

Distributionally Robust Optimization over Time

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Joint work with Kevin-Martin Aigner, Kristin Braun, Sebastian Pokutta, Frauke Liers, Andreas Barman, Oscar Schneider, and Sebastian Tschupik





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- DRO problems are difficult to solve
- Dualization removes increases the size of the problem
- It also removes any special structure



Existing Work

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DRO over Time

- Uses recomputation of exact solution with updated ambiguity set
(Bayraksan and Love 2015, Esfahani and Kuhn 2018 ...)

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Faster Solutions Methods

- Convert DRO problem into a regularisation problem (Namkoong and Duchi 2016, Chen et al 2017, Levy et al 2020 ...)



Model

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Learning

- Observe samples from a distribution p^* (**finite dimension**)
- Use samples to construct ambiguity set \mathcal{P}_0 containing p^* with high probability

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Optimization

- Leverage ambiguity set \mathcal{P}_t to make robust decisions x_t

Learning

- Use new observations to update the set \mathcal{P}_{t+1} and repeat

Ambiguity Sets

- Confidence Interval Ambiguity Set
- ℓ_2 -norm Ambiguity Set
- Kernel based Ambiguity Set

$$\mathcal{P}_t = \left\{ p \in \mathcal{P}_0 \mid |p - \hat{p}_t| \leq \frac{z_{\delta_t/2}}{\sqrt{t}} \right\}$$

$$\mathcal{P}_t = \left\{ p \in \mathcal{P}_0 \mid \|p - \hat{p}_t\|_2 \leq \frac{\sqrt{2|\mathcal{S} \log(2/\delta_t)}}{\sqrt{t}} \right\}$$

$$\mathcal{P}_t = \left\{ p \in \mathcal{P}_0 \mid \|p - \hat{p}_t\|_M \leq \frac{\sqrt{C}}{\sqrt{t}} (2 + \sqrt{2 \log(1/\delta_t)}) \right\}$$



Standard Reformulation of DRO

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$$\widehat{J}_t := \min_{x \in \mathcal{X}} \max_{p \in \mathcal{P}_t} \mathbb{E}_{s \sim p} [f(x, s)]$$

- Dual Reformulation for Interval Ambiguity Sets

$$\begin{aligned} & \min_{x, z, \alpha, \beta} z - \langle l_t, \alpha \rangle + \langle u_t, \beta \rangle \\ \text{s.t. } & z - \alpha_k + \beta_k \geq f(x, s_k) \quad \forall k = 1, \dots, |\mathcal{S}|, \end{aligned}$$

$$\alpha, \beta \geq 0,$$

$$x \in \mathcal{X}, z \in \mathbb{R}, \alpha, \beta \in \mathbb{R}^{|\mathcal{S}|}.$$

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$$x \in \mathcal{X}, \quad z \in \mathbb{R}, \quad \alpha, \beta \in \mathbb{R}^{|\mathcal{S}|}.$$

Algorithm 1 DRO over Time

- 1: **Input:** functions $f(\cdot, s)$ for $s \in \mathcal{S}$, feasible set \mathcal{X} , initial ambiguity set \mathcal{P}_0
- 2: **Output:** sequence of DRO solutions x_1, \dots, x_T
- 3: **for** $t = 1$ **to** T **do**
- 4: $x_t \leftarrow$ solve Problem (1) or (2) for \mathcal{P}_{t-1}
- 5: $\mathcal{P}_t \leftarrow$ observe data and update set parameters such as interval width and ambiguity set.
- 6: **end for**



Online Optimization

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- Initial solutions x_0 and p_0

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- Step by adversary p

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$$p_t = \arg \min_{p \in \mathcal{P}_{t-1}} \left\langle -\eta \nabla_p \mathbb{E}_{s \sim p_{t-1}} [f(x_{t-1}, s)], p \right\rangle + \frac{1}{2} \|p - p_{t-1}\|^2.$$
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- Step by decision maker x

$$x_t = \max_{p \in \mathcal{P}_{t-1}} \mathbb{E}_{s \sim p} [f(x_{t-1}, s)]$$

Online Optimization

Algorithm 2 DRO over Time with Online Projected Gradient Descent

- 1: **Input:** functions $f(\cdot, s)$ for $s \in \mathcal{S}$, feasible set \mathcal{X} , initial ambiguity set \mathcal{P}_0
- 2: **Output:** x_1, \dots, x_T

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- 5: Set $x_0 = \min_{x \in \mathcal{X}} f(x, s_1)$
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Proof technique:

- True distribution inside ambiguity set
- The ambiguity set converges to the true distribution

Regret Bound on Solutions

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With probability at least $1 - \delta$ we have

$$\frac{1}{T} \sum_{t=1}^T \left(\max_{p \in \mathcal{P}_t} \mathbb{E}_{s \sim p} [f(x_t, s)] - \min_{x \in \mathcal{X}} \max_{p \in \mathcal{P}_t} \mathbb{E}_{s \sim p} [f(x, s)] \right) \leq G \sqrt{\frac{|\mathcal{S}| h(T)}{2T}} + \frac{2G}{T},$$

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Here

$$h(T) \in \mathcal{O}(|\mathcal{S}| \log^2(T))$$

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Confidence Intervals

ℓ_2 -norm Sets

Kernel based Sets

$$h(T) = 8|\mathcal{S}| \log(\pi T)(1 + \log T)$$

$$h(T) = 8|\mathcal{S}| \log \frac{\pi T}{\sqrt{3\delta}}(1 + \log T)$$

$$h(T) = \frac{32C}{\lambda^2} + \frac{32C}{\lambda^2} \log \frac{\pi T}{\sqrt{6\delta}}(1 + \log T)$$

Numerical Experiments

Benchmark Libraries

- MILPs and MIQPs from the MIPLIB set of benchmark Instances
- Comparisons against other methods

Optimal Route Choice

- *ChicagoSketch* model from Transportation Networks library
- Illustration of impact on solutions

