MECE 6388: HW #2

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2.1-1

Let the scalar plant

$$x_{k+1} = x_k u_k + 1$$

have performance index

$$J = \frac{1}{2}\sum_{k=0}^{N-1} u_k^2$$

with final time N=2. Given x_0, it is desired to make x_2=0.

(a) Write state and costate equations with u_k eliminated.

(b) Assume the final costate λ_2 is known. Solve for λ_0, λ_1 in terms of λ_2, and the state. Use this to express x_2 in terms of λ_2 and x_0. Hence, find a quartic equation for λ_2 in terms of initial state x_0.

(c) If x_0=1, find the optimal state and costate sequences, the optimal control, and the optimal value of the performance index.

Solution

The state equation, costate equation, and stationary condition are given respectively as

$$\begin{aligned} x_{k+1} &= x_k u_k + 1 && (1.1)\\ \lambda_k &= \lambda_{k+1} u_k && (1.2)\\ 0 &= u_k + \lambda_{k+1} x_k && (1.3) \end{aligned}$$

(1.3) can be solved for u_k and plugged into (1.1) and (1.2) to eliminate u_k from the state and costate equations

$$\boxed{\begin{aligned} \lambda_k &= -\lambda_{k+1}^2 x_k \\ x_{k+1} &= -\lambda_{k+1} x_k^2 + 1 \end{aligned}} \qquad \begin{aligned} (1.4)\\ (1.5) \end{aligned}$$

b) Plugging in k=0,1 into equation (1.4) gives us

$$\begin{aligned} \lambda_1 &= -\lambda_2^2 x_1 \\ \lambda_0 &= -\lambda_1^2 x_0 \\ \lambda_0 &= -(\lambda_2^2 x_1)^2 x_0 \\ \lambda_0 &= -\lambda_2^4 x_1^2 x_0 \end{aligned}$$

This gives us $\lambda_{1,2}$ in terms of λ_2 (known), x_0 (known), and x_1 (unknown). Therefore we need to find a way to relate x_0 to our known variables (x_0, λ_2).
We do this by plugging in k=0,1 to equation (1.5) that we solved in part a).
First for k=0

$$\begin{aligned}
x_1 &= -\lambda_1 x_0^2 + 1 \\
\lambda_1 &= -\lambda_2^2 x_1 \\
x_1 &= \lambda_2^2 x_1 x_0^2 + 1 \\
x_1(1 - \lambda_2^2 x_0^2) &= 1 \\
x_1 &= \frac{1}{1 - \lambda_2^2 x_0^2}
\end{aligned}$$

Now for k=1

$$\begin{aligned}
x_2 = 0 &= -\lambda_2 x_1^2 + 1 \\
0 &= -\lambda_2 (\frac{1}{1 - \lambda_2^2 x_0^2})^2 + 1 \\
1 &= \frac{\lambda_2}{(1 - \lambda_2^2 x_0^2)^2} \\
\lambda_2 &= 1 - 2\lambda_2^2 x_0^2 + \lambda_2^4 x_0^4
\end{aligned}$$

$$\boxed{\lambda_2^4 x_0^4 - 2\lambda_2^2 x_0^2 - \lambda_2 + 1 = 0} \tag{1.6}$$

c) If x_0=1, then we can use the quartic equation (1.6) found in part b) to find λ_2

$$\begin{aligned}
\lambda_2^4 - 2\lambda_2^2 - \lambda_2 + 1 = 0 \\
\lambda_2 = 0.525, 1.490
\end{aligned}$$

We will examine the optimal control sequence for both values of λ_2 and choose the control sequence with the lower performance index.

