MECE 6374: Fun Work #9

Eric Eldridge (1561585)

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Problem 1

For the nonlinear system

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = -sin(x_1) + x_1^3 + u$$

find a feedback control law

$$u = u(x_1, x_2)$$

that results in a feedback linearized system. Design the control law such that the closed-loop system is equivalent to a linear system with closed-loop poles at -1 and -2.

Solution

We can see that $\dot{x}_1$ is already linearized and we only need to linearize $\dot{x}_2$ which has our control u in it. If we choose

$$u = sin(x_1) - x_1^3 + v$$

where we will choose $v = Kx$ later when placing our eigenvalues.
Our system can now be represented as

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

The simplest choice to achieve our desired eigenvalues is $v = -2x_2$. This gives the new state space representation of the system as

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} x$$

$$\boxed{\boldsymbol{u = sin(x_1) - x_1^3 - 2x_2}}$$

Problem 2

Consider the system:

$$\begin{aligned}\dot{x}_1 &= sin(x_2)\\ \dot{x}_2 &= cos(x_3)\\ \dot{x}_3 &= u\end{aligned}$$

(a) Show that the following state transformation is a feedback linearizing transformation for the system

$$\begin{aligned}z_1 &= x_1\\ z_2 &= sin(x_2)\\ z_3 &= cos(x_2)cos(x_3)\end{aligned}$$

(b) Design a state feedback control law so that the closed loop poles of the equivalent linearized system are -1, -2 and -3.

Solution

(a) Taking the derivative of the state transformation gives the following set of equations

$$\begin{aligned}\dot{z}_1 &= \dot{x}_1 = sin(x_2)\\ \dot{z}_2 &= \dot{x}_2 cos(x_2) = cos(x_2)cos(x_3)\\ \dot{z}_3 &= -sin(x_2)cos(x_3)\dot{x}_2 - cos(x_2)sin(x_3)\dot{x}_3\\ \dot{z}_3 &= -sin(x_2)cos^2(x_3) + cos(x_2)sin(x_3)u\end{aligned}$$

Next we want to describe the RHS of the equations in terms of z_1, z_2, z_3.

$$\begin{aligned}x_1 &= z_1\\ x_2 &= sin^{-1}(z_2)\\ x_3 &= cos^{-1}(\frac{z_3}{cos(sin^{-1}(z_2))})\\ x_3 &= cos^{-1}(\frac{z_3}{\sqrt{1-z_2^2})})\end{aligned}$$

$$\begin{aligned}\dot{z}_1 &= z_2\\ \dot{z}_2 &= z_3\\ \dot{z}_3 &= -z_2 cos^2(x_3) + cos(x_2)sin(x_3)u\\ \dot{z}_3 &= -\frac{z_2 z_3^2}{1-z_2^2} + \cancel{\sqrt{1-z_2^2}} \cdot \frac{\sqrt{1-z_2^2-z_3^2}}{\cancel{\sqrt{1-z_2^2}}} u\\ \dot{z}_3 &= -\frac{z_2 z_3^2}{1-z_2^2} + \sqrt{1-z_2^2-z_3^2}\,u\end{aligned}$$

We have now linearized the system for every state variable except the one containing our control effort, u. We can now choose u s.t. $\dot{z}_3 = v$

$$\begin{aligned}\dot{z}_3 &= -\frac{z_2 z_3^2}{1-z_2^2} + \sqrt{1-z_2^2-z_3^2}\,u = v\\ u &= \frac{v + \frac{z_2 z_3^2}{1-z_2^2}}{\sqrt{1-z_2^2-z_3^2}} u = \frac{sin(x_2)cos^2(x_3)}{cos(x_2)sin(x_3)} + v\end{aligned}$$

where we will choose $v = Kz$ later when placing our eigenvalues.
Our system can now be represented as

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v$$

We now seek to choose our K (from $v = Kz$) s.t. we have poles at -1, -2, -3. Using the place function in Matlab, we find

$$K = \begin{bmatrix} -6 & -11 & -6 \end{bmatrix}$$

We can now use this information to solve for u

$$u = \frac{sin(x_2)cos^2(x_3)}{cos(x_2)sin(x_3)} + v$$
$$u = \frac{sin(x_2)cos^2(x_3)}{cos(x_2)sin(x_3)} - 6z_1 - 11z_2 - 6z_3$$

$$\boxed{u = tan(x_2)\frac{cos^2(x_3)}{sin(x_3)} - 6x_1 - 11sin(x_2) - 6cos((x_2)cos(x_3))}$$

Problem 3

Consider the system:

$$\dot{x}_1 = e^{x_2} - 1$$
$$\dot{x}_2 = ax_1^2 + u$$

(1) Is the system feedback linearizable?