Benchmark Libraries

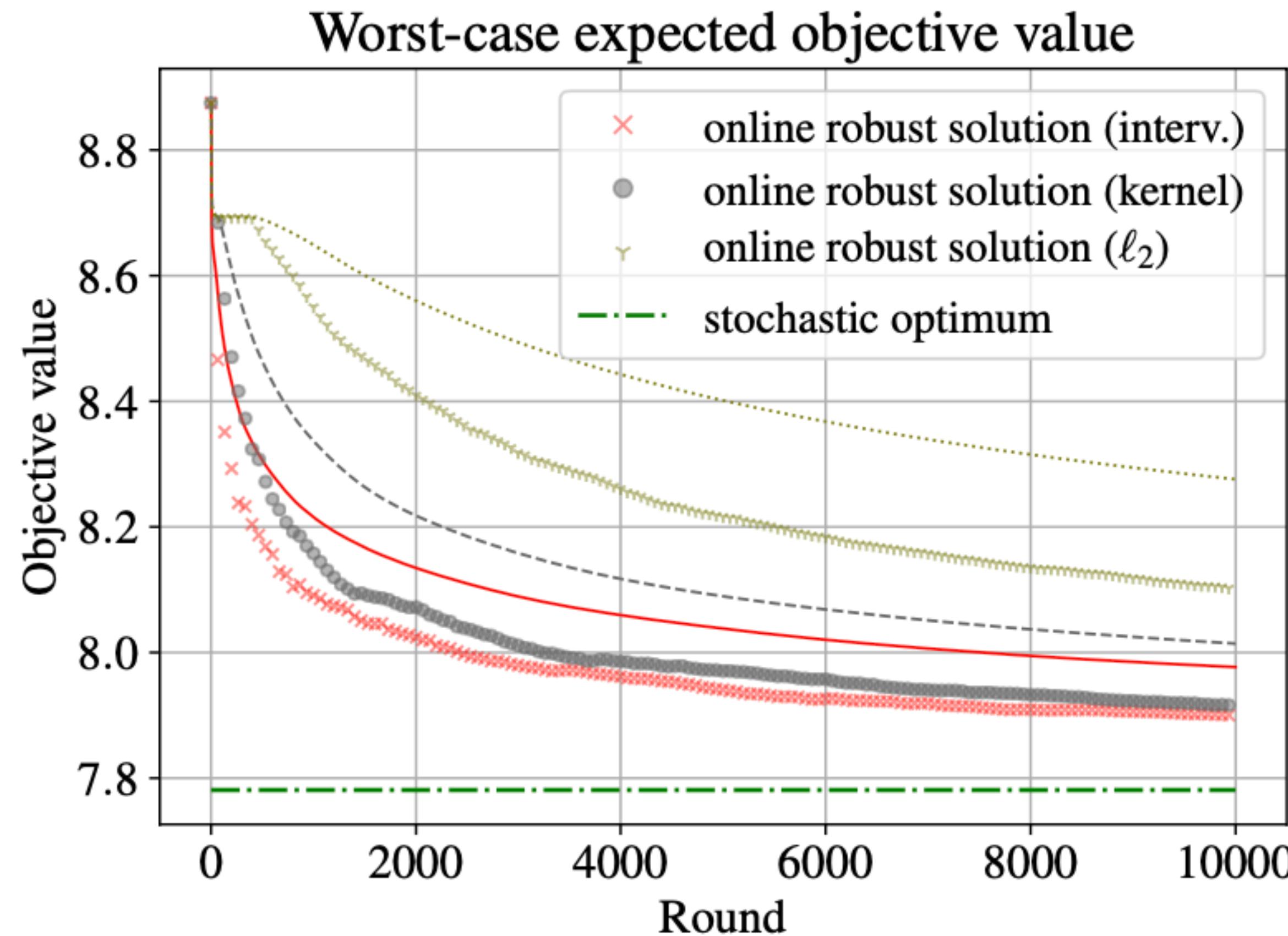
- Takes MILP and MIQP instances from MIPLIB library
- Instances are of the form

$$f(x, s) = x^\top Qx + (c + s)^\top x + d$$

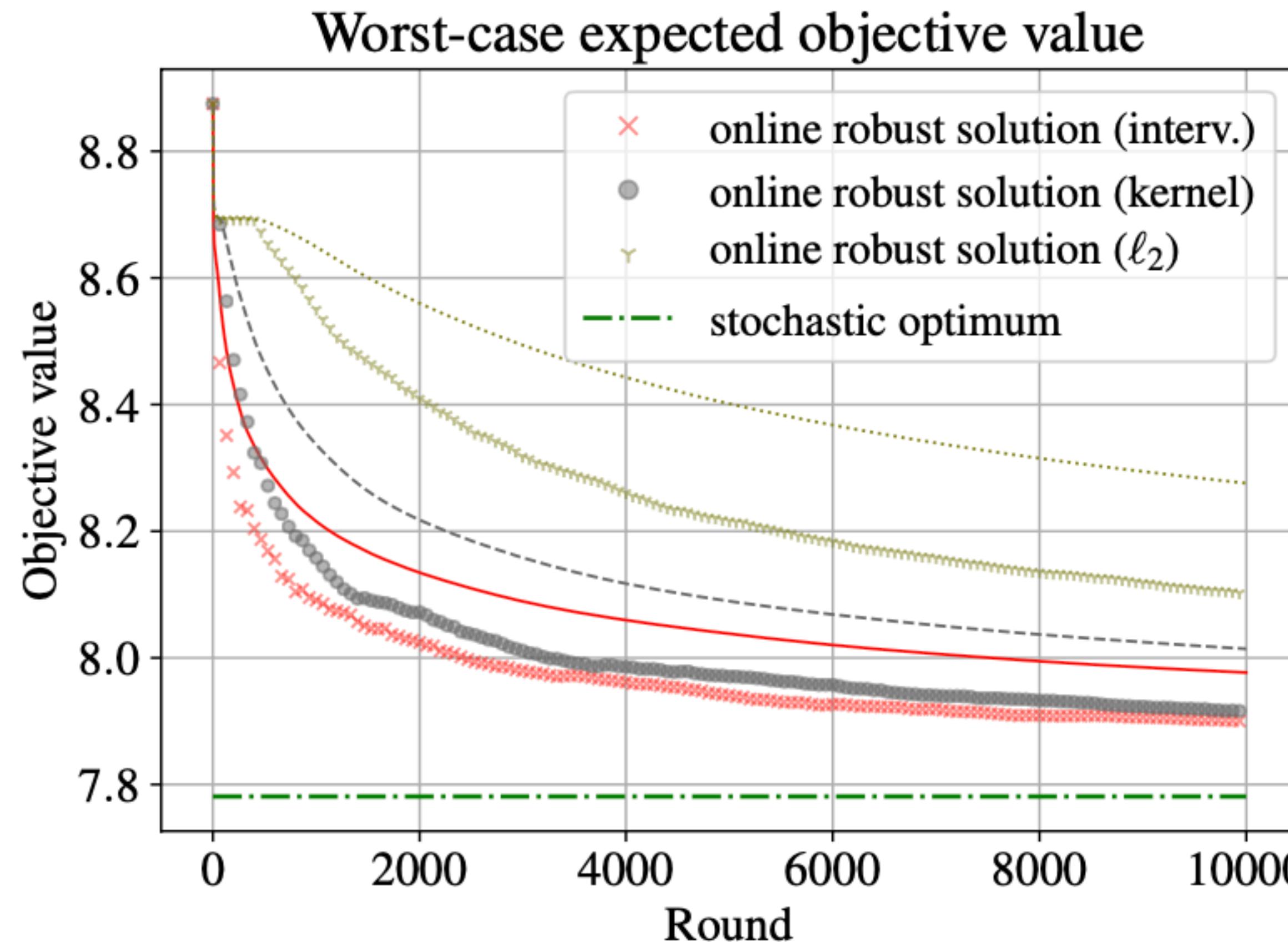
- **Objective** uncertainty in the instances through scenarios $s \in \mathcal{S}$ with $|\mathcal{S}| = (2, 10, 15)$

