EE3025 Assignment-1

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Download all python codes from

https://github.com/kartikeyajaiswal/EE3025/tree/main/assignment1/codes

and latex-tikz codes from

https://github.com/kartikeyajaiswal/EE3025/tree/main/assignment1

1 Problem

1.1. Let

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 4, 2, 1\right\} \tag{1.1.1}$$

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2) \tag{1.1.2}$$

1.2. Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{1.2.1}$$

and $H(k)$ using h(n).

1.3. Compute $X(k)$, $H(k)$ and $y(n)$ using FFT and IFFT methods.

2 Solution

2.1. To compute h(n), find the Y(z) by applying Z-transform on equation i.e.,

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{2.1.1}$$

$$\implies Y(z) = \frac{2(z^2+1)}{z(2z+1)}X(z) \tag{2.1.2}$$

Now H(z)

$$H(z) = \frac{Y(z)}{X(z)} \tag{2.1.3}$$

$$H(z) = \frac{2(z^2+1)}{z(2z+1)} \tag{2.1.4}$$

$$H(z) = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{2.1.5}$$

applying inverse Z-transform to compute $h(n)$

$$h(n) = Z^{-1}\left[\frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}}\right] \tag{2.1.6}$$

$$h(n) = \left[\frac{-1}{2}\right]^n u(n) + \left[\frac{-1}{2}\right]^{n-2} u(n-2) \tag{2.1.7}$$

2.2. X can be expressed as Matrix Multiplication of DFT Matrix and x.

$$X(k) = \left[e^{-j2\pi n/N}\right]_{1\times N} x, \quad n = 0, 1, \ldots, N-1 \tag{2.2.1}$$

i.e.

$$X = \left[e^{-j2\pi nk/N}\right]_{N\times N} x, \quad n, k = 0, 1, \ldots, N-1 \tag{2.2.2}$$

H can be calculated in a similar manner; also

$$Y(k) = X(k)H(k) \tag{2.2.3}$$

2.3. Let $e^{-j2\pi/N} = W_N$ and $e^{-j2\pi nk/N} = W_N^{nk}$

2.4. Consider:

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{2.4.1}$$

$$= \sum_{n=0}^{N-1} x(n)W_N^{kn}, \quad k = 0, 1, \ldots, N-1 \tag{2.4.2}$$

Now, using the following properties of W_N,

a) $W_N^{k+N} = W_N^k$

b) $W_N^2 = W_{N/2}$

c) $W_N^{k+N/2} = -W_N^k$

to compute FFT from DFT:

$$\mathcal{X}(k) = \sum_{n=even} x(n)W_N^{kn} + \sum_{n=odd} x(n)W_N^{kn} \tag{2.4.3}$$

$$= \sum_{m=0}^{2} x(2m)W_N^{2mk} + \sum_{m=0}^{2} x(2m+1)W_N^{(2m+1)k} \tag{2.4.4}$$

Let first term of the above be $X_e(k)$ and the second be $X_o(k)$, which are basically DFTs of x(2m) and x(2m+1) for m=0,1,2.

2.5. X_e and X_o can be written as

$$X_e(k) = \sum_{m=0}^{2} x(2m)W_3^{mk} \tag{2.5.1}$$

$$X_o(k) = \sum_{m=0}^{2} x(2m+1)W_3^{mk} \tag{2.5.2}$$

Here, N=3 $\because$ m takes three values

written in matrix form,

$$\begin{bmatrix} X_e(0) \\ X_e(1) \\ X_e(2) \end{bmatrix} = \begin{bmatrix} W_3^0 & W_3^0 & W_3^0 \\ W_3^0 & W_3^1 & W_3^2 \\ W_3^0 & W_3^2 & W_3^4 \end{bmatrix} \begin{bmatrix} x(0) \\ x(2) \\ x(4) \end{bmatrix} \tag{2.5.3}$$

and

$$\begin{bmatrix} X_o(0) \\ X_o(1) \\ X_o(2) \end{bmatrix} = \begin{bmatrix} W_3^0 & W_3^0 & W_3^0 \\ W_3^0 & W_3^1 & W_3^2 \\ W_3^0 & W_3^2 & W_3^4 \end{bmatrix} \begin{bmatrix} x(1) \\ x(3) \\ x(5) \end{bmatrix} \tag{2.5.4}$$

combining the two:

$$\begin{bmatrix} X_e(0) \\ X_e(1) \\ X_e(2) \\ X_o(0) \\ X_o(1) \\ X_o(2) \end{bmatrix} = \begin{bmatrix} W_3^0 & W_3^0 & W_3^0 & 0 & 0 & 0 \\ W_3^0 & W_3^1 & W_3^2 & 0 & 0 & 0 \\ W_3^0 & W_3^2 & W_3^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & W_3^0 & W_3^0 & W_3^0 \\ 0 & 0 & 0 & W_3^0 & W_3^1 & W_3^2 \\ 0 & 0 & 0 & W_3^0 & W_3^2 & W_3^4 \end{bmatrix} \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(1) \\ x(3) \\ x(5) \end{bmatrix} \tag{2.5.5}$$

