Z4

zad. 1.

a) $\sum_{n=1}^{\infty} \frac{e^{-nx^2}}{1+n^2} \qquad x\in R$

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$$\left|\frac{e^{-nx^2}}{1+n^2}\right| \le \left|\frac{1}{1+n^2}\right| \le \left|\frac{1}{n^2}\right| = \frac{1}{n^2} \text{ - zbieżny}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{e^{-nx^2}}{1+n^2} \text{ - jest zbieżny jednostajnie}$$

b) $\sum_{n=1}^{\infty} \frac{\ln(1+nx)}{(n^2+x^2)^2} \qquad x\in(0,+\infty) \qquad \ln(1+x)\le x$

$$\left|\frac{\ln(1+nx)}{(n^2+x^2)^2}\right| \le \left|\frac{nx}{(n^2+x^2)^2}\right| \le \frac{1}{2}\left|\frac{x^2+n^2}{(n^2+x^2)^2}\right| \le \frac{1}{2}\left|\frac{1}{n^2+x^2}\right| \le \frac{1}{n^2}$$

$nx \le \frac{1}{2}(n^2+x^2)$

$\sum_{n=1}^{\infty}\frac{1}{n^2}$ - zbieżny $\Rightarrow$ zbieżność jednostajna szeregu $\sum_{n=1}^{\infty} \frac{\ln(1+nx)}{(n^2+x^2)^2}$

zad. 2.

a) $\sum_{n=1}^{\infty} \frac{2^n}{n+1}x^n$

$a_n = \frac{2^n}{n+1}$

$\sum a_n(x-x_0)^n$

$R \quad R$

x_0

$R = \frac{1}{\lambda}$

$\lambda = \lim \frac{a_{n+1}}{a_n}$

$\lambda = \lim \sqrt[n]{a_n}$

$$R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$$

$$R = \lim_{n\to\infty} \frac{2^n}{n+1} \cdot \frac{n+2}{2^{n+1}} = \lim_{n\to\infty} \frac{1}{2} \cdot \frac{n+2}{n+1} = \frac{1}{2}$$

dla $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ szereg $\sum_{n=1}^{\infty} \frac{2^n}{n+1} x^2$ - zbieżny

badamy dla $x = \frac{1}{2}$

$$\sum_{n=1}^{\infty} \frac{2^n}{n+1} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n+1} \text{ - rozbieżne}$$

badamy dla $x = -\frac{1}{2}$

$$\sum_{n=1}^{\infty} \frac{2^n}{n+1} \left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} = \frac{(-1)^n}{n+1}$$

z Leibnitza:

$$\left.\begin{array}{l} \lim_{n\to\infty} \frac{1}{n+1} = 0 \\ \left\{\frac{1}{n+1}\right\} \text{ - nierosnący} \end{array}\right\} \overset{k.L}{\Longrightarrow} \text{zbieżny}$$

$\Rightarrow$ dla $x \in \left\langle -\frac{1}{2}, \frac{1}{2}\right)$ $\sum_{n=1}^{\infty} \frac{2^n}{n+1} x^n$ - zbieżny

b) $\sum_{n=1}^{\infty} \frac{\ln n}{n} x^n$

$R = \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \cdot \frac{n+1}{n} = 1$

$f(x) = \frac{\ln x}{\ln(x+1)}$

$\lim_{x\to\infty} \frac{\ln x}{\ln(x+1)} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x+1}} = \lim_{x\to\infty} \frac{x+1}{x} = 1$

dla $x \in (-1,1)$ $\sum_{n=1}^{\infty} \frac{\ln n}{n} x^n$ - zbieżny

dla $x = 1$

$\sum_{n=1}^{\infty} \frac{\ln n}{n} \cdot 1^n$, $\lim_{n\to\infty} \frac{\ln n}{n} = 0$, bo:

$f(x) = \frac{\ln x}{x}$, $\lim_{x\to\infty} \frac{\ln x}{x} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$

