{for } u = 0,1,2,\ldots,M-1, v = 0,1,2,\ldots,N-1$$

  - Inverse Fourier transform:

$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$$

$$\text{for } x = 0,1,2,\ldots,M-1, y = 0,1,2,\ldots,N-1$$

    - *u, v* : the transform or frequency variables
    - *x, y* : the spatial or image variables

Basics of Filtering in the Frequency Domain

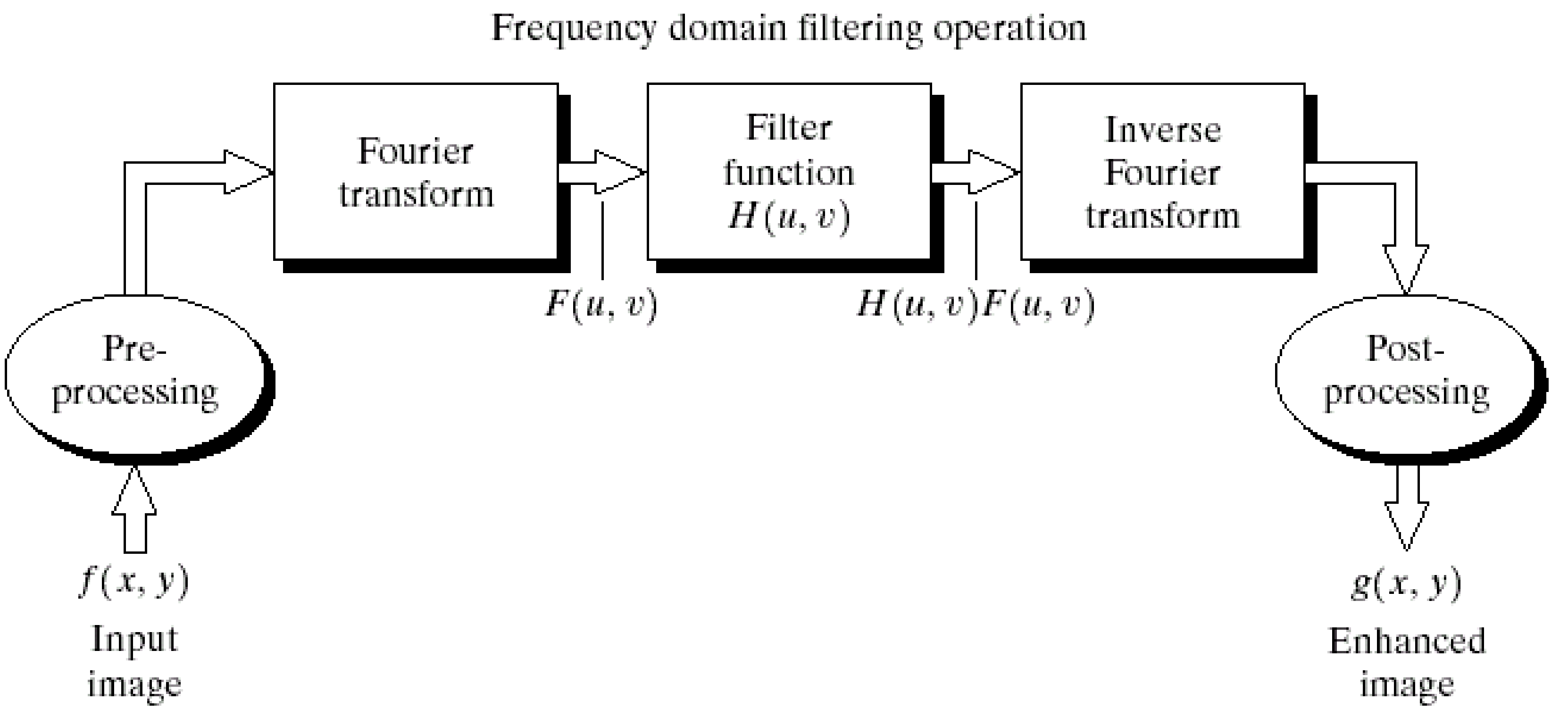
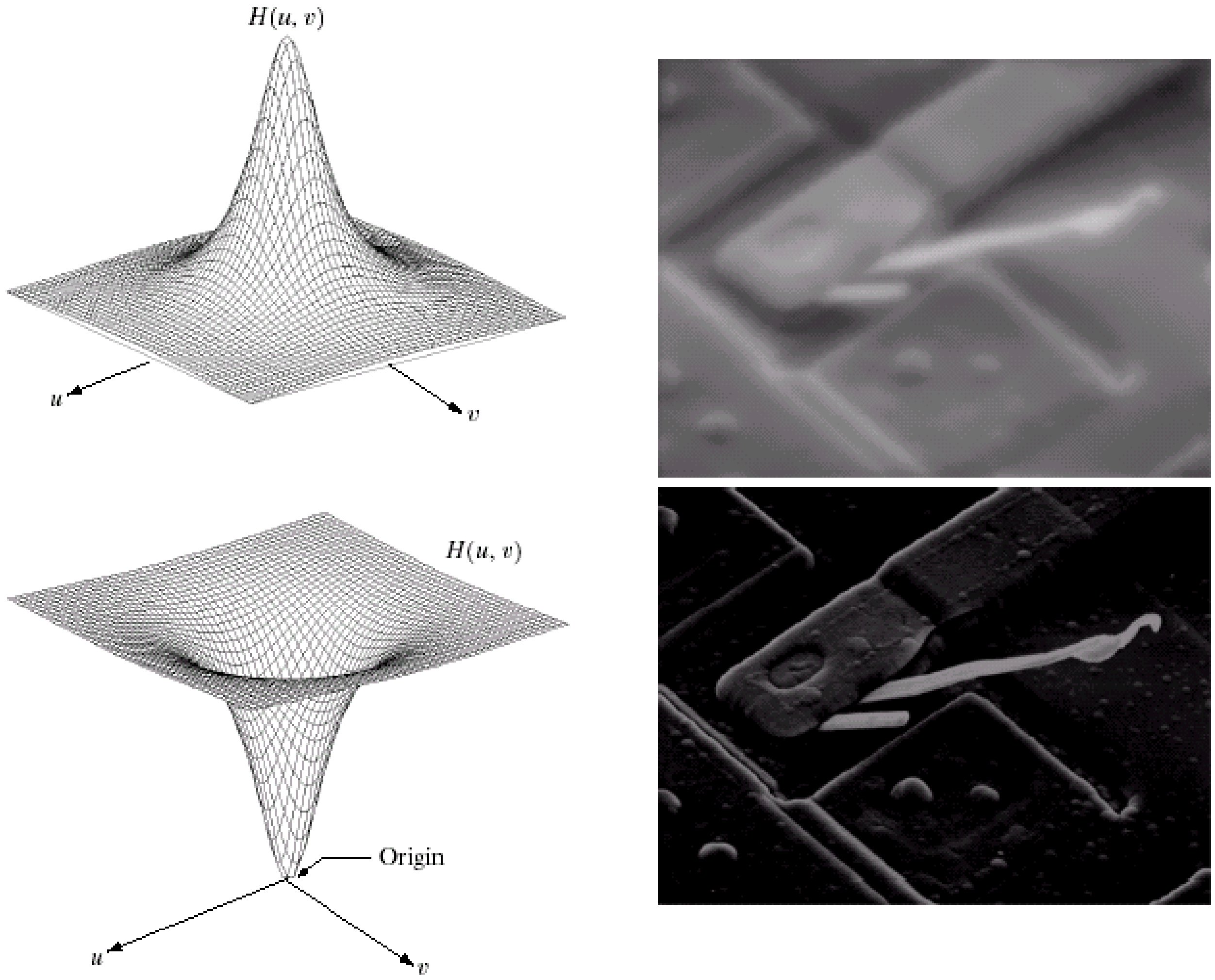


FIGURE 4.5 Basic steps for filtering in the frequency domain.

