Optimization under Decision Dependent Uncertainty

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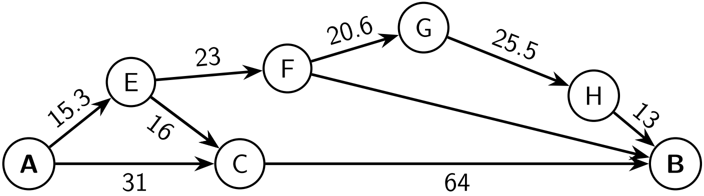
What is the problem?



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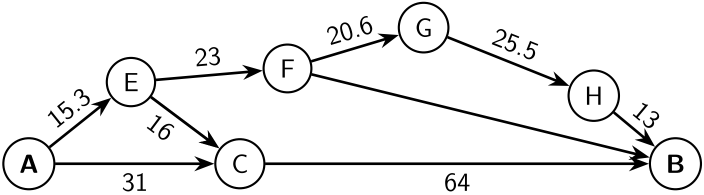

$$d_e = \bar{d}_e(1 + 0.5\xi_e) \qquad \boldsymbol{\xi} \in \mathcal{U} = \{\boldsymbol{\xi} \mid \xi_e \in [0,1]\ \forall e\}$$

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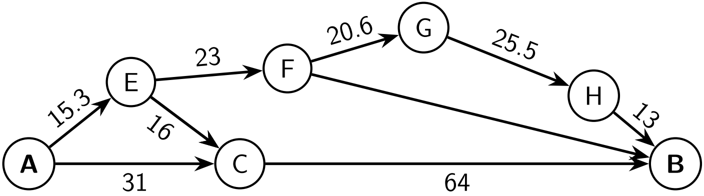

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$\xi_e \sim \text{Unif}[0,1]$

$$\min_{\mathbf{y}} \mathbb{E}_{\boldsymbol{\xi} \in \mathcal{U}}[\mathbf{d}(\boldsymbol{\xi})^\top \mathbf{y}]$$

s.t. $\mathbf{y} \in Y$

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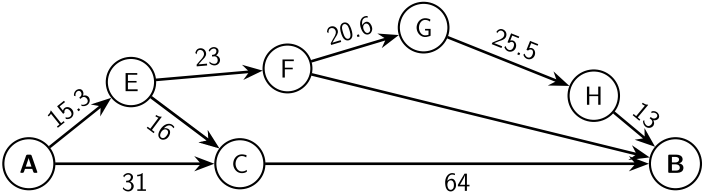

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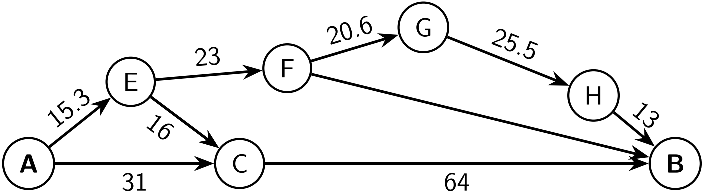

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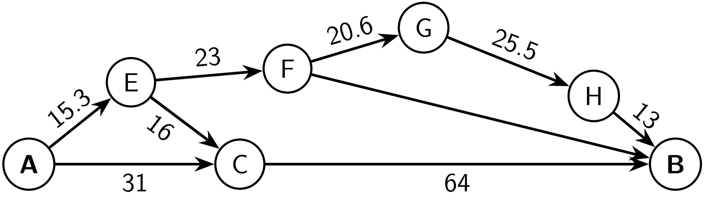

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$$\min_{\mathbf{y},\, \mathbf{x}} \max_{\boldsymbol{\xi}\in\mathcal{U}(\mathbf{x})} \mathbf{d}(\boldsymbol{\xi})^\top \mathbf{y} + \mathbf{c}^\top \mathbf{x} \quad \text{s.t. } \mathbf{y} \in Y$$

Stochastic Opt: Jonsbråten et al., 1998, Goel and Grossmann, 2004, 2006, Novoa et al., 2016, Gutin et al. 2015

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Decision Dependent Uncertainty