(2) Find a nonlinear state transformation that results in a feedback linearization of this system

(3) Find the feedback control law that places the closed loop poles of the feedback linearized system at -1 and -2.

Solution

(1) The NL system $\dot{x} = f(x) + g(x)u$ is feedback linearizable iff

(a) The set

$$\{g, ad_f g, ad_f^2 g, \cdots, ad_f^{n-1} g\}$$

is linearly independent

(b) The set

$$\{g, ad_f g, ad_f^2 g, \cdots, ad_f^{n-2} g\}$$

is involutive.

We can see that $f(x) = \begin{bmatrix} e^{x_2} - 1 \\ ax_1^2 \end{bmatrix}$ and $g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$ad_f g = \cancelto{0}{\nabla g} \cdot f - \nabla f \cdot g$$
$$ad_f g = -\begin{bmatrix} 0 & e^{x_2} \\ 2ax_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$ad_f g = \begin{bmatrix} e^{x_2} \\ 0 \end{bmatrix}$$

Now we check the rank of $\begin{bmatrix} g & ad_f g \end{bmatrix} = \begin{bmatrix} 0 & e^{x_2} \\ 1 & 0 \end{bmatrix}$ to verify linear independence.

From observation we can see that rank=2 and our first criteria is satisfied.

For the second criteria, we know that for a constant vector, g(x), involutivity is satisfied.

Therefore, since both criteria are satisfied, we have shown that **this system is feedback linearizable**.

(2) Now that we've shown the system is feedback linearizable, we seek to find the transformation $z = z(x)$.

$$
\begin{aligned}
\nabla z_1 \cdot g = 0 &\implies \begin{bmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \\
\implies \frac{\partial z_1}{\partial x_2} &= 0 \\
\nabla z_1 \cdot ad_f g \neq 0 &\implies \begin{bmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} \end{bmatrix} \begin{bmatrix} e^{x_2} \\ 0 \end{bmatrix} \neq 0 \\
\frac{\partial z_1}{\partial x_1} &\neq 0 \\
Choose\ z_1 &= x_1 \\
z_2 &= L_f z_1 = \nabla z_1 \cdot f \\
z_2 &= \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} e^{x_2} - 1 \\ a x_1^2 \end{bmatrix} \\
z_2 &= e^{x_2} - 1
\end{aligned}
$$

This gives us the transformation

$$
\begin{aligned}
z_1 &= x_1 \\
z_2 &= e^{x_2} - 1 \\
\dot{z}_1 &= \dot{x}_1 = e^{x_2} - 1 = z_2 \\
\dot{z}_2 &= e^{x_2} \dot{x}_2 = e^{x_2}(a x_1^2 + u) \\
\dot{z}_2 &= (1 + z_2)(a z_1^2 + u)
\end{aligned}
$$

We have now linearized the system for every state variable except the one containing our control effort, u. We can now choose u s.t. $\dot{z}_2 = v$ is linearized.

$$
\begin{aligned}
v &= (1 + z_2)(a z_1^2 + u) \\
u &= \frac{v}{1 + z_2} - a z_1^2
\end{aligned}
$$

$$
\dot{z} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v
$$

The simplest choice to achieve our desired eigenvalues is $v = Kx$. This gives the new state space representation of the system as

$$
\dot{z} = \begin{bmatrix} 0 & 1 \\ k_1 & k_2 \end{bmatrix} z
$$

(3) To set our eigenvalues at -1,-2, we want the characteristic equation to

$$
(\lambda + 1)(\lambda + 2) = \lambda^2 + 3\lambda + 2 = 0
$$

The characteristic equation of our system is

$$
\begin{aligned}
det(A - \lambda I) &= 0 \\
(\lambda)(\lambda - k_2) - k_1 &= 0 \\
\lambda^2 - k_2 \lambda - k_1 &= 0 = \lambda^2 + 3\lambda + 2 \\
\implies k_1 = -2, k_2 &= -3
\end{aligned}
$$

Therefore, we have found $z = -2z_1 - 3z_2$. We can plug this in to find our u that will provide feedback linearization.

$$u = \frac{v}{1 + z_2} - az_1^2$$
$$u = \frac{-2z_1 - 3z_2}{1 + z_2} - az_1^2$$
$$u = \frac{-2x_1 - 3(e^{x_2} - 1)}{e^{x_2}} - ax_1^2$$

$$\boxed{u = \frac{-2x_1 - 3e^{x_2} + 3}{e^{x_2}} - ax_1^2}$$