Benchmark Instances: Different Ambiguity Sets

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Benchmark Instances: Different Ambiguity Sets



The objective value shrinks for all set types

Fastest reduction for confidence intervals

Benchmark Instances: Running Times

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	$ \mathcal{S} $	ONLINE ROBUST	EXACT DRO
MIP (I)	10	52.4s	115.8s
MIP (ℓ_2)	10	49.4s	127.5s
MIP (K)	10	56.3s	129.5s
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MIP (ℓ_2)	50	60.4s	206.1s*
MIP (K)	50	67.0s	244.4s*
MIQP (I)	2	170.2s	271.4s*
MIQP (ℓ_2)	2	186.3s	329.5s*
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More time savings for large and non linear problems

Benchmark Instances: Running Times

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	MIP $ \mathcal{S} = 10$	MIP $ \mathcal{S} = 50$	MIQP $ \mathcal{S} = 2$
DRO	45.6s	55.9s*	271.4s*
Wasserstein	52.3s	59.1s	299.9s
DRBO	42.7s**	66.1s**	738.3s*
Online robust	26.8s	27.1s	170.2s
Running SO	26.6s	26.9s	172.6s

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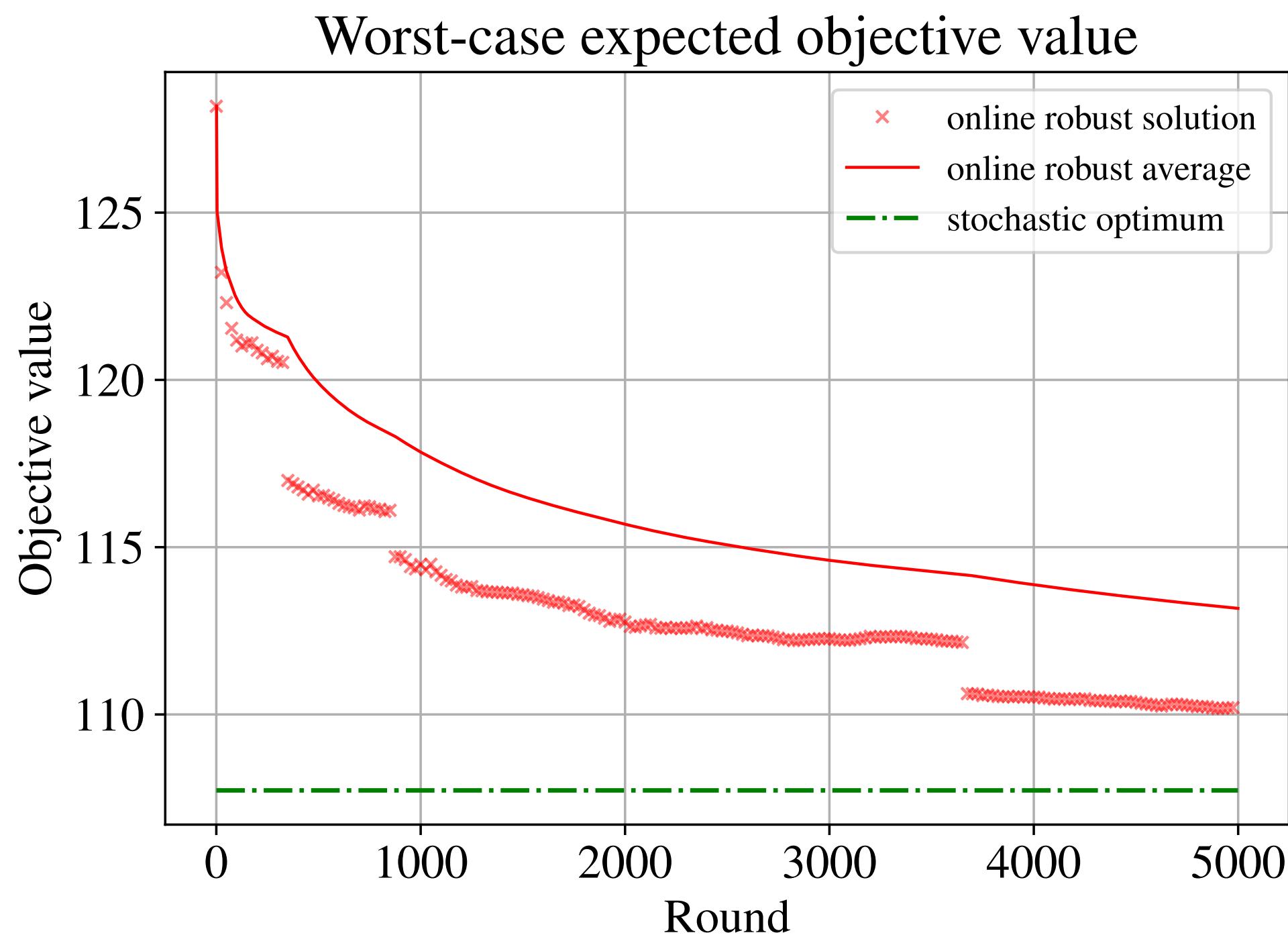
Online robust methods are significantly faster