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Robust Opt: Poss, 2013, 2014, Bertsimas and Vayanos, 2015, Vujanic et al., 2016, Zhang et al., 2016

$$\min_{\mathbf{x},\mathbf{y}} \mathbf{c}^\top \mathbf{x} + \mathbf{d}^\top \mathbf{y}$$

$$\text{s.t. } \mathbf{A}\mathbf{x} + \boldsymbol{\Xi}\mathbf{y} \leq \mathbf{b} \ \ \forall \boldsymbol{\Xi} \in \mathcal{U}(\mathbf{x})$$

- Model interpretation.
  - Proactive uncertainty control
  - Natural effects
- Possible dependencies :
  - $\mathcal{U}(\mathbf{x}) = \{\boldsymbol{\Xi} \mid \mathbf{G} \cdot vec(\boldsymbol{\Xi}) \leq \mathbf{g} + \boldsymbol{\Delta}\mathbf{x}\}$
  - $\mathcal{U}(\mathbf{x}) = \{\boldsymbol{\Xi} \mid vec(\boldsymbol{\Xi}) = vec(\bar{\boldsymbol{\Xi}}) + \mathbf{L}\mathbf{u},\ \|\mathbf{u}\|_2 \leq 1 - \gamma x\}$

Complexity

$$\begin{aligned}\min_{\mathbf{x},\mathbf{y}} \ & \mathbf{c}^\top \mathbf{x} + \mathbf{d}^\top \mathbf{y} \\ \text{s.t.} \ & \mathbf{a}_i^\top \mathbf{x} + \boldsymbol{\xi}_i^\top \mathbf{y} \le b_i \quad \forall \boldsymbol{\xi}_i \in \mathcal{U}_i^{\mathsf{P}}(\mathbf{x})\end{aligned}$$

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- Reformulation leads to a bilinear program.
- Indicates difficulty of problem.

Complexity

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Theorem

The robust linear problem with uncertainty set $\mathcal{U}^P$ is NP-complete.

Leveraging the set

- If $\mathbf{x}$ is binary, Big-M leads to MILP reformulation.
- Poor numerical performance
- Linearization does not leverage set structure
- Imposing structure allows better reformulations.

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Let $\mathbf{x} \in \{0,1\}^n$

$$\mathcal{U}^{\overline{\mathbf{\Pi}}}(\mathbf{x}) = \{\boldsymbol{\xi} \mid \mathbf{G}\boldsymbol{\xi} \leq \mathbf{g},\ \ \boldsymbol{\xi} \leq \mathbf{v} + \mathbf{W}(\mathbf{e} - \mathbf{x}),\ \ \boldsymbol{\xi} \geq \mathbf{0}\}$$

Constraint to be reformulated:

$$\mathbf{y}^\top \boldsymbol{\xi} \leq b\ \ \forall \boldsymbol{\xi} \in \mathcal{U}^{\overline{\mathbf{\Pi}}}(\mathbf{x}).$$

$\overline{\mathbf{\Pi}}$-Reformulation

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$$\begin{aligned} \max_{\boldsymbol{z},\boldsymbol{\zeta}}\; & (\mathbf{y} - \overline{\boldsymbol{\Pi}}\mathbf{x})^\top \boldsymbol{z} + \mathbf{y}^\top \boldsymbol{\zeta} \\ \text{s.t. }\; & \mathbf{G}(\boldsymbol{z} + \boldsymbol{\zeta}) \leq \mathbf{g} \\ & \boldsymbol{z} \leq \mathbf{W}\mathbf{e} \\ & \boldsymbol{\zeta} \leq \mathbf{v} \\ & \boldsymbol{z}, \boldsymbol{\zeta} \geq \mathbf{0} \end{aligned}$$

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- $\overline{\mathbf{\Pi}}$: of upper bounds on dual variables. Similar to Big-M.

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- $\overline{\mathbf{\Pi}}$: of upper bounds on dual variables. Similar to Big-M.
- Problem convex in $\mathbf{x}$ and $\mathbf{y}$.
- Network interdiction (Cormican et al. 1996).