<u>If $\lambda_2 = 0.525$</u> From (1.4)

$$\begin{aligned}
\lambda_1 &= -\lambda_2^2 x_1 \\
\lambda_1 &= -\lambda_2^2 (\frac{1}{1 - \lambda_2^2 x_0^2}) \\
\lambda_1 &= -(0.525)^2 (\frac{1}{1 - (0.525)^2 (1)^2}) \\
\lambda_1 &= -0.3805 \\
\lambda_0 &= -\lambda_1^2 x_1 \\
\lambda_0 &= -(-0.3805)^2 (1) \\
\lambda_0 &= -0.1448
\end{aligned}$$

This gives us the costate sequence for $\lambda_2 = 0.525$

$$\lambda_k = \begin{bmatrix} -0.1448, & -0.3805, & 0.525 \end{bmatrix}$$

To get the state sequence, we use equation (1.5)

$$
\begin{aligned}
x_{k+1} &= -\lambda_{k+1} x_k^2 + 1 \\
x_1 &= -\lambda_1 x_0^2 + 1 \\
x_1 &= -(-0.3805)(1)^2 + 1 \\
x_1 &= 1.3805 \\
x_2 &= -\lambda_2 x_1^2 + 1 \\
x_2 &= -(0.525)(1.3805)^2 + 1 \\
x_2 &= 0 \ \ (expected) \\
x_k &= \begin{bmatrix} 1, & 1.3805, & 0 \end{bmatrix}
\end{aligned}
$$

To get the optimal control sequence, we solve equation (1.3)

$$
\begin{aligned}
u_k &= -\lambda_{k+1} x_k \\
u_1 &= -\lambda_2 x_1 \\
u_1 &= -(0.525)(1.3805) \\
u_1 &= -0.7248 \\
u_0 &= -\lambda_1 x_0 \\
u_0 &= -(-0.3805)(1) \\
u_0 &= 0.3805 \\
u_k &= \begin{bmatrix} 0.3805, & -0.7248 \end{bmatrix}
\end{aligned}
$$

Finally we need to calculate the performance index

$$
\begin{aligned}
J_2 &= 0 \\
J_1 &= \frac{1}{2} u_1^2 \\
J_1 &= 0.2627 \\
J_0 &= \frac{1}{2}(u_1^2 + u_2^2) \\
J_0 &= 0.3351 \\
J_k &= \begin{bmatrix} 0.3351, & 0.2627, & 0 \end{bmatrix}
\end{aligned}
$$

<u>If $\lambda_2 = 1.490$</u>

From (1.4)

$$
\begin{aligned}
\lambda_1 &= -\lambda_2^2 x_1 \\
\lambda_1 &= -\lambda_2^2 \left(\frac{1}{1 - \lambda_2^2 x_0^2}\right) \\
\lambda_1 &= -(1.490)^2 \left(\frac{1}{1 - (1.490)^2 (1)^2}\right) \\
\lambda_1 &= -1.8196 \\
\lambda_0 &= -\lambda_1^2 x_1 \\
\lambda_0 &= -(-1.8196)^2 (1) \\
\lambda_0 &= -3.3109
\end{aligned}
$$

This gives us the costate sequence for $\lambda_2 = 0.525$

$$
\lambda_k = \begin{bmatrix} -3.3109, & -1.8196, & 1.490 \end{bmatrix}
$$

To get the state sequence, we use equation (1.5)

$$
\begin{aligned}
x_1 &= \frac{1}{1-\lambda_2^2 x_0^2} \\
x_1 &= \frac{1}{1-(1.49)^2(1)^2} \\
x_1 &= -0.8196 \\
x_2 &= 0 \\
x_k &= \begin{bmatrix} 1, & -0.8196, & 0 \end{bmatrix}
\end{aligned}
$$

To get the optimal control sequence, we solve equation (1.3)

$$
\begin{aligned}
u_k &= -\lambda_{k+1} x_k \\
u_1 &= -\lambda_2 x_1 \\
u_1 &= -(1.49)(-0.8196) \\
u_1 &= 1.2212 \\
u_0 &= -\lambda_1 x_0 \\
u_0 &= -(-1.8196)(1) \\
u_0 &= 1.8196 \\
u_k &= \begin{bmatrix} 1.8196, & 1.2212 \end{bmatrix}
\end{aligned}
$$

Finally we need to calculate the performance index

$$
\begin{aligned}
J_2 &= 0 \\
J_1 &= \frac{1}{2} u_1^2 \\
J_1 &= 1.6555 \\
J_0 &= \frac{1}{2}(u_1^2 + u_2^2) \\
J_0 &= 2.4007 \\
J_k &= \begin{bmatrix} 2.4007, & 1.6555, & 0 \end{bmatrix}
\end{aligned}
$$

