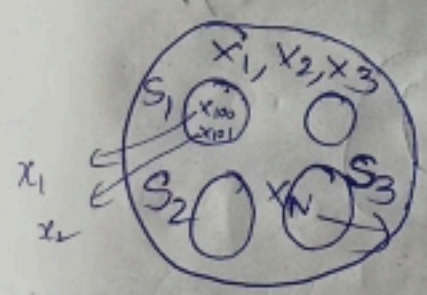
Unit-4

Estimation

sampling

~~Ex- If A Po Commision~~ :

If we want to find the average salary of Delhi, then it is not possible to collect data from each and every Person in delhi. So we will calculate some samples and from that samples we will estimate the average salary of Delhi People.



Population units

$x_1, x_2, x_3, \ldots x_N$ → large

$x_1, x_2, \ldots x_n$ → sample units.

Avg. Data usage of Punjab

↓

Population

$x_1, x_2 \ldots x_N$ → Population units.

* if N is small then we can directly calculate.
* Avg. Marks of K19EG.

Population mean $= \dfrac{x_1 + x_2 + \cdots + x_N}{N} = \mu$

Sample mean $= \dfrac{x_1 + x_2 + \cdots + x_n}{n} = \bar{x}$

Population varience $= \dfrac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2$
$= \sigma^2$

Sample varience $= \dfrac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$
$= s^2$

$(\because V(Y) = E[(Y - E(Y))^2])$ $\left(\because \text{var}(y) = \dfrac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2\right)$

$E(x_i) = \Sigma$ (values taken by x_i) $\times$ (Probability x_i takes that value).

$$\boxed{\therefore E(Y) = \Sigma y_i P[y = y_i]}$$

$$= x_1\left(\frac{1}{N}\right) + x_2\left(\frac{1}{N}\right) + \cdots + x_N\left(\frac{1}{N}\right)$$

$$= \frac{x_1 + x_2 + \cdots + x_N}{N}$$

$$\boxed{E(x_i) = \mu}$$

$$* E(\bar{x}) = E\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)$$

$$= \frac{1}{n}\left[E(x_1) + E(x_2) + \cdots + E(x_n)\right]$$

$$= \frac{1}{n}\left[\mu + \mu + \mu + \cdots \mu\right]$$

$$= \frac{n\mu}{n} = \mu$$

$$\cancel{E = \mu} \quad \boxed{E(\bar{x}) = \mu}$$

$$* V(x_i) = E\left[x_i - E(x_i)\right)^2\right]$$

$$= E\left[(x_i - \mu)^2\right]$$

$$= (x_1 - \mu)^2\left(\frac{1}{N}\right) + (x_1 - \mu)^2\left(\frac{1}{N}\right) + \cdots + (x_N - \mu)^2\left(\frac{1}{N}\right)$$

$$= \sum_{i=1}^{N} (x_i - \mu)^2\left(\frac{1}{N}\right)$$

$$\boxed{V(x_i) = \sigma^2} \qquad \boxed{(\because V(ax) = a^2 V(x))}$$

$$* V(\bar{x}) = V\left[\frac{x_1 + x_2 + \cdots + x_n}{n}\right]$$

$$= \frac{1}{n^2}\left[V(x_1) + V(x_2) + \cdots + V(x_n)\right]$$

$(\because V(a_1x_1+a_2x_2) = a_1^2V(x_1) + a_2^2V(x_2) + \cancel{2a_1a_2cov}$
$+ 2a_1a_2 cov(x_1,x_2))$

* x_1, x_2 are independent then
$cov(x_1,x_2) = 0$.

* $V[a_1x_1 + a_2x_2 + \cdots + a_nx_n] = a_1^2V(x_1)$
$+ a_2^2V(x_2) + \cdots + a_n^2V(x_n)$
$x_1, x_2, \cdots, x_n$ are independent.

$(\because V(x_i) = \sigma^2)$

$= \frac{1}{n^2}[\sigma^2 + \sigma^2 + \cdots + \sigma^2]$

$= \frac{1}{n^2}(n\sigma^2)$

$\boxed{\therefore V(\bar{x}) = \frac{\sigma^2}{n}}$

* $E(s^2) = ?$

$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n}\sum x_i^2 - (\bar{x})^2$

* $E\left[\frac{1}{n}\sum x_i^2 - (\bar{x})^2\right] = \frac{1}{n}E(\sum x_i^2) - E[(\bar{x})^2]$

$= \frac{1}{n}E[x_1^2 + x_2^2 + \cdots + x_n^2] - E((\bar{x})^2)$

$$= \frac{1}{n}\left[E(x_1^2) + E(x_2^2) + \cdots + E(x_n^2)\right] - E((\bar{x})^2)$$

$$\boxed{E(S^2) = \frac{1}{n}\left[\sum E(x_i^2)\right] - E((\bar{x})^2) \quad \text{--- (1)}}$$

$$\boxed{* \; E\left(\sum x_i^2\right) = \sum_{i=1}^{n} E(x_i^2)}$$

Expectation of sum of the squares is ~~sum~~ equal to sum of expectation of the squares.

$$V(\bar{x}) = E((\bar{x})^2) - (E(\bar{x}))^2 = E\left((\bar{x} - E(\bar{x}))^2\right)$$

so. $E((\bar{x})^2) = V(\bar{x}) + (E(\bar{x}))^2$

$$E((\bar{x})^2) = \frac{\sigma^2}{n} + \mu^2$$

$$\boxed{\text{so } (E(x_i^2)) = V(x_i) + (E(x_i))^2}$$

so. ~~$E(S^2) = \frac{1}{n}$~~

$$\boxed{E(x_i^2) = \sigma^2 + \mu^2}$$

$$\text{so } E(S^2) = \frac{1}{n}\left(\sum_{i=1}^{n}(\sigma^2 + \mu^2)\right) - \left(\frac{\sigma^2}{n} + \mu^2\right)$$

$$= \frac{1}{n}\left(n(\sigma^2 + \mu^2)\right) - \left(\frac{\sigma^2}{n} + \mu^2\right)$$

$$= \frac{1}{n}\left(n(\sigma^2 + \mu^2)\right) - \left(\frac{\sigma^2}{n} + \mu^2\right)$$

$$= \sigma^2 + \cancel{\mu^2} - \frac{\sigma^2}{n} - \cancel{\mu^2}$$

$$= \sigma^2 - \frac{\sigma^2}{n}$$

$$\boxed{E(s^2) = \left(1 - \frac{1}{n}\right)\sigma^2} = \frac{n-1}{n}\sigma^2$$

$$\boxed{\therefore s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$$\boxed{\therefore S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

