MECE 6374: Fun Work #8

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Problem 1

Consider the nonlinear systems

$$\dot{x}_1 = x_1 x_2$$
$$\dot{x}_2 = x_1^5 + u$$

(a) Using a Lyapunov function $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ to find a control law $u = u(x)$ such that the closed-loop system is globally stable.

(b) Can you show Global Asymptotic Stability?

Solution

a) We can see from inspection that V(x) is globally positive definite and radially unbounded. We now want to check the properties of $\dot{V}(x)$.

$$\begin{aligned}
V(x) &= \frac{1}{2}(x_1^2 + x_2^2) \\
\dot{V}(x) &= x_1\dot{x}_1 + x_2\dot{x}_2 \\
\dot{V}(x) &= x_1(x_1x_2) + x_2(x_1^5 + u) \\
\dot{V}(x) &= x_1^2x_2 + x_1^5x_2 + x_2u \\
\dot{V}(x) &= x_2(x_1^2 + x_1^5 + u)
\end{aligned}$$

To show global stability, we need to choose u such that $\dot{V}$ is at least global negative semi-definite. Let us choose u such that $\dot{V}(x) = -x_2^2$.

$$\begin{aligned}
\dot{V}(x) &= x_2(x_1^2 + x_1^5 + u) \\
-x_2^2 &= x_2(x_1^2 + x_1^5 + u) \\
-x_2 &= x_1^2 + x_1^5 + u \\
u &= -x_2 - x_1^2 - x_1^5 \\
\implies \dot{V}(x) &= -x_2^2
\end{aligned}$$

This shows that if we set $u = -x_2 - x_1^2 - x_1^5$, then the closed-loop system is globally stable.

b) We can attempt to show Global Asymptotic Stability by applying LaSalle's theorem. LaSalle's Theorem states that for a stable system, if the set of x s.t. $\dot{V}(x) = 0$ consists strictly of $x = 0$ for any time,t, the the system is not only stable, but asymptotically stable.

$$
\begin{aligned}
\dot{V}(x) &= -x_2^2 = 0 \\
\implies x_2 &= 0 \; \forall \; t \\
\implies \dot{x}_2 &= 0 \\
\cancel{x_1^5} - \cancelto{0}{x_2} - x_1^2 - \cancel{x_1^5} &= 0 \\
-x_1^2 &= 0 \\
\implies x_1 &= 0
\end{aligned}
$$

Therefore the tenants of LaSalle's theorem are met and we have shown **the system is Globally Asymptotically Stable**.

Problem 2

Consider the following controlled nonlinear system

$$\begin{aligned}\dot{x}_1 &= sin(x_2)cos(x_1) + x_1^3 + u \\ \dot{x}_2 &= -x_1 x_2 cos(x_1)\end{aligned}$$

Use a Lyapunov function candidate $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ to design a feedback control law u(x) guaranteeing that the origin of the closed loop system is asymptotically stable.

Solution

Solution

We can see from inspection that V(x) is globally positive definite and radially unbounded. We now want to check the properties of $\dot{V}(x)$.

$$\begin{aligned}V(x) &= \frac{1}{2}(x_1^2 + x_2^2) \\ \dot{V}(x) &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ \dot{V}(x) &= x_1(sin(x_2)cos(x_1) + x_1^3 + u) + x_2(-x_1x_2cos(x_1)) \\ \dot{V}(x) &= x_1[sin(x_2)cos(x_1) + x_1^3 + u - x_2^2cos(x_1)]\end{aligned}$$

To show global stability, we need to choose u such that $\dot{V}$ is at least global negative semi-definite. Let us choose u such that $\dot{V}(x) = -x_2^2$.

$$\begin{aligned}\dot{V}(x) &= x_1[sin(x_2)cos(x_1) + x_1^3 + u - x_2^2cos(x_1)] \\ -x_1^2 &= x_1[sin(x_2)cos(x_1) + x_1^3 + u - x_2^2cos(x_1)] \\ -x_1 &= sin(x_2)cos(x_1) + x_1^3 + u - x_2^2cos(x_1) \\ u &= -x_1 - sin(x_2)cos(x_1) - x_1^3 + x_2^2cos(x_1)\end{aligned}$$

This shows that if we set $u = -x_1 - sin(x_2)cos(x_1) - x_1^3 + x_2^2cos(x_1)$, then the closed-loop system is globally stable.

We can attempt to show Global Asymptotic Stability by applying LaSalle's theorem.

$$\begin{aligned}\dot{V}(x) &= -x_1^2 = 0 \\ \implies x_1 &= 0 \; \forall \, t \\ \implies \dot{x}_1 &= 0 \\ \cancel{sin(x_2)cos}(x_1) + \cancel{x_1^3} - \cancelto{0}{x_1} - \cancel{sin(x_2)cos}(x_1) - \cancel{x_1^3} + x_2^2\cancelto{1}{cos(x_1)} &= 0 \\ x_2^2 &= 0 \\ \implies x_2 &= 0\end{aligned}$$

Therefore the tenants of LaSalle's theorem are met and we have shown **the system is Globally Asymptotically Stable**.