$\left| \frac{\ln n}{} \right.$

c) $\sum_{n=1}^{\infty} \frac{n(2n+1)}{6^n} \cdot x^{2n}$

$$\lambda = \lim_{n\to\infty} \sqrt[2n]{a_{2n}} = \lim_{n\to\infty} \sqrt[2n]{\frac{n(2n+1)}{6^n}} = \lim_{n\to\infty} \frac{\sqrt[2n]{n} \cdot \sqrt[2n]{2n+1}}{\sqrt[2n]{6^n}} = \frac{1}{\sqrt{6}}$$

$R = \frac{1}{\lambda} = \sqrt{6}$, dla $x \in (-\sqrt{6}, \sqrt{6})$ $\sum_{n=1}^{\infty} \frac{n(2n+1)}{6^n} x^{2n}$ - zbieżny

dla $x = \pm\sqrt{6}$

$\sum_{n=1}^{\infty} \frac{n(2n+1)}{6^n} \cdot \left(\pm\frac{1}{\sqrt{6}}\right)^{2n} = \sum_{n=1}^{\infty} n(2n+1)$ - rozbieżny (nie jest spełniony warunek konieczny)

dla $x \in (-\sqrt{6}, \sqrt{6})$ - $\sum_{n=1}^{\infty} \frac{n(2n+1)}{6^n} x^{2n}$ - jest zbieżny

zad. 3.

a) $\sum_{n=0}^{\infty} \frac{x^{4n+1}}{4n+1} = \sum_{n=0}^{\infty} \int_0^x t^{4n}\,dt = \int_0^x \left[\sum_{n=0}^{\infty} t^{4n}\right] dt = \int_0^x \frac{1}{1-t^4}\,dt$

z tw. o całkowaniu etc.

dla $|t^4| < 1$
$|t| < 1$
$t \in (-1, 1)$

$\left[\frac{x^{4n+1}}{4n+1}\right]' = x^{4n}$

$$\frac{1}{1-t^4} = \frac{1}{(1-t)(1+t)(1+t^2)} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{Ct+D}{1+t^2}$$

$1 = A(1+t)(1+t^2) + B(1-t)(t^2+1) + Ct\cdot(1-t)(1+t) + D(1-t)(1+t)$

$t=0 \quad 1 = A+B+D$

$t=-1 \quad 1 = 4B \Rightarrow B = \frac{1}{4}$

$t=1 \quad 1 = 4A \Rightarrow A = \frac{1}{4} \Rightarrow D = \frac{1}{2}$

$t=2 \quad 1 = 15A + B\cdot 5 - 6C - 3C \Rightarrow C = 0$

$$\frac{1}{4}\int_0^x \frac{1}{1-t} + \frac{1}{4}\int_0^x \frac{1}{1+t}\,dt + \frac{1}{2}\int_0^x \frac{1}{t^2+1}\,dt = \Big(\frac{1}{4}\ln\frac{1+t}{1-t} + \cancel{\dots}$$

$$+\frac{1}{2}\operatorname{arctg} x\Big)\Big|_0^x = \underbrace{\frac{1}{4}\ln\left|\frac{1+x}{1-x}\right| + \frac{1}{2}\operatorname{arctg} x}_{\text{to jest szukana suma}}$$

II sposób (Paw).

$$\left(\frac{x^{4n+1}}{4n+1}\right)' = x^{4n}$$

$$\sum \frac{x^{4n+1}}{4n+1} = f(x)$$

$$f'(x) = \left(\sum \frac{x^{4n+1}}{4n+1}\right)' = \sum\left(\frac{x^{4n+1}}{4n+1}\right)' =$$

$$= \sum_{n=0}^{\infty} x^{4n} \underset{|x|<1}{=} \frac{1}{1-x^4}$$

$$f'(x) = \frac{1}{1-x^4} \qquad |x|<1$$

$$\int f'(x)\,dx = \int \frac{1}{1-x^4}\,dx = \dots + C$$

(czyli podstawiamy np. dla $x=0$

$\Rightarrow C=0$