Relation b/w s^2 & S^2

$$\boxed{n s^2 = (n-1) S^2}$$

$$S^2 = \frac{n}{n-1} s^2$$

$$E(S^2) = E\left(\frac{n}{n-1} s^2\right)$$

$$= \frac{n}{n-1} E(s^2) \qquad \left(\because E(s^2) = \frac{n-1}{n}\sigma^2\right)$$

$$= \frac{\cancel{n}}{\cancel{n-1}} \times \frac{\cancel{n-1}}{\cancel{n}}\sigma^2$$

$$\boxed{\therefore E(S^2) = \sigma^2}$$

* if $n \to \infty$ $E(s^2) \to \sigma^2$

because

$$E(s^2) = \left(1-\frac{1}{n}\right)\sigma^2$$
$$= \left(1-\frac{1}{\infty}\right)\sigma^2$$
$$= (1-0)\sigma^2$$
$$E(s^2) = \sigma^2$$

"n" is sample size
$x_1, x_2, \ldots x_n$.

* if sample size is large the expectation of sample varience tends towards population varience.
* if bigger is sample size better will be our estimation.
* so it is recommended to have large sample size

So

* $E(x_i) = \mu$
* $E(\bar{x}) = \mu$
* $V(x_i) = \sigma^2$
* $V(\bar{x}) = \frac{\sigma^2}{n}$
* $E(s^2) = \left(\frac{n-1}{n}\right)\sigma^2$
* $E(S^2) = \sigma^2$

as $n \to \infty$

* $E(s^2) \to \sigma^2$

$$\boxed{* \; ns^2 = (n-1)S^2}$$

Parameter (μ, σ^2)

Any statistical value associated with population

Ex:- Population mean, Population varience

statistic $(\bar{x}, s^2)$

Any statistical value associated with sample.

Ex:- Sample mean, Sample varience

for statistic we can use symbol

$T_n(x_1, x_2, \ldots, x_n)$ it is a function of sample units, n is size of sample

$T_n(x_1, x_2, \ldots x_n)$ like $= f(x_1, x_2 \ldots x_n)$

our sample mean is

$$\bar{x} = \frac{x_1 + x_2 + x_3 \ldots x_n}{n}$$

Sample varience

$$s^2 = \frac{1}{n}\left((x_1 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2\right)$$

These both are functions of $f(x_1, x_2 \ldots x_n)$.

so we gave new notation $T_n(x_1, x_2, \ldots, x_n)$

Estimator

It is a statistic is when used to estimate a Parameter.

Ex:- When you use sample mean to estimate Population mean.

So $\bar{x}$ as a estimator. and same with sample varience. to estimate population varience

symbol of estimator $T_n(x_1, x_2, \ldots, x_n)$

T_n is function of $x_1, x_2, \ldots x_n$.

* Parameter is used with symbol r, $r(\theta)$ means this r further depends on unknown factors like θ.

* for Good estimator some characteristics are requirements.

Good estimator

Some characteristics are

* Unbiased
* Consistent
* Efficient
* Sufficient

Unbiased estimator

An estimator $T_n = T_n(x_1, x_2 \ldots x_n)$ is said to be an unbiased estimator of $\gamma(\theta)$ if $E(T_n) = \gamma(\theta)$

Ex:-

* Sample mean is an unbiased estimator of population mean.

* ~~Sample varience~~ S^2 is an unbiased estimator of population varience.

$\boxed{E(\bar{x}) = \mu}$ $\boxed{E(S^2) = \sigma^2}$

* Let $x_1, x_2 \ldots x_n$ be a sample drawn from a Normal population $N(\mu, 1)$

then $t = \frac{1}{n}\sum_{i=1}^{n} x_i^2$ is unbiased estimator of μ^2+1.

A) here distribution of population is normal distribution.

population mean $= \mu$

population varience $= 1 = \sigma^2$

population S.D $= 1 = \sigma$

} Normal distribution

$$E(t) = E\left[\frac{1}{n}\sum x_i^2\right]$$

$$= \frac{1}{n}E\left[\sum x_i^2\right] \quad (\because V(x_i) = E(x_i^2) - (E(x_i))^2)$$

$$= \frac{1}{n}\sum E(x_i^2)$$

$$= \frac{1}{n}\sum\left[V(x_i) + (E(x_i))^2\right]$$

$$= \frac{1}{n}\sum\left(\sigma^2 + \mu^2\right)$$

$$\boxed{\because E(x_i) = \mu}$$

$$= \frac{1}{n}\sum\left[(1)^2 + \mu^2\right]$$

$$= \frac{1}{n}\sum\left[1 + \mu^2\right]$$

$$= \frac{1}{\cancel{n}} \times \left[\cancel{n}(1+\mu^2)\right]$$

$$E(t) = \mu^2 + 1$$

$$\boxed{\therefore E\left[\frac{1}{n}\sum x_i^2\right] = \mu^2 + 1}$$

* If $t = \frac{1}{n^2}\sum_{i=1}^{n} x_i^2$ then "t" is unbiased estimator of ?