Let the above 6x6 matrix be Z_1 and 6x1 be X_f
Now, a matrix P can be found such that,

$$P \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(1) \\ x(3) \\ x(5) \end{bmatrix} \tag{2.5.6}$$

where $P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Then, the equation can be written as,

$$X_f = Z_1 P x \tag{2.5.7}$$

2.6. To compute X

$$X(k) = X_e(k) + W_N^k X_o(k) \tag{2.6.1}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & W_6^0 & 0 & 0 \\ 0 & 1 & 0 & 0 & W_6^1 & 0 \\ 0 & 0 & 1 & 0 & 0 & W_6^2 \\ 1 & 0 & 0 & W_6^3 & 0 & 0 \\ 0 & 1 & 0 & 0 & W_6^4 & 0 \\ 0 & 0 & 1 & 0 & 0 & W_6^5 \end{bmatrix} \begin{bmatrix} X_e(0) \\ X_e(1) \\ X_e(2) \\ X_o(0) \\ X_o(1) \\ X_o(2) \end{bmatrix} \tag{2.6.2}$$

$$X = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \end{bmatrix} = Z_2 \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_2(0) \\ X_2(1) \\ X_2(2) \end{bmatrix} \tag{2.6.3}$$

using equation (2.5.7), we get

$$\mathcal{X} = Z_2 Z_1 P x \tag{2.6.4}$$

H can be calculated using the same formula as above,

$$\mathcal{H} = Z_2 Z_1 P h \tag{2.6.5}$$

$$H = \begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \\ H(4) \\ H(5) \end{bmatrix} = \begin{bmatrix} 1.28125 \\ 0.515625 - j0.5142 \\ -0.078125 + j1.1095 \\ 3.84375 + j4.97x10^{-16} \\ -0.078125 - j1.10959 \\ 0.515625 + j0.5142 \end{bmatrix} \tag{2.6.6}$$

2.7. To compute Y, we can do elementwise multiplication Y = H.X

$$Y = \begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \\ Y(4) \\ Y(5) \end{bmatrix} = \begin{bmatrix} 16.65625 \\ -2.95312 + j1.1637 \\ -0.07812 + j1.10959 \\ -3.8437 - j1.0953x10^{-14} \\ -0.078125 - j1.10959 \\ -2.953125 - j1.16372 \end{bmatrix} \tag{2.7.1}$$

2.8. Now IFFT can be computed as;

$$y = \frac{1}{N}(Z_1 Z_2 P)^H Y \tag{2.8.1}$$

where ^{H} denotes hermitian of a matrix. The above can be used to calculate y.

2.9. The following code computes Y and generates magnitude and phase plots of X, H, Y

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https://github.com/kartikeyajaiswal/EE3025/
    tree/main/assignment1/codes
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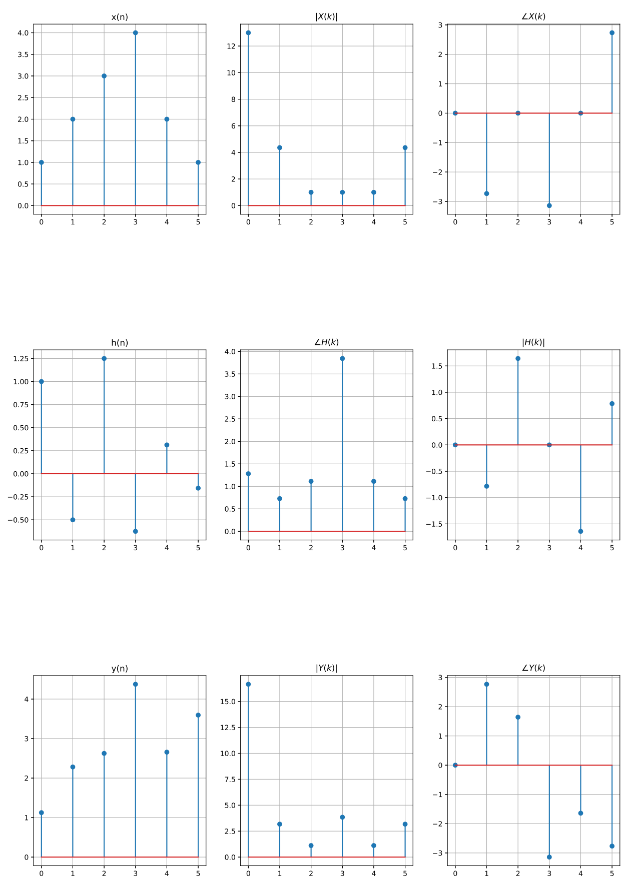
2.10. The following plots are obtained



