Deterministic Optimization

Illustration of the optimization process

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Formulating a portfolio optimization model

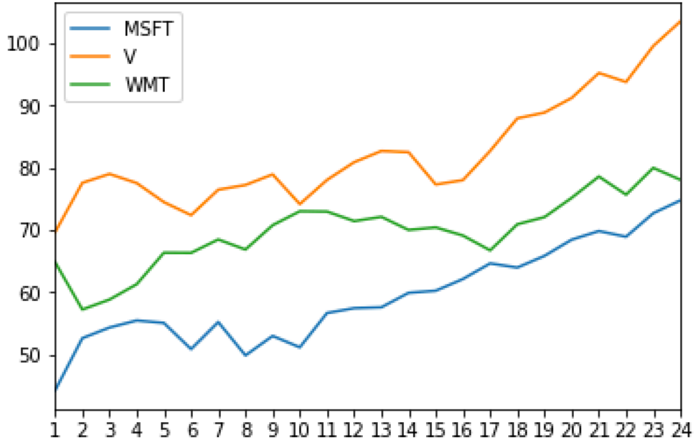
Formulating a Portfolio Optimization Model

Learning Objectives

- Observe the process of building an optimization model
- Recognize the associated considerations

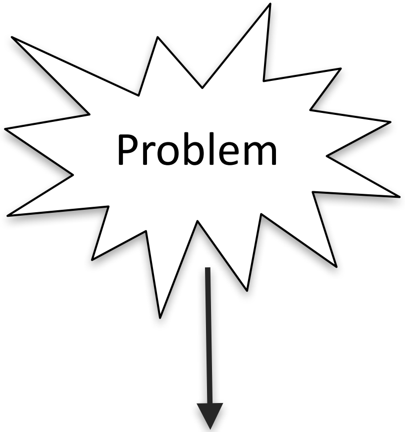
Our Portfolio Optimization Problem

- Want to invest $1000 in three stocks: MSFT, V, WMT for a month
- Make sure expected return exceeds a given threshold
- Minimize risk of the portfolio return



Modeling

- Involves approximating the underlying decision problem using mathematical expressions suitable for quantitative analysis
- Key trade-off: How much of the detail of the actual problem to consider while maintaining computational tractability of the mathematical model?
- Requires making simplifying assumptions, either because some of the problem characteristics are not well-defined mathematically, or because we wish to develop a model that can actually be solved
- Need to exercise great caution in these assumptions and not loose sight of the true underlying problem



$$\begin{array}{lll} \min & f(x) & \\ \text{s.t} & g_i(x) \leq b_i & \forall i \end{array}$$

Model components

- The **Decision variables** represent the actual decisions we are seeking. In our portfolio optimization example, these represent the investment levels
- The **Constraints** specify the restrictions and interactions between the decision variables, thus defining the set of possible decisions. In our example, one constraint corresponds to the restriction that our investment should provide a specified minimum expected return
- The **Objective function** quantifies the criteria for choosing a best decision. The values of the decision variables that maximize or minimize the objective function is "best" among the set of decision values defined by the constraints in the optimization model. In our example, the objective function is the variance of the portfolio

Building the optimization model - 1

- Decision variables:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ where } x_i \text{ is the \$ invested in stock } i$$

$$\text{Here } i = 1 : \text{MSFT}, \; i = 2 : \text{V}, \; i = 3 : \text{WMT}$$

- **Assumption**: We can buy any continuum of shares

- This implies the non-negativity constraint:

$$x_i \geq 0 \; \forall \; i = 1, 2, 3 \quad \text{or} \quad \mathbf{x} \geq \mathbf{0} \quad \text{or} \quad \mathbf{x} \in \mathbb{R}^3_+$$

Building the optimization model - 2

- Budget Constraint:

$$\sum_{i=1}^{3} x_i \leq 10000 \quad \text{or} \quad \mathbf{e}^\top \mathbf{x} \leq 1000$$

$$\text{where } \mathbf{e} \in \mathbb{R}^3 \text{ is a vector of ones}$$

- **Assumption**: There are no transaction costs

- Return on portfolio:

$$\sum_{i=1}^{3} \tilde{r}_i x_i \quad \text{or} \quad \tilde{\mathbf{r}}^\top \mathbf{x}$$

$$\text{where } \tilde{\mathbf{r}} \text{ is the vector of random returns}$$

Building the optimization model - 3

- The expected return on our portfolio is:

$$\mathbb{E}\left[\sum_{i=1}^{3} \tilde{r}_i x_i\right] = \sum_{i=1}^{3} \mathbb{E}[\tilde{r}_i] x_i = \sum_{i=1}^{3} \bar{r}_i x_i = \bar{\mathbf{r}}^\top \mathbf{x}$$

$$\text{where } \bar{r}_i \text{ is expected return of stock } i$$

- Supposed we want a minimum expected return of $r_{\min}$ then we have expected return constraint:

$$\sum_{i=1}^{3} \bar{r}_i x_i \geq r_{\min} \quad \text{or} \quad \bar{\mathbf{r}}^\top \mathbf{x} \geq r_{\min}$$

Building the optimization model - 4

- Objective: Minimize risk or variance of portfolio returns

$$
\begin{aligned}
\mathrm{Var}[\sum_{i=1}^{3} \tilde{r}_i x_i] &= \mathbb{E}\left[\left(\sum_{i=1}^{3} \tilde{r}_i x_i - \sum_{i=1}^{3} \bar{r}_i x_i\right)^2\right] \\
&= \mathbb{E}\left[\left(\sum_{i=1}^{3} (\tilde{r}_i - \bar{r}_i) x_i\right)\left(\sum_{j=1}^{3} (\tilde{r}_j - \bar{r}_j) x_j\right)\right] \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} x_i x_j \mathbb{E}[(\tilde{r}_i - \bar{r}_i)(\tilde{r}_j - \bar{r}_j)] = \sum_{i=1}^{3}\sum_{j=1}^{3} x_i x_j \sigma_{ij} \\
&= \mathbf{x}^\top Q \mathbf{x}
\end{aligned}
$$

where σ_{ij} is the covariance of the return of stock i, with stock j, and Q is the variance-covariance matrix, i.e. $Q_{ij} = \sigma_{ij}$.

Optimization model

Sum notation:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{3}\sum_{j=1}^{3} x_i x_j \sigma_{ij} \\
\text{s.t.} \quad & \sum_{i=1}^{3} x_i \leq 1000.00, \\
& \sum_{i=1}^{3} \bar{r}_i x_i \geq r_{\min}, \\
& x_i \geq 0 \qquad i = 1, 2, 3.
\end{aligned}
$$

Matrix notation:

$$
\begin{aligned}
\min \quad & \mathbf{x}^\top Q \mathbf{x} \\
\text{s.t.} \quad & \mathbf{e}^\top \mathbf{x} \leq 1000.00, \\
& \bar{\mathbf{r}}^\top \mathbf{x} \geq r_{\min}, \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned}
$$

Summary

- We have gone through the process of building an optimization model
- Pay careful attention to the assumptions made on the way

- Before we can attempt to solve the model we need to specify the required data or model parameters:

$$\bar{\mathbf{r}},\ Q \text{ and } r_{\min}$$

GTx