(a) μ^2 (b) μ^2+1 (c) $n(\mu^2+1)$ (d) NOT

A) $t = \frac{1}{n^2}\sum_{i=1}^{n} x_i^2$

$$E(t) = E\left[\frac{1}{n^2}\sum_{i=1}^{n} x_i^2\right]$$

$$= \frac{1}{n^2}E\left[\sum_{i=1}^{n} x_i^2\right] \quad \left(\because E\left(\sum x_i^2\right) = \sum_{i=1}^{n} E(x_i^2)\right)$$

$$= \cancel{\frac{1}{n}(x_1^2 + x_2^2 + \cdots} \; x_n^2)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} E(x_i^2)$$

$$\left(\because V(x_i) = E(x_i^2) - (E(x_i))^2\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\left(V(x_i) + (E(x_i))^2\right)$$

$$= \frac{1}{n^2}\sum\left(\sigma^2 + \mu^2\right)$$

$$= \frac{1}{n^2}\left(n(1+\mu^2)\right)$$

$$\boxed{\therefore E(t) = \frac{1}{n}(1+\mu^2)}$$

If $N(\mu, 2)$ then varience

$V(x_i) = \sigma^2 = 2$ then it is $= \frac{1}{n}(2+\mu^2)$

* If T is unbiased estimator of 'θ' then T^2 is U.E of θ^2? true or false. Ans) false.

A) $E(T) = \theta$, $E(T^2) = V(T) + [E(T)]^2$

$= V(T) + \theta^2 \quad V(T) \geq 0$

$= V(T) + \theta^2 > \theta^2$ if $V(T) = 4$ tno.

So $E(T^2) \neq \theta^2$.

Bernoulli Population

Binomial Distribution $B(n, P)$ n is no. of trails, P is ~~no~~ Probability of success.

If $n=1$, then Binomial Distribution becomes Bernoulli distribution.

So Parameters ~~are~~ is P.

* sum of 'n' independent Bernoulli random variable is a binomial variable if we repeat Bernoulli trails for 'n' ~~no~~ no. of times you will get binomial.

$1+1+1+\cdots+1 = n.$

* in Bernoulli, p is Probability of success in a single trail.

* If $x_1, x_2, \ldots x_n$ is a sample drawn on 'x' which population takes value '1' or '0' with respective Probabilites θ & $1-\theta$. then P.T $\frac{\Sigma x_i(\Sigma x_i - 1)}{n(n-1)}$ is U.E of θ^2.

A) So here x is following Bernoulli

$$X \sim Ber(\theta).$$

$$X \begin{cases} = 1 \rightarrow \text{succ} \rightarrow P(S) = \theta \quad \text{(success)} \\ = 0 \rightarrow \text{fail} \rightarrow P(F) = 1-\theta. \quad \text{(failure.)} \end{cases}$$

Let $T = \sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots x_n$ — n independent Bernoulli Variate.

$\sim Bin(n, \theta)$

$\boxed{x_i \sim Ber(\theta)}$

~~Sum of 'n'~~

* sum of 'n' independent Bernoulli variates is Binomial variate.

$E(T) = n\theta \quad (\because E(x) = \mu \text{ (mean)})$

$E(T^2) = V(T) + (E(T))^2$

$= n\theta(1-\theta) + \theta^2 n^2$

$T \sim Bin(n, p)$

$E(T) = np$

$V(T) = npq$

let $t = \dfrac{\Sigma x_i(\Sigma x_i - 1)}{n(n-1)}$

$t = \dfrac{T(T-1)}{n(n-1)}$

$E(t) = E\left[\dfrac{T(T-1)}{n(n-1)}\right]$

$= \dfrac{1}{n(n-1)} E(T^2 - T)$

$= \dfrac{1}{n(n-1)}\left[E(T^2) - E(T)\right]$

$= \dfrac{1}{n(n-1)}\left[n\theta(1-\theta) + \theta^2 n^2 - n\theta\right]$

$= \dfrac{1}{n(n-1)}\left[n\theta - n\theta^2 + \theta^2 n^2 - n\theta\right]$

$= \dfrac{1}{n(n-1)} n(n-1)\theta^2$

$\boxed{E(T) = \theta^2}$

$E\left[\dfrac{\Sigma x_i(\Sigma x_i - 1)}{n(n-1)}\right] = \theta^2$

hence Proved.

Let 'x' be distributed in the Poisson form with Parameter θ. Prove that only unbiased estimator of $\exp(-(k+1)\theta)$, $k>0$ is $T(x)=(-k)^x$ so that $T(x)>0$ if x is even, <0 if x is odd.

A) distribution is Poisson distribution

$X \sim P(\theta)$

~~P.d.f of~~ P.d.f of x is

$$= \frac{e^{-\theta}\theta^{x}}{x!}$$

hence Proved.

Let 'x' be distributed in the Poisson form with Parameter θ. Prove that only unbiased estimator of $\exp(-(k+1)\theta)$, $k>0$ is $T(x)=(-k)^x$ so that $T(x)>0$ if x is even, <0 if x is odd.

A) distribution is Poisson distribution

$X \sim P(\theta)$

~~P.d.f of~~ P.d.f of x is

$$= \frac{e^{-\theta}\theta^x}{x!}$$

So $T(x)=(-k)^x$.

$$E(T(x)) = E((-k)^x) = \exp(-(k+1)\theta)$$

$$= e^{-(k+1)\theta}$$

$$= \Sigma(-k)^x P(X=x)$$

$$= \Sigma(-k)^x \left(\frac{e^{-\theta}\cdot\theta^x}{x!}\right)$$

$$= e^{-\theta} \sum_{x=0}^{\infty} \frac{(-k\theta)^x}{x!}$$

$$\boxed{e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots}$$

$$= e^{-\theta} \cdot e^{-k\theta}$$

$$\boxed{\sum_{x=0}^{\infty} \frac{y^x}{x!} = e^y}$$

$$\boxed{E(T(x)) = e^{-(k+1)\theta}}$$

Consistent estimators

$T_n = T_n(x_1, x_2, \ldots, x_n)$.

It can be sequenced estimator.