Some Basic Filters and Their Functions

Lowpass filter

Highpass filter



a b
c d

FIGURE 4.7 (a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).

Smoothing Frequency-Domain Filters

- The basic model for filtering in the frequency domain

  $$G(u,v) = H(u,v)F(u,v)$$

  where $F(u,v)$: the Fourier transform of the image to be smoothed

  $H(u,v)$: a filter transfer function

- Smoothing is fundamentally a lowpass operation in the frequency domain.
- There are several standard forms of lowpass filters (LPF).
  - Ideal lowpass filter
  - Butterworth lowpass filter
  - Gaussian lowpass filter

Ideal Lowpass Filters (ILPFs)

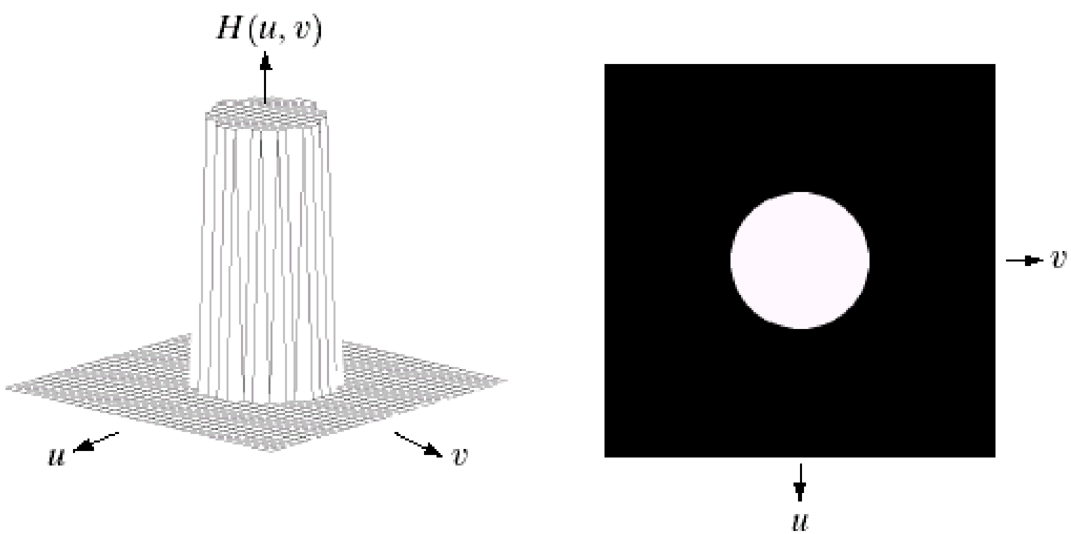
- The simplest lowpass filter is a filter that "cuts off" all high-frequency components of the Fourier transform that are at a distance greater than a specified distance D_0 from the origin of the transform.
- The transfer function of an ideal lowpass filter

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$$

where $D(u,v)$: the distance from point (u,v) to the center of ther frequency rectangle

$$D(u,v) = \left[(u - M/2)^2 + (v - N/2)^2\right]^{\frac{1}{2}}$$

Ideal Lowpass Filters (ILPFs)



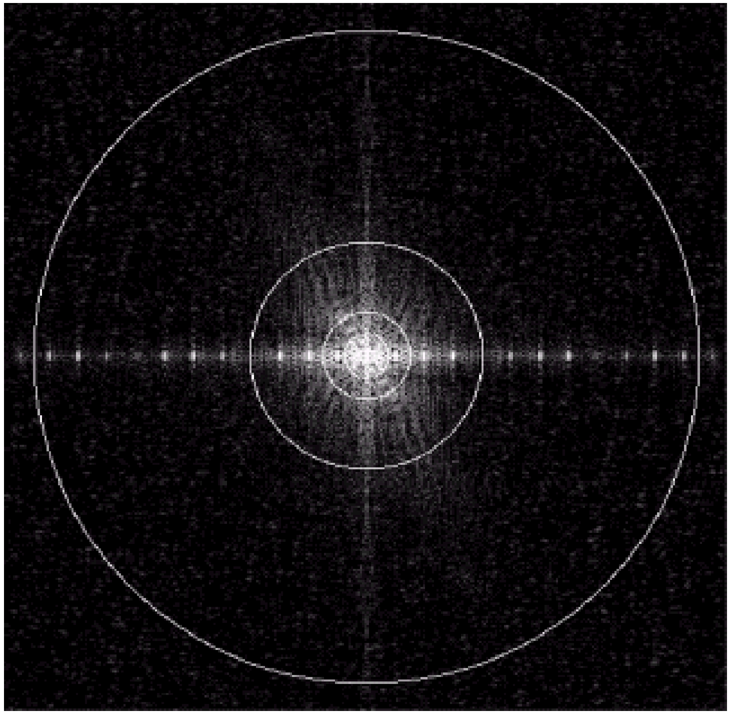
Here all the frequencies on or inside the circle of radious D_0 are passed without attenuation, where as all the frequencies outside the circle are completely attenuated

a b c

FIGURE 4.10 (a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

Ideal Lowpass Filters (ILPFs)

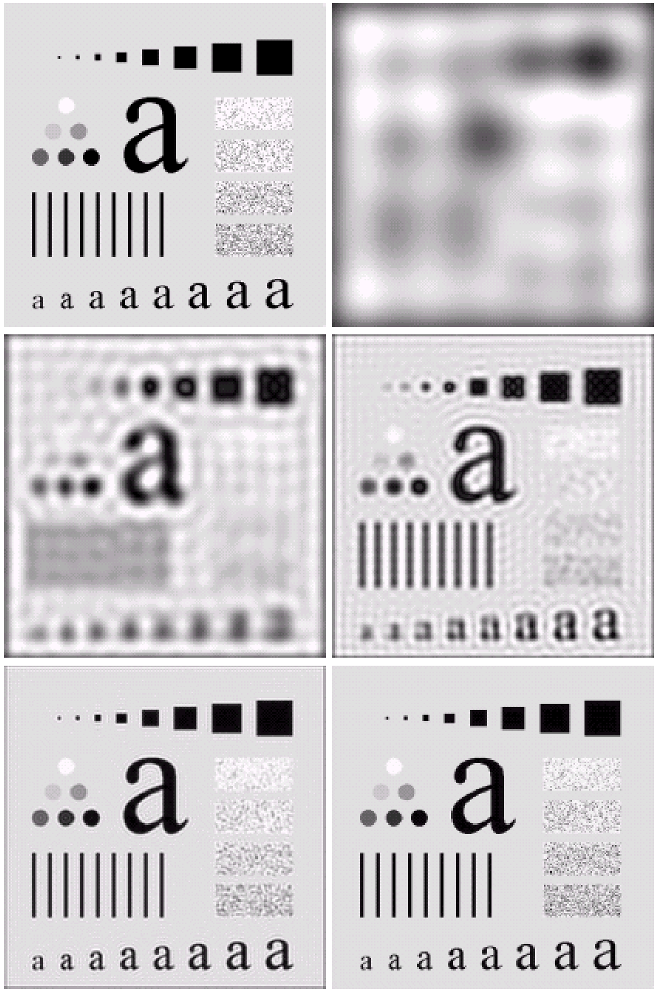
This is a spectrum of image. The circles superimposed in the spectrum have radii of 5, 15, 30, 80 and 230. This circle encloses α percent of the image, for α = 92.0, 94.6, 96.4, 98 and 99.5% respectively. The spectrum falls off rapidly with 92% of the total power being enclosed by relatively small circle of radius 5.