Optimal Route Choice

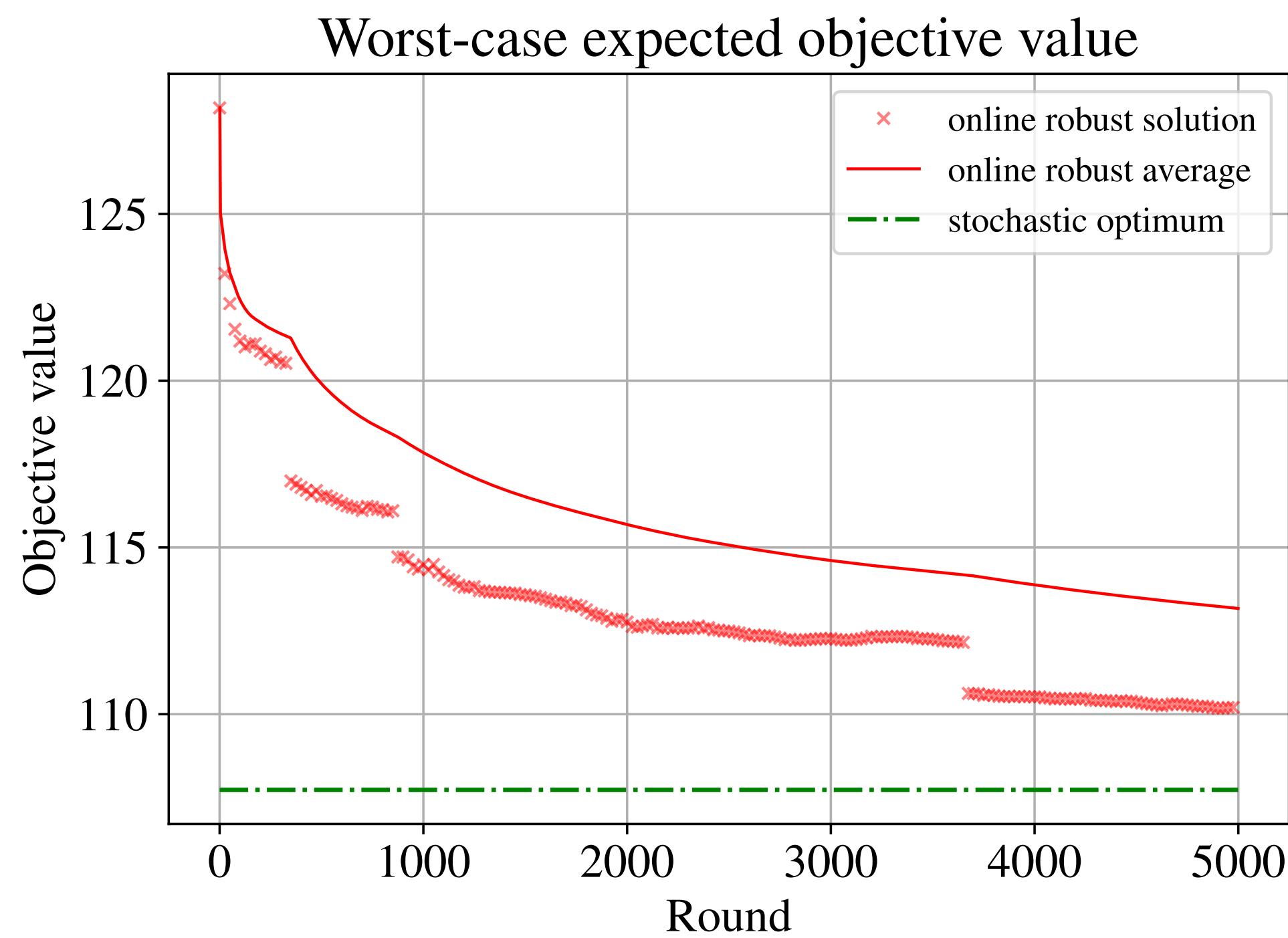
- Choose the shortest paths in a street network with uncertain arc times
- Model: *ChicagoSketch* with 933 nodes and 2950 arcs
- Randomly generated *true* probability distribution for arcs lengths
- Solving directly eliminates structure



