AUTOMATYKA I ROBOTYKA - SEMESTR 2

ANA2. ZESTAW 9. - Rozwiązania

Zad. 1. Rozwinąć następujące funkcje w szereg Taylora wokół punktu $z_0 = 1$

$$\text{(a)} \quad f(z) = \frac{z}{z+2}$$

$f(z) = \frac{z}{z+2} = \frac{(z-1)+1}{(z-1)+3}$

$\frac{1}{(z-1)+3} = \frac{1}{3\left(1+\frac{z-1}{3}\right)} = \left\| \begin{array}{c} \left|\frac{z-1}{3}\right| < 1 \\ |z-1| < 3 \end{array} \right\| = \sum_{n=0}^{\infty}(-1)^n \frac{(z-1)^n}{3^{n+1}}$

$f(z) = \sum_{n=0}^{\infty}(-1)^n \frac{(z-1)^{n+1}}{3^{n+1}} + \sum_{n=0}^{\infty}(-1)^n \frac{(z-1)^n}{3^{n+1}}$

$$\text{(b)} \quad f(z) = \frac{z}{z^2-2z+5}$$

$f(z) = \frac{(z-1)+1}{(z-1)^2+4}$

$\frac{1}{(z-1)^2+4} = \frac{1}{4\left(1+\frac{(z-1)^2}{4}\right)} = \left\| \begin{array}{c} \left|\frac{z-1}{2}\right|^2 < 1 \\ |z-1| < 2 \end{array} \right\| = \sum_{n=0}^{\infty}(-1)^n \frac{(z-1)^{2n}}{4^{n+1}}$

$f(z) = \sum_{n=0}^{\infty}(-1)^n \frac{(z-1)^{2n+1}}{4^{n+1}} + \sum_{n=0}^{\infty}(-1)^n \frac{(z-1)^{2n}}{4^{n+1}}$

$$\text{(c)} \quad f(z) = \frac{z^2}{(z+1)^2}$$

$f(z) = \frac{[(z-1)+1]^2}{[(z-1)+2]^2} = \frac{(z-1)^2+2(z-1)+1}{[(z-1)+2]^2}$

$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \Rightarrow \sum_{n=1}^{\infty} nz^{n-1} = \frac{1}{(1-z)^2}\,, \quad |z| < 1$

$\sum_{n=0}^{\infty}(-1)^n z^n = \frac{1}{1+z} \Rightarrow \sum_{n=1}^{\infty}(-1)^{n+1} nz^{n-1} = \frac{1}{(1+z)^2}\,, \quad |z| < 1$

$\frac{1}{[(z-1)+2]^2} = \frac{1}{4\left[1+\frac{z-1}{2}\right]^2} = \left\| \begin{array}{c} \left|\frac{z-1}{2}\right| < 1 \\ |z-1| < 2 \end{array} \right\| = \sum_{n=1}^{\infty}(-1)^{n+1} n \frac{(z-1)^{n-1}}{2^{n+1}}$

$f(z) = \sum_{n=1}^{\infty}(-1)^{n+1} n \frac{(z-1)^{n+1}}{2^{n+1}} + \sum_{n=1}^{\infty}(-1)^{n+1} n \frac{(z-1)^{n}}{2^{n}} + \sum_{n=1}^{\infty}(-1)^{n+1} n \frac{(z-1)^{n-1}}{2^{n+1}}$

$$\text{(d)} \quad f(z) = z \cdot e^z$$

$f(z) = [(z-1)+1]e^{(z-1)+1} = e[(z-1)+1] \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} =$
$= e \sum_{n=0}^{\infty} \frac{(z-1)^{n+1}}{n!} + e \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \qquad z \in \mathbb{C}$

Zad. 2. Rozwinąć następujące funkcje w szereg Laurenta w podanych pierścieniach $P(z_0; r, R)$ oraz podać wartości współczynników a_{-1}, a_0, a_1 dla każdego rozwinięcia

$$\text{(a)} \quad f(z) = z \cdot \sin\frac{1}{z}, \quad P(0; 0, \infty)$$

$P(0; 0, \infty) = \{z \in \mathbb{C} : |z| > 0\}$

$f(z) = z \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n}(2n+1)!}$

$-2n = -1 \Rightarrow a_{-1} = 0$

$-2n = 0 \Rightarrow a_0 = 1$

$-2n = 1 \Rightarrow a_1 = 0$

$$\text{(b)} \quad f(z) = \frac{2z}{z^2+1}, \quad P(i; 0, 2)$$

$f(z) = \frac{1}{z-i} + \frac{1}{z+i}, \qquad P(i; 0, 2) = \{z : 0 < |z-i| < 2\}$

$\frac{1}{z+i} = \frac{1}{(z-i)+2i} = \frac{1}{2i\left(1+\frac{z-i}{2i}\right)} = \left\| \begin{array}{c} \left|\frac{z-i}{2i}\right| < 1 \\ |z-i| < 2 \end{array} \right\| = \sum_{n=0}^{\infty}(-1)^n \frac{(z-i)^n}{(2i)^{n+1}}$