Because we are trying to minimize the performance index, we can see that we want to choose the optimal control sequence associated with λ_2=0.525

$$
\boxed{
\begin{aligned}
\lambda_k &= \begin{bmatrix} -0.1448, & -0.3805, & 0.525 \end{bmatrix} \\
x_k &= \begin{bmatrix} 1, & 1.3805, & 0 \end{bmatrix} \\
u_k &= \begin{bmatrix} 0.3805, & -0.7248 \end{bmatrix} \\
J_k &= \begin{bmatrix} 0.3351, & 0.2627, & 0 \end{bmatrix}
\end{aligned}
}
$$

2.1-2 Optimal control of a bilinear system

Consider the bilinear system

$$x_{k+1} = Ax_k + Dx_k u_k + bu_k \tag{2.1}$$

where $x_k \epsilon \mathbb{R}^n$, $u_k \epsilon \mathbb{R}$, with quadratic performance index

$$J = \frac{1}{2} x_N^T S_N x_N + \frac{1}{2} \sum_{k=0}^{N-1} (x_k^T S_k x_k + r u_k^2) \tag{2.2}$$

where $S_N \geq 0$, $Q \geq 0$, $r > 0$. Show that the optimal control is the bilinear state–costate feedback,

$$u_k = (-b + Dx_k)^T \lambda_{k+1}/r \tag{2.3}$$

and that the state and costate equations after eliminating u_k are

$$x_{k+1} = Ax_k - (b + Dx_k)(b + Dx_k)^T \lambda_{k+1}/r \tag{2.4}$$

$$\lambda_k = Qx_k + A^T \lambda_{k+1} - (b + Dx_k)^T \lambda_{k+1} D^T \lambda_{k+1}/r \tag{2.5}$$

Solution

We start by defining the Hamiltonian function, H^k,

$$H^k = L^k + \lambda_{k+1}^T f^k$$

where

$$L^k = x_k^T Q x_k + r u_k^2$$

$$f^k = Ax_k + dx_k u_k + bu_k$$

The state, costate, and stationary equations give us the following 3 sets of equations

$$x_{k+1} = \frac{\partial H^k}{\partial \lambda_{k+1}} = Ax_k + Dx_k u_k + bu_k$$

$$\lambda_k = \frac{\partial H^k}{\partial x_k} = \frac{1}{2}(Qx_k + Q^T x_k) + \frac{\partial}{\partial x_k}(\lambda_{k+1}^T A x_k + \lambda_{k+1}^T D x_k u_k)$$

$$\lambda_k = \frac{1}{2}(Qx_k + Q^T x_k) + A^T \lambda_{k+1} + D^T \lambda_{k+1} u_k \tag{2.6}$$

$$0 = \frac{\partial H^k}{\partial u_k} = r u_k + (Dx_k + b)^T \lambda_{k+1} \tag{2.7}$$

We can solve (2.7) for u_k

$$\boxed{u_k = -\frac{1}{r}(Dx_k + b)^T \lambda_{k+1}}$$

Plugging u_k, (2.3) into (2.1) gives

$$x_{k+1} = Ax_k + Dx_k((-\frac{1}{r}(Dx_k + b)^T \lambda_{k+1})) + b(-\frac{1}{r}(Dx_k + b)^T \lambda_{k+1})$$

$$\boxed{x_{k+1} = Ax_k - \frac{1}{r}(Dx_k + b)(Dx_k + b)^T \lambda_{k+1}}$$

Plugging u_k, (2.3) into (2.6)

$$\lambda_k = \frac{1}{2}(Qx_k + Q^T x_k) + A^T \lambda_{k+1} + D^T \lambda_{k+1}(-\frac{1}{r}(Dx_k + b)^T \lambda_{k+1})$$

$$\boxed{\frac{1}{2}(Qx_k + Q^T x_k) + A^T \lambda_{k+1} - \frac{1}{r}(Dx_k + b)^T \lambda_{k+1} D^T \lambda_{k+1}}$$

We have shown (2.3), (2.4), and (2.5) to be true which is what we wanted to do.