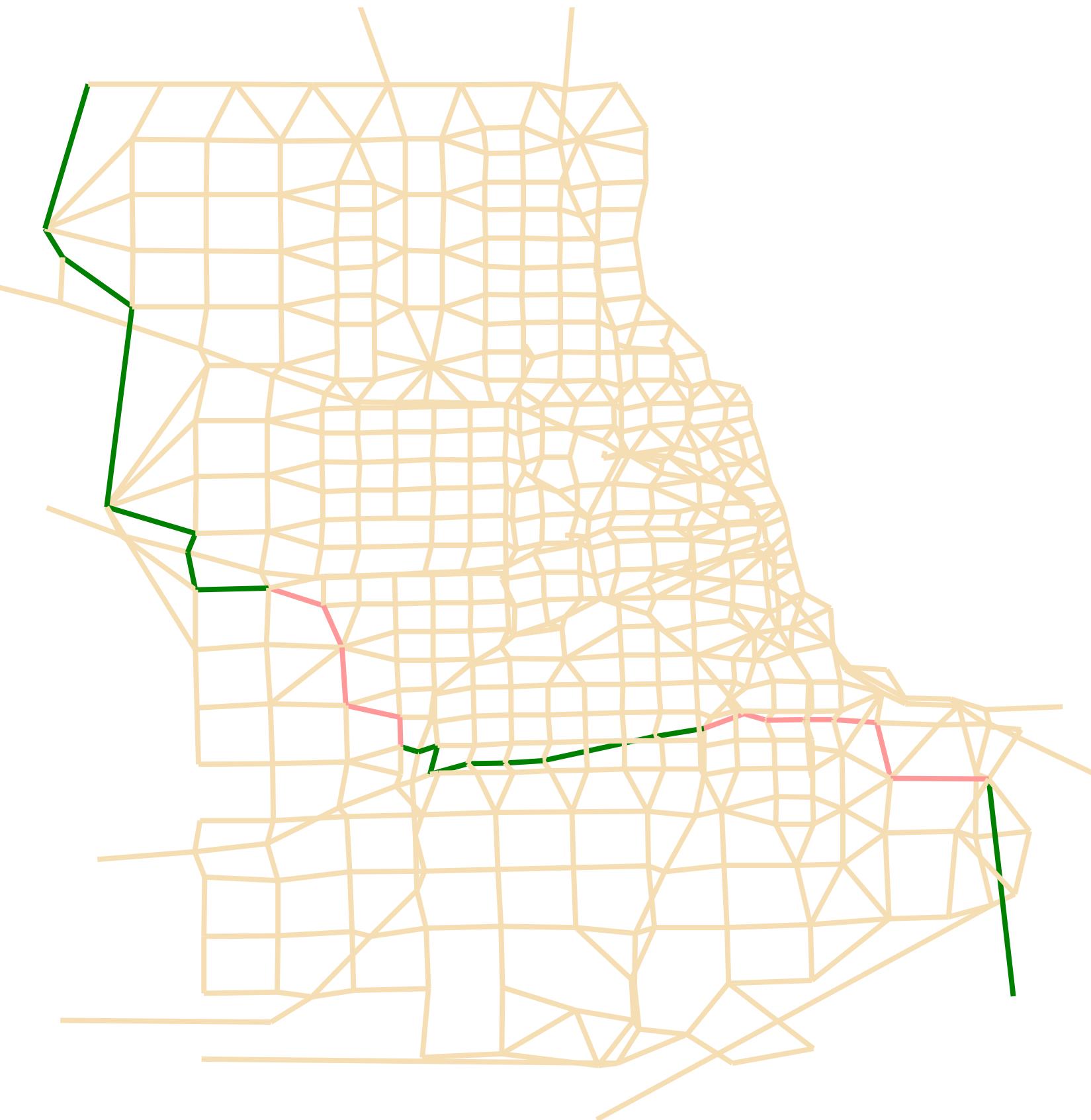
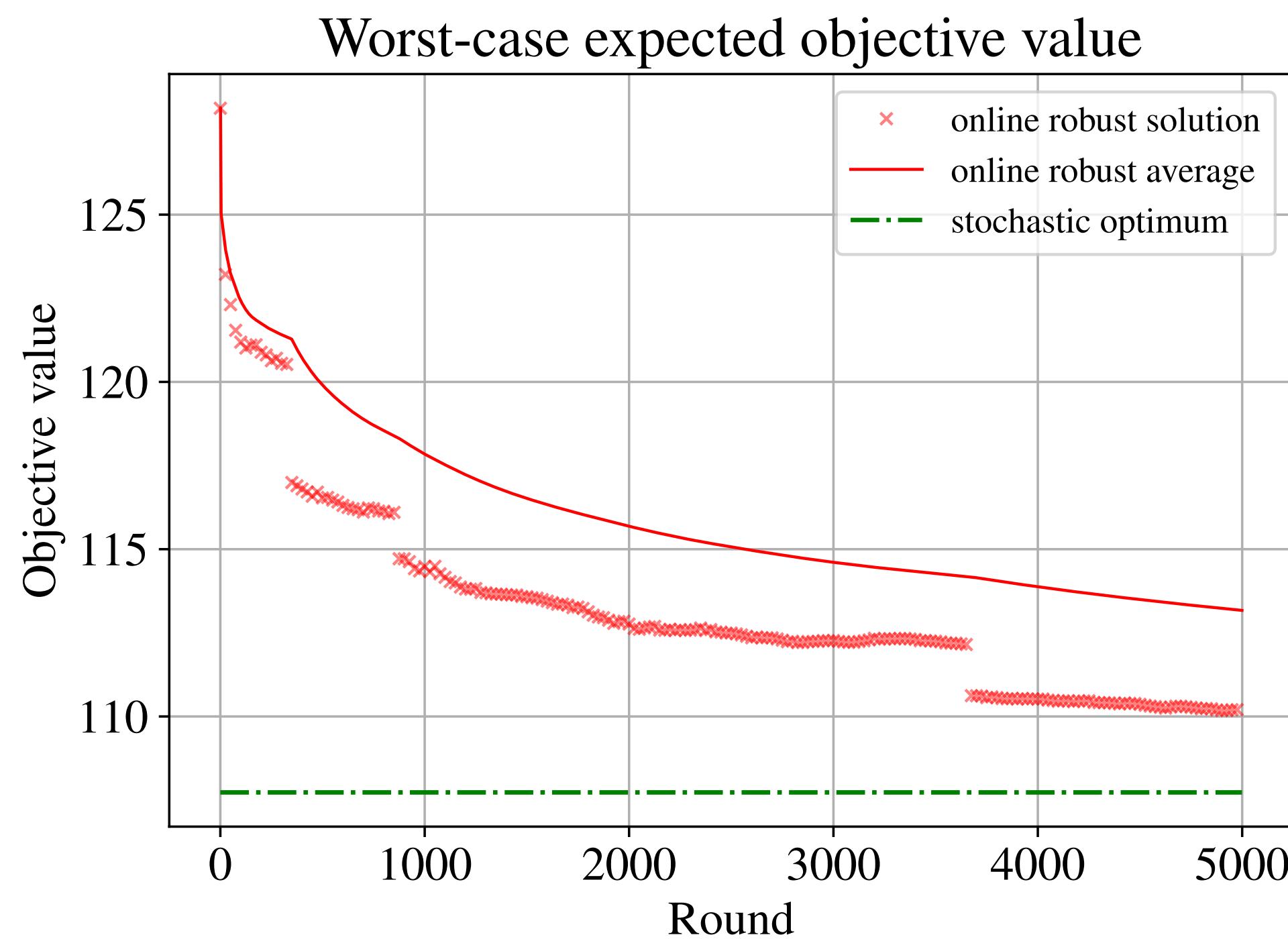
Optimal Route Choice



