The Josephus Problem

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Abstract

We deal with the approach to solving the Josephus problem using recursion, give a general formula with proof for a special case($k = 2$) of the problem. We will see an efficient way to compute the $J(n, 2)$ using binary representations and build an algorithm to solve the problem for the general case.

1 Brief History of The Problem

This problem is based on an account by the Historian Flavius Josephus, who was part of a band of 41 Jewish rebels trapped in a cave by the Romans during the Jewish- Roman war of the first century. The rebels preferred suicide to capture; they decided to form a circle and to repeatedly count off around the circle, killing every third rebel left alive. However, Josephus and another rebel did not want to be killed this way; they determined the positions where they should stand to be the last two rebels remaining alive. Source:[Wik22]

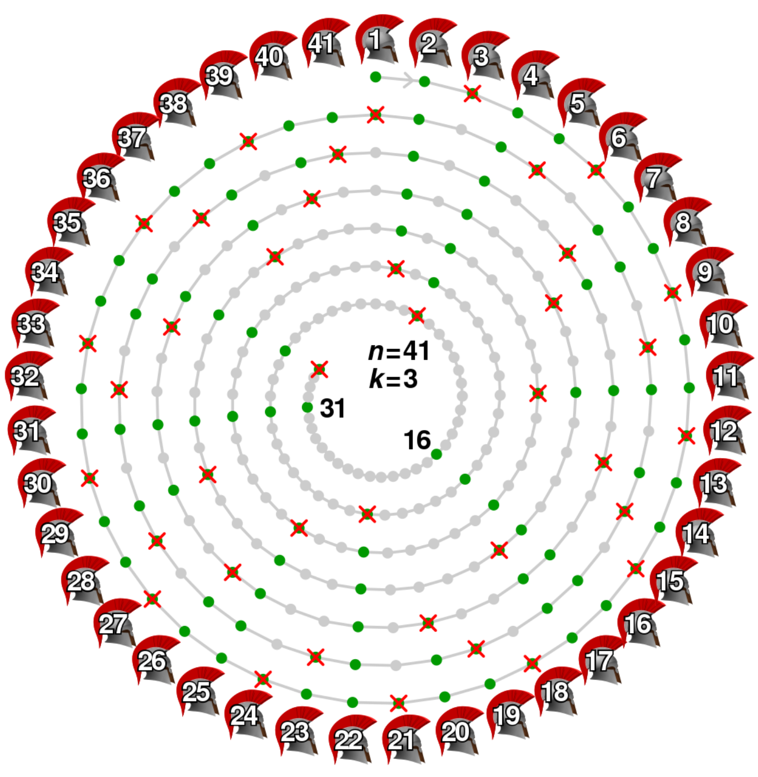

Figure 1: The Josephus Problem with $n = 41$ and $k = 3$

2 Special Cases of the Problem

2.1 k = 2

The variation we consider begins with n people, numbered 1 to n, standing around a circle. In each stage, every second person still left alive is eliminated until only one survives.We denote the number of the survivor by $J(n)$ (The problem describing the variation has been described under the exercises in Chapter 8.1 [Ros88]

2.1.1 How do we approach the problem?

When we are dealing with puzzles, The best problem solving strategy is to work out a few special cases and get a "Feel" of what the problem is asking. So let us solve the problem for a few values of n and tabulate the results

n	$J(n)$
1	1
2	1
3	3
4	1
5	3
6	5
7	7
8	1
9	3
10	5
11	7
12	9
13	11
14	13
15	15
16	1

2.1.2 A pattern!

Voila! We have a pattern! The pattern of 1,3,5,7.. and all the odd numbers appearing between consecutive powers of 2. Experimenting was the right way to approach the problem! More formally, the observations yield the following equation $\forall k \leq 2^m - 1$

$$J(2^m + k) = 2k + 1$$

2.1.3 The Main Idea

Since the function $J(n)$ seems to be repeating itself again and again, it is possible that problems with a larger number of people may be linked to a smaller sub problem! Let us investigate this for $n = 8$. Refer to Figure 2 for the diagram. Let p_i denote the warriors. Consider the first round of deaths, this will include the deaths of p_2, p_4, p_6 and p_8. They have been marked in red in the inner circle. Let us call this state of the puzzle the "Halting State"- where each one of the people has been visited and skipped once or has been killed.

Here is my claim: I claim that the halting state (after the removal of the dead bodies :))will look exactly similar to the Josephus problem of smaller size (size $n/2$ to be precise).More formally there is going to be a bijection between the arrangements of remaining people in the halting state and the standard 4 sized Josephus Problem.

Notation: Consider the Josephus problem amongst $2n$ people $p_1, p_2 \ldots p_n$. Consider the halting state where each one of $p_1, p_2 \ldots p_{2n}$ has either been skipped once or has been killed. Let the set of people remaining people be denoted by P_{left}.