$n=1, \quad T_1 = T_1(x_1)$

$n=2, \quad T_2 = T_2(x_1, x_2)$

$n=3, \quad T_3 = T_3(x_1, x_2, x_3)$

$\vdots$

The squence of estimators $\{T_n\}_{n=1}^{\infty}$ said to be a consistent estimator of a parameter $\gamma(\theta)$

if ① $E(T_n) \to \gamma(\theta)$ as $n \to \infty$

② $V(T_n) \to 0$

$\forall n,\ a_n = 1 \quad \{a_n\}_{n=1}^{\infty} = \{(1)^n\}_{n=1}^{\infty}$

$= \{1, 1, 1 \ldots\}$.

$n \to \infty,\ a_n \to 1$

$\forall n,\ b_n \neq 1 \quad \{b_n\} = \{1 + \frac{1}{n}\}_{n=1}^{\infty} = 2, (1+\frac{1}{2}), (1+\frac{1}{3}) \ldots\}$

$n \to \infty,\ b_n \to 1$

* if $E(T_n) = r(\theta)$ then $E(T_n) \to r(\theta)$ is correct.
* But if $E(T_n) \to r(\theta)$ then $E(T_n) = r(\theta)$ is not correct.

*

	19.5	20.5
	19	21
Student 1	20	

E(Marks.

	18	24
	19	22
student 2	20	

$x_i - \bar{x} \not> 0$.

So student 1 is more Consistent towards 20 marks.

$(x_i - \bar{x}) \to 0$ for i^{st} student deviation of 'x_i' from $\bar{x}$

$v(x) = \frac{1}{n} \Sigma (x_i - \bar{x})^2 \to 0$ also.

if variations are small is consistent.

* which of the following is a consistent estimator of population mean.

(a) s^2 (b) $\bar{x}$ (c) S^2 (d) Not.

A) $s^2 = \frac{1}{n} \Sigma (x_i - \bar{x})^2$

$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$

$\bar{x} = \frac{\Sigma x_i}{n}$

$E(\bar{x}) = \mu$ $\quad E(\bar{x}) \to \mu, n \to \infty$

$v(\bar{x}) = \sigma^2/n$ $\quad v(\bar{x}) \to 0 \; n \to \infty$

$E(s^2) = \left(\frac{n-1}{n}\right)\sigma^2$

$E(S^2) = \sigma^2$

So. $\bar{x}$ is c.e of P.oP mean.

$\bar{x}$ is u.e of " " (μ)

Invariance Property of Consistent estimator

If $\{T_n\}$ is a c.e of $r(\theta)$ & $\psi(x)$ (సమాన్య) is a continuous function of 'x' then $\{\psi(T_n)\}$ is c.e of $\psi(r(\theta))$

$T_n = \bar{x}$ is c.e of $\mu \Rightarrow r(\theta)$

$\psi(x) = \sin x \Rightarrow$ (contious function)

$$\boxed{\psi(T_n) = \psi(\bar{x}) = \sin\bar{x} \text{ is c.e of } \psi(r(\theta)) = \psi(\mu) = \sin\mu.}$$

Ex:-

* $\psi(x) = 3x^2 - 4x$

$\psi(T_n) = \psi(\bar{x}) = 3(\bar{x})^2 - 4\bar{x}$ is a c.e of

$\psi(r(\theta)) = \psi(\mu) = 3\mu^2 - 4\mu$

* $\psi(x) = \log x$ $\quad x>0, x>0.$

$= \psi(\bar{x}) = \log\bar{x}$ is a c.e of $\psi(r(\theta))$

$= \psi(\mu) = \log\mu.$

* If $x_1, x_2, \ldots x_n$ is a random sample drawn from 'x' which takes the values '1' & '0' with respec. Probabilities θ & $1-\theta$. then $\frac{\Sigma x_i}{n}\left(\frac{\Sigma x_i}{n}-1\right)$ is C.E of (a) θ (b) $1-\theta$, (c) $\theta(\theta-1)$ (d) Not

A) here it is following Bernoulli distribution.

if $Pop \sim Bin(n, \theta)$
$\mu = n\theta$

So. Population $\sim Ber(\theta)$

$n = 1$.

mean $= np$
$= (1)\theta$.
varience $= npq$
$= (1)\theta(1-\theta)$.

So $E(\bar{x}) =$ Pop Mean
$= \theta$.

$V(\bar{x}) = \frac{\sigma^2}{n} = \frac{\text{Pop. var}}{n} = \frac{\theta(1-\theta)}{n}$

$(\because T = \frac{\Sigma x_i}{n} = \bar{x}$ C.E of θ.

So here

$E(\bar{x}) = \theta$ then $E(\bar{x}) \to \theta$.

$V(\bar{x}) = \frac{\theta(1-\theta)}{n} \to 0$ } if $n \to \infty$

$\boxed{\text{So } T = \frac{\Sigma x_i}{n} = \bar{x} \text{ is C.E of } \theta}$

if $\psi(x) = x(x-1)$
$\psi(T) = T(T-1)$ is C.E of $\psi(\theta) = \theta^2$

$\psi(T) = T(T-1)$ is c.f of θ.

$\frac{\Sigma x_i}{n}\left(\frac{\Sigma x_i}{n} - 1\right)$ So. Ans is $\theta(\theta-1)$

if $3\left(\frac{\Sigma x_i}{n}\right)^2 - 4\left(\frac{\Sigma x_i}{n}\right)$ is c.f of $3\theta^2 - 4\theta$.

Efficient estimator

Let t_1, t_2 estimators on the same Parameter $\gamma(\theta)$

both t_1, t_2 are unbiased and consistent estimators of $\gamma(\theta)$

let $t_1 = \bar{x}$ ✓ it is U.E C.E of Pop. mean

Population can be Normal
Binomial.
Bernoulli
Poission

t_2 = Sample Meadian

Population is Normal $N(\mu, \sigma^2)$

$E(t_2) = \mu \Rightarrow E(t_2) \to \mu$

$V(t_2) = \frac{\pi\sigma^2}{2n} \quad V(t_2) \to 0$ $\Big\}\, n \to \infty$

so this t_2 is both U.E and C.E of Population mean.