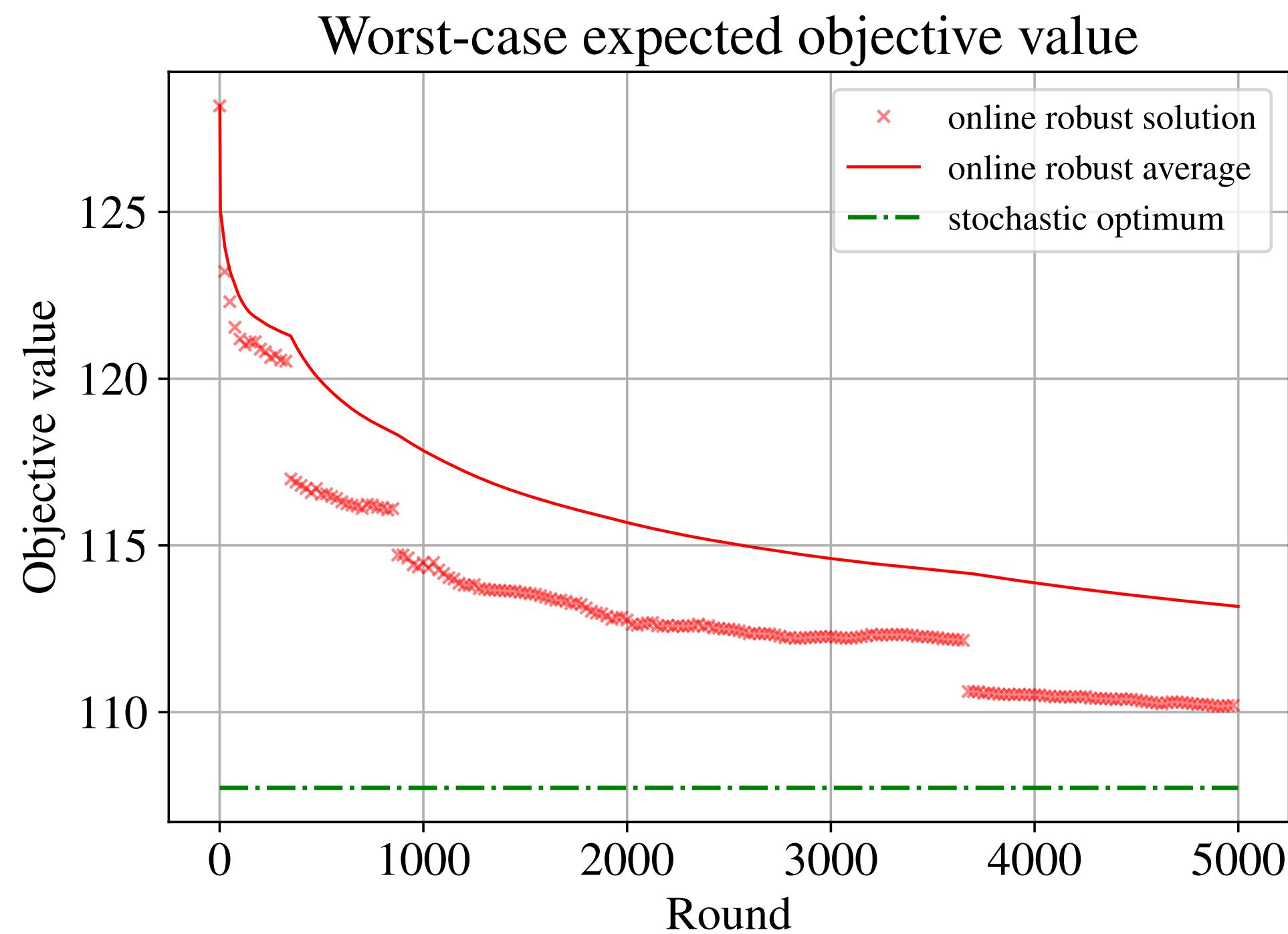
Optimal Route Choice



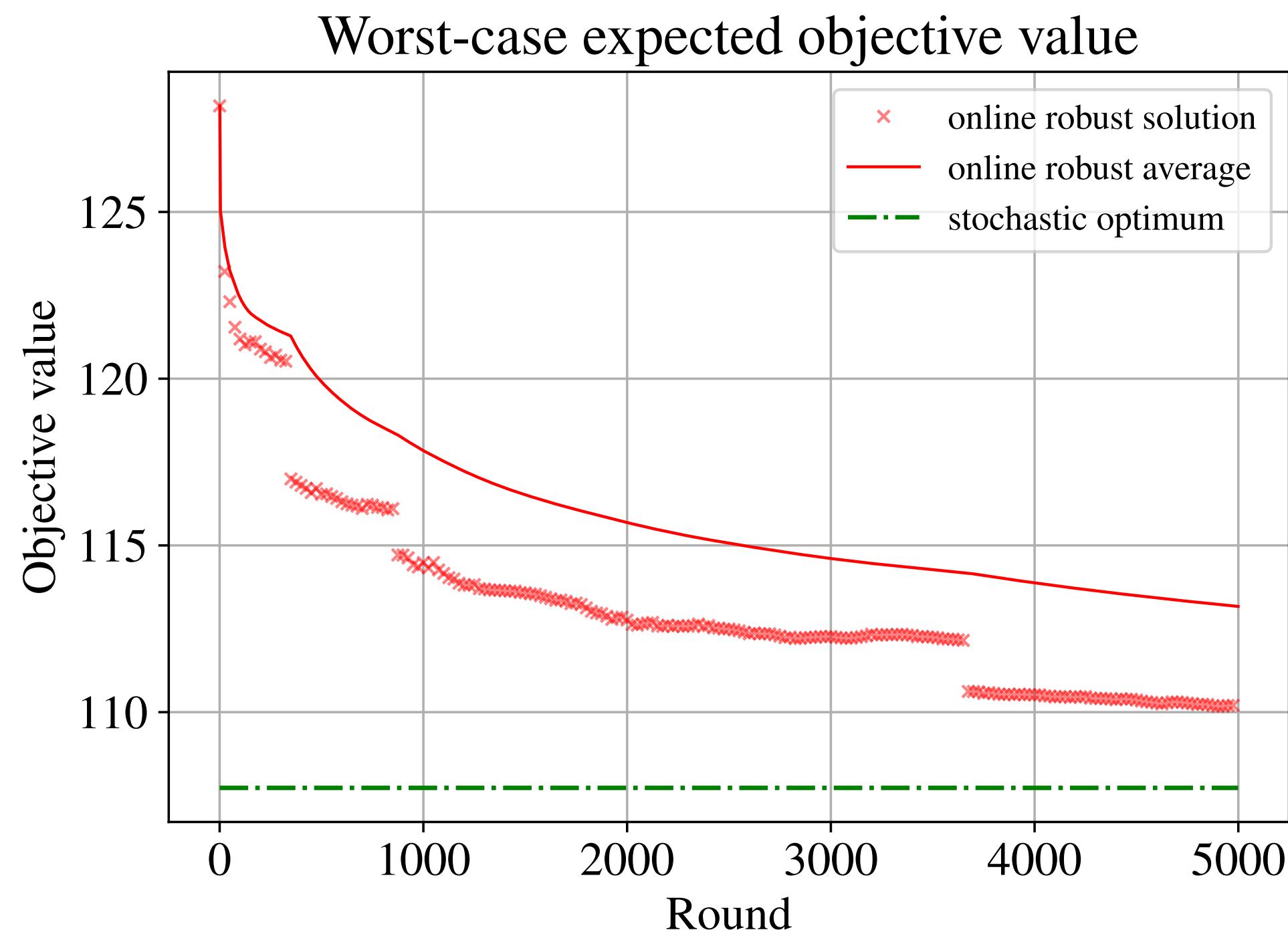
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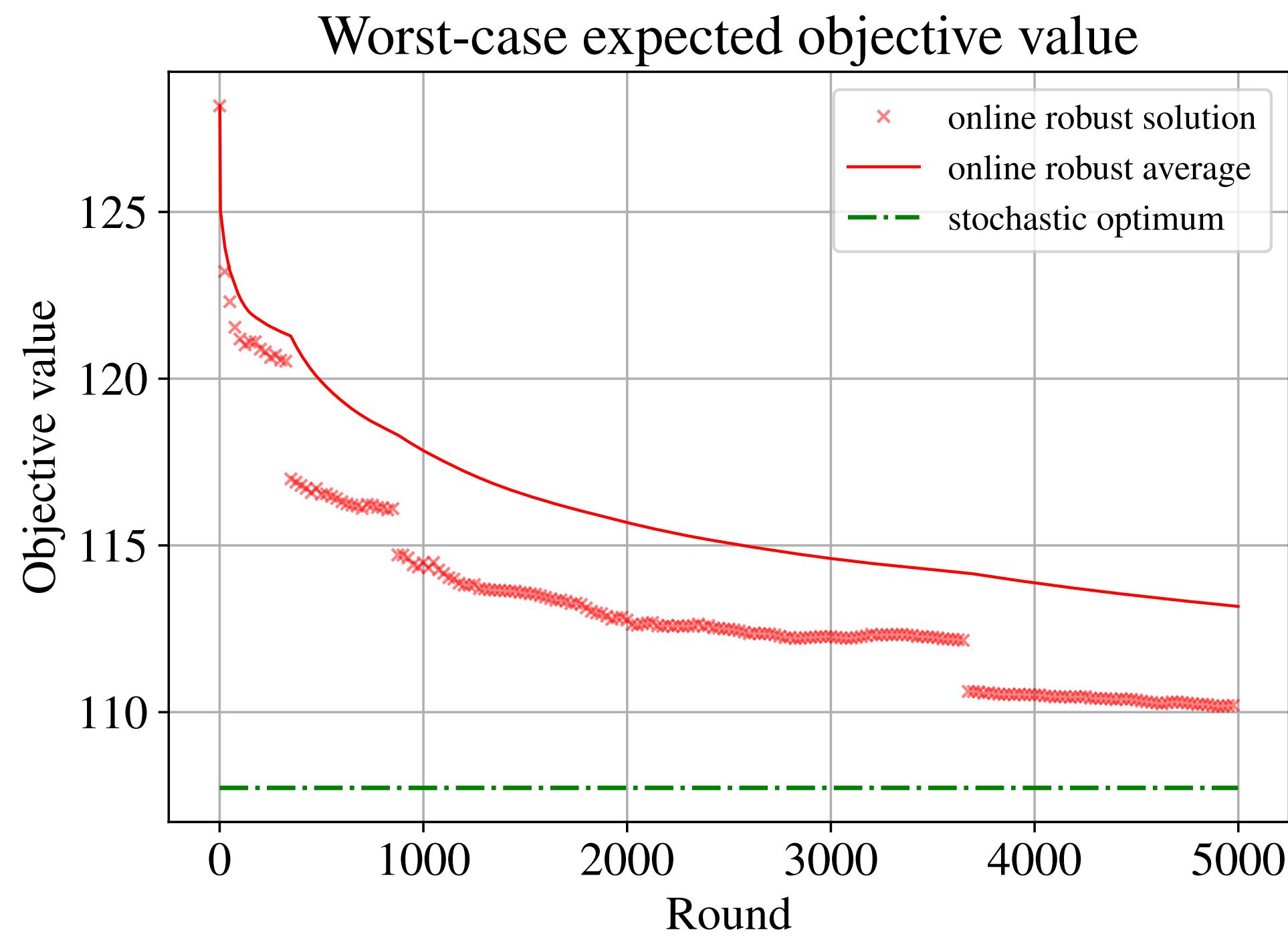
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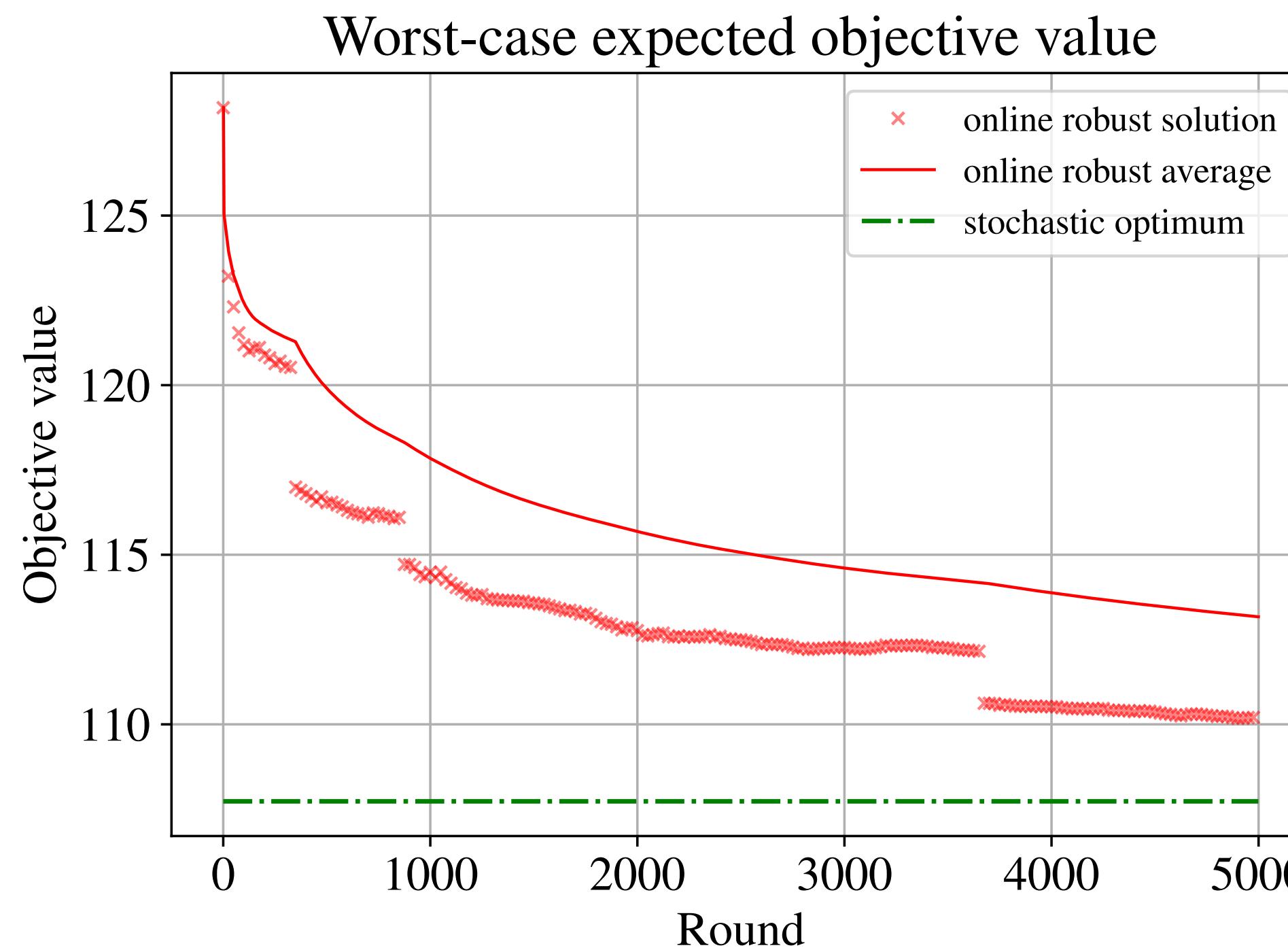