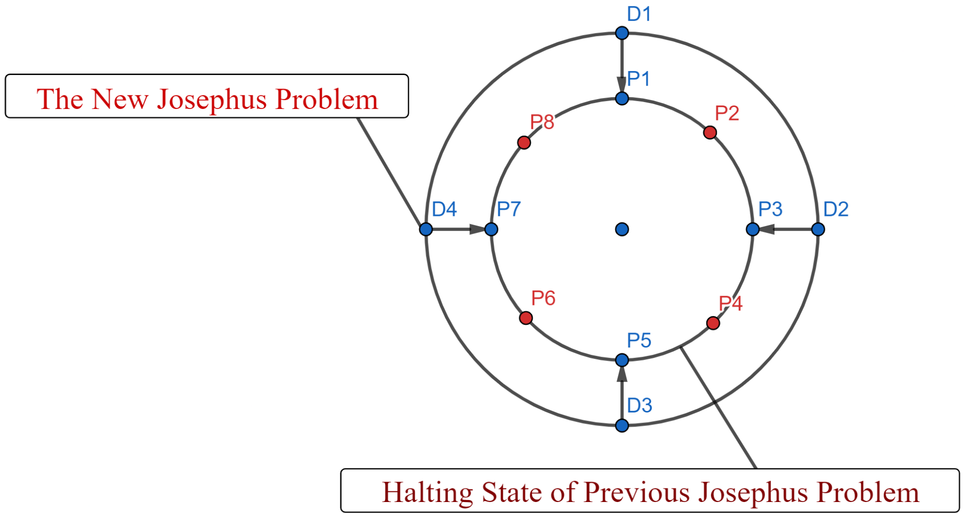

Figure 2: The Recurrence for $n = 8$

Observation: Consider another instance of the Josephus problem with n members $d_1, d_2 \dots d_n \equiv D$. Then there is a bijective mapping from the set P_{left} to D, of the following form:

$$\forall\ i|\ 1 \leq i \leq n \quad d_i \iff p_{2i-1}$$

Similarly for odd number of p's namely $p_1, p_2 \dots p_{2n+1}$ the following bijection holds:

$$\forall\ i|\ 1 \leq i \leq n \quad d_i \iff p_{2i+1}$$

Corollary 2.0.1. *Let $J(n)$ denote the index (i) of the only survivor amongst the n people $p_1, p_2 \dots p_i \dots p_n$ during the process. Then*

$$J(2n) = 2J(n) - 1$$

$$J(2n+1) = 2J(n) + 1$$

Proof. Consider the bijection from D to P_{left} for persons $P_1, P_2, \dots P_{2n}$ then the bijection is

$$\forall i = 1 \text{ to } n$$
$$d_i \equiv P_{2i-1}$$

So the answer for the josephus problem of the n elements $d_1, d_2 \dots d_n$ is

$$d_{J(n)} = P_{2J(n)-1}$$

A similar argument can be made for odd integers. □

Theorem 2.1. *For n persons where $n = 2^m + k$ and $k < 2^m$, the value of $J(n)$ is given by the following equation:*

$$J(2^m + k) = 2k + 1$$

Proof. We prove via strong Induction

$$\text{Let } P(x) = \forall k \mid \ 0 \leq k < 2^x;\ J(2^x + k) = 2k + 1$$

Induction Hypothesis:

$$H(m-1) = P(1) \wedge P(2) \wedge \dots P(m-1)$$

we need to show $H(1)$ and $H(m-1) \implies H(m)$

Base case

$$H(1) \equiv P(0) \wedge P(1)$$

$$J(k) \text{ for } k = 0, 1 \text{ must be verified for x} = 1$$

$$J(2+0) = 2(0) + 1 = 1$$

$$J(2+1) = 2(1) + 1 = 3$$

Induction Step:

Let us use the corollary

Case 1: $k = 2\lambda$

$$J(2^m + k) = 2J(2^{m-1} + \lambda) - 1 \dots\dots (1)$$

$$k < 2^m \implies \lambda < 2^{m-1}$$

$$\text{By induction hypothesis } J(2^{m-1} + \lambda) = 2\lambda + 1$$

$$\text{Re substituting in (1) we get } J(2^m + k) = 2(2\lambda + 1) - 1 = 2k + 1$$

Case 2: $k = 2\lambda + 1$

$$J(2^m + 2\lambda + 1) = 2J(2^{m-1} + \lambda) + 1 \dots\dots (2)$$

$$k < 2^m \implies \lambda < 2^{m-1}$$

$$\text{By induction hypothesis } J(2^{m-1} + \lambda) = 2\lambda + 1$$

$$\text{Re substituting in (2) we get } J(2^m + k) = 2(2\lambda + 1) + 1 = 2k + 1$$

This completes the Induction Step □

2.2 A Neat Trick

For this special case, there is a very simple trick to obtain the solution $J(n)$ quickly. We write the number n in its binary format, and then we perform a circular left shift operation, the number represented by this new string...will be our value of $J(n)$! See : [STB+15]