$f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{(2i)^{n+1}} + \frac{1}{z-i}$

$a_{-1} = 1\,, \quad a_0 = \frac{1}{2i}\,, \quad a_1 = \frac{1}{4}$

$$\text{(c)} \quad f(z) = \frac{2(z+i)}{z^2-1}\,, \quad P(1+i;1,\sqrt{5})$$

$f(z) = \frac{1+i}{z-1} + \frac{1-i}{z+1}\,, \qquad P(1+i;1,\sqrt{5}) = \{z : 1 < |z-1-i| < \sqrt{5}\}$

$\frac{1}{z-1} = \frac{1}{(z-1-i)+i} = \frac{1}{(z-1-i)\left(1+\frac{i}{z-1-i}\right)} = \left\| \begin{array}{c} \left|\frac{i}{z-1-i}\right| < 1 \\ |z-1-i| > 1 \end{array} \right\| = \sum_{n=0}^{\infty} (-1)^n \frac{i^n}{(z-1-i)^{n+1}}$

$\frac{1}{z+1} = \frac{1}{(z-1-i)+(i+2)} = \frac{1}{(i+2)\left(1+\frac{z-1-i}{i+2}\right)} = \left\| \begin{array}{c} \left|\frac{z-1-i}{i+2}\right| < 1 \\ |z-1-i| < \sqrt{5} \end{array} \right\| = \sum_{n=0}^{\infty} (-1)^n \frac{(z-1-i)^n}{(i+2)^{n+1}}$

$f(z) = (1+i) \sum_{n=0}^{\infty} (-1)^n \frac{i^n}{(z-1-i)^{n+1}} + (1-i) \sum_{n=0}^{\infty} (-1)^n \frac{(z-1-i)^n}{(i+2)^{n+1}}$

$a_{-1} = 1+i\,, \quad a_0 = \frac{1-i}{i+2}\,, \quad a_1 = \frac{i-1}{(i+2)^2}$

$$\text{(d)} \quad f(z) = \frac{z^2-2z+5}{(z^2+1)(z-2)}\,, \quad P(i;2,\sqrt{5})$$

$f(z) = \frac{1}{z-2} + \frac{i}{z-i} - \frac{i}{z+i}\,, \qquad P(i;2,\sqrt{5}) = \{z : 2 < |z-i| < \sqrt{5}\}$

$\frac{1}{z-2} = \frac{1}{(z-i)+(i-2)} = \frac{1}{(i-2)\left(1+\frac{z-i}{i-2}\right)} = \left\| \begin{array}{c} \left|\frac{z-i}{i-2}\right| < 1 \\ |z-i| < \sqrt{5} \end{array} \right\| = \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{(i-2)^{n+1}}$

$\frac{1}{z+i} = \frac{1}{(z-i)+2i} = \frac{1}{(z-i)\left(1+\frac{2i}{z-i}\right)} = \left\| \begin{array}{c} \left|\frac{2i}{z-i}\right| < 1 \\ |z-i| > 2 \end{array} \right\| = \sum_{n=0}^{\infty} (-1)^n \frac{(2i)^n}{(z-i)^{n+1}}$

$f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{(i-2)^{n+1}} + \frac{i}{z-i} - i \sum_{n=0}^{\infty} (-1)^n \frac{(2i)^n}{(z-i)^{n+1}}$

$a_{-1} = 0\,, \quad a_0 = \frac{1}{i-2}\,, \quad a_1 = \frac{-1}{(i-2)^2}$

Zad. 3. Przedstawić funkcję

$$f(z) = \frac{z}{z^2 - 3z + 2}$$

w postaci szeregu (Taylora lub Laurenta) zbieżnego w obszarze

$$\text{(a)} \quad |z| < 1$$

$f(z) = \frac{2}{z-2} - \frac{1}{z-1} = \frac{2}{z-2} + \frac{1}{1-z}$

$\frac{1}{z-2} = \frac{1}{-2\left(1-\frac{z}{2}\right)} = \left\| \begin{array}{l} \left|\frac{z}{2}\right| < 1 \\ |z| < 2 \end{array} \right\| = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$

$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad |z| < 1$

$f(z) = \sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty} \frac{z^n}{2^n} = \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^n}\right) z^n$

$$\text{(b)} \quad 1 < |z| < 2$$

$\frac{1}{z-2}$ - jak wyżej

$\frac{1}{1-z} = -\frac{1}{z-1} = -\frac{1}{z\left(1-\frac{1}{z}\right)} = \left\| \begin{array}{l} \left|\frac{1}{z}\right| < 1 \\ |z| > 1 \end{array} \right\| = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$

$f(z) = -\sum_{n=0}^{\infty} \frac{z^n}{2^n} - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$

$$\text{(c)} \quad |z| > 2$$

$\frac{1}{1-z}$ - jak wyżej

$\frac{1}{z-2} = \frac{1}{z\left(1-\frac{2}{z}\right)} = \left\| \begin{array}{l} \left|\frac{2}{z}\right| < 1 \\ |z| > 2 \end{array} \right\| = \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}}$

