25 16.04.2015

zad. 1.

a) $f(x) = x^4 \cdot e^{-2x}$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad x \in \mathbb{R}$

$e^{-2x} = \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \cdot 2^n \cdot \frac{x^n}{n!} \quad x \in \mathbb{R}$

$f(x) = x^4 \cdot \sum_{n=0}^{\infty} (-1)^n \cdot 2^n \cdot \frac{x^n}{n!} =$

$= \sum_{n=0}^{\infty} (-1)^n \cdot 2^n \cdot \frac{x^{n+4}}{n!}$

wzór Taylora

$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$

dla $x_0 = 0 \rightarrow$ Maclaurena

b) $f(x) = \frac{1}{1+a^2x^2} \quad a > 0$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad |x| < 1$

$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$

$|x| < 1$

$\frac{1}{1+a^2x^2} = \sum_{n=0}^{\infty} (-1)^n \cdot (a^2x^2)^n =$

$|a^2x^2| < 1$

$|x| < \frac{1}{a}$

$= \sum_{n=0}^{\infty} (-1)^n \cdot a^{2n} x^{2n}$

c) $f(x) = 2\sin x \sin 3x = 2 \cdot \frac{1}{2} (\cos(x-3x) - \cos(x+3x)) =$

$= \cos(-2x) - \cos 4x = \cos 2x - \cos 4x$

$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad x \in \mathbb{R}$

$\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4^n \cdot x^{2n}}{(2n)!}$

$$\cos 4x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (4x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 16^n \cdot x^{2n}}{(2n)!}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n \cdot x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 16^n x^{2n}}{(2n)!} =$$

$$= \sum_{n=0}^{\infty} x^{2n} \frac{(-1)^n}{(2n)!} \left(4^n - 16^n\right) \qquad x \in \mathbb{R}$$

d) $f(x) = \ln \frac{1+x}{1-x} = \ln(1+x) - \ln(1-x)$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}, \quad x \in (-1,1]$$

$$\ln(1-x) = \ln(1+(-x)) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot (-x)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n} \cdot (-1)^1 \cdot x^n}{n} =$$

$$= \sum_{n=1}^{\infty} \frac{-x^n}{n} = -\sum_{n=1}^{\infty} \frac{x^n}{n} \qquad -x \in (-1,1]$$

$$x \in [-1,1)$$

dla $x \in (-1,1)$:

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} + \sum_{n=1}^{\infty} \frac{x^n}{n} = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) +$$

$$+ \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right) =$$

$$= 2 \cdot \sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}$$

e) $f(x) = \frac{x}{1+x-2x^2}$

$\cancel{f(x) = \frac{x}{1}}$

na ułamki proste:

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad |x|<1$

$$f(x) = -\frac{1}{2}\,\frac{x}{x^2-\frac{1}{2}x-\frac{1}{2}} = -\frac{1}{2}\,\frac{x}{(x-1)(x+\frac{1}{2})} = (*)$$

$$\frac{x}{(x-1)(x+\frac{1}{2})} = \frac{A}{x-1} + \frac{B}{x+\frac{1}{2}}$$

$$A\left(x+\tfrac{1}{2}\right) + B(x-1) = x$$

$x=1 \qquad \frac{3}{2}A = 1 \qquad A = \frac{2}{3}$

$x = -\frac{1}{2} \qquad -\frac{3}{2}B = -\frac{1}{2} \Rightarrow B = \frac{1}{3}$

$$(*) = -\frac{1}{2}\left(\frac{\frac{2}{3}}{x-1} + \frac{\frac{1}{3}}{x+\frac{1}{2}}\right) = -\frac{1}{3}\,\frac{1}{x-1} - \frac{1}{6}\,\frac{1}{x+\frac{1}{2}} =$$

$$= \frac{1}{3}\cdot\frac{1}{1-x} - \frac{1}{3}\cdot\frac{1}{1+2x} =$$

$$= \frac{1}{3}\sum_{n=0}^{\infty} x^n - \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n(2x)^n = \frac{1}{3}\left(\sum_{n=0}^{\infty} x^n\left(1-(-1)^n\cdot 2^n\right)\right)$$

$\begin{cases} |2x|<1 \Rightarrow |x|<\frac{1}{2} \\ |x|<1 \end{cases}$

zad. 2.

a) $f(x) = \ln x$, $x_0 = 1$

$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} \quad x\in(-1,1]$

$$\ln(x) = \ln(1+(x-1)) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-1)^n}{n}$$

$x-1 \in (-1,1] \Rightarrow x\in(0,2]$

b) $f(x) = \frac{1}{x}$, $x_0 = 3$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!} \cdot (x-3)^n$$

$$\frac{1}{x} = \frac{1}{3+(x-3)} = \frac{1}{3} \cdot \frac{1}{1+\frac{x-3}{3}} =$$

$$= \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \cdot \left(\frac{x-3}{3}\right)^n = \frac{1}{3} \cdot \sum_{n=0}^{\infty} (-1)^n \cdot (x-3)^n \cdot \frac{1}{3^n} =$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot (x-3)^n \cdot \frac{1}{3^{n+1}}$$

$\left|\frac{x-3}{3}\right| < 1$

$|x-3| < 3$

$-3 < x-3 < 3$

$0 < x < 6$

$x \in (0,6)$

$\boxed{Z6}$