a b

FIGURE 4.11 (a) An image of size 500 × 500 pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.

Ideal Lowpass Filters

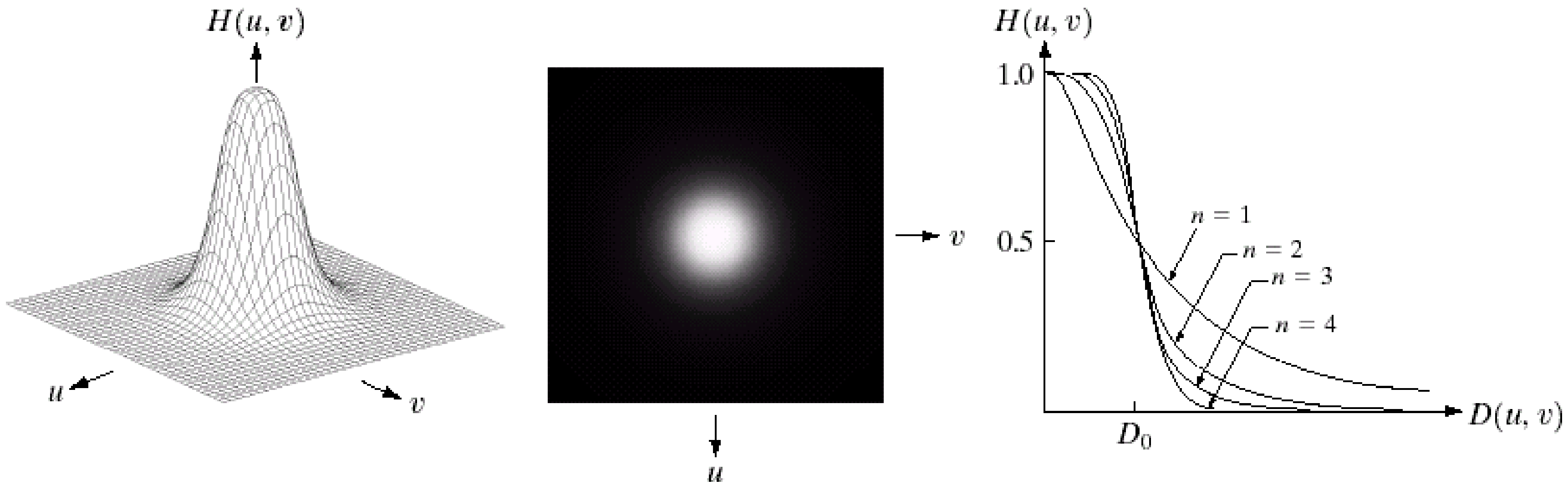


a b
c d
e f

FIGURE 4.12 (a) Original image. (b)–(f) Results of ideal lowpass filtering with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). The power removed by these filters was 8, 5.4, 3.6, 2, and 0.5% of the total, respectively.

Butterworth Lowpass Filters (BLPFs) With order *n*

$$H(u,v) = \frac{1}{1 + \left[D(u,v)/D_0\right]^{2n}}$$



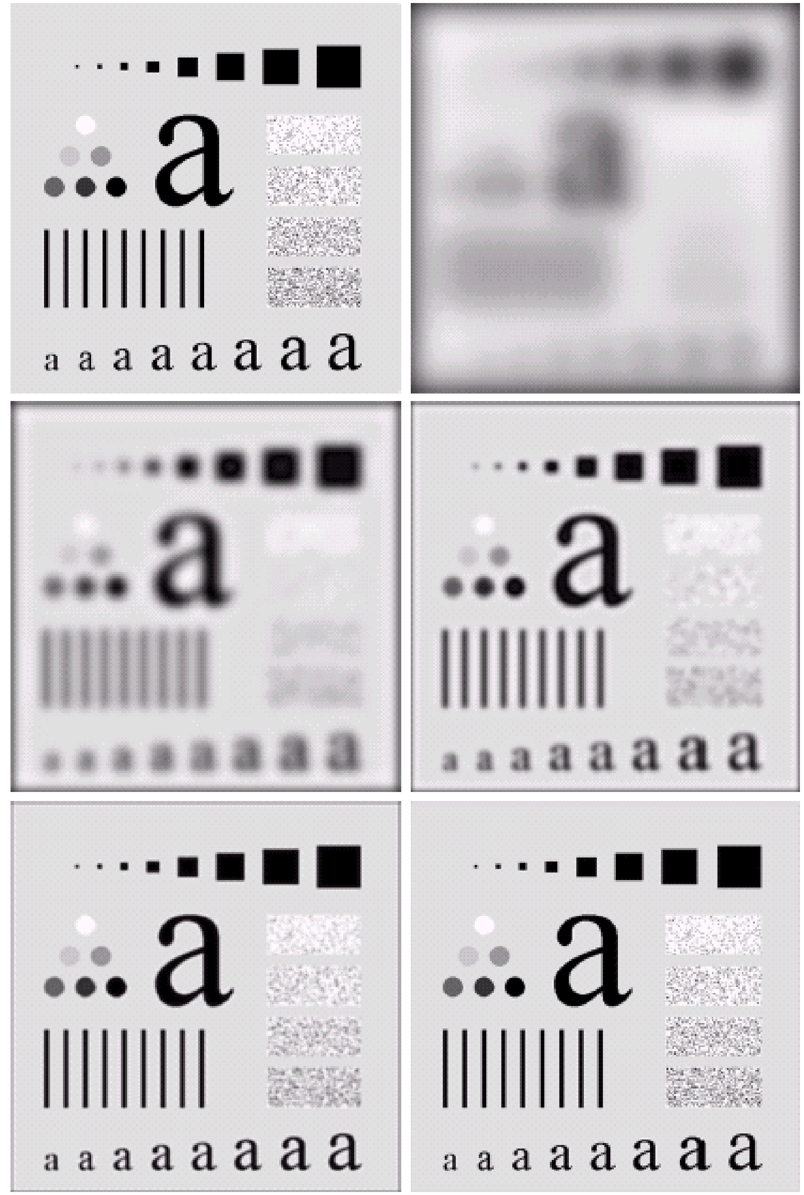
a b c

FIGURE 4.14 (a) Perspective plot of a Butterworth lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