$f(z) = \sum_{n=0}^{\infty} \frac{2^{n+1}}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = \sum_{n=0}^{\infty} (2^{n+1} - 1) \frac{1}{z^{n+1}}$

Zad. 4. Rozwinąć funkcję

$$f(z) = \frac{z+1}{z^2+1}$$

(a) w szereg Laurenta w $P(-2i; 1, 3)$

$f(z) = \frac{1-i}{2} \cdot \frac{1}{z-i} + \frac{1+i}{2} \cdot \frac{1}{z+i}\,, \qquad P(-2i; 1, 3) = \{z : 1 < |z+2i| < 3\}$

$\frac{1}{z-i} = \frac{1}{(z+2i)-3i} = \frac{1}{-3i\left(1-\frac{z+2i}{3i}\right)} = \left\| \begin{array}{c} \left|\frac{z+2i}{3i}\right| < 1 \\ |z+2i| < 3 \end{array} \right\| = -\sum_{n=0}^{\infty} \frac{(z+2i)^n}{(3i)^{n+1}}$

$\frac{1}{z+i} = \frac{1}{(z+2i)-i} = \frac{1}{(z+2i)\left(1-\frac{i}{z+2i}\right)} = \left\| \begin{array}{c} \left|\frac{i}{z+2i}\right| < 1 \\ |z+2i| > 1 \end{array} \right\| = \sum_{n=0}^{\infty} \frac{i^n}{(z+2i)^{n+1}}$

$f(z) = \frac{i-1}{2} \sum_{n=0}^{\infty} \frac{(z+2i)^n}{(3i)^{n+1}} + \frac{1+i}{2} \sum_{n=0}^{\infty} \frac{i^n}{(z+2i)^{n+1}}$

(b) w szereg Laurenta w sąsiedztwie punktu ∞: $P(-2i; 3, \infty)$

$P(-2i; 3, \infty) = \{z : |z+2i| > 3\}$

$\frac{1}{z-i} = \frac{1}{(z+2i)-3i} = \frac{1}{(z+2i)\left(1-\frac{3i}{z+2i}\right)} = \left\| \begin{array}{c} \left|\frac{3i}{z+2i}\right| < 1 \\ |z+2i| > 3 \end{array} \right\| = \sum_{n=0}^{\infty} \frac{(3i)^n}{(z+2i)^{n+1}}$

$\frac{1}{z+i}$ - jak wyżej

$f(z) = \frac{1-i}{2} \sum_{n=0}^{\infty} \frac{(3i)^n}{(z+2i)^{n+1}} + \frac{1+i}{2} \sum_{n=0}^{\infty} \frac{i^n}{(z+2i)^{n+1}}$

(c) w szereg Taylora wokół punktu $z_0 = 1$

$\frac{1}{z-i} = \frac{1}{(z-1)+(1-i)} = \frac{1}{(1-i)\left(1+\frac{z-1}{1-i}\right)} = \left\| \begin{array}{c} \left|\frac{z-1}{1-i}\right| < 1 \\ |z-1| < \sqrt{2} \end{array} \right\| = \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^n}{(1-i)^{n+1}}$

$\frac{1}{z+i} = \frac{1}{(z-1)+(1+i)} = \frac{1}{(1+i)\left(1+\frac{z-1}{1+i}\right)} = \left\| \begin{array}{c} \left|\frac{z-1}{1+i}\right| < 1 \\ |z-1| < \sqrt{2} \end{array} \right\| = \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^n}{(1+i)^{n+1}}$

$f(z) = \frac{1-i}{2} \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^n}{(1-i)^{n+1}} + \frac{1+i}{2} \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^n}{(1+i)^{n+1}} =$

$= \frac{1}{2}\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{(1-i)^n}+\frac{(-1)^n}{(1+i)^n}\right](z-1)^n\,, \qquad |z-1|<\sqrt{2}$

Zad. 5. Wyznaczyć obsszar zbieżności i sumę szeregu Laurenta

$$\sum_{n=-\infty}^{\infty} a_n z^n$$

gdzie

$$a_n = \begin{cases} \frac{1}{2^{n+1}}\,, & n \geq 0 \\ 0\,, & n=-2 \\ -1\,, & -2 \neq n < 0 \end{cases}$$

$n \geq 0$: $\quad \frac{1}{2}\sum_{n=0}^{\infty}(\frac{z}{2})^n = \frac{1}{2}\cdot\frac{1}{1-\frac{z}{2}} = \frac{1}{2-z}\,, \qquad |z|<2$

$n < 0$: $\quad -\sum_{n=1}^{\infty}\left(\frac{1}{z}\right)^n + \frac{1}{z^2} = -\frac{\frac{1}{z}}{1-\frac{1}{z}} + \frac{1}{z^2} = -\frac{1}{z-1}+\frac{1}{z^2}\,, \qquad |z|>1$

$f(z) = \frac{1}{2-z} - \frac{1}{z-1} + \frac{1}{z^2}\,, \qquad 1<|z|<2$