Z2

zad. 1

a) $y'' - 4y' = 8x$

RJ: $y'' - 4y' = 0$

równanie charakterystyczne: $r^2 - 4r = 0$

$r = 0 \vee r = 4$

$UPC = \{e^{0x}, e^{4x}\}$

CORJ: $y = C_1 + C_2 e^{4x}$

RN: $f(x) = 8x$, $\alpha = 0, \beta = 0$, $\alpha + \beta i = 0$ - pierwiastek r. char. (1-krotny) $\Rightarrow k = 1$

$W(x) = Ax + B$

przewidywanie:

$y = x \cdot (Ax + B)$

$\begin{cases} y' = 2Ax + B \\ y'' = 2A \end{cases}$

do RN: $2A - 8Ax - 4B = 8x$

$\begin{cases} A = -1 \\ B = \frac{1}{2} \end{cases}$

CSRN:

$y = -x(x + \frac{1}{2})$

CORN: $y = C_1 + C_2 e^{4x} - x(x + \frac{1}{2})$

warunek początkowy:

$y(0) = 1$, $y'(0) = -1$

$y' = 4C_2 e^{4x} - 2x - \frac{1}{2}$

$\begin{cases} 1 = C_1 + C_2 + 0 \\ -1 = 4C_2 - \frac{1}{2} \end{cases} \Rightarrow \begin{matrix} C_2 = -\frac{1}{8} \\ C_1 = \frac{9}{8} \end{matrix}$

zatem

CS $\quad y = \frac{9}{8} - \frac{1}{8}e^{4x} - x(x + \frac{1}{2})$

zad 2

b) $y'' + 2y' + y = \frac{e^x}{x^2+1}$

RJ: $y'' - 2y' + y = 0$

r. ch. $\quad r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1$ - pierwiastek podwójny

CORJ: $y = C_1 \cdot e^x + C_2 x \cdot e^x$

RN: (MUS)

$y = C_1(x) \cdot e^x + C_2(x) \cdot x \cdot e^x$

$$\begin{cases} C_1'(x) \cdot e^x + C_2'(x) \cdot x \cdot e^x = 0 \\ C_1'(x) \cdot e^x + C_2'(x) \cdot x \cdot e^x = \frac{e^x}{x^2+1} \end{cases}$$

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^x(xe^x + e^x) - xe^{2x} = e^{2x}$$

$$W_{C_1'(x)} = \begin{vmatrix} 0 & xe^x \\ \frac{e^x}{x^2+1} & xe^x + e^x \end{vmatrix} = -\frac{xe^{2x}}{x^2+1}$$

$C_1'(x) = -\frac{x}{x^2+1}$

$$C_1(x) = -\int \frac{x}{x^2+1}\,dx = -\frac{1}{2}\ln|x^2+1| + C$$

$W_{C_2'(x)} =$

a) $y'' + y = \operatorname{tg}x$

RJ: $y'' + y = 0$

$r^2 + 1 = 0$ ← równanie charakterystyczne

$(r-i)(r+i) = 0 \qquad r_1 = \hat{0} + \hat{1}\cdot i$

UPC: $\{e^{0\cdot x}\cos(1\cdot x),\ e^{0x}\sin(1\cdot x)\}$

CORJ: $C_1 \cdot \cos x + C_2 \cdot \sin x = y$

RN (MUS) $C_1(x)\cdot\cos x + C_2(x)\cdot\sin x = y$

$$\begin{cases} C_1'(x)\cos x + C_2'(x)\sin x = 0 \\ -C_1'(x)\sin x + C_2'(x)\cos x = \dfrac{\sin x}{\cos x} \end{cases}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x - (-\sin^2 x) = 1$$

$$W_{C_1'(x)} = \begin{vmatrix} 0 & \sin x \\ \dfrac{\sin x}{\cos x} & \cos x \end{vmatrix} = -\frac{\sin^2 x}{\cos x}$$

$C_1'(x) = -\frac{\sin^2 x}{\cos x}$

$C_1(x) = \int -\frac{\sin^2 x}{\cos x}\,dx = \int -\frac{\sin^2 x \cdot \cos x}{\cos^2 x}\,dx = \Big\{ \int \frac{-\sin^2 x \cdot \cos x}{1-\sin^2 x}\,dx =$

$= \left\{ \begin{array}{l} \sin x = t \\ dt = \cos x dx \end{array} \right\} = \int -\frac{t^2}{1-t^2}\,dt = \int \frac{1-t^2-1}{1-t^2}\,dt = \int dt - \int \frac{1}{1-t^2}\,dt =$