Butterworth Lowpass Filters (BLPFs)

n=2

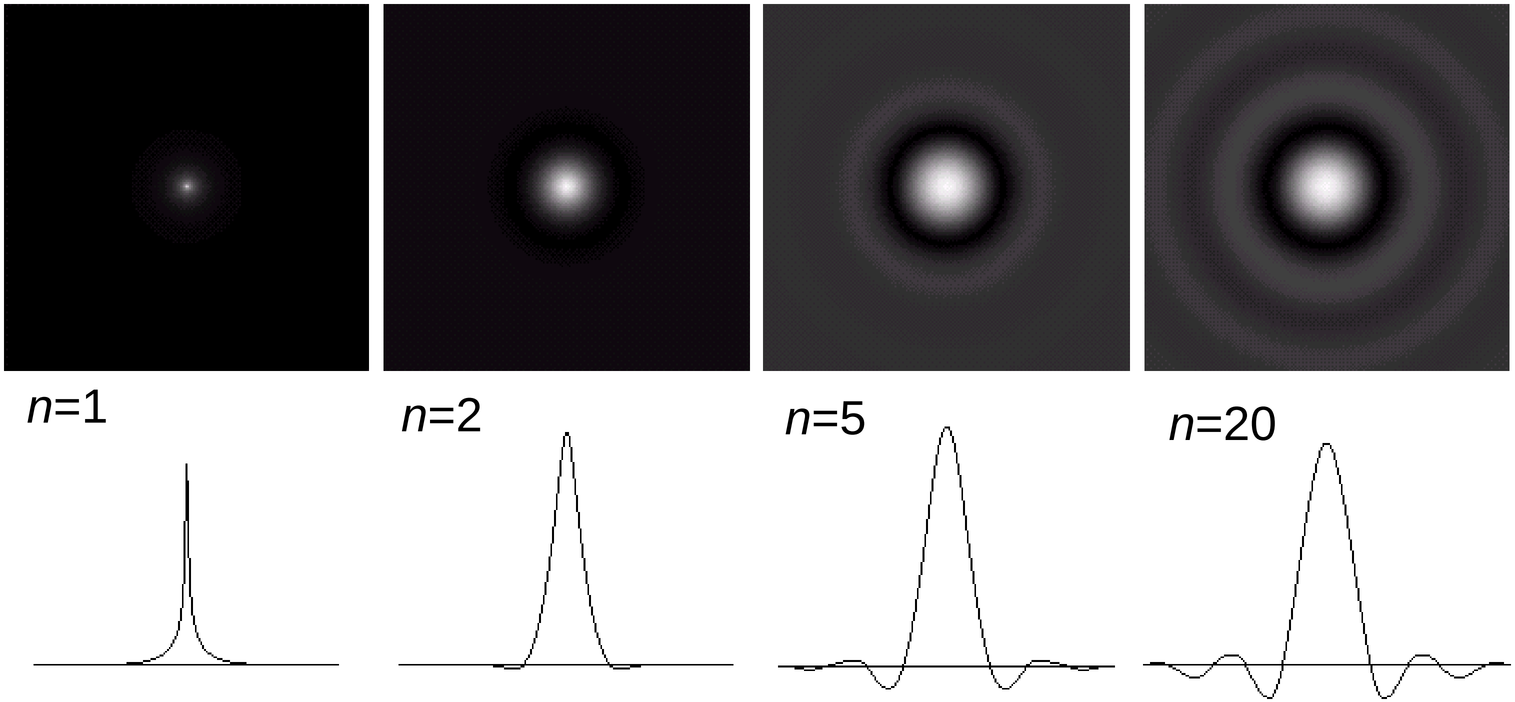
D_0=5,15,30,80,and 230



a b
c d
e f

FIGURE 4.15 (a) Original image. (b)–(f) Results of filtering with BLPFs of order 2, with cutoff frequencies at radii of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Fig. 4.12.

Butterworth Lowpass Filters (BLPFs)
Spatial Representation



a b c d

FIGURE 4.16 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding gray-level profiles through the center of the filters (all filters have a cutoff frequency of 5). Note that ringing increases as a function of filter order.

Gaussian Lowpass Filters (FLPFs)

$$H(u,v) = e^{-D^2(u,v)/2D_0^2}$$

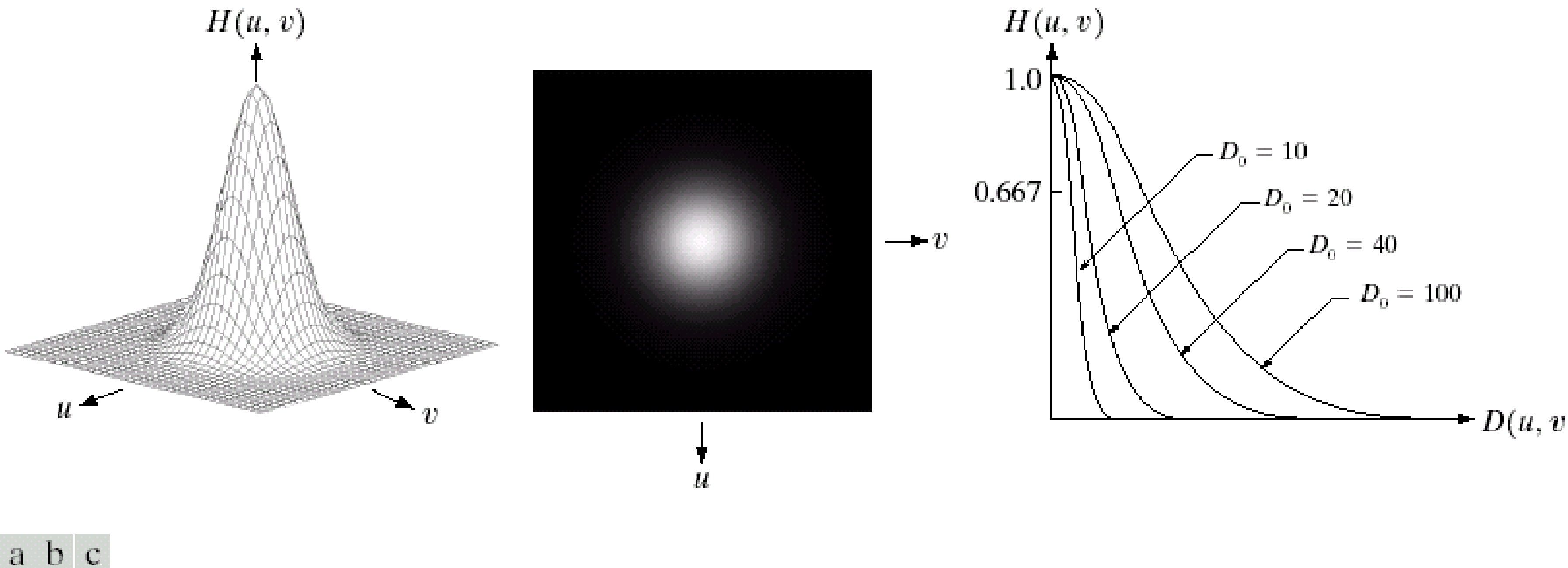


FIGURE 4.17 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

Gaussian Lowpass Filters (FLPFs)

