Deterministic Optimization

Introduction

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Mathematical Ingredients

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Mathematical Ingredients

Learning Objectives

- Identify the main mathematical ingredients of an optimization
- Recognize the restrictions on the class of problems under consideration
- Visualize mathematical notation involved in a generic optimization model

Review

- Optimization = Mathematical approach for selection a **best decision/action** from a **set of alternatives**
- **Decision/Actions** are modeled as decision variables whose values we are seeking
- The **set of alternatives** is modeled by constraints on the values the decision variables can take
- The notion of **best** is modeled by an objective function (of the decision variables) that needs to be maximized or minimized
- An optimization model also has specified problem data to describe the constraints and objective function

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Ingredients of an Optimization Model

Minimize or Maximize an **objective function** of **decision variables** subject to **constraints** on the values of the decision variables

$$
\begin{array}{rlr}
\text{min or max} & f(x_1, x_2, \ldots, x_n) & \\
\text{s.t.} & g_i(x_1, x_2, \ldots, x_n) \leq b_i & i = 1, \ldots, m \\
& x_j \text{ is continuous or discrete} & j = 1, \ldots, n
\end{array}
$$

The functions describing the objective function and constraints involve problem data

The Problem Setting

- Finite number of decision variables (problem dimension)
- A single objective function of decision variables and problem data
  - Multiple objective functions are handled by either taking a weighted combination of them or by optimizing one of the objectives while ensuring the other objectives meet target requirements
- The constraints are defined by a finite number of inequalities or equalities involving functions of the decision variables and problem data
- There may be domain restrictions (continuous or discrete) on some of the variables
- The functions defining the objective and constraints are algebraic (typically with rational coefficients)

A Generic Formulation

- Express all decision variables as one decision vector $\mathbf{x} \in \mathbb{R}^n$
- The objective function has a vector argument $f : \mathbb{R}^n \to \mathbb{R}$
- The constraints define a set of values the decision vector can take, e.g. $X = \{\mathbf{x} \in \mathbb{R}^{n-p} \times \mathbb{Z}^p : g_i(\mathbf{x}) \leq b_i \; i = 1, \ldots, m\}$

Generic formulation: $\begin{array}{ll} \min & f(\mathbf{x}) \\ \text{s.t.} & \mathbf{x} \in X \end{array}$ or $\min_{\mathbf{x}}\{f(\mathbf{x}) : \; \mathbf{x} \in X\}$

Notation: $\mathbb{R}$ is the set of real numbers and $\mathbb{Z}$ is the set of integers. The domain restriction $\mathbf{x} \in \mathbb{R}^{n-p} \times \mathbb{Z}^p$ means that the first $n-p$ components of $\mathbf{x}$ are continuous variables and the remaining p are discrete variables

Visualizing the constraint set

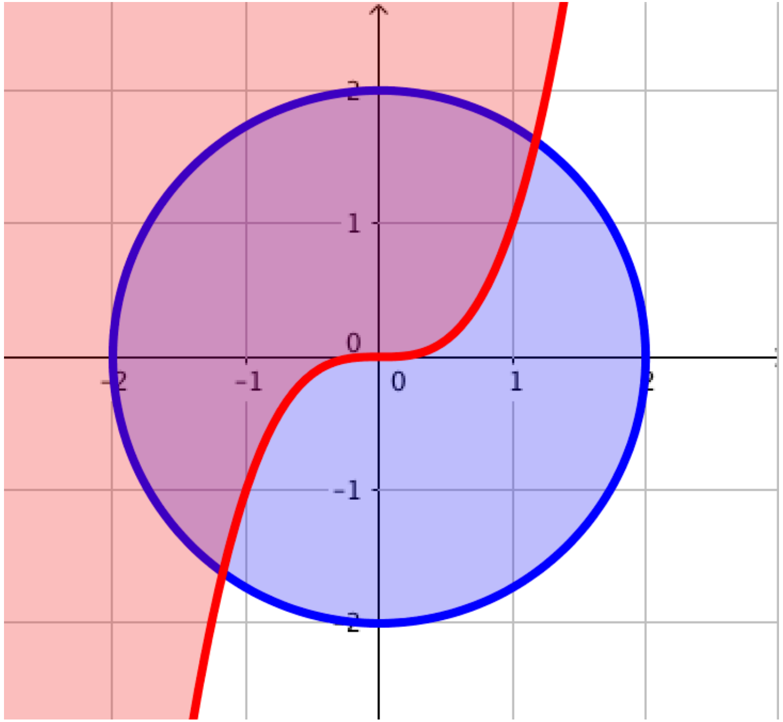
Example:

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$g_1(\mathbf{x}) := x^2 + y^2$$

$$g_2(\mathbf{x}) := x^3 - y$$

$$X = \{\mathbf{x} : g_1(\mathbf{x}) \leq 4, g_2(\mathbf{x}) \leq 0\}$$



Minimization vs Maximization

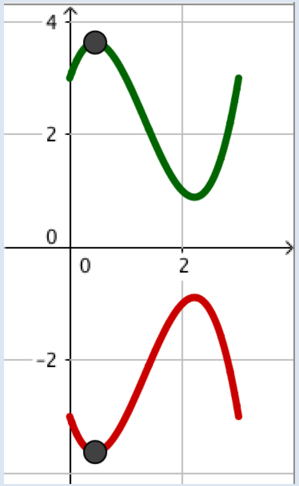
Without loss of generality, it is sufficient to consider a minimization objective since $\max_{\mathbf{x}}\{f(\mathbf{x}) : \ \mathbf{x} \in X\} \equiv -\min_{\mathbf{x}}\{-f(\mathbf{x}) : \ \mathbf{x} \in X\}$

Example:

$$\max\{4 - x^2 + (x-1)^3 : 0 \le x \le 3\}$$

$$\equiv$$

$$-\min\{-4 + x^2 - (x-1)^3 : 0 \le x \le 3\}$$



Thus to develop the theory we will only consider minimization problems. When actually solving problems we can use the actual min or max objective

Summary

- Optimization problem = maximizing/minimize an objective function of the decision variables over a set of values the variables can take
- The set of values is defined by inequalities/equalities involving functions of the decision variables
- Setting: Finite number of variables and constraints, single objective function, all functions are algebraic
- Sufficient to study minimization problems