Z 8

zad. 1

a) $f(z) = z|z|^2 = (x+iy)(x^2+y^2) = x^3 + ix^2y + xy^2 + iy^3 = x^3 + xy^2 + i(y^3 + x^2y)$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial x} = 3x^2 + y^2 \qquad \frac{\partial v}{\partial y} = 2xy$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \frac{\partial v}{\partial y} = 3y^2 + x^2 \qquad \frac{\partial v}{\partial x} = 2xy$$

$$\begin{cases} 3x^2 + y^2 = 3y^2 + x^2 \\ 2xy = -2xy \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

holomorficzność - spełnione C-R i u i v są klasy C^1

zad. 2

a) $f(z) = \Im m(z+i)^2 = \Im m(x^2 + 2i(y+1)x - (y+1)^2) = 2x(y+1) = 2xy + 2x$

$u(x,y) = 2xy + 2x$

$\frac{\partial u}{\partial x} = 2y + 2$ $\quad \begin{cases} 2y+2=0 \\ 2x=0 \end{cases} \Rightarrow \begin{cases} y=-1 \\ x=0 \end{cases}$

$\frac{\partial u}{\partial y} = 2x$

C-R spełnione dla $z = -i$

$u(x,y)$ i $v(x,y) \in C^1$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2y + 2$

$f'(-i) = 2\cdot(-1) + 2 = 0 \rightarrow$

funkcja nie jest holomorficzna w $z=-i$ i w ogóle

b) $f(z) = \frac{|z|^2}{z} = \frac{x^2+y^2}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x^2+y^2}{x^2+y^2}(x-iy) = x - iy = \bar{z}$

$u(x,y) = x$ $\quad \frac{\partial u}{\partial x} = 1$ $\quad \frac{\partial v}{\partial x} = 0$

$v(x,y) = -y$ $\quad \frac{\partial u}{\partial y} = 0$ $\quad \frac{\partial v}{\partial y} = -1$

C-R: $\begin{cases} 1 = -1 \\ 0 = 0 \end{cases}$ ↯ $\Rightarrow$ warunki C-R nie są spełnione

$\Rightarrow$ nigdzie nie istnieje pochodna $\Rightarrow$
$f(z)$ nigdzie nie jest holomorficzna

c) $f(z) = e^{\bar{z}} = e^{x-iy} = e^{x} \cdot e^{-iy} = e^{x}(\cos(-y) + i\sin(-y)) =$

$$= e^{x}(\cos y - i\sin y) = \underbrace{e^{x}\cos y}_{u(x,y)} + i \cdot \underbrace{(-e^{x}\sin y)}_{v(x,y)}$$

$$\frac{\partial u}{\partial x} = e^{x}\cos y \qquad \frac{\partial v}{\partial x} = -e^{x}\sin y$$

$$\frac{\partial u}{\partial y} = -\sin y\, e^{x} \qquad \frac{\partial v}{\partial y} = -e^{x}\cos y$$

$$\begin{cases} e^{x}\cos y = -e^{x}\cos y \\ -e^{x}\sin y = e^{x}\sin y \end{cases} \Rightarrow \begin{cases} e^{x}\cos y = 0 \\ e^{x}\sin y = 0 \end{cases} \Rightarrow \begin{cases} \cos y = 0 \\ \sin y = 0 \end{cases} \lightning$$

$\Rightarrow$ warunki C-R nie są spełnione $\Rightarrow$

$f'(z)$ nie istnieje w żadnym punkcie $\Rightarrow$

$\Rightarrow$ $f(z)$ nie jest holomorficzna

d) $f(z) = x(2-x) + y^2 + i\,2y(1-x)$

$u(x,y) = 2x - x^2 + y^2$

$v(x,y) = 2y - 2x$

$u_x = 2 - 2x \qquad v_x = -2y \qquad \cancel{v(x,y) = 2y - 2yx}$

$u_y = 2y \qquad v_y = 2 - 2x$

$$\cancel{u_x =} \begin{cases} 2 - 2x = 2 - 2x \\ 2y = 2y \end{cases} \Rightarrow \text{C-R spełnione } \forall_{x,y}$$

$u, v \in C^1$

zatem $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = 2 - 2x - i2y$

holomorficzna na całej płaszczyźnie zespolonej

zad. 3

a) $u(x,y) = x^3 - 3xy^2 + x \qquad f(0) = i$

$f(z) = u(x,y) + i\,v(x,y)$

funkcja $u(x,y)$ musi być harmoniczna, wtedy istnieje $v(x,y)$,

czyli $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 1 \qquad \frac{\partial^2 u}{\partial x^2} = 6x$

$6x - 6x = 0$ ✓

$\frac{\partial u}{\partial y} = -6xy \qquad \frac{\partial^2 u}{\partial y^2} = -6x$

$\Rightarrow \exists_{v(x,y)}$ u i v są sprzężone za pomocą C-R

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \qquad \begin{cases} 3x^2 - 3y^2 + 1 = \frac{\partial v}{\partial y} \\ -6xy = -\frac{\partial v}{\partial x} \end{cases}$$

$v(x,y) = \int (3x^2 - 3y^2 + 1)\,dy = 3x^2y - y^3 + y + C(x)$

$v(x,y)$: $\frac{\partial v}{\partial x} = 6xy + C'(x) \overset{C-R}{=} 6xy \Rightarrow C'(x) = 0 \Rightarrow C(x) = C$

$f(z) = (x^3 - 3xy^2 + x) + i\,(\cancel{6xy + C}\ 3x^2y - y^3 + y + 1)$ do warunku pocz.

$f(0) = i \Rightarrow x^3 - 3xy^2 + x = 0$

$\cancel{6xy + C = 1}$

$3x^2y - y^3 + y + C$