Ex: sample mean or sample median which is more efficient? for that we have efficient estimator.

let T_1, T_2 ~~est~~ two estimators of same parameter $\tau(\theta)$, i can san T_1 is more efficient than T_2 if

$$\boxed{V(T_1) < V(T_2) \ \forall n.}$$

$$V(t_1) = V(\bar{x}) = \frac{\sigma^2}{n}.$$

$$V(t_2) = V(\text{Median}) = \frac{\pi\sigma^2}{2n} = \frac{1.57\sigma^2}{n}.$$

So sample mean is more efficient than sample Median.

$T, T_1, T_2, \ldots T_n$ ~~of~~ estimators of $\tau(\theta)$. we can say T is Most efficient estimator of $\tau(\theta)$ if

$$V(T) < V(T_i) \ \forall i = 1, 2, \ldots n.$$

Efficiency of T_i $\quad E_{T_i} = \frac{V(T)}{V(T_i)} < 1$

* $x_1, x_2, \ldots x_5$ ~~r.v.s~~ random samples of size 5. of $N(\mu, \sigma^2)$

3 estimators:

$$t_1 = \frac{x_1 + x_2 + \cdots + x_5}{5}, \quad t_2 = \frac{x_1 + x}{2} + x_3.$$

$$t_3 = \frac{2x_1 + x_2 + \lambda x_3}{3}$$

(a) for what λ, t_3 is U.E of Pop. Mean.

(a) 1 (b) 0 (c) -1 (d) NOT.

$E(t_3) = \mu$ (in question given)

$$E\left[\frac{2x_1 + x_2 + \lambda x_3}{3}\right] = \mu.$$

$$\frac{2}{3}E(x_1) + \frac{1}{3}E(x_2) + \frac{\lambda}{3}E(x_3) = \mu.$$

$$\frac{2}{3}\mu + \frac{1}{3}\mu + \frac{\lambda}{3}\mu = \mu. \quad (\because E(x_i) = \mu)$$

$$\frac{3}{3} + \frac{\lambda}{3} = 1$$

$$\cancel{1} + \frac{\lambda}{3} = \cancel{1}$$

$$\boxed{\lambda = 0}$$

(B) which of t_1, t_2 is U.E of Population mean.

A) $t_1 = \frac{x_1 + x_2 + \cdots + x_5}{5}$

$E(t_1) = E\left(\frac{x_1 + x_2 + \cdots + x_5}{5}\right)$

$= \frac{1}{5}E(x_1) + \frac{1}{5}E(x_2) + \frac{1}{5}E(x_3) + \frac{1}{5}E(x_4) + \frac{1}{5}E(x_5)$

$= \frac{1}{5}\mu + \frac{1}{5}\mu + \frac{1}{5}\mu + \frac{1}{5}\mu + \frac{1}{5}\mu$

$= \frac{5}{5}\mu$

$\boxed{E(t_1) = \mu.}$

$E(t_2) = E\left(\left(\frac{x_1 + x_2}{2}\right) + x_3\right)$

$= \frac{1}{2}E(x_1) + \frac{1}{2}E(x_2) + E(x_3)$

$= \frac{1}{2}\mu + \frac{1}{2}\mu + \mu$

$= \mu + \mu$

$\boxed{\therefore E(t_2) = 2\mu}$

$\boxed{\therefore \text{So } t_1 \text{ is U.E of } \mu}$

Q) Which of t_1, t_2, t_3 is best estimator (most efficient). $V(x_i) = \sigma^2$

A) $V(t_1) = V\left(\frac{1}{5}(x_1 + x_2 + \cdots + x_5)\right)$

$= \frac{1}{25}\left(V(x_1) + V(x_2) + \cdots + V(x_5)\right)$

$$= \frac{\sigma^2 + \sigma^2 + \cdots + \sigma^2}{25}$$

$$v(t_1) = \frac{\sigma^2}{5}$$

$$v(t_2) = v\left(\frac{x_1 + x_2}{2} + x_3\right)$$

$$= \frac{1}{4}\left(v(x_1) + v(x_2)\right) + v(x_3)$$

$$= \frac{1}{4}(\sigma^2 + \sigma^2) + \sigma^2$$

$$= \frac{2\sigma^2}{4} + \sigma^2$$

$$= \frac{\sigma^2}{2} + \sigma^2 = \frac{3\sigma^2}{2}$$

$$v(t_3) = v\left(\frac{2x_1 + x_2 + 0(x_3)}{3}\right)$$

$$= v\left(\frac{2x_1}{3}\right) + v\left(\frac{x_2}{3}\right)$$

$$= \frac{4}{9} v(x_1) + \frac{1}{9} v(x_2)$$

$$= \frac{4}{9}\sigma^2 + \frac{1}{9}\sigma^2$$

$$v(t_3) = \frac{5}{9}\sigma^2$$

So $v(t_1) < v(t_2), v(t_3)$

So. $v(t_1)$ is efficient.

efficiency

of t_2.

$$E_{t_2} = \frac{V(t_1)}{V(t_2)} = \frac{\frac{\sigma^2}{5}}{\frac{3\sigma^2}{2}} = \frac{2}{5} < 1$$

$$E_{t_3} = \frac{V(t_1)}{V(t_3)} = \frac{\sigma^2/5}{\frac{5\sigma^2}{9}} = \frac{9}{25} < 1$$

MLE [Maximum Likelihood estimator]

let $x_1, x_2, \ldots x_n$ be a random sample from a Population whose P.d.f is $f(x,\theta)$. Unknown Parameter which we are going to estimate.

if Population is Normal $N(\mu, \sigma^2)$

so $f(x,\mu)$

P.d.f is $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

if $\theta = \mu$ (Pop. mean).

if $\theta = \sigma^2$ (Pop. σ^2)

then $f(x,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

if $\theta = (\mu, \sigma^2)$ both are unknown then.

$\theta = (\theta_1, \theta_2)$

then $f(x,(\mu,\sigma^2)) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$\theta = (\theta_1, \theta_2)$ is Two dimensional Parameter
(or)
vector valued Parameter.