2.1-3

Rederive the equations in Table 2.1-1 to find the optimal controller for the nonlinear generalized state-space (or descriptor) system

$$Ex_{k+1} = f^k(x_k, u_k)$$

where E is singular. These systems often arise in circuit analysis, economics, and similar areas.

Solution

The system model is given to us as

$$Ex_{k+1} = f^k(x_k, u_k)$$

Assuming we use the same performance index

$$J_i = \phi(N, x_N) + \sum_{k=i}^{N-1} L_k(x_k, u_k)$$

Let $\lambda \epsilon \mathbb{R}^n$ and append the constraints to the performance index

$$J' = \phi(N, x_N) + \sum_{k=i}^{N-1} [L_k(x_k, u_k) + \lambda_{k+1}^T (f^k(x_k, u_k) - Ex_{k+1})]$$

Next, we define the Hamiltonian, H^k as

$$H^k(x_k, u_k) = L^k(x_k, u_k) + \lambda_{k+1}^T (f^k(x_k, u_k))$$

We can write

$$J' = \phi(N, x_N) + H^i - \lambda_N^T E x_N + \sum_{k=i+1}^{N-1} (H^k - \lambda_k^T E x_k)$$

Now we will look at the increment dJ' due to increment in the variables x_k, u_k, λ_k

$$dJ' = (\phi_{x_N} - E^T \lambda_N)^T dx_N + (H_{x_i}^i)^T dx_i + (H_{u_i}^i)^T du_i + \sum_{k=i+1}^{N-1} [(H_{x_k}^k - \lambda_k^T E)^T dx_k + (H_{u_k}^k)^T du_k + (H_{\lambda_k}^{k-1} - x_k^T E)^T d\lambda_k]$$

Necessary conditions for a constrained minimum are thus given by

$$\begin{aligned}
(\phi_{x_N} - E^T \lambda_N)^T dx_N &= 0 \\
\frac{\partial H^k}{\partial x_k} - \lambda_k^T E &= 0 \\
\frac{\partial H^{k-1}}{\partial \lambda_k} - x_k^T E &= 0 \\
\frac{\partial H^k}{\partial \lambda_{k+1}} - x_{k+1}^T E &= 0 \\
\frac{\partial H^k}{\partial u_k} &= 0
\end{aligned}$$

Based on the above set of equations, we can rewrite Table 2.1-1 as

System model	$Ex_{k+1} = f^k(x_k, u_k)$
Performance Index	$J_i = \phi(N, x_N) + \sum_{k=i}^{N-1} L_k(x_k, u_k)$
Hamiltonian	$H^k(x_k, u_k) = L^k(x_k, u_k) + \lambda_{k+1}^T(f^k(x_k, u_k))$
Optimal Controller State Equation:	$Ex_{k+1} = \frac{\partial H^k}{\partial \lambda_{k+1}} = f^k(x_k, u_k)$
Costate Equation:	$\lambda_k^T E = \frac{\partial H^k}{\partial x_k} = (\frac{\partial f^k}{\partial x_k})^T \lambda_{k+1} + \frac{\partial L^k}{\partial x_k}$
Stationarity Condition:	$0 = \frac{\partial H^k}{\partial u_k} = (\frac{\partial f^k}{\partial u_k})^T \lambda_{k+1} + \frac{\partial L^k}{\partial u_k}$
Boundary Conditions:	$(\phi_{x_N} - E^T \lambda_N)^T dx_N = 0$ $(\frac{\partial L^i}{\partial x_i} + (\frac{\partial f^i}{\partial x_i})^T \lambda_{i+1})^T dx_i = 0$.

2.2-2 Solutions to the algebraic Lyapunov equation

(a) Find all possible solutions to (2.2-26) if
$A = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}$, $C = \begin{bmatrix} 2 & 0 \end{bmatrix}$, $Q = C^T C$
Hint: Let

$$P = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}$$

substitute into (2.2-26), and solve for the scalars, p_i. Alternatively, the results of problem 2.1-1 can be used.