$\overline{\mathbf{\Pi}}$-Reformulation

Theorem

The constraint $\mathbf{y}^\top \boldsymbol{\xi} \leq b \ \ \forall \boldsymbol{\xi} \in \mathcal{U}^{\overline{\mathbf{\Pi}}}(\mathbf{x})$ *can be reformulated as*

$$
\begin{aligned}
&\mathbf{t}^\top \mathbf{g} + \mathbf{r}^\top \mathbf{W}\mathbf{e} + \mathbf{s}^\top \mathbf{v} \leq b \\
&\mathbf{s}^\top + \mathbf{t}^\top \mathbf{G} \geq \mathbf{y}^\top \\
&\mathbf{r}^\top + \mathbf{t}^\top \mathbf{G} \geq \mathbf{y}^\top - \mathbf{x}^\top \overline{\mathbf{\Pi}} \\
&\mathbf{r}, \mathbf{s}, \mathbf{t} \geq \mathbf{0}.
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$\overline{\mathbf{\Pi}}$-Reformulation

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\end{aligned}$$

- Fewer constraints than Big-M reformulation.
- Convex problem : use of cut-generating methods.
- Better solution times.

$$\min_{\mathbf{x},\mathbf{y}} \max_{\boldsymbol{\xi}\in\mathcal{U}^{SP}(\mathbf{x})} \sum_{(i,j)\in\mathcal{A}} cx_{ij} + \sum_{(i,j)\in\mathcal{A}} d_{ij}(\boldsymbol{\xi})y_{ij}$$

$$\text{s.t. } \mathbf{x}\in\{0,1\}^{|\mathcal{A}|},\ \mathbf{y}\in Y,$$

$$\mathcal{U}^{SP}(\mathbf{x}) = \left\{\boldsymbol{\xi} \mid \sum_{(i,j)\in\mathcal{A}} \xi_{ij} \leq \Gamma,\ \xi_{ij} \leq\ 1-\gamma x_{ij},\ \xi_{ij}\geq 0\ \ \forall(i,j)\in\mathcal{A}\right\}$$

Appliction : Shortest Path

c : cost of reduction

$$\min_{\mathbf{x},\mathbf{y}} \max_{\boldsymbol{\xi}\in\mathcal{U}^{SP}(\mathbf{x})} \sum_{(i,j)\in\mathcal{A}} c x_{ij} + \sum_{(i,j)\in\mathcal{A}} d_{ij}(\boldsymbol{\xi}) y_{ij}$$

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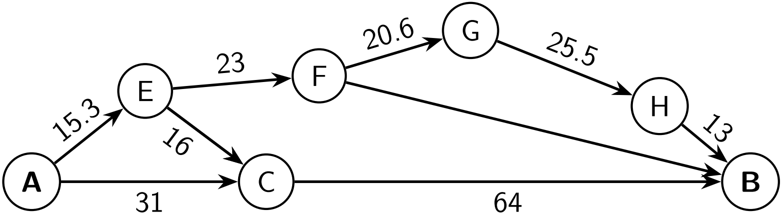
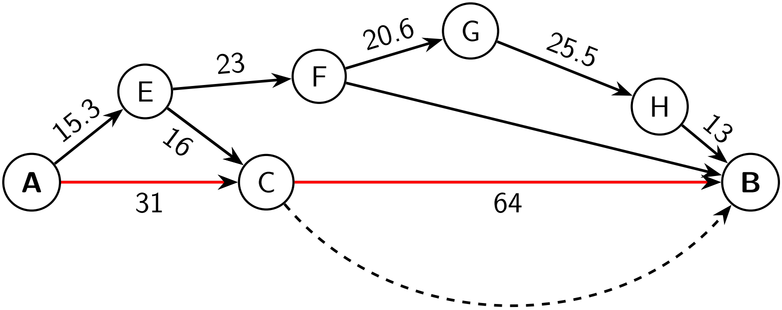
γ : uncertainty reduction