D_0=5,15,30,80,and 230

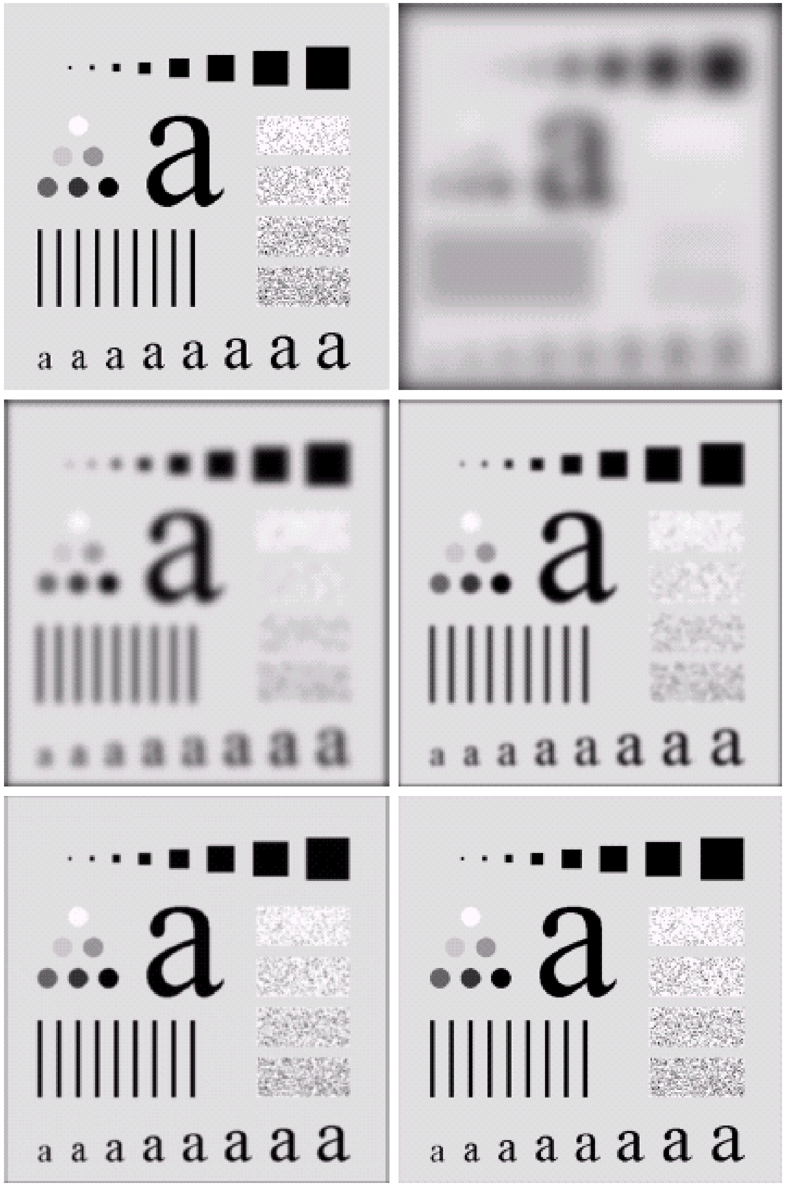
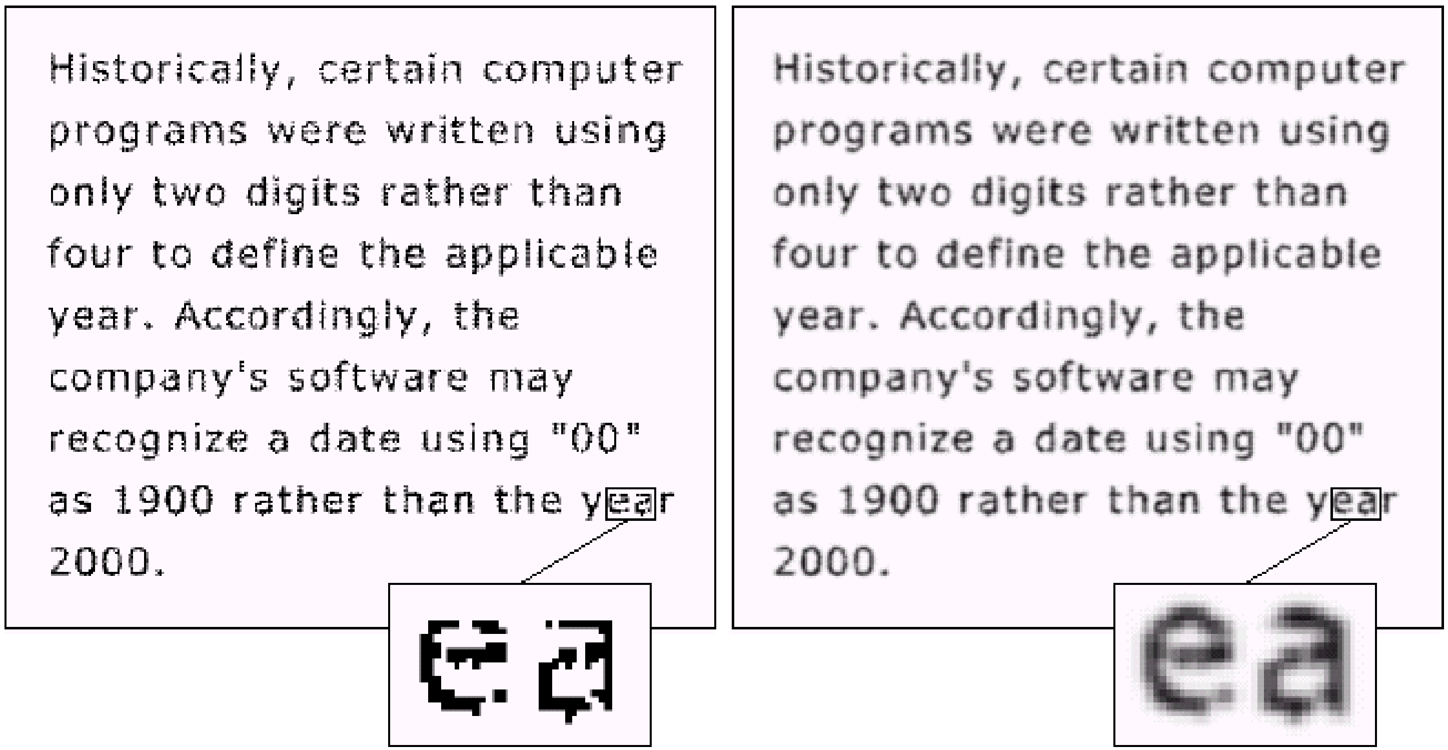


FIGURE 4.18 (a) Original image. (b)–(f) Results of filtering with Gaussian lowpass filters with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Figs. 4.12 and 4.15.