Theorem 2.2. *Let n be any natural number, and let the binary representation of n be denoted by $\bar{x} = x_{m+1}x_m x_{m-1}..x_1$ such that $x_{m+1} = 1$.*
Let

$$n = 2^m + k \mid 0 \leq k \leq 2^m - 1$$

Then the number represented by the left rotation of n is $2k + 1$

$$left(\bar{x}) = x_m x_{m-1} \dots x_1 x_{m+1} \iff 2k + 1$$

Proof. First note that the binary string for n can be broken into 2 parts by addition,

$$\bar{x} = 1000\dots0 + x_m x_{m-1} x_{m-2} \dots x_1 \Longleftrightarrow 2^m + k$$

Left shifting $\bar{x}$ is equivalent to multiplying the $'k'$ part on the right by two and concatenating the one at the end, i.e.

$$left(\bar{x}) = \text{left shift}(x_m x_{m-1} \dots x_1) + 00\dots1$$

$$= 2\ \times (x_m x_{m-1} \dots x_1) + 000\dots1 = 2k+1$$

$\square$

Example:

$$41 = 101001$$

$$left(\bar{41}) = 010011 = 19 = J(41)!$$

3 The General Case

Let $J(n,k)$ describe the index of the survivor when every k'th member is killed in a circle of n people. Finding exact values for the Josephus problem for higher values of k tends to be slightly difficult. No exact answers have been found, however there have been good estimates for the $k = 3$ case.

3.1 K = 3

Lorenz Halbeisen gave the following result:-
Let α be a constant that can be determined to arbitrary precision

$$\alpha \approx 0.8111$$

Let m be an integer such that the value of $\alpha \times (\frac{3}{2})^m$ is smaller than n but also closest to n, i.e.

$$round(\alpha \times (\tfrac{3}{2})^m) \leq n \text{ ......(1)}$$

Then the result $J(n,3)$ is given by

$$J(n,3) = 3(n - round(\alpha \times (\tfrac{3}{2})^m)) + 2$$

Note: The value of m for which this happens can be easily determined, since we can take the logarithm of the inequality (1) (log to the base 3/2 is monotonic increasing) we get,

$$m = round(\log_{3/2}(n/\alpha)) \text{ } or \text{ } round(\log_{3/2}(n/\alpha)) - 1$$

For more information see [EHH97]

3.2 A Generalized Recursion

Our main goal in this section is to extend the recursion idea we built in the previous sections to the more general case of the k'th person being killed. We will try to observe the state of the Josephus problem after a single killing at the k'th position.
For the sake of Convenience let us assume that the indices of the persons are taken from 0 to $n-1$ instead of 1 to n.

Let us denote the index of survivor when there are n people $p_0, p_1 \dots p_{n-1}$ and every k'th person is killed by $J(n,k)$.

Theorem 3.1. *If J(n,k) is the index of the survivor when there are n people $p_0, p_1 \dots p_{n-1}$ and every k'th person is killed,then*

$$J(n,k) = (J(n-1,k) + k) mod\ n \tag{1}$$
$$J(1,k) = 0 \tag{2}$$

Proof. Consider the state when the first(k'th person p_{k-1}) is killed. The number of remaining people is clearly $n-1$. We will attempt to find a Bijection between the indices of the remaining people P_{left} and the indices of a new Josephus problem of size $n-1$.

Idea: The remaining problem after the first death has p_k as the new zeroth element and each of the remaining $n-1$ elements $p_k, p_{k+1} \dots .. p_n, p_1 \dots p_{k-2}$ form the elements of a new Josephus Problem with persons $D_0, D_1 \dots D_{n-1}$

The index of the Survivor in the new problem is $J(n-1,k)$,But since this order is shifted by k, this index actually represents the original element who is k further in the circular order.

$$p_{(x+k) mod\ n} \longleftrightarrow D_x$$

$$\therefore D_{J(n-1,k)} \longleftrightarrow p_{(J(n-1,k)+k) mod\ n} = p_{J(n,k)}$$

□

For a similar version dealing with indices from 1 to n see [EHH97]

3.3 An Algorithm

The approach to create an algorithm to solve this problem is exactly the same, we will try to use the recursion we built in the previous section! The algorithm is not only just recursive in nature, but bigger sub-problems have a direct dependency on the smaller sub - problems! This means it would be efficient to compute the smaller sub-problems first(This technique is called Dynamic Programming) i.e. if $dp[i]$ represents $J(i,k)$ then we get the following form for an input k

$$dp[i] = (dp[i-1] + k) mod\ n \tag{3}$$
$$dp[1] = 0 \tag{4}$$

We will iterate a simple for loop which will start calculating $dp[2]$, $dp[3]$ successively. We will calculate $dp[i+1]$ using the already calculated value $dp[i]$ using the recursion (3).
Running Time :-
Since calculation of each $dp[i]$ takes only $O(1)$ time, it is not surprising that the total time taken by the algorithm is $O(n)$ Creating the dp array itself will take $O(n)$ time as well.

References

[EHH97] Lorenz Halbeisen Eth, Lorenz Halbeisen, and Norbert Hungerbuhler. The josephus problem. In *J. Thor. Nombres Bordeaux*. Citeseer, 1997.

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[Wik22] Wikipedia contributors. Josephus problem — Wikipedia, the free encyclopedia, 2022. [Online; accessed 15-March-2022].