$\Gamma = 1, \ \ \gamma = 0.8$

Network



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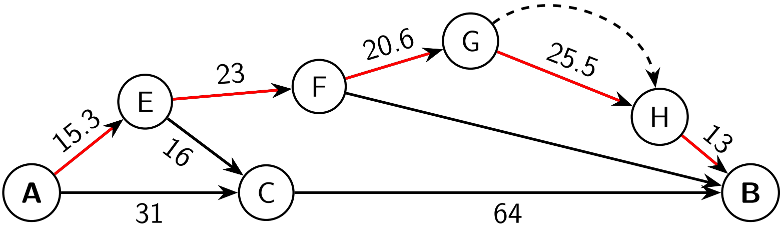
SP **Nominal** $= 95$ **Worst Case** $= 127$



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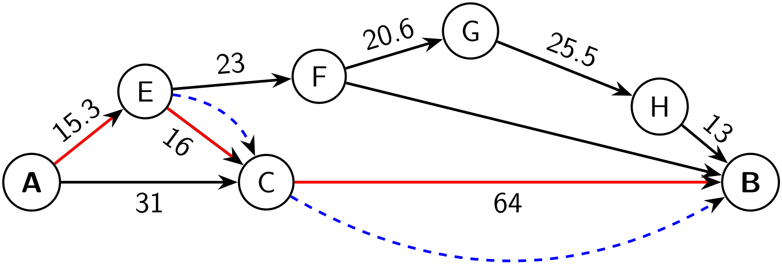
RSP **Nominal** $= 97.4$ **Worst Case** $= 110.15$



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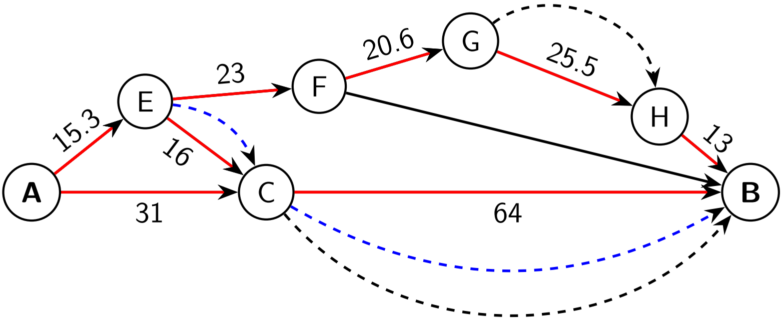
DDRSP **Nominal** $= 95.6 + c$ **Worst Case** $= 108.5 + c$

Network



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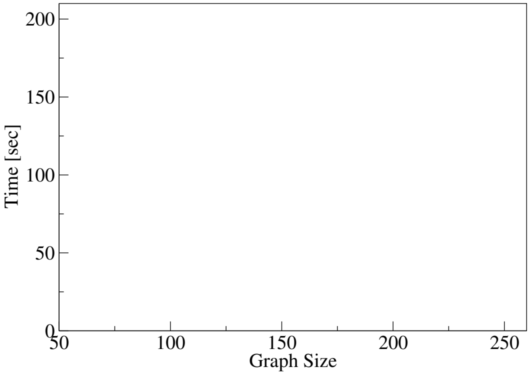
DDRSP **Nominal** $= 95.6 + \mathrm{c}$ **Worst Case** $= 108.5 + \mathrm{c}$