$= \int dt - \int \frac{\frac{1}{2}}{1-t} + \frac{\frac{1}{2}}{1+t}\,dt = t + \frac{1}{2}\ln|1-t| - \frac{1}{2}\ln|1+t| + C =$

$= t + \frac{1}{2}\ln\left|\frac{1-t}{1+t}\right| + C = t + \ln\sqrt{\frac{1-t}{1+t}} + C$

$C_1(x) = \sin x + \ln\sqrt{\frac{1-\sin x}{1+\sin x}} + C_1$

$W_{C_2'(x)} = \begin{vmatrix} \cos x & 0 \\ -\sin x & \frac{\sin x}{\cos x} \end{vmatrix} = \sin x$

$C_2'(x) = \sin x$

$C_2(x) = -\cos x + C_2$

CORN:

$y = \left(\sin x + \ln\sqrt{\frac{1-\sin x}{1+\sin x}} + C_1\right)\cos x + (-\cos x + C_2)\sin x$

zad. 1.

b) $y'' - 2y' + y = 4\sin^2\left(\frac{x}{2}\right) = 4\cdot\left(\frac{1-\cos x}{2}\right) = \underbrace{2}_{f_1} \underbrace{-2\cos x}_{f_2}$, $\quad y(0) = 2, \quad y'(0) = 1$

RJ: $y'' - 2y' + y = 0$

$r^2 - 2r + 1 = 0 \Rightarrow r_{1,2} = 1$

UPC: $\{e^x, xe^x\}$

$\sin^2 x = \frac{1-\cos 2x}{2}$

CORJ: $y_0(x) = C_1 e^x + C_2 x e^x$

$f(x) = e^{\alpha x}(W_1(x)\cdot\cos\beta x + W_2(x)\cdot\sin\beta x)$

f_1: $\alpha = 0$, $\beta = 0$ st. $w_1(x) = 0$

$\alpha + \beta i = 0 \Rightarrow k = 0$ $\quad \lfloor 2$

$y_1(x) = A$

$y_1'(x) = 0$ $\quad A = 2$

$y_1''(x) = 0$

f_2: $\alpha = 0$

$\beta = 1$ $\quad \lfloor -2\cos x$

$\alpha + \beta i = i \Rightarrow k = 0$ st $w_2(x) = 0$

$y_2(x) = \cancel{B\cos x}\ B\cos x + C\sin x$

$y_2'(x) - B\sin x + C\cos x$

$y_2''(x) = -B\cos x - C\sin x$

wracamy do równania początkowego

$-B\cos x - C\sin x + 2B\sin x - 2C\cos x + B\cos x + C\sin x = -2\cos x$

$2B\sin x - 2C\cos x = -2\cos x$

$B = 0,\ C = 1$

$y_2(x) = \sin x$

CSRN: $y(x) = 2 + \sin x$

CORN: $y(x) = 2 + \sin x + C_1 e^{x} + C_2 x e^{x}$

zad. 3.

a) $y''' + y'' - 4y' - 4y = e^{x} + 3xe^{2x} + \cos x$

RJ: $y''' + y'' - 4y' - 4y = 0$

$r^3 + r^2 - 4r - 4 = 0$

$r^2(r+1) - 4(r+1) = 0$ $\quad r_1 = -2, \ r_2 = 2, \ r_3 = -1$

CORJ: $C_1 e^{-2x} + C_2 e^{2x} + C_3 e^{-x}$

$A_1 e^{\alpha x}$

$C_1 \sin\beta x + C_2 \cos\beta x$

RN: ~~[illegible]~~ $\quad f_1: e^{x}$

$\alpha = 1, \ \beta = 0$

$\alpha + \beta i = 1 \neq r_1 \Rightarrow k = 0$ $\quad$ st. $w_1(x) = 0$

$y_1 = A e^{x}$

$f_2: \ 3xe^{2x}$ $\quad \alpha = 2, \ \beta = 0$ $\quad$ st. $w_2(x) = 1$

$\alpha + \beta i = 2 = r_2 \Rightarrow k = 1$

$y_2 = (Bx + C) \cdot x^1 \cdot e^{2x}$

$f_3: \ \cos x$ $\quad \alpha = 0, \ \beta = 1$ $\quad$ st. $w_3(x) = 0$

$\alpha + \beta i = i \neq r_i \Rightarrow k = 0$

$y_3 = D\cos x + E\sin x$

CSRN: $A \cdot e^{x} + (Bx + C)xe^{2x} + D\cos x + E\sin x$

CORN: $C_1 e^{-2x} + C_2 e^{2x} + C_3 e^{-x} + Ae^{x} + (Bx + C)xe^{2x} + D\cos x$
$+ E\sin x$