(b) Now find the symmetric solutions.

Solution

First, we solve for Q,

$$Q = C^T C = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \end{bmatrix}$$
$$Q = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

Now we solve the equation

$$S = A^T S A + Q$$

Per the hint we choose $S = P = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}$ which gives us the following equation to solve

$$\begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}p_1 & \frac{1}{2}p_2 \\ p_1 - \frac{1}{2}p_3 & p_2 - \frac{1}{2}p_4 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}p_1 & \frac{1}{2}p_1 - \frac{1}{4}p_2 \\ \frac{1}{2}p_1 - \frac{1}{4}p_3 & p_1 - \frac{1}{2}p_2 - \frac{1}{2}p_3 + \frac{1}{4}p_4 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} \frac{3}{4}p_1 & -\frac{1}{2}p_1 + \frac{5}{4}p_2 \\ -\frac{1}{2}p_1 + \frac{5}{4}p_3 & -p_1 + \frac{1}{2}p_2 + \frac{1}{2}p_3 + \frac{3}{4}p_4 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

This gives us 4 equations with 4 unknowns

$$\frac{3}{4}p_1 = 4$$
$$-\frac{1}{2}p_1 + \frac{5}{4}p_2 = 0$$
$$-\frac{1}{2}p_1 + \frac{5}{4}p_3 = 0$$
$$-p_1 + \frac{1}{2}p_2 + \frac{1}{2}p_3 + \frac{3}{4}p_4 = 0$$

This can be written as

$$\begin{bmatrix} \frac{3}{4} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{5}{4} & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{5}{4} & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{5}{4} & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{5}{4} & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 5.333 \\ 2.133 \\ 2.133 \\ 4.267 \end{bmatrix}$$

This gives us

$$\boxed{P = \begin{bmatrix} 5.333 & 2.133 \\ 2.133 & 4.267 \end{bmatrix}}$$

This is a symmetric solution.

2.2-4 Control of a scalar system

Let

$$x_{k+1} = 2x_k + u_k$$

(a) Find the homogeneous solution x_k for k = 0, 5 if $x_0 = 3$

(b) Find the minimum-energy control sequence u_k required to drive $x_0 = 3$ to $x_5 = 0$. Check your answer by finding the resulting state trajectory.

(c) Find the optimal feedback gain sequence K_k to minimize the performance index

$$J_0 = 5x_5^2 + \frac{1}{2}\sum_{k=0}^{4}(x_k^2 + u_k^2)$$

Find the resulting state trajectory and the costs to go J_k^* for k = 0, 5.

Solution

a) The homogenous solution implies u_k=0 which gives

$$x_{k+1} = 2x_k$$

Knowing x_0=3, we can solve for x_k by plugging in for k=0,1,2,3,4

$$\boxed{x_k = \begin{bmatrix} 3, & 6, & 12, & 24, & 48 \end{bmatrix}}$$

b) We wish to find the minimum-energy solution with a fixed final state so we use the performance index

$$J = \frac{1}{2}\sum_{k=0}^{4}(u_k^2)$$

From the performance index, we see that r=1.
Comparing our system model $x_{k+1} = 2x_k + u_k$ to the generic system model $x_{k+1} = ax_k + bu_k$, we see that a=2 and b=1
The optimal control sequence is given by (2.2-38) in the textbook.

$$u_k^* = \frac{1}{r}(b)(a)^{N-k-1}G_{0,N}^{-1}(r_N - A^N x_0)$$
$$u_k^* = \frac{1}{1}(1)(2)^{5-k-1}G_{0,5}^{-1}(1 - (2^5)(3))$$
$$u_k^* = (2)^{4-k}G_{0,5}^{-1}(-95)$$

where $G_{0,N}$ is defined as

$$U_N \begin{bmatrix} \frac{1}{r} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{r} \end{bmatrix} U_N^T$$

where U_N is defined as

$$\begin{bmatrix} b & ab & \cdots & a^{N-1}b \end{bmatrix}$$

Now, we can solve U_5 as

$$\begin{bmatrix}1 & 2 & 4 & 8 & 16\end{bmatrix}$$

We plug this in to find $G_{0,5}$

$$\begin{aligned}G_{0,5} &= \begin{bmatrix}1 & 2 & 4 & 8 & 16\end{bmatrix} I \begin{bmatrix}1 & 2 & 4 & 8 & 16\end{bmatrix}^T \\ G_{0,5} &= 341\end{aligned}$$