100 random graphs, $c = 1.0, \gamma = 0.2, \Gamma = 2$



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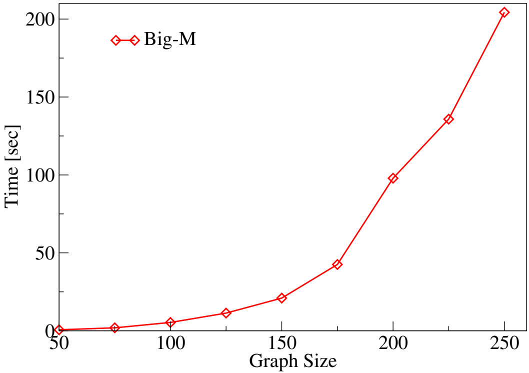
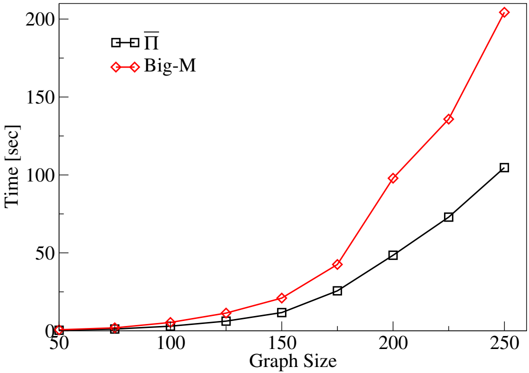


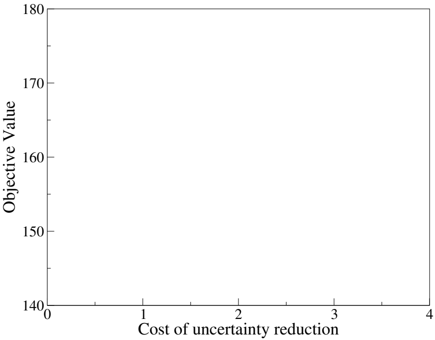
Numerical Results : Speed

100 random graphs, $c = 1.0, \gamma = 0.2, \Gamma = 2$



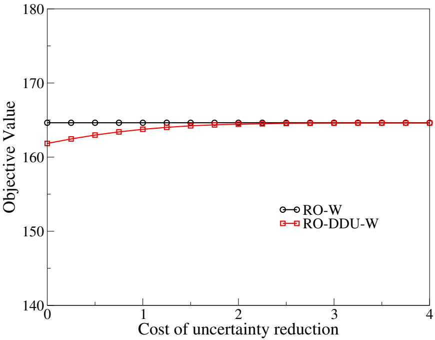
$\Longrightarrow$ $\overline{\mathbf{\Pi}}$ formulation better than Big-M

100 random graphs, 50 samples, $|\mathcal{V}| = 30, \gamma = 0.2, \quad \Gamma = 3$



Numerical Results : Performance

100 random graphs, 50 samples, $|\mathcal{V}| = 30, \gamma = 0.2, \ \ \Gamma = 3$



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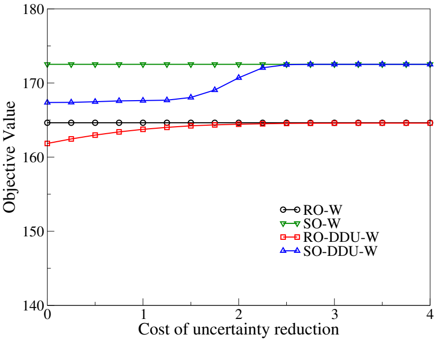

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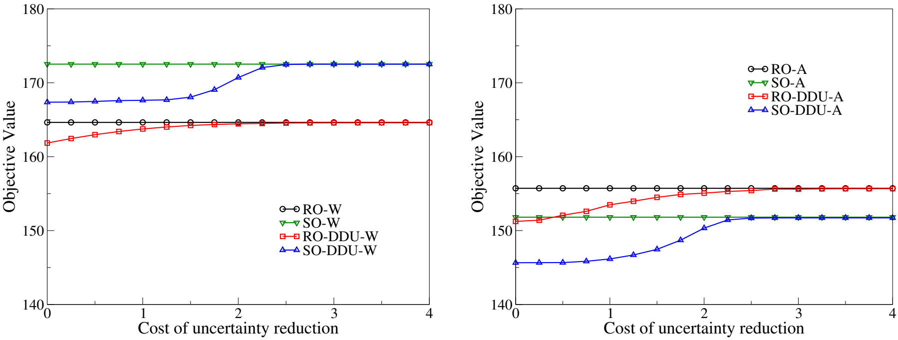
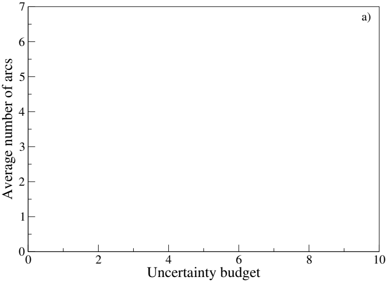


$\Longrightarrow$ Performance of DDU improves with lower reduction costs.