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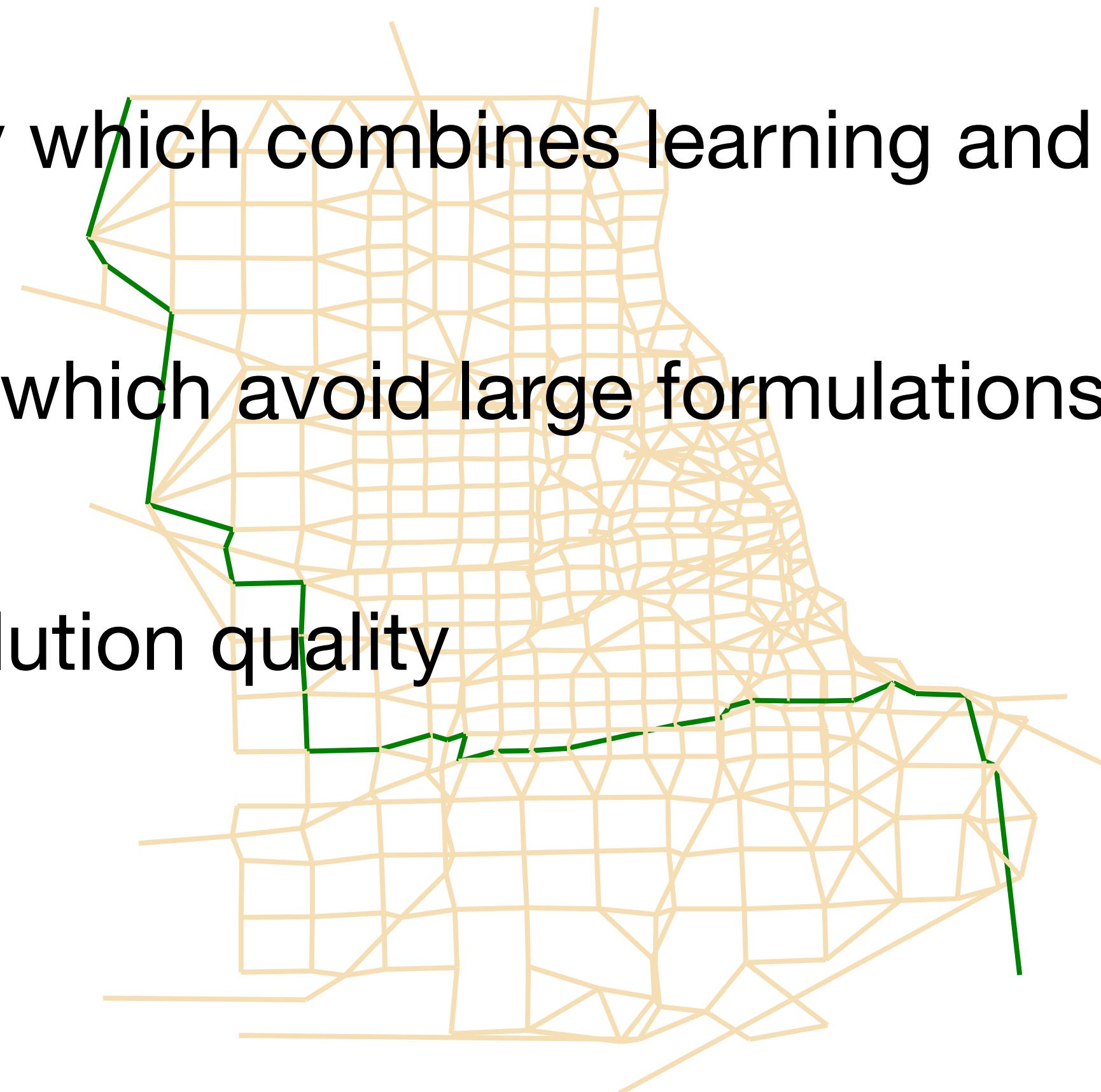
Optimal Route Choice



Continuous decrease in objective value but discrete jumps due to changing solution

Conclusions

- Method for optimization under uncertainty which combines learning and Distributionally Robust Optimization
- Iterative algorithm for solution of problem which avoid large formulations and maintains structure
- Theoretical proofs of convergence and solution quality
- Numerical illustrations for the results



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