Likelihood function

$$\boxed{L = \prod_{i=1}^{n} f(x_i, \theta)}$$ p.d.f

this gives likelihood (Probability) that the r.v. 'x' assumes a Particular set of sample values $x_1, x_2, \ldots x_n$.

Maximum Likelihood estimator.

we have to find the estimator which maximizes the likelihood function.

if we have x_i values (sample values) then

$$\boxed{L = \prod_{i=1}^{n} f(x_i, \theta) = L(\theta).}$$

$$\boxed{\frac{\partial L}{\partial \theta} = 0, \quad \frac{\partial^2 L}{\partial \theta^2} < 0}$$

so ~~ran~~ it is better to find Maximum value for $\log L$ than L (non-decreasing func).

$$\boxed{\begin{aligned} &\frac{\partial L}{\partial \theta}(\log L) = 0 \\ &\Downarrow \quad \frac{1}{L}\frac{\partial L}{\partial \theta} = 0. \end{aligned}}$$

* for a random sample from a Normal population $N(\mu, \sigma^2)$ find the MLE for

(a) μ when σ^2 is known

(b) σ^2 " μ " "

(c) simultaneous estimation of μ & σ^2 (unknown).

A) $L = \prod_{i=1}^{n} f(x_i, \theta)$.

$$= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right)^2}$$

$L = \frac{1}{\sigma^2}$

$$L = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_1-\mu}{\sigma}\right)^2} \times \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_2-\mu}{\sigma}\right)^2} \times \cdots$$

$$\cdots \times \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_n-\mu}{\sigma}\right)^2}$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\left[\left(\frac{x_1-\mu}{\sigma}\right)^2 + \left(\frac{x_2-\mu}{\sigma}\right)^2 + \cdots + \left(\frac{x_n-\mu}{\sigma}\right)^2\right]}$$

$$\boxed{L = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2} \quad \text{--- (1)}$$

it is now difficult to find.

$\frac{\partial L}{\partial \mu}$, $\frac{\partial L}{\partial \sigma^2}$ for L.

So

$$\log L = \log\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n + \log e^{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2}$$

$$= n\log\left(\sigma(2\pi)^{\frac{1}{2}}\right)^{-1} - \frac{1}{2}\sum\left(\frac{x_i-\mu}{\sigma}\right)^2(\log e)$$

$$= -n\left(\log\sigma + \log(2\pi)^{1/2}\right) - \frac{1}{2}(\quad)\times$$

$$\log L = -n\log\sigma - \frac{n}{2}\log 2\pi - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2$$

$$\log L = -n\log\sigma - \frac{n}{2}\log 2\pi - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 \quad \text{--- (2)}$$

$$\log L = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 \quad \text{--- (3)}$$

a) MLE for $\mu = 0$ when σ^2 is known

$$\frac{\partial}{\partial \mu}(\log L) = 0.$$

We will differentiat this $\log L$ w.r.t "μ":

$$0 - 0 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^{2-1} \times (-1) = 0.$$

$$\Rightarrow \sum_{i=1}^{n}(x_i - \mu) = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i - n\mu = 0.$$

$$\Rightarrow \sum_{i=1}^{n} x_i = n\mu.$$

$$\boxed{\Rightarrow \mu = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}}$$

MLE for μ is $\bar{x}$

(ii) when μ is known, the likelihood eq for estimating σ^2 is

let $\boxed{\sigma^2 = y}$

$$\boxed{\frac{\partial}{\partial \sigma^2}(\log L) = 0.}$$

$$\frac{\partial}{\partial y}(\log L) = 0.$$

$$\log L = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log y - \frac{1}{2y}\sum_{i=1}^{n}(x_i - \mu)^2$$

$$\Rightarrow -0 - \frac{n}{2} \times \frac{1}{y} - \left(\frac{-1}{2y^2}\right) \sum_{i=1}^{n} (x_i - \mu)^2 = 0$$

$$\Rightarrow \frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^{n} (x_i - \mu)^2 = 0 \quad \text{--- (4)}$$

$$\frac{n}{\cancel{2\sigma^2}} = \frac{1}{2\sigma^4} \Sigma (x_i - \mu)^2$$

$$\boxed{\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2}$$

(iii) MLE for σ^2 is $\frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2$

$$\frac{\partial^2}{\partial \sigma^2} (\text{Log } L) < 0$$

(iii) $\frac{-n}{2} \times \frac{1}{y} + \frac{1}{2y^2} \Sigma (x_i - \mu)^2 = 0$

Now differiente again

$$-\frac{n}{2} \times \left(\frac{-1}{y^2}\right) + \frac{(-2)}{2y^3} \Sigma (x_i - \mu)^2$$

$$\frac{n}{2y^2} - \frac{1}{y^3} \sum_{i=1}^{n} (x_i - \mu)^2 < 0 \quad \text{--- (5)}$$

$$\frac{n}{2y^2} < \frac{1}{y^3} \Sigma (x_i - \mu)^2$$

$$\frac{2}{\sigma^2} (y) < \frac{2}{n} \Sigma (x_i - \mu)^2$$

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2 < \frac{2}{n}\sum_{i=1}^{n}(x_i-\mu)^2$$

so

$$\boxed{\frac{\partial^2}{\partial\sigma^2}\log L < 0}$$

* When μ is known MLE for σ^2 is $= \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$

* When σ^2 is known MLE for μ is $\bar{x}$

* $\frac{\partial^2}{\partial\mu^2}\log L < 0.$

so $\frac{\partial}{\partial\mu}\log L = \frac{1}{\sigma^2}\sum(x_i-\mu)$

$$\frac{\partial^2}{\partial\mu^2}\log L = \frac{1}{\sigma^2}\sum_{i=1}^{n}(-1)$$

$$= -\frac{n}{\sigma^2} < 0$$

(c) Simultaneous estimation of μ & σ^2

$\theta = (\mu, \sigma^2) = (\theta_1, \theta_2)$

$\frac{\partial}{\partial\mu}\log L = 0$

$\hat{\mu} = \bar{x}$

$\frac{\partial}{\partial\sigma^2}\log L = 0$

$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\hat{\mu})^2$

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \quad (\because \hat{\mu} = \bar{x}).$$

$$= s^2$$

MLE for $\hat{\mu} = \bar{x}$

MLE for $\sigma^2 = s^2$

P.T the MLE of parameter 'α' of a population having density function $f(x) = \frac{2}{\alpha^2}(\alpha - x)$, $0 < x < \alpha$ for a sample of unit size is $2x$.

