Tutorial Sheet 1: COL726

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1. **Discretization**

   (a) The proof for this is trivial. It can be shown by dividing the region into small strips and computing the area of each strip by assuming it to be a trapezium.
   Code for this problem is in the *Codes* folder.
   Result comparison:

   Table 1: Variation of h with result

Value of h	Result
0.1	1.99662925254
0.01	1.99994799207
0.001	1.99999975037

   (b) i. Solution of Differential Equation

   $$\begin{aligned} y(x) &= e^{x^2} + x \\ \implies LHS &= y'(x) = 2xe^{x^2} + 1 \\ \implies RHS &= 2xy(x) - 2x^2 + 1 \\ &= 2xe^{x^2} + 2x^2 - 2x^2 + 1 \\ &= 2xe^{x^2} + 1 \\ \implies LHS &= RHS \end{aligned}$$

   ii. Divided difference
   Substituting $y'_k = \frac{y_{k+1} - y_k}{h}$ we get

   $$\begin{aligned} \frac{y_{k+1} - y_k}{h} &= 2x_k y(x_k) - 2{x_k}^2 + 1 \\ y_{k+1} - y_k &= h * (2x_k y(x_k) - 2{x_k}^2 + 1) \\ \implies y_{k+1} &= y_k + h * (2x_k y(x_k) - 2{x_k}^2 + 1) \end{aligned}$$

iii. Graph and analysis
Refer to *Codes* folder.

(c) i. Deriving square root

$$
\begin{aligned}
y(x) &= x^2 - c \\
\implies y'(x) &= 2x \\
\implies x_{k+1} &= x_k - \frac{f(x)}{f'(x)} \\
&= x_k - \frac{{x_k}^2 - c}{2x_k} \\
c = 2 \implies x_{k+1} &= \frac{x_k}{2} + \frac{1}{x_k}
\end{aligned}
$$

ii. Proof for digits of accuracy
Let α be the root of the equation.

$$
\begin{aligned}
&\implies f(\alpha) = 0 \\
&\quad f(\alpha) = f(x_n) + f'(x_n)(\alpha - x_n) + 0.5 * f"(z_n)(\alpha - x_n)^2 \\
&\implies 0 = f(\alpha) = f(x_n) + f'(x_n)(\alpha - x_n) + 0.5 * f"(z_n)(\alpha - x_n)^2 \\
&\implies \frac{f(x_n)}{f'(x_n)} + (\alpha - x_n) = -\frac{f"(z_n) * (\alpha - x_n)^2}{2f'(x_n)} \\
&\implies \alpha - x_{n+1} = -\frac{f"(z_n)}{2 * f'(x_n)}(\alpha - x_n)^2 \\
&\implies \epsilon_{n+1} = \frac{|f"(z_n)|}{2|f'(x_n)|}\epsilon_n^2
\end{aligned}
$$

iii. Table of convergence rate
Code can be found in the *Codes* folder.

(d) i. Truncation Error
ii. Rounding Error
iii. Truncation Error
iv. Truncation Error

2. **Unstable and Ill-conditioned problems**

Table 2: Variation of digits of accuracy with iterations

Number of Iterations	Value Guessed	Digits of Accuracy
0	1	0
1	1.5	1
2	1.416666666666666666666666666	2
3	1.414215686274509803921568628	5
4	1.414213562374689910626295579	10
5	1.414213562373095048801689624	23
6	1.414213562373095048801688724	27+

(a) i.

$$\begin{aligned} y'(x) &= (2/\pi)xy(y-\pi) \\ y(0) &= y_0 \\ \implies (\frac{1}{y-\pi} - \frac{1}{y})dy &= 2xdx \\ \implies \log\frac{y-\pi}{y} &= x^2 + c \\ \implies \frac{y-\pi}{y} &= \frac{y_0-\pi}{y_0} * e^{x^2} \\ \implies y &= \frac{\pi * y_0}{y_0 + (\pi - y_0) * e^{x^2}} \end{aligned}$$

(b) For the given equations:

$$\begin{aligned} x &= 2y + 0.5 \\ cx &= ay - 2 \end{aligned}$$

The solution is: $x = \frac{8-a}{4c-2a}; y = \frac{4-c}{4c-2a}$.
This problem is not stable.

Table 3: Variation of solution

	x	y
c=2.998 , a=6.001	-199.9	-100.2
c=2.998, a=6	-250	-125.25

(c) The roots of the equation can be seen as follows:
When we vary the value of c it can be seen that the roots first converge to 1 and then turn complex.

Table 4: Variation of roots

Value of c vs Roots	x_1	x_2	x_3
203	99.9796	1.1527	0.8677
202	99.9898	1.1057	0.9045
201	100	1	1
200	100.01	0.99+0.1i	0.99-0.1i
199	100.02	0.99+0.14i	0.99-0.14i

3. **Unstable methods**

(a) The number of digits of accuracy in x will then be 2 as the value of $\sqrt{b^2 - 4ac}$ correctly to 8 digits is . The source for this error is that the value of b and $\sqrt{b^2 - 4ac}$ are very close to each other hence we lose precision while subtracting.

(b) Roots of a quadratic equation are given by

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} * \frac{-b \mp \sqrt{b^2 - 4ac}}{-b \mp \sqrt{b^2 - 4ac}} \\ &= \frac{2c}{-b \mp \sqrt{b^2 - 4ac}} \end{aligned}$$

The program for this can be found in the *Codes* folder. The results of this are as follows:

Table 5: Comparison of both methods

	$10^1 < a, c < 10^2; 10^3 < b < 10^6$	$10^1 < a, c < 10^2; 10^6 < b < 10^8$
Mean of difference in roots	0.00049	3573.818
Maximum difference in roots	0.0033	33072.863

(c) The graph and code can be found in the *codes* folder.
There is a large deviation between the real and value computed in this case because the value of error is amplifying with every iteration. The error is increasing by a factor of k while computing the I_k .

(d) i. When the x_i are close to each other, the valye of x_i and $\bar{x}$ is close to each other. On squaring these 2 individually, because of limited digits of accuracy, the squared values will be close to each other. Hence the value of the result will be 0 or something very small.

ii. Refer to *Codes* folder.

(e) i. No, $sum1$ is not equal to the true value.
The difference between the values is very small. This is because the number being added is not 0.1 but a number slightly different from 0.1 .

ii. No $sum2$ and $sum3$ are not equal to the true values.
In this case the mathematical values of the $sum2$ and $sum3$ are different. On fixing this, the values of $sum2$ and $sum3$ becomes closer. The difference now is because of the method of adding them. When big numbers are added initially, while adding small numbers in the end, there is more error.

iii. No. $sum4$ and $sum5$ are not equal to their true values.
In this case as well, the mathematical values of $sum4$ and $sum5$ are different. On correcting this, the values become much closer.
The value of $sum4$ is absolute 0. The value of $sum5$ on the other hand is first the exact value and an error term and then it becomes the sum of 2 error terms.