zad. 1.

$= 1 + i - 2i - 1 + 2i + 1 + 2i - 2$
-4

a) $$\frac{z}{z+2} = \frac{(z-1)+1}{z-1+3} = \frac{z-1}{z-1+3} + \frac{1}{z-1+3} = (*)$$

$$\frac{1}{z-1+3} = \frac{1}{3} \cdot \frac{1}{1-\left(\frac{z-1}{3}\right)} = \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-1}{3}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{n+1}} (z-1)^n$$

$$\left|\frac{z-1}{3}\right| < 1 \Rightarrow |z-1| < 3$$

$$(*) = (z-1+1) \cdot \sum_{n=0}^{\infty} (-1)^n \cdot (z-1)^n \cdot \frac{1}{3^{n+1}} = \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^{n+1}}{3^{n+1}} +$$

$$+ \sum_{n=0}^{\infty} \frac{(-1)^n (z-1)^n}{3^{n+1}} \qquad \text{dla } |z-1| < 3$$

zad. 2.

a) $f(z) = z \cdot \sum_{n=0}^{\infty} (-1)^n \cdot \left(\frac{1}{z}\right)^{2n+1} \cdot \frac{1}{(2n+1)!} = (*)$ $\quad P(0;0,\infty)$ $\quad \left\{ \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot z^{2n+1}}{(2n+1)!} \right.$

$0 < |z| < \infty$

$(*) = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n} \cdot (2n+1)!}$ $\qquad a_{-1} = 0, \; a_1 = 0, \; a_0 = 1$

b) $f(z) = \frac{2z}{z^2+1} = (*)$ $\qquad P(i;0,2) \Leftrightarrow 0 < |z-i| < 2$

$\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}$

$f(z) = \frac{2z}{(z-i)(z+i)} = \frac{A}{z-i} + \frac{B}{z+i}$ $\qquad A(z+i) + B(z-i) = 2z$

$A + B = 2 \Rightarrow A = 1$

$A - B = 0 \quad\;\; B = 1$

$f(z) = \frac{1}{z-i} + \frac{1}{z+i} \Rightarrow a_{-1} = 1$ bo $(z-i)^{-1}$

$\cancel{\frac{1}{z+i}}$

$\frac{1}{z+i} = \frac{1}{2i + (z-i)}$

$\nearrow \frac{1}{2i} \left(\frac{1}{1 + \frac{z-i}{2i}} \right) = \sum (*)$

$\left|\frac{z-i}{2i}\right| < 1 \Leftrightarrow |z-i| < |2i| \Leftrightarrow |z-i| < 2$

$\searrow \frac{1}{z-i} \frac{1}{1 + \frac{2i}{z-i}} = \sum \ldots$

$\left|\frac{2i}{z-i}\right| < 1 \Leftrightarrow |2i| < |z+i| \Leftrightarrow |z-i| > 2$

$$(*) = \frac{1}{2i}\sum_{n=0}^{\infty}\left(\frac{z-i}{2i}\right)^n(-1)^n = \sum_{n=0}^{\infty}(-1)^n\cdot(z-i)^n\cdot\frac{1}{(2i)^{n+1}}$$

$$f(z) = \frac{1}{z-i} + \sum_{n=0}^{\infty}(-1)^n\cdot(z-i)^n\cdot\frac{1}{(2i)^{n+1}}$$

$$a_{-1} = 1$$

$$a_0 = (-1)^0\cdot\frac{1}{(2i)^1} = \frac{1}{2i} =$$

$$a_1 = (-1)^1\cdot\frac{1}{(2i)^2} = \frac{1}{4}$$

c) $f(z) = \frac{2(z+i)}{z^2-1}$ $\quad P(i+1;\,1,\,\sqrt{5}) \equiv 1 < |z-(i+1)| < \sqrt{5}$