Plugging this into u_k^*, we find the optimal control sequence we are seeking.

$$u_k^* = -\frac{95}{341}(2)^{4-k}$$

Plugging in for k=0,1,2,3,4 gives

$$\boxed{u_k = \begin{bmatrix}-4.4575, & -2.2287, & -1.1144, & -0.5572, & -0.2786\end{bmatrix}} \tag{1}$$

c) This is a free final state with closed loop control. From the given performance index, we can see that s_N=10, $q_k = r_k$=1.
Since we know s_N, we can use(2.2-52) from the textbook to solve backwards in time for s_k.
The equation is given as

$$\begin{aligned}s_k &= a_k s_{k+1}(1 + b_k \frac{1}{r_k} b_k s_{k+1})^{-1} a_k + q_k \\ s_k &= \frac{(2)s_{k+1}(2)}{1 + (1)(\frac{1}{1})(1)s_{k+1}} + 1 \\ s_k &= \frac{4s_{k+1}}{1 + s_{k+1}} + 1\end{aligned}$$

Plugging in k=0,1,2,3,4 gives us

$$s_k = \begin{bmatrix}4.2362, & 4.2372, & 4.2439, & 4.2903, & 4.6364, & 10\end{bmatrix}$$

Now that we have the sequence s_k, we can find the optimal feedback gain sequence K_k.
The equation for the optimal feedback gain sequence is given by (2.2-57) in the textbook.

$$\begin{aligned}K_k &= (b_k s_{k+1} b_k + r_k)^{-1} b_k s_{k+1} a_k \\ K_k &= ((1)s_{k+1}(1) + 1)^{-1}(1)s_{k+1}(2) \\ K_k &= \frac{2s_{k+1}}{s_{k+1} + 1}\end{aligned}$$

Plugging in k=0,1,2,3,4 gives us

$$K_k = \begin{bmatrix}1.6181, & 1.6186, & 1.6219, & 1.6452, & 1.8182\end{bmatrix}$$

To find the optimal state trajectory, we use (2.2-50) from the textbook and iterate over k=0,1,2,3,4

$$\begin{aligned}x_k &= (a - bK_k)x_k \\ x_k &= \begin{bmatrix}3.0, & 1.1457, & 0.4370, & 0.1652, & 0.0586, & 0.0107\end{bmatrix}\end{aligned}$$

We can solve for u_k using the relation $u_k = -K_k x_k$. Iterating over k=0,1,2,3,4 gives us

$$u_k = \begin{bmatrix}-4.8543, & -1.8544, & -0.7088, & -0.2718, & -0.1065\end{bmatrix}$$

This can now be plugged into our performance index (shown below) to calculated J_k^*

$$J_i = 5x_5^2 + \frac{1}{2}\sum_{k=i}^{4}(x_k^2 + u_k^2)$$

Plugging in and solving for J_k^* for k=0,1,2,3,4,5 gives

$$J_k^* = \begin{bmatrix}5.7245E-4, & 0.008, & 0.0586, & 0.4053, & 2.781, & 19.0631\end{bmatrix}$$

Therefore we have found the optimal state trajectory, the optimal control sequence, and the performance indices

$$\boxed{\begin{aligned} x_k &= \begin{bmatrix}3.0, & 1.1457, & 0.4370, & 0.1652, & 0.0586, & 0.0107\end{bmatrix} \\ u_k &= \begin{bmatrix}-4.8543, & -1.8544, & -0.7088, & -0.2718, & -0.1065\end{bmatrix} \\ J_k^* &= \begin{bmatrix}5.7245E-4, & 0.008, & 0.0586, & 0.4053, & 2.781, & 19.0631\end{bmatrix}\end{aligned}}$$