A) $$L = \prod_{i=1}^{n} f(x_i, \theta) \qquad f(x_1, \theta) = f(x_1)$$

$$= \prod_{i=1}^{n} f(x_i, \alpha)$$

$$= \prod_{i=1}^{n} \frac{2}{\alpha}(\alpha - x_i)$$

here sample size is unit size

$n = 1$. "1" sample value $(x_1 = x)$

$$\Rightarrow L = \frac{2}{\alpha^2}(\alpha - x)$$

$\frac{\partial}{\partial\alpha}(\log L) = 0, \quad \frac{\partial^2}{\partial\alpha^2}(\log L) < 0.$

$\log L = \log 2 - 2\log\alpha + \log(\alpha - x)$

$\frac{\partial}{\partial\alpha}(\log L) = 0 \Rightarrow -\frac{2}{\alpha} + \frac{1}{\alpha - x} = 0$

$\Rightarrow \frac{2}{\alpha} = \frac{1}{\alpha - x}$

$\Rightarrow 2\alpha - 2x = \alpha.$

$\boxed{\alpha = 2x}$

$\boxed{\therefore \frac{\partial^2}{\partial\alpha^2}(\log L) < 0}$ log is for a non-decreasing function 2nd derivative is always -ve (<0)

* Is this MLE of α, an U.E of α?

is $E(\hat{\alpha}) = \alpha$?

$E(\hat{\alpha}) = E(2x)$

$= \int_0^\alpha 2x f(x, \alpha)$

$= \int_0^\alpha 2x \times \frac{2}{\alpha^2}(\alpha - x)\,dx$

$= \frac{4}{\alpha^2}\int_0^\alpha (\alpha x - x^2)\,dx$

$= \frac{2\alpha}{3}$.

$E(\hat{\alpha}) \neq \alpha$.

So it M.L.E of α is not an unbiased estimator of α.

Pmf of Binomial Distribution is

$$P[x = x_i] = {}^nC_{x_i} \, p^{x_i} (1-p)^{n-x_i}$$

So

Pmf of Bernoulli distribution is

$n=1$ for Binomial.

$$P[x = x_i] = {}^1C_{x_i} \, p^{x_i} (1-p)^{1-x_i}$$

$${}^1C_{x_i} = 1$$

$$\boxed{P(x=x_i) = p^{x_i} (1-p)^{1-x_i}}$$

for $x_i = 0, 1$.

Sufficient estimator

An estimator T_n is a function of sample values $T_n = T_n(x_1, x_2, \ldots, x_n)$ is a sufficient estimator of parameter $\gamma(\theta)$ if

T_n has all the information regarding $\tau(\theta)$.

means even if you forget the values of $x_1, x_2, \ldots x_n$ if you miss sample values but you have value of $\boxed{T_n = k}$ still you can estimate the unknown Parameter $\tau(\theta)$.

---

* If we don't have the knowledge of the distribution of sample values $x_1, x_2 \ldots x_n$ but we have knowledge of T_n, still we will be able to estimate $\tau(\theta)$

---

* $P[x_1 \cap x_2 \cap \ldots \cap x_n / T_n = k]$ is independent of $\tau(\theta)$.

(meaning)

* T_n is sufficient estimator of $\tau(\theta)$, if the conditional distribution/Probability of $x_1, x_2, \ldots x_n$ given T_n is independent of $\tau(\theta)$

* If $x_1, x_2, \ldots x_n$ is a random sample from a Bernoulli Population Ber.(P) then $\Sigma x_i = T$ is sufficient estimator of "p"

$\Downarrow T = \sum_{i=1}^{n} (x_i) = x_1 + x_2 + \ldots + x_n.$

* $P[x_1 \cap x_2 \cap \cdots \cap x_n / T_n = k]$ is independent of 'p'

$$= \frac{P[x_1 \cap x_2 \cdots \cap x_n \cap T_n = k]}{P[T_n = k]}$$

$$= \frac{p^k (1-p)^{n-k}}{{}^nC_k \, p^k (1-p)^{n-k}}$$

$T_n = \sum_{i=1}^{n} x_i$

$= x_1 + x_2 + \cdots + x_n$

Sum of 'n' indepent Bernolli variates is Binomial variate

$T_n \sim Bin(n, p)$

$x_1 \cap x_2 \cap \cdots \cap x_n \cap T_n = k$

exactly 'k' of x_n's = 1

&

exactly $(n-k)$ of x_n's = 0

$K = T_n = \sum x_i = x_1 + \cdots + x_n$

$1 + 1 + 1 + \cdots 1$ (k times) $+ 0 + 0 + \cdots + 0$ $(n-k)$ times

So $P[\cancel{x_1 = 1}]^k$

$P[x_1 \cap x_2 \cap \cdots \cap x_n \cap T_n] = k$.

~~So P[x~~ $= P \times P \times P \cdots \times P$ (k times) $\times (1-P) \times (1-P) \times \cdots (n-k)$ times

Probability of success $\therefore P = P[x_i = 1]$

" " failure $\therefore 1-P = P[x_i = 0]$

$= P^k (1-P)^{n-k}$.

So.

$$= \frac{\cancel{P^k (1-P)^{n-k}}}{{}^nC_k \cancel{P^k (1-P)^{n-k}}}$$

$$= \frac{1}{{}^nC_k}$$

* if x_1, x_2, x_3 3 sample values and.