$$\cancel{f(z) = \frac{2(z+i)}{(z-i)(z+i)} = \frac{A}{z-i} + \frac{B}{z+i}}$$

$$f(z) = \frac{2(z+i)}{(z-1)(z+1)} = \frac{A}{z-1} + \frac{B}{z+1} =$$

$$A(z+1) + B(z-1) = 2z+2i$$
$$z(A+B) + A - B = 2z+2i$$
$$+\begin{cases} A+B = 2 \\ A-B = 2i \end{cases}$$
$$2A = 2i+2$$
$$A = i+1$$
$$B = 1-i$$

$$= \frac{1-j}{z+1} + \frac{1+j}{z-1} = (1-j)\frac{1}{z+1} + (1-j)\frac{1}{z-1}$$

$$\frac{1}{z+1} = \frac{1}{(z-(1+j))+2+j} = \frac{1}{z-(1+j)}\cdot\frac{1}{1+\frac{2+j}{z-(1+j)}} =$$

$$\left|\frac{2+j}{z-(1+j)}\right| < 1$$
$$|2+j| < |z-(1+j)|$$
$$\sqrt{5} < |z-(1+j)|$$

czyli wyciągamy $2+j$

$$= \frac{1}{2+j}\cdot\frac{1}{1+\frac{z-(1+j)}{2+j}} =$$

$$= \frac{1}{2+j}\cdot\sum_{n=0}^{\infty}(-1)^n\cdot\frac{(z-(1+j))^n}{(2+j)^n} = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{(z-(1+j))^n}{(2+j)^{n+1}}$$

$$\frac{1}{z-1}=\frac{1}{z-(1+j)+j}=\frac{1}{z-(1+j)}\cdot\frac{1}{1+\frac{j}{z-(1+j)}}= \left\{ \begin{array}{l} \left|\frac{j}{z-(1+j)}\right|<1 \\ |j|<|z-(1+j)| \\ 1<|z-(1+j)| \end{array} \right.$$

$$=\frac{1}{z-(1+j)}\cdot\sum_{n=0}^{\infty}(-1)^n\frac{j^n}{(z-(1+j))^n}=$$

$$=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{j^n}{(z-(1+j))^{n+1}}$$

$$f(z)=(1+j)\sum_{n=0}^{\infty}(-1)^n\cdot\frac{(z-(1+j))^n}{(2+j)^{n+1}}+(1+j)\sum_{n=0}^{\infty}(-1)^n\cdot\frac{j^n}{(z-(1+j))^{n+1}}$$

$$(z-(1+j))^{-(n+1)}$$

$$a_{-1}=(-1)^0\cdot j^0\cdot(1+j)=(1+j)$$

$$a_0=(-1)^0\cdot\frac{1}{(2+j)^1}\cdot(1+j)=\frac{1+j}{2+j}$$

$$a_1=-\frac{1}{(2+j)^2}\cdot(1+j)$$

zad. 3.

a) $f(z)=z^2(e^{z^2}-1)$ b) $=6\sin z^3+z^3(z^6-6)$

a)

$$f(z)=z^2(e^{z^2}-1)=z^2\cdot\left(\sum_{n=0}^{\infty}\frac{z^{2n}}{n!}-1\right)=z^2\left(1+\frac{z^2}{1!}+\frac{z^4}{2!}+\ldots+\frac{z^{2n}}{n!}+\ldots-1\right)=$$

$$=z^4\left(1+\frac{z^2}{2!}+\ldots+\frac{z^{2n+2}}{n!}\right)\quad n=4$$

b)

$$f(z) = 6\sin z^3 + z^3(z^6 - 6) =$$

$$= 6 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot (z^3)^{2n+1} + z^3(z^6 - 6) =$$

$$\cancel{+6 \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{6n+3}} =$$

$$= 6\left(z^3 - \frac{z^9}{6} + \frac{z^{15}}{5!} - \ldots\right) + z^9 - 6z^3 =$$

$$= 6\left(\frac{z^{15}}{5!} - \frac{z^{21}}{7!} + \ldots\right) = z^{15} \cdot 6\left(\frac{1}{5!} - \frac{z^6}{7!} + \ldots\right)$$

$$n = 15$$