2.11. Benefits of FFT:
- The DFT would have n^2 operations for the computation, whereas in the modified algorithm (FFT) the number of operations are $2(\frac{n}{2})^2$ since two smaller matrices are combined instead of one large matrix.
- Also there is a computational benefit since the matrices are sparse (most elements are zeros or ones).
- If the above is recursively performed, the complexity of the algorithm will be $\frac{n}{2}$logn.

2.12. Generalised for an N-point DFT, where N is a power of 2.

consider $N = 8 \implies N = 2^3$
, say n=3
If the dft is recursively broken down to 2-point dft, we get $W_2 = -1$, which gives

$$\begin{bmatrix} X(0) \\ X(1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \end{bmatrix} = \begin{bmatrix} x(0) + x(1) \\ x(0) - x(1) \end{bmatrix} \tag{2.12.1}$$

So, the 8-point dft has to be recursively brought down to multiple 2-point dfts.

Using Eq. (2.6.1): breaking the 8-point dft to 4-point dfts

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_e(0) \\ X_e(1) \\ X_e(2) \\ X_e(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_o(0) \\ X_o(1) \\ X_o(2) \\ X_o(3) \end{bmatrix} \tag{2.12.2}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_e(0) \\ X_e(1) \\ X_e(2) \\ X_e(3) \end{bmatrix} + \begin{bmatrix} W_8^4 & 0 & 0 & 0 \\ 0 & W_8^5 & 0 & 0 \\ 0 & 0 & W_8^6 & 0 \\ 0 & 0 & 0 & W_8^7 \end{bmatrix} \begin{bmatrix} X_o(0) \\ X_o(1) \\ X_o(2) \\ X_o(3) \end{bmatrix} \tag{2.12.3}$$

now again breaking down 4-point dft to 2-point dfts, we get

$$\begin{bmatrix} X_e(0) \\ X_e(1) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} \tag{2.12.4}$$

$$\begin{bmatrix} X_e(2) \\ X_e(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} + \begin{bmatrix} W_4^2 & 0 \\ 0 & W_4^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} \tag{2.12.5}$$

$$\begin{bmatrix} X_o(0) \\ X_o(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{2.12.6}$$

$$\begin{bmatrix} X_o(2) \\ X_o(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix}. + \begin{bmatrix} W_4^2 & 0 \\ 0 & W_4^3 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{2.12.7}$$

where,

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{2.12.8}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{2.12.9}$$

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{2.12.10}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{2.12.11}$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{2.12.12}$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{2.12.13}$$

2.13. above equations when written in matrix form,

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & W_8^0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & W_8^1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & W_8^3 \\ 1 & 0 & 0 & 0 & W_8^4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & W_8^5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & W_8^6 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & W_8^7 \end{bmatrix} \tag{2.13.1}$$

$$T_2 = \begin{bmatrix} 1 & 0 & W_8^0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & W_8^2 & 0 & 0 & 0 & 0 \\ 1 & 0 & W_8^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & W_8^6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & W_8^0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & W_8^2 \\ 0 & 0 & 0 & 0 & 1 & 0 & W_8^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & W_8^6 \end{bmatrix} \tag{2.13.2}$$

$$T_3 = \begin{bmatrix} 1 & W_8^0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & W_8^4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & W_8^0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & W_8^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & W_8^0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & W_8^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & W_8^0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & W_8^4 \end{bmatrix} \tag{2.13.3}$$

So, for an N = 2^r, the N-point dft (with one dense matrix of NxN), can be converted to multiplication of r (NxN) sparse matrices.

Now, the fft will be (in matrix notation):

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = T_1 T_2 T_3 \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \\ x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{2.13.4}$$

2.14. The matrix

$$P_8 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.14.1}$$

and combining the two P_4 matrices:

$$P_c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.14.2}$$

Therefore,

$$P_c P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(4) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \\ x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{2.14.3}$$

The FFT can be then written as:

$$X = T_1 T_2 T_3 P_c P_8 x \tag{2.14.4}$$