$2 = T_n = x_1 + x_2 + x_3$

Ans)

$2 = 1 + 0 + 1$

$P(x_1 \cap x_2 \cap x_3 \cap T_2 = 2)$

$= P[x_1 = 1] P[x_2 = 0] P[x_3 = 1]$

$= P \times P \times (1-P)$

* $P[x_1 \cap x_2 \cap \cdots \cap x_n / T_n = k] = \frac{1}{{}^nC_k}$.

it is independent of P.

So $T_n = \Sigma x_i$ is sufficient estimator of P.

$$P[x_1 \cap x_2 \cap \cdots \cap x_n / T_n = k] = \begin{cases} \frac{1}{{}^nC_k} & \text{if } \Sigma x_i = k \\ 0 & \text{if } \Sigma x_i \neq k \end{cases}$$

$T_n = \Sigma x_i \neq k$ but exactly k of x_n's $= 1$

& " " $n-k$ " " $= 0$

is not possible. So

function of Sample values.

P.m.f of Ber(P) distribution.

A) $^nC_x P^x (1-P)^{n-x}$

if $n=1$.

$\Rightarrow \underbrace{^1C_x}_{=1} P^x (1-P)^{1-x}$

$= P^x (1-P)^{1-x}$.

P.m.f

Bernoulli ~~Parameter~~ Parameter.

$L = \prod f(x_i, \theta) = \prod f(x_i, P)$

$= \prod_{i=1}^{n} P^{x_i} (1-P)^{1-x_i}$.

$L = \{P^{x_1}(1-P)^{1-x_1}\}\{P^{x_2}(1-P)^{1-x_2}\} \cdots \{P^{x_n}(1-P)^{1-x_n}\}$

$= P^{x_1+x_2+\cdots+x_n} (1-P)^{1-x_1+1-x_2+\cdots 1-x_n}$.

$$\boxed{L = P^{\Sigma x_i} (1-P)^{n-\Sigma x_i} \times} \quad \text{--- (1)}$$

$= g_\theta(t(x)) \times h(x)$.

$= g_P(\Sigma x_i) \times h(x)$ --- (2)

Compare (1) & (2)

$g_\theta(t(x))$
$= g_p(\Sigma x_i)$

$= p^{\Sigma x_i}(1-p)^{n-\Sigma x_i}$

$\boxed{h(x) = 1}$

So. $t(x) = \Sigma x_i$ is sufficient estimator of $\theta = p$

* $x_1, x_2, \ldots x_n$ are random sample ~~with~~ from population with density function $= \theta x^{\theta-1}$, $0 < x < 1$, $\theta > 0$.
what is sufficient estimator of θ?
(a) Σx_i (b) $\Sigma x_i - 1$ (c) $\Pi x_i - 1$ (d) Πx_i

A) $L = \Pi f(x_i, \theta)$

$= \prod_{i=1}^{n} \theta x_i^{\theta-1}$

$= (\theta x_1^{\theta-1})(\theta x_2^{\theta-1}) \ldots (\theta x_n^{\theta-1})$

$= \theta^n (x_1^{\theta-1} x_2^{\theta-1} \ldots x_n^{\theta-1})$

$= \theta^n (x_1^{\theta} x_2^{\theta} \ldots x_n^{\theta})(x_1^{-1} x_2^{-1} \ldots x_n^{-1})$

$$= \theta^n \prod_{i=1}^{n} x_i^{\theta} \left(\prod_{i=1}^{n} x_i^{-1} \right)$$

$$L = \theta^n \prod_{i=1}^{n} x_i^{\theta} \left(\frac{1}{\prod_{i=1}^{n} x_i} \right) \quad (1)$$

compare

$$= g_\theta(t(x)) \times h(x).$$

if $= g_\theta\left(\prod_{i=1}^{n} x_i\right) \times \frac{1}{\prod x_i}$

$$g_\theta(t(x)) = \theta^n \prod_{i=1}^{n} x_i^{\theta}.$$

$$\boxed{\text{So } t(x) = \prod_{i=1}^{n} x_i \text{ s.f of } \theta = P}$$

anotherway

from Eq (1)

$$L = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1} \times 1$$

$$L = g_\theta\left(\prod_{i=1}^{n} x_i\right) \times h(x).$$

$$\boxed{t(x) = \prod_{i=1}^{n} x_i} \text{ s.f of } \theta = P.$$

* $x_1, x_2, \ldots x_n$ random sample from Normal Population $N(\mu, \sigma^2)$ S.F of μ & σ^2.

A) $\theta = (\mu, \sigma^2)$ θ is Parameter

$= (\theta_1, \theta_2)$ } two dimentional

$t(x) = (t_1(x), t_2(x))$

$$L = \prod f(x_i \theta) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right)^2}$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2}$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\Sigma(x_i^2 + \mu^2 - 2\mu x_i)}$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}(\Sigma x_i^2 + n\mu^2 - 2\mu\Sigma x_i)}$$

$$= g(\mu, \sigma^2)(t_1(x), t_2(x)) \times h(x).$$

$$= g_\theta(t(x) \times h(x)$$

$t_1(x) = \Sigma x_i$ S.F of μ

$t_2(x) = \Sigma x_i^2$ S.F of σ^2.

$h(x) = 1$

here by change Σx_i can be negative
$1-3+1=-1$, and $\Sigma x_i^2 > 0$ $1^2+(-3)^2+1^2$
So here $\sigma^2 > 0$ so positive estimator
Σx_i^2 can estimate the σ^2

* $x_1, x_2, \ldots x_n$ is an random sample from $x_1, x_2, \ldots x_N$

$E(x_i) = \mu$ $\quad$ $E(s^2) = \left(\frac{n-1}{n}\right)\sigma^2$

$V(x_i) = \sigma^2$ $\quad$ $E(s^2) = \sigma^2$

$E(\bar{x}) = \mu$

$V(\bar{x}) = \frac{\sigma^2}{n}$

$\bar{x}$ is U.E of μ.

$\bar{x}$ is C.E of μ.

If T is C.E of $\tau(\theta)$, $\psi(x)$ is continuous of x then $\psi(T)$ is continuous function of $\psi(\tau(\theta))$