100 random graphs, $c = 1.0, \gamma = 0.2$, nodes $= 30$



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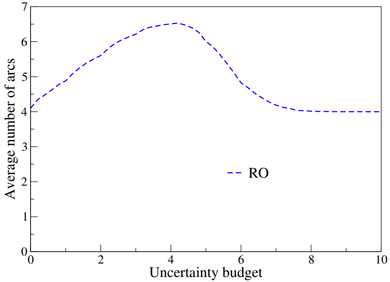
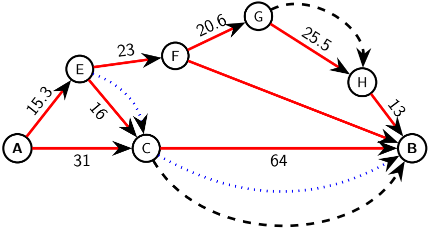
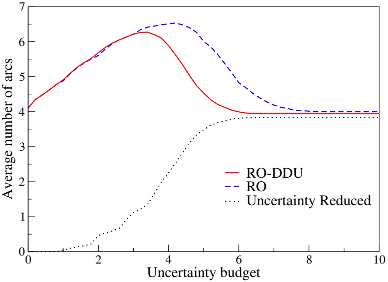
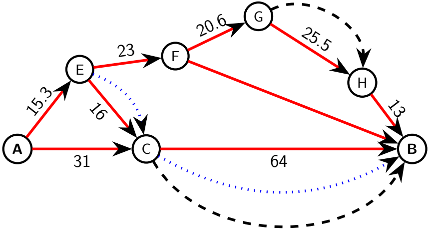




Path Behavior

100 random graphs, $c = 1.0, \gamma = 0.2$, nodes $= 30$





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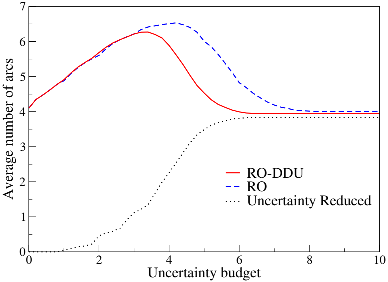
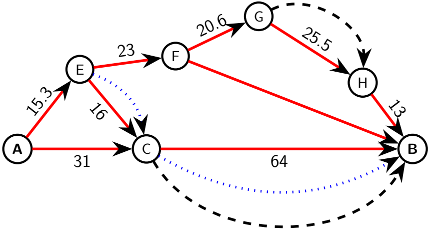
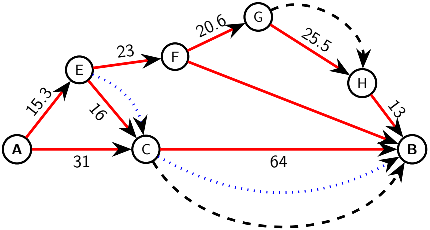




The number of arcs in the path increases and then decreases with increase in the total amount of uncertainty.

Conclusion

- Decision-dependent uncertainty allows to reduce conservatism by proactive control or better models
- Problems with decision-dependent uncertainty are NP-complete.
- Leveraging set structure allows to improve performance.



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Omid Nohadani and Kartikey Sharma. "Optimization under decision-dependent uncertainty." SIAM Journal on Optimization 28.2 (2018): 1773-1795.

Lemma

If $\mathbf{G}$ *and* $\mathbf{y}$ *are nonnegative, then* $\pi_i(\mathbf{x}, \mathbf{y}) \leq y_i \; \forall (\mathbf{x}, \mathbf{y})$.

