Z1 26.02.2015

zad. 1.

a) $y' = y - y^2 \qquad y(0) = 0,5$

$$\frac{dy}{dx} = y - y^2$$

$$\int \frac{dy}{y - y^2} = \int dx$$

$$\int \frac{dy}{y} + \int \frac{dy}{1-y} = \int dx$$

$$\ln|y| - \ln|1-y| = x + C$$

$$\ln\left|\frac{y}{-y+1}\right| = x + C$$

$$\ln \frac{0,5}{0,5} = 0 + C \Rightarrow C = 0$$

zatem $\ln\left|\frac{y}{1-y}\right| = x + 0$

$$\left|\frac{y}{1-y}\right| = e^x$$

$$y = e^x - y \cdot e^x$$

$y = \frac{e^x}{1+e^x}$ → rozwiązanie szczególne spełniające warunek początkowy $y(0) = 0,5$).

$y = \frac{e^{x+c}}{1+e^{x+c}}$ → rozwiązanie ogólne

b) $x\cdot y' = \operatorname{tg} y$, $y(\tfrac{1}{2}) = \tfrac{5}{6}\pi$

$$x\cdot\frac{dy}{dx} = \operatorname{tg} y$$

$$\int\frac{dy}{\operatorname{tg} y} = \int\frac{dx}{x} \equiv \int\frac{\cos y}{\sin y}\,dy = \int\frac{dx}{x}$$

$$C_1 + \ln|\sin y| = \ln|x| + C_2$$

$$\ln|\sin y| = \ln|x| + C \equiv \ln|\sin y| = \ln|xC|$$

zatem $\sin y = x\cdot C$ ~~rozwiązanie ogólne~~

~~[illegible]~~

~~$\arcsin(x\cdot C) = y$~~

~~$\frac{5}{6}\pi = \arcsin(\frac{1}{2}C) \Rightarrow C=1$~~

~~$\sin y = x$~~

~~$\arcsin x = y$ - rozwiązanie szczególne~~

CO: $\pi - \arcsin x\cdot C = y$

$$\sin\frac{5}{6}\pi = \frac{1}{2}C \Rightarrow C=1$$

CS: $y = \pi - \arcsin x$

c) $y' = \frac{y-x}{x}$ $\qquad y(1) = -2$

$\cancel{\frac{dy}{dx} = \frac{y-x}{x}}$

~~[illegible]~~

$$y' = \frac{y}{x} - 1$$

$$u(x) = \frac{y(x)}{x}$$

czyli

$$u'(x)\cdot x + u(x) = u(x) - 1$$

$$u'(x) = -\frac{1}{x}$$

$$\frac{du}{dx} = -\frac{1}{x}$$

$$\int du = -\int \frac{dx}{x}$$

$$u = -\ln|x| + C$$

$$u = \ln\left|\frac{C}{x}\right|$$

$$\frac{y}{x} = \ln\left|\frac{C}{x}\right|$$

CO: $y = x \cdot \ln\left|\frac{C}{x}\right|$

$$-2 = \ln C \Rightarrow C = e^{-2}$$

$$y = x \cdot \ln\left|\frac{e^{-2}}{x}\right|$$

$$y = x \cdot (\ln e^{-2} - \ln x)$$

CS: $y = -2x - x\ln x$

zad. 2.

a) $y' - \frac{2x}{1+x^2} \cdot y = 1+x^2$

nie może być metoda przewidywań, bo nie ma stałych współczynników $\left(\frac{2x}{1+x^2}\right)$, więc uzmiennienie stałej

CORN = CORJ + CSRN

RJ: $y' - \frac{2x}{1+x^2} \cdot y = 0$

$$\frac{dy}{dx} = \frac{2x}{1+x^2} \cdot y$$

$$\frac{dy}{y} = \frac{2x}{1+x^2}\, dx$$

$$\int \frac{dy}{y} = \int \frac{2x}{1+x^2}\, dx$$

$$\ln|y| = \ln|1+x^2| + \ln C$$

$$y = (1+x^2) \cdot C \leftarrow \text{CORJ}$$

RN: MUS

$$y = (1+x^2)\, C(x)$$

$$y' = (1+x^2) \cdot C'(x) + C(x) \cdot 2x$$

wracamy do RN:

$$(1+x^2)\, C'(x) + 2x \cdot C(x) - \frac{2x}{1+x^2}(1+x^2) \cdot C(x) = 1+x^2$$

$$(1+x^2) \cdot C'(x) + 2x \cdot C(x) - 2x \cdot C(x) = 1+x^2$$

$$(1+x^2) \cdot (C'(x) - 1) = 0$$

$$C'(x) = 1$$

$$C(x) = x + C$$

CORN: $y = (1+x^2)(x+C)$

$$y = \underbrace{(1+x^2)x}_{CSRN} + \underbrace{(1+x^2)\cdot C}_{CORJ}$$

b) $y' + 2xy = x \cdot e^{-x^2}$

CORJ: $\frac{dy}{dx} + 2xy = 0 \Rightarrow \frac{dy}{dx} = -2xy$

$$\int \frac{dy}{y} = \int -2x\,dx \Rightarrow \ln|y| = -x^2 + C$$

$$y = e^{-x^2+C} = C \cdot e^{-x^2} \leftarrow CORJ$$

RN: (MUS)

$$y = C(x) \cdot e^{-x^2} \Rightarrow y' = C'(x) e^{-x^2} + C(x) \cdot e^{-x^2} \cdot (-2x) =$$
$$= C'(x) \cdot e^{-x^2} - 2x \cdot C(x) \cdot e^{-x^2}$$