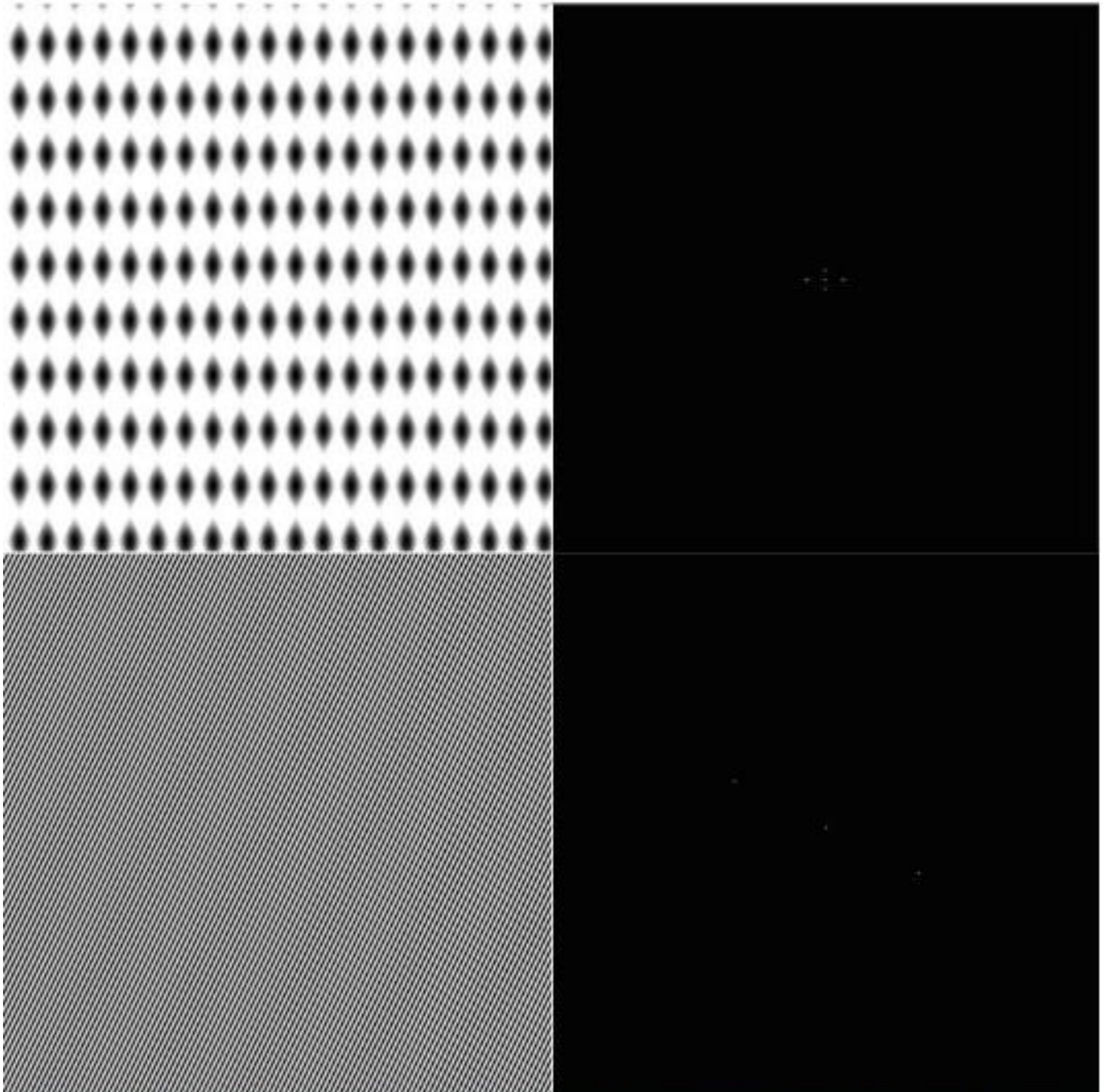
a b
c d
e f

Additional Examples of Lowpass Filtering



a b

FIGURE 4.19 (a) Sample text of poor resolution (note broken characters in magnified view). (b) Result of filtering with a GLPF (broken character segments were joined).



Additional Examples of Lowpass Filtering

a b c

FIGURE 4.20 (a) Original image (1028×732 pixels). (b) Result of filtering with a GLPF with $D_0 = 100$. (c) Result of filtering with a GLPF with $D_0 = 80$. Note reduction in skin fine lines in the magnified sections of (b) and (c).

