MECE 6374: Fun Work #9

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Problem 1

For the nonlinear system

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = -sin(x_1) + x_1^3 + u$$

find a feedback control law

$$u = u(x_1, x_2)$$

that results in a feedback linearized system. Design the control law such that the closed-loop system is equivalent to a linear system with closed-loop poles at -1 and -2.

Solution

We can see that $\dot{x}_1$ is already linearized and we only need to linearize $\dot{x}_2$ which has our control u in it. If we choose

$$u = sin(x_1) - x_1^3 + v$$

where we will choose $v = Kx$ later when placing our eigenvalues.
Our system can now be represented as

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

The simplest choice to achieve our desired eigenvalues is $v = -2x_2$. This gives the new state space representation of the system as

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} x$$

$$\boxed{\boldsymbol{u = sin(x_1) - x_1^3 - 2x_2}}$$

Problem 2

Consider the system:

$$\dot{x}_1 = sin(x_2)$$
$$\dot{x}_2 = cos(x_3)$$
$$\dot{x}_3 = u$$

(a) Show that the following state transformation is a feedback linearizing transformation for the system

$$z_1 = x_1$$
$$z_2 = sin(x_2)$$
$$z_3 = cos(x_2)cos(x_3)$$

(b) Design a state feedback control law so that the closed loop poles of the equivalent linearized system are -1, -2 and -3.

Solution

(a) Taking the derivative of the state transformation gives the following set of equations

$$\dot{z}_1 = \dot{x}_1$$
$$\dot{z}_2 = \dot{x}_2 cos(x_2) = cos(x_2)cos(x_3)$$
$$\dot{z}_3 = -sin(x_2)cos(x_3)\dot{x_2} - cos(x_2)sin(x_3)\dot{x_3}$$
$$\dot{z}_3 = -sin(x_2)cos^2(x_3) + cos(x_2)sin(x_3)u$$

Next we want to describe the RHS of the equations in terms of z_1, z_2, z_3.

$$x_1 = z_1$$
$$x_2 = sin^{-1}(z_2)$$
$$x_3 = cos^{-1}(\frac{z_3}{cos(sin^{-1}(z_2))})$$

$$\dot{z}_1 = z_1$$
$$\dot{z}_2 = z_3$$
$$\dot{z}_3 = -z_2 cos^2(x_3) + cos(x_2)sin(x_3)u$$

We have now linearized the system for every state variable except the one containing our control effort, u. We can now choose u s.t. $\dot{z}_3$ is linearized.

$$u = \frac{z_2 cos^2(x_3)}{cos(x_2)sin(x_3)} + v$$

where we will choose $v = Kx$ later when placing our eigenvalues.
Our system can now be represented as

$$\dot{z} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v$$

Problem 3

Consider the system:

$$\dot{x}_1 = e^{x_2} - 1$$
$$\dot{x}_2 = ax_1^2 + u$$

(1) Is the system feedback linearizable?

(2) Find a nonlinear state transformation that results in a feedback linearization of this system

(3) Find the feedback control law that places the closed loop poles of the feedback linearized system at -1 and -2.

Solution

(1) The NL system is feedback $\dot{x} = f(x) + g(x)u$ linearizable iff

(a) The set

$$\{g, ad_f g, ad_f^2 g, \cdots, ad_f^{n-1} g\}$$

is linearly independent

(b) The set

$$\{g, ad_f g, ad_f^2 g, \cdots, ad_f^{n-2} g\}$$

is involutive.

We can see that $f(x) = \begin{bmatrix} e^{x_2} - 1 \\ ax_1^2 \end{bmatrix}$ and $g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$ad_f g = \cancelto{0}{\nabla g} \cdot f - \nabla f \cdot g$$
$$ad_f g = -\begin{bmatrix} 0 & e^{x_2} \\ 2ax_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$ad_f g = \begin{bmatrix} e^{x_2} \\ 0 \end{bmatrix}$$

Now we check the rank of $\begin{bmatrix} g & ad_f g \end{bmatrix} = \begin{bmatrix} 0 & e^{x_2} \\ 1 & 0 \end{bmatrix}$
From observation we can see that rank=2 and our first criteria is satisfied.

For the second criteria, we know that for a constant vector, g(x), involutivity is satisfied.

Therefore, since both criteria are satisfied, we have shown that **this system is feedback linearizable**.

(2) Now that we've shown the system is feedback linearizable, we now seek to find the transformation $z = z(x)$.

$$
\begin{aligned}
\nabla z_1 \cdot g = 0 &\implies \begin{bmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \\
&\implies \frac{\partial z_1}{\partial x_2} = 0 \\
\nabla z_1 \cdot ad_f g \neq 0 &\implies \begin{bmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} \end{bmatrix} \begin{bmatrix} e^{x_2} \\ 0 \end{bmatrix} \neq 0 \\
&\frac{\partial z_1}{\partial x_1} \neq 0 \\
Choose\ z_1 &= x_1 \\
&z_2 = L_f z_1 = \nabla z_1 \cdot f \\
&z_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} e^{x_2} - 1 \\ a x_1^2 \end{bmatrix} \\
&z_2 = e^{x_2} - 1
\end{aligned}
$$

This gives us the transformation

$$
\begin{aligned}
z_1 &= x_1 \\
z_2 &= e^{x_2} - 1 \\
\dot{z}_1 &= \dot{x}_1 = e^{x_2} - 1 = z_2 \\
\dot{z}_2 &= e^{x_2}\dot{x}_2 = e^{x_2}(a x_1^2 + u) \\
\dot{z}_2 &= (1 + z_2)(a z_1^2 + u)
\end{aligned}
$$

We have now linearized the system for every state variable except the one containing our control effort, u. We can now choose u s.t. $\dot{z}_2 = v$ is linearized.

$$
\begin{aligned}
v &= (1 + z_2)(a z_1^2 + u) \\
u &= \frac{v}{1 + z_2} - a z_1^2
\end{aligned}
$$

$$
\dot{z} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v
$$

The simplest choice to achieve our desired eigenvalues is $v = Kx$. This gives the new state space representation of the system as

$$
\dot{z} = \begin{bmatrix} 0 & 1 \\ k_1 & k_2 \end{bmatrix} z
$$

To set our eigenvalues at -1,-2, we want the characteristic equation to

$$
(\lambda + 1)(\lambda + 2) = \lambda^2 + 3\lambda + 2 = 0
$$

The characteristic equation of our system is

$$
\begin{aligned}
det(A - \lambda I) &= 0 \\
(\lambda)(lambda - k_2) - k_1 &= 0 \\
\lambda^2 - k_2\lambda - k_1 &= 0 = \lambda^2 + 3\lambda + 2 \\
\implies k_2 = -3, k_1 = -2
\end{aligned}
$$