- Consider an instance of the 3-Satisfiability problem (3-SAT) for a set $N = \{1, 2, \ldots, n\}$ of literals and m clauses, which tries to find a solution $\mathbf{x} \in \{0, 1\}^n$ that satisfies

$$x_{i_1} + x_{i_2} + (1 - x_{i_3}) \geq 1 \ \ \forall i = 1, \ldots, m.$$

- Next, consider the following special decision dependent problem with $\mathbf{x} \in \Re^n$, $\mathbf{y} \in \Re^m$, $z \in \Re$

$$\min_{\mathbf{x}, \mathbf{y}, z \geq 0} \left\{ -z \mid z - \mathbf{a}^\top \mathbf{y} \leq 0, \ \ \forall \mathbf{a} \in \mathcal{U}(\mathbf{x}), \ \ \mathbf{x}, \mathbf{y} \leq \mathbf{1}, \ \ -\mathbf{y} \leq -\mathbf{1} \right\},$$

$$\mathcal{U}(\mathbf{x}) = \{(a_1, \ldots, a_m) \mid a_i \geq x_{i_1} \ , \ a_i \geq x_{i_2} \ , \ a_i \geq 1 - x_{i_3} \ , \ a_i \leq 1\}$$

Note that the 3-SAT problem is embedded in this set.

Standard Reformulations

Theorem

The constraint $\mathbf{y}^\top \boldsymbol{\xi} \leq b \ \ \forall \boldsymbol{\xi} \in \mathcal{U}^{\overline{\mathbf{\Pi}}}(\mathbf{x})$ *has the reformulation*

$$
\begin{aligned}
&\mathbf{t}^\top \mathbf{d} + \mathbf{s}^\top \mathbf{v} + \boldsymbol{s}^\top \mathbf{We} - \sum_i r_i \leq b \\
&\mathbf{s}^\top + \mathbf{t}^\top \mathbf{D} \geq \mathbf{y}^\top \\
&w_i s_i - M(1 - x_i) \leq r_i \leq M x_i \\
&r_i \leq w_i s_i \\
&\mathbf{r}, \mathbf{s}, \mathbf{t} \geq \mathbf{0}
\end{aligned}
$$

Standard Reformulations

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$$
\begin{aligned}
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&\mathbf{s}^\top + \mathbf{t}^\top \mathbf{D} \geq \mathbf{y}^\top \\
&w_i s_i - M(1 - x_i) \leq r_i \leq M x_i \\
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&\mathbf{r}, \mathbf{s}, \mathbf{t} \geq \mathbf{0}
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$$

- Large number of constraints and poor numerical performance

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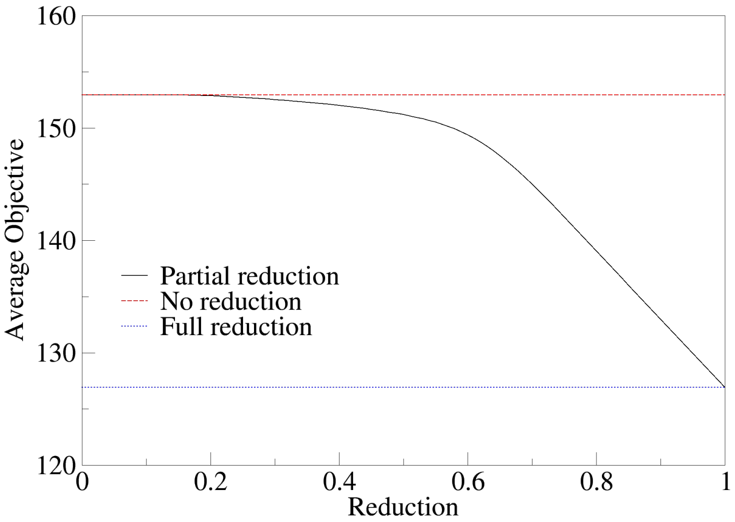
\end{aligned}
$$

- Large number of constraints and poor numerical performance
- Does not leverage the structure of the uncertainty set


160
150
140
130
120
0
0.2
0.4
0.6
0.8
1
Average Objective
Reduction
Partial reduction
No reduction
Full reduction

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