Sharpening Frequency Domain Filter

$$H_{hp}(u,v) = H_{lp}(u,v)$$

Ideal highpass filter

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$$

Butterworth highpass filter

$$H(u,v) = \frac{1}{1+\left[D_0 / D(u,v)\right]^{2n}}$$

Gaussian highpass filter

$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$$

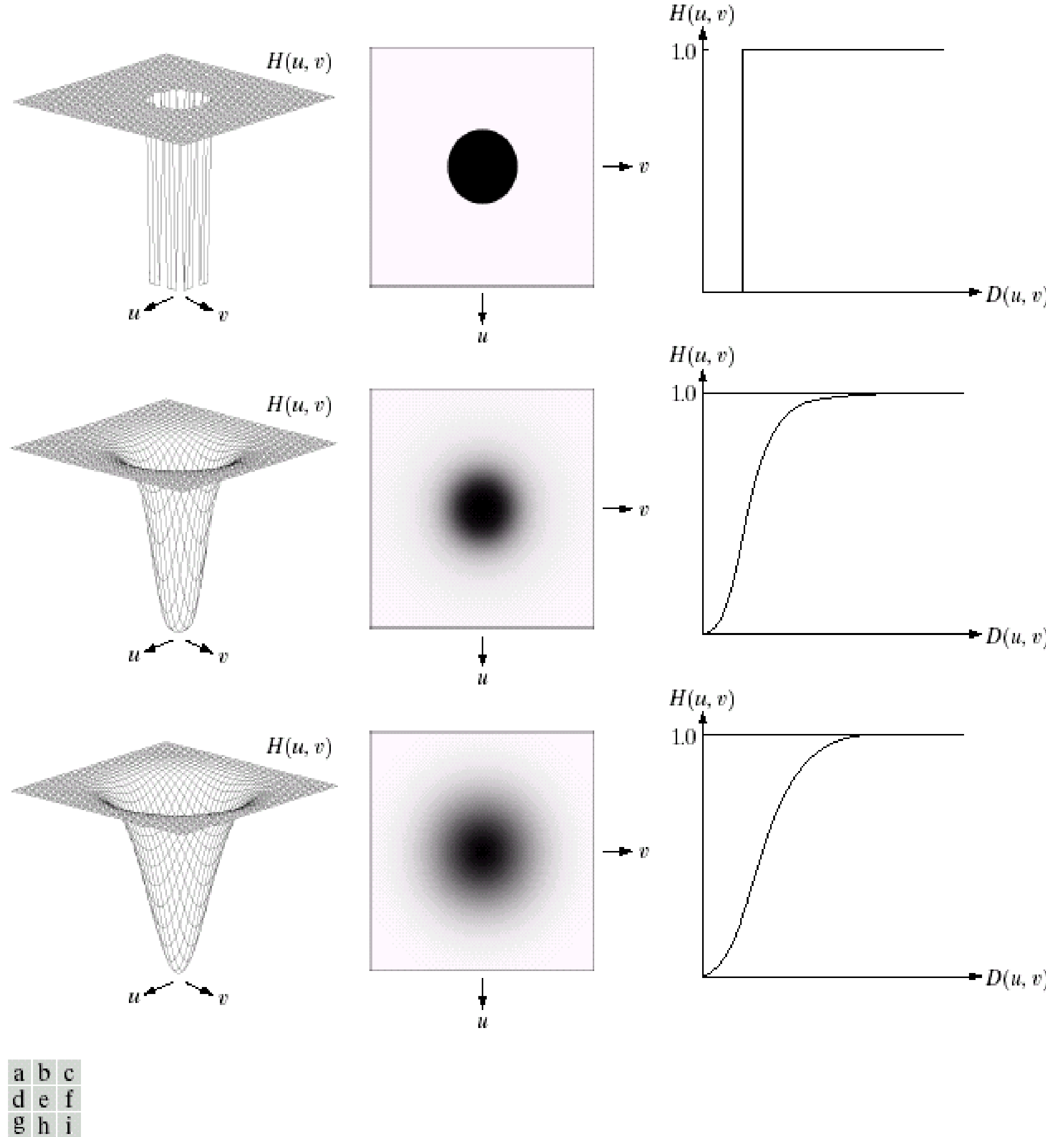


FIGURE 4.22 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

Highpass Filters
Spatial Representations

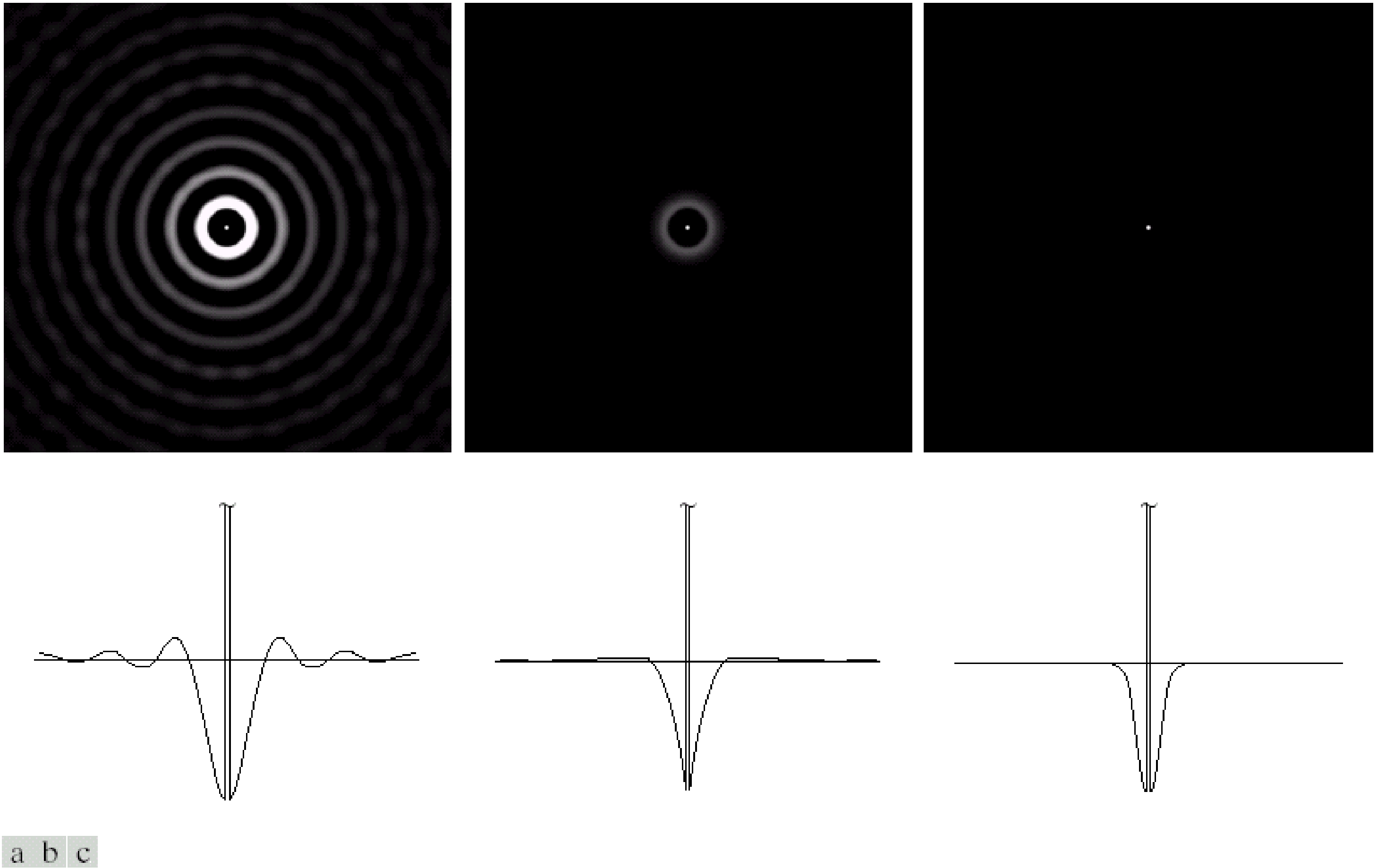
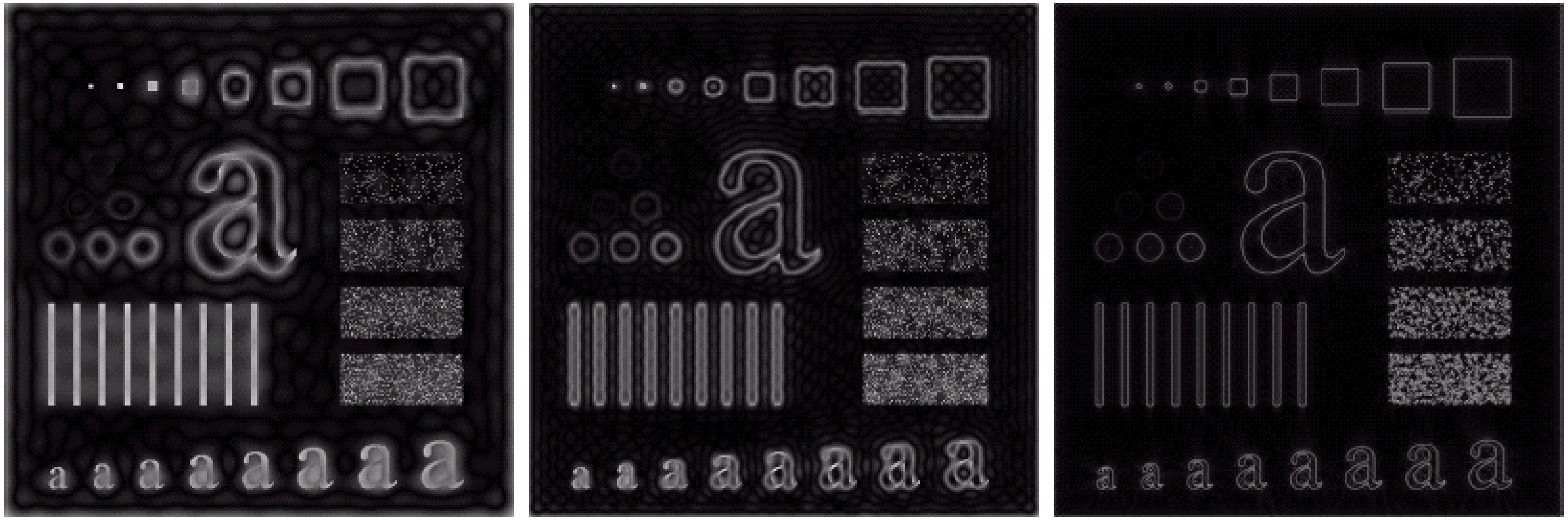