do rownania

$$C'(x) \cdot e^{-x^2} - 2x \cdot C(x) e^{-x^2} + 2x \cdot C(x) \cdot e^{-x^2} = x \cdot e^{-x^2}$$
$$C'(x) \cdot e^{-x^2} = x \cdot e^{-x^2}$$

$$C'(x) = x \Rightarrow C(x) = \frac{1}{2}x^2 + C$$

CORN: $y = \left(\frac{1}{2}x^2 + C\right) \cdot e^{-x^2}$

05.03.2015 | Rad. 3

a) $y' + 2y = 5\cos x \qquad y\left(\frac{\pi}{2}\right) = 1.$

RJ: $y' + 2y = 0$

$\frac{dy}{dx} = -2y \Rightarrow \int \frac{dy}{y} = -2\int dx$

$\ln|y| = -2x + C \quad , \quad C \in \mathbb{R}$

CORJ: $y = C \cdot e^{-2x}$

(MP) $f(x) = 5\cos x$

$f(x) = e^{\alpha x}(W_1(x) \cdot \cos\beta x + W_2(x)\sin\beta x)$

$\alpha = 0, \quad \beta = 1, \quad \alpha + \beta i = i \neq -p = -2 \Rightarrow k = 0$

$\underbrace{\deg W_1(x) = 0 \qquad \deg W_2(x) = 0}$

$\deg V_1 = \deg V_2 = \max(\deg W_1(x), \deg W_2(x))$

$y = x^k \cdot e^{\alpha x}(V_1(x)\cos\beta x + V_2(x)\sin\beta x)$

$y = A\cos x + B\sin x$

$y' = A(-\sin x) + B\cos x$

$-A\sin x + B\cos x + 2A\cos x + 2B\sin x = 5\cos x$

$\begin{cases} -A + 2B = 0 \\ B + 2A = 5 \end{cases} \Rightarrow B = 1, A = 2$

CSRN: $y = 2\cos x + \sin x$

CORN: $y = C \cdot e^{-2x} + 2\cos x + \sin x$

$$1 = Ce^{-2\cdot\frac{\pi}{2}} + 2\cos\frac{\pi}{2} + \sin\frac{\pi}{2}$$

$$Ce^{-\pi} = 0 \Rightarrow C = 0$$

czyli: $y = 2\cos x + \sin x$

b) $y' - \frac{xy}{2(x^2-1)} = \frac{x}{2y}$ $\qquad y(2) = \sqrt{3}$

Bernoulliego:

$$\frac{dy}{dx} = p(x)\cdot y + q(x)\cdot y^r$$

$$u(x) = y^{1-r}$$

$$u(x) = [y(x)]^{1-r}$$

$$u(x) = y(x)^2 \Rightarrow y(x) = \sqrt{u(x)}$$

$$\Rightarrow y'(x) = \frac{u'(x)}{2\sqrt{u(x)}}$$

$$\frac{u'(x)}{2\sqrt{u(x)}} - \frac{x\sqrt{u(x)}}{2(x^2-1)} = \frac{x}{2\sqrt{u(x)}} \quad / \cdot 2\sqrt{u(x)}$$

$$u'(x) - \frac{x\cdot u(x)}{x^2-1} = x$$

$$\frac{du}{dx} - \frac{x\cdot u(x)}{x^2-1} = x$$

RJ: $\frac{du}{dx} - \frac{x\,u(x)}{x^2-1} = 0$

$$\frac{du}{dx} = \frac{x\cdot u(x)}{x^2-1}$$

$$\int\frac{du}{u(x)} = \int\frac{x}{x^2-1}\,dx$$

$$\ln|u(x)| = \frac{1}{2}\ln|x^2-1| + C$$

CORJ: $u(x) = C\sqrt{x^2-1}$

RN:

(MUS) $u(x) = c(x) \cdot \sqrt{x^2-1}$

$$u'(x) = c'(x)\sqrt{x^2-1} + \frac{1}{2\sqrt{x^2-1}} \cdot 2x \cdot c(x)$$

$$u'(x) = c'(x) \cdot \sqrt{x^2-1} + \frac{x}{\sqrt{x^2-1}} \cdot c(x)$$

$$c'(x) \cdot \sqrt{x^2-1} + \frac{x}{\sqrt{x^2-1}} \cdot c(x) - \frac{x}{x^2-1} \cdot c(x) \cdot \sqrt{x^2-1} = x$$

$$c'(x) \cdot \sqrt{x^2-1} = x$$

$$c'(x) = \frac{x}{\sqrt{x^2-1}}$$

$$\frac{dc}{dx} = \frac{x}{\sqrt{x^2-1}}$$

$$\int dc = \int \frac{x}{\sqrt{x^2-1}}\, dx$$

$$c(x) = \sqrt{x^2-1} + C$$

CORN: $u(x) = (\sqrt{x^2-1} + C) \cdot \sqrt{x^2-1}$

warunek pocz: $y(2) = \sqrt{3}$

$$u(x) = y(x)^2$$

$$y = \cancel{4} \pm \sqrt{x^2-1 + C\sqrt{x^2-1}}$$

$$\sqrt{3} = \cancel{4}\sqrt{3 + C\sqrt{3}}$$

$$3 = 3 + C\sqrt{3}$$

$$C\sqrt{3} = 0$$

$$C = 0$$

$$y(x) = \sqrt{x^2-1}$$