$\boxed{Z11}$

11.06.2015 zad. 1

residuum to a_{-1} w rozwinięciu w szeregu Laurenta

a) $f(z) = \frac{1+\cos z}{z-\pi}$

$z_0 = \pi$ - punkt pozornie osobliwy

$$\cos z = \sum_0^\infty (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots$$

$\cos(z-\pi) = -\cos z$

zatem

$$f(z) = \frac{1+\cos z}{z-\pi} = \frac{1-\cos(z-\pi)}{z-\pi} = \frac{1-\left(1-\frac{(z-\pi)^2}{2!}+\frac{(z-\pi)^4}{4!}-\ldots\right)}{z-\pi} =$$

$$= \frac{(z-\pi)}{2!} - \frac{(z-\pi)^3}{4!} + \ldots$$

$$\operatorname{res}_\pi f(z) = 0$$

albo można:

$\frac{\text{rząd } 1}{\text{rząd } 1} \to$ punkt pozornie osobliwy $\Rightarrow \operatorname{res}_{z_0} f(z) = 0$

b) $f(z) = \frac{e^z}{z^2+1} = \frac{e^z}{(z+i)(z-i)}$

$z = i \to$ biegun jednokrotny

$z = -i \to$ biegun jednokrotny

$$\operatorname{res}_{-i} f(z) = \lim_{z\to -i} (z+i)\frac{e^z}{(z+i)(z-i)} = \frac{e^{-i}}{-2i}$$

$$\operatorname{res}_{i} f(z) = \lim_{z\to -i} (z-i)\cdot\frac{e^z}{(z+i)(z-i)} = \frac{e^{i}}{2i}$$

c) $f(z) = \dfrac{z}{\sin^2 z} \qquad z=0$

$$\operatorname{res}_0 f(z) = \lim_{z\to 0} z \cdot \frac{z}{\sin^2 z} = \lim_{z\to 0} \frac{z^2}{\sin^2 z} = \lim_{z\to 0} \frac{1}{\underbrace{\left(\frac{\sin z}{z}\right)^2}_{\to 1}} = 1$$

d) $f(z) = z^3 \cos\dfrac{1}{z} = z^3 \cdot \displaystyle\sum_{n=0}^{\infty} (-1)^n \cdot \frac{\left(\frac{1}{z}\right)^{2n}}{(2n)!} =$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! \cdot z^{2n-3}}$$

$$\operatorname{res}_0\left(z^3 \cos\frac{1}{z}\right) = a_{-1}$$

$2n-3 = 1$

$2n = 4$

$n = 2$

$$a_{-1} = \frac{(-1)^2}{4!} = \frac{1}{24} = \operatorname{res}_0 f(z)$$

e) $f(z) = \dfrac{\sin z}{z^2} = \dfrac{1}{z^2} \cdot \displaystyle\sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n-1}}{(2n+1)!}$

$\Rightarrow \cancel{2n-1=-1} \quad \cancel{2n=0} \quad \cancel{n=1}$

$2n-1 = -1$

$n = 0$

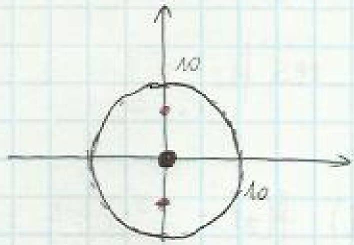
$$a_{-1} = (-1)^0 \cdot \frac{1}{1} = 1$$

można też z granicy

$$\lim_{z\to 0} z \cdot \frac{\sin z}{z^2} = 1$$

zad. 2.

a) $$\oint_K \frac{e^z+1}{e^z-1}\,dz = (*) \qquad K(0,10)$$



ze wzoru na sumę residuów

$$(*) = 2\pi i \sum_{k=1}^{n} \operatorname{res}_{z_k} f(z)$$

sprawdzamy zerowanie mianownika

$$e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\sin y) = 1$$

punkty os.

$$e^x = 1 \quad \wedge \quad \cos y + i\sin y = 1$$

$$x = 0 \quad \wedge \quad \cos y = 1 \quad \wedge \sin y = 0$$

$$y = 2l\pi, \quad l \in \mathbb{Z}$$

$$z = 0 + i2l\pi = i2\pi l$$

jak liczyć

$$\frac{P(z)}{Q(z)} \qquad P(z_0) \neq 0 \quad Q(z_0) = 0 \qquad \operatorname{res}_{z_0} \frac{P(z)}{Q(z)} = \frac{P(z_0)}{Q'(z_0)}$$

$z_0 = 0$

$z_1 = -2\pi i$

$z_2 = 2\pi i$

$$\sum_{k=0}^{2} \operatorname{res}_{z_k} f(z_k) =$$

$$\operatorname{res}_0 \frac{e^z+1}{e^z-1} = \frac{e^z+1}{(e^z-1)'}\Big|_{z=0} =$$

$$= \frac{e^z+1}{e^z}\Big|_{z=0} = 2$$

$$\operatorname{res}_{2\pi i} \frac{e^z+1}{e^z-1} = \frac{e^z+1}{e^z}\Big|_{z=2\pi i} = 2$$

$$\operatorname{res}_{-2\pi i} \frac{e^z+1}{e^z-1} = \frac{e^z+1}{e^z}\Big|_{z=-2\pi i} = 2$$

$$= 2\pi i(2+2+2) = 12\pi i$$

b) $\oint\limits_K z^2 e^{\frac{1}{z-i}}\, dz = (**)$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad e^{\frac{1}{z-i}} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z-i}\right)^n}{n!} =$$

$$= \sum_{n=0}^{\infty} \frac{1}{(z-i)^n \cdot n!}$$

$$f(z) = z^2 \cdot \sum_{n=0}^{\infty} \frac{1}{(z-i)^n \cdot n!} = (*)$$

$$z^2 = \left((z-i)+i\right)^2 = (z-i)^2 + 2i(z-i) - 1$$

$$(*) = \left[(z-i)^2 + 2i(z-i) - 1\right] \cdot \sum_{n=0}^{\infty} \frac{1}{(z-i)^n \cdot n!} =$$

$$= \sum_{n=0}^{\infty} \frac{1}{(z-i)^{n-2} \cdot n!} + 2i \sum_{n=0}^{\infty} \frac{1}{(z-1)^{n-1} \cdot n!} - \sum_{n=0}^{\infty} \frac{1}{(z-i)^n \cdot n!}$$

$$a_{-1} = \frac{1}{3!} + 2i \cdot \frac{1}{2!} - \frac{1}{1!} = \frac{1}{6} - 1 + i = -\frac{5}{6} + i$$

$$\| \qquad \operatorname{res}_i \left(z^2 e^{\frac{1}{z-i}}\right)$$

$$(**) = 2\pi i \cdot \left(-\frac{5}{6} + i\right) = -\frac{5}{3}\pi i - 2\pi = -2\pi - \frac{5}{3}\pi i$$

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residuum – albo z rozwinięcia a_{-1}
– albo jeśli można to z lh
– albo jeśli krotność >1 to inny wzór (?)
– albo jeśli $\frac{P(z)}{Q(z)}$ to ... (patrz strona obok)