FIGURE 4.23 Spatial representations of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding gray-level profiles.

Ideal Highpass Filters

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$$

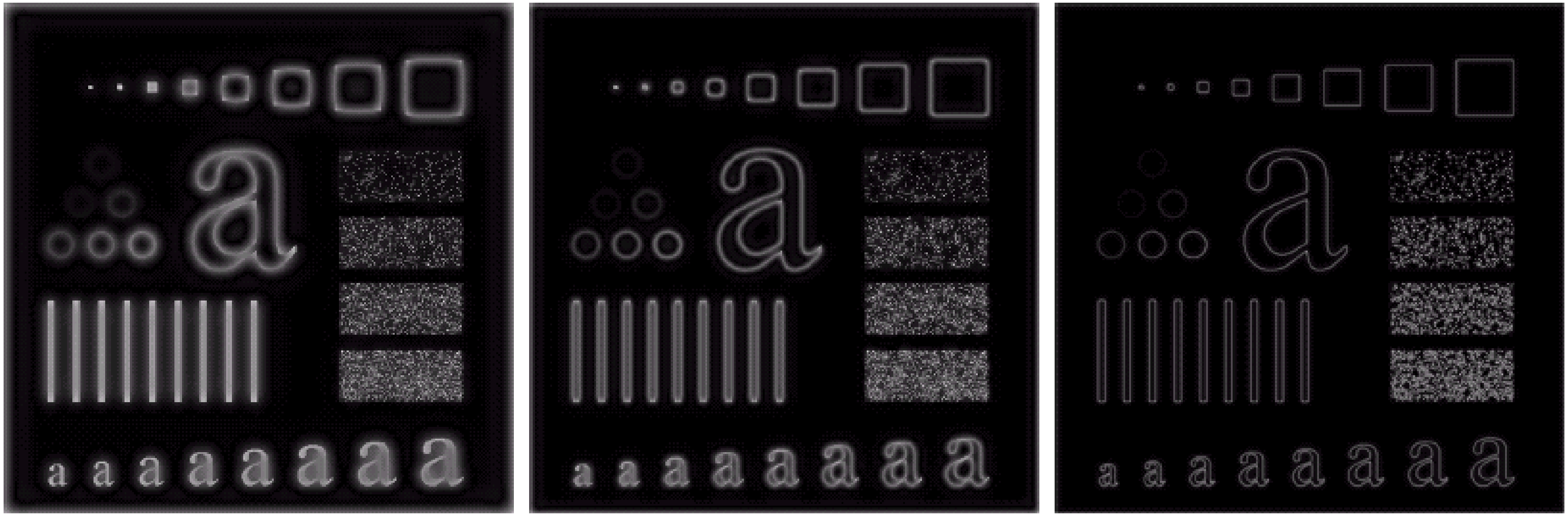


a b c

FIGURE 4.24 Results of ideal highpass filtering the image in Fig. 4.11(a) with $D_0 = 15$, 30, and 80, respectively. Problems with ringing are quite evident in (a) and (b).

Butterworth Highpass Filters

$$H(u,v) = \frac{1}{1 + [D_0 / D(u,v)]^{2n}}$$

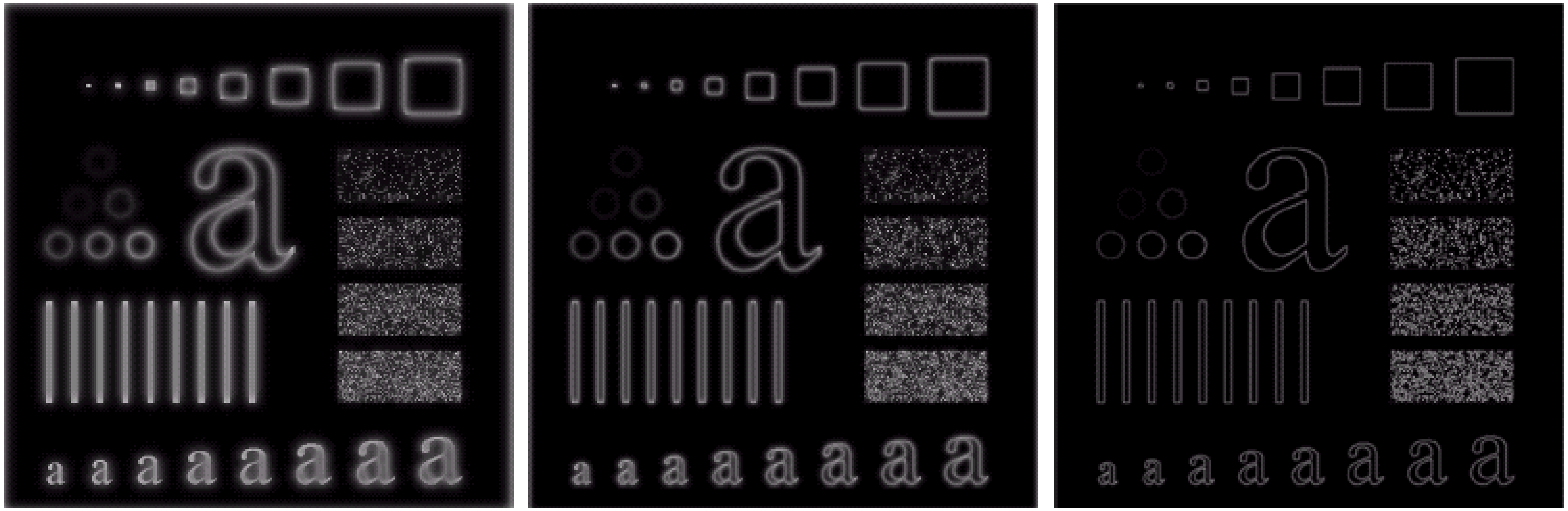


a b c

FIGURE 4.25 Results of highpass filtering the image in Fig. 4.11(a) using a BHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. These results are much smoother than those obtained with an ILPF.

Gaussian Highpass Filters

$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$$



a b c

FIGURE 4.26 Results of highpass filtering the image of Fig. 4.11(a) using a GHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. Compare with Figs. 4.24 and 4.25.