ASSIGNMENT 4

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a) $P = A^T A$ ($n \times n$ matrix as A is of size $m \times n$)

$Q = A A^T$ ($m \times m$ matrix)

Let $Ay = x$ ($n \times 1$ vector)

$y^T P y = y^T A^T A y$ (where y is $n \times 1$ vector)

$\Rightarrow y^T P y = \cancel{(Ay)^T (Ay)} = (Ay)^T Ay = x^T x$

and $x^T x = \|x\|^2 \geq 0$

$\Rightarrow y^T P y = x^T x \geq 0 \quad \forall y$

Similarly for any real valued vector z of size $m \times 1$,

$z^T Q z = z^T A A^T z = (A^T z)^T (A^T z) = w^T w$ (where $w = A^T z$ is a $m \times 1$ vector)

$\Rightarrow z^T Q z = w^T w = \|w\|^2 \geq 0$

$\Rightarrow z^T Q z \geq 0 \quad \forall z$

Eigenvalues of P and Q are non negative.

Proof: Let P have a non-zero eigen vector v with eigenvalue λ

$$P v = \lambda v$$

$$v^T (P v) = v^T (\lambda v) = \lambda (v^T v) = \lambda \|v\|^2$$

since $\|v\| \neq 0$ as v is non-zero

$$\Rightarrow \lambda = \frac{v^T P v}{\|v\|^2} \geq 0$$

as $v^T P v \geq 0$ (proven earlier) and so is $\|v\|^2$

Hence any eigenvalue of P is ≥ 0.

similarly for Q (eigenvalue is γ, eigenvector is u)

$$Q u = \gamma u$$

$$u^T Q u = u^T \gamma u = \gamma (u^T u) = \gamma \|u\|^2$$

$$\gamma = \frac{u^T Q u}{\|u\|^2} \geq 0$$

Since $u^T Q u \geq 0$ (proven earlier) and $\|u\|^2 \geq 0$ (by definition). Hence γ is non-negative and all eigenvalues of Q are non-negative.

b) A is a $m \times n$ real valued matrix

$\Rightarrow A^T A$ is of size $n \times n$.

Let P's eigenvector be u with eigenvalue λ.

u has size $n \times 1$.

$$P u = \lambda u$$

$$\Rightarrow A P u = A(\lambda u) = \lambda A u$$

$$P = A^T A \Rightarrow A A^T A u = \lambda A u$$

$$(A A^T) A u = \lambda A u$$

$$Q (Au) = \lambda (Au) \quad (\text{since } Q = AA^T)$$

$$Q v' = \lambda v' \quad (\text{where } v' = Au \text{ is } m \times 1 \text{ vector})$$

$\Rightarrow v'$ or Au is eigenvector of Q with eigenvalue λ.

Similarly, for Q, let it's eigenvector be ~~...~~ v ~~...~~ with eigenvalue y.

$\Rightarrow v$ is of size $m \times 1$

$$Q v = y v$$

$$A^T Q v = A^T y v = y (A^T v)$$

$$\Rightarrow A^T A A^T v = (A^T A) A^T v = y (A^T v)$$

(since $Q = A A^T$)

Since $P = A^T A$ and let $A^T v = w$ ($n \times 1$)

$$\Rightarrow P(A^T v) = P w = y(A^T v) = y w$$

Hence $A^T v$ or w is an eigenvector of P with eigenvalue y

Number of elements of u are n and of v are m.

c) v_i is an eigenvector of Q (with eigenvalue α_i) ①

$$Q v_i = A A^T v_i = \alpha_i v_i$$

$$u_i = \frac{A^T v_i}{\|A^T v_i\|_2} \quad \text{(given)}$$

$$\Rightarrow A u_i = \frac{A A^T v_i}{\|A^T v_i\|_2} \quad (\text{since } Q = A A^T)$$

$$\Rightarrow A u_i = \frac{Q v_i}{\|A^T v_i\|_2} = \frac{\alpha_i v_i}{\|A^T v_i\|_2} = \left(\frac{\alpha_i}{\|A^T v_i\|_2}\right) v_i$$

$$\gamma_i = \frac{\alpha_i}{\|A^T v_i\|_2} \Rightarrow A u_i = \gamma_i v_i$$

Since α_i is non negative since any eigenvalue of Q is non negative (proven earlier) and $\|A^T v_i\|_2$ is positive ~~non negative~~ by definition. (not even zero else $Q v_i = A A^T v_i = 0$ if $\|A^T v_i\|_2 = 0$)

Hence γ_i is non-negative

There exists a non negative γ_i, s.t,

$$A u_i = \gamma_i v_i.$$

d) $$u_i^T u_j = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$$

as

$$u_i^T u_i = \frac{(A^T v_i)^T (A^T v_i)}{\|A^T v_i\|^2} = 1$$

Hence matrices $V = [u_1 | u_2 | \dots | u_n]$ and similarly $U = [v_1 | v_2 | v_3 \dots | v_m]$ are orthonormal as $V V^T = V^T V = I_n$ and $U U^T = U^T U = I_m$

Now, $U^TAV = U^TA[u_1|u_2|\ldots|u_n]$

from previous part we have $Au_i = \gamma_i v_i$

$\Rightarrow U^TAV = U^T[\gamma_1 v_1|\gamma_2 v_2|\ldots|\gamma_n v_n]$

$$= \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_m^T \end{bmatrix} [\gamma_1 v_1|\gamma_2 v_2|\ldots|\gamma_n v_n]$$

$= \Gamma$ ($m \times n$ matrix)

$$\Gamma_{ij} = v_i^T \gamma_j v_j = \begin{cases} 0 & \text{if } i \neq j \\ \gamma_j & \text{if } i = j \end{cases}$$

Hence $U^TAV = \Gamma$ is a diagonal matrix with ith diagonal element $= \gamma_i$

$\Rightarrow (UU^T)A(VV^T) = U\Gamma V^T$

$\Rightarrow A = U\Gamma V^T$ ($UU^T = I_m$ and $VV^T = I_n$ since U and V are orthonormal)

Hence ~~Proved~~, $A = U\Gamma V^T$ where Γ is a diagonal matrix containing the non-negative values $\gamma_1, \gamma_2, \ldots, \gamma_m$.