MECE 6374: Fun Work #6

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Problem 1

Consider the following system

$$\begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= -2x_2 + x_1 x_2^2 \end{aligned}$$

Use the variable gradient method (VGM) to construct a Lyapunov function for the system and determine the stability of the equilibrium point at the origin.

Solution

With VGM, we first assume $\nabla V = \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix} = \begin{bmatrix} \alpha_{11}x_1 + \alpha_{12}x_2 \\ \alpha_{21}x_1 + \alpha_{22}x_2 \end{bmatrix}$

Next step is find α_{ij} using the curl condition

$$\begin{aligned} \frac{\partial g_1}{\partial x_2} &= \frac{\partial g_2}{\partial x_1} \\ \implies \alpha_{12} &= \alpha 21 \end{aligned}$$

Next step is to find $\dot{V}$ and restrict the coefficients α_{ij} s.t. $\dot{V}$ is at least negative semi-definite.

$$\begin{aligned} \dot{V} &= \frac{dV}{dt} = \frac{\partial V}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial V}{\partial x_2} \cdot \frac{dx_2}{dt} \\ \dot{V} &= (\alpha_{11}x_1 + \alpha_{12}x_2)(-x_1) + (\alpha_{12}x_1 + \alpha_{22}x_2)(-2x_2 + x_1x_2^2) \\ \dot{V} &= -\alpha_{11}x_1^2 - \alpha_{12}x_1x_2 - 2\alpha_{12}x_1x_2 - 2\alpha_{22}x_2^2 + \alpha_{22}x_1x_2^3 \\ \dot{V} &= -\alpha_{11}x_1^2 - 2\alpha_{22}x_2^2 - 3\alpha_{12}x_1x_2 + \alpha_{22}x_1x_2^3 \\ \dot{V} &= -\alpha_{11}x_1^2 - 2\alpha_{22}x_2^2 + x_1x_2(\alpha_{22}x_2^2 - 3\alpha_{12}) \end{aligned}$$

Let $\alpha_{11} = \alpha_{22} = 1, \alpha_{12} = \alpha_{21} = 0$. Then

$$\dot{V} = -x_1^2 - 2x_2^2 + x_1x_2^3$$

We can see that $\dot{V}$ **is locally negative definite.**

Now that we know the properties of $\dot{V}$, we will construct V(x) so that we can examine what properties the Lyapunov function has.

$$
\begin{aligned}
V(x) &= \int_0^{x_1} g_1(x)dx_1 + \int_0^{x_2} g_2(x)dx_2 \\
V(x) &= \int_0^{x_1} x_1 dx_1 + \int_0^{x_2} x_2 dx_2 \\
V(x) &= \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2
\end{aligned}
$$

It's clear that **V(x) is globally positive definite and radially unbounded.**

Combined with what we know about $\dot{V}$(x), we can see that **the equilibrium point (0,0) is locally, asymptotically stable**

Problem 2

Consider the following two nonlinear systems:

$$\dot{x}_1 = -x_1^3 + x_2$$
$$\dot{x}_2 = -x_1^3 - x_2^3$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1^3$$

(i) Can you determine the stability of the equilibrium point at (0, 0) using linearization?

(ii) Can you determine stability of the equilibrium point at (0, 0) using a quadratic type Lyapunov function?

(iii) Can you determine the stability of the equilibrium point at (0, 0) using a Lyapunov function $V(x_1, x_2) = x_1^4 + 2x_2^2$

Solution

(i) The first system can be linearized as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$$

The second system can also be linearized as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$$

Both systems have repeated eignevalues of 0. Therefore, we cannot determine the stability of the equilibrium points using linearization.

(ii) System 1: A quadratic type Lyapunov function takes the form

$$V(x) = \alpha x_1^2 + \beta x_2^2$$

We can see that this function is gpd and radially unbounded. Now we want to examine the properties of $\dot{V}$.

$$\dot{V}(x) = 2\alpha x_1 \dot{x}_1 + 2\beta x_2 \dot{x}_2$$
$$\dot{V}(x) = 2\alpha x_1(-x_1^3 + x_2) + 2\beta x_2(-x_1^3 - x_2^3)$$
$$\dot{V}(x) = -2\alpha x_1^4 + 2\alpha x_1 x_2 - 2\beta x_2^4 - 2\beta x_1^3 x_2$$

To find the properties of $\dot{V}(x)$, we must look at $\frac{\partial \dot{V}}{\partial x}$ and $\frac{\partial^2 \dot{V}}{\partial x^2}$

$$\frac{\partial \dot{V}}{\partial x} = \begin{bmatrix} -8\alpha x_1^3 + 2\alpha x_2 - 6\beta x_1^2 x_2 \\ 2\alpha x_1 - 8\beta x_2^3 - 2\beta x_1^3 \end{bmatrix}$$
$$\frac{\partial \dot{V}}{\partial x}|_{(0,0)} = \begin{bmatrix} -8\alpha x_1^3 + 2\alpha x_2 - 6\beta x_1^2 x_2 \\ 2\alpha x_1 - 8\beta x_2^3 - 2\beta x_1^3 \end{bmatrix}|_{(0,0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\frac{\partial^2 \dot{V}}{\partial x^2} = \begin{bmatrix} -24\alpha x_1^2 - 12\beta x_1 x_2 & 2\alpha - 6\beta x_1^2 \\ 2\alpha - 6\beta x_1^2 & -24\beta x_2^2 \end{bmatrix}$$
$$\frac{\partial^2 \dot{V}}{\partial x^2}|_{(0,0)} = \begin{bmatrix} -24\alpha x_1^2 - 12\beta x_1 x_2 & 2\alpha - 6\beta x_1^2 \\ 2\alpha - 6\beta x_1^2 & -24\beta x_2^2 \end{bmatrix}|_{(0,0)} = \begin{bmatrix} 0 & 2\alpha \\ 2\alpha & 0 \end{bmatrix}$$

We see that $\frac{\partial \dot{V}}{\partial x} \leq 0$ and $\frac{\partial^2 \dot{V}}{\partial x^2} \leq 0$. Therefore, we can conclude that $\dot{V}$ is locally negative semi-definite.

This implies that **the first system is locally stable at the equilibrium point (0,0)**.

System 2: A quadratic type Lyapunov function takes the form

$$V(x) = \alpha x_1^2 + \beta x_2^2$$

We can see that this function is globally positive definite and radially unbounded. Now we want to examine the properties of $\dot{V}$.

$$\begin{aligned}
\dot{V}(x) &= 2\alpha x_1 \dot{x}_1 + 2\beta x_2 \dot{x}_2 \\
\dot{V}(x) &= 2\alpha x_1 (x_2) + 2\beta x_2 (-x_1^3) \\
\dot{V}(x) &= 2\alpha x_1 x_2 - 2\beta x_1^3 x_2
\end{aligned}$$

To find the properties of $\dot{V}(x)$, we must look at $\frac{\partial \dot{V}}{\partial x}$ and $\frac{\partial^2 \dot{V}}{\partial x^2}$

$$\begin{aligned}
\frac{\partial \dot{V}}{\partial x} &= \begin{bmatrix} 2\alpha x_2 - 6\beta x_1^2 x_2 \\ 2\alpha x_1 - 2\beta x_1^3 \end{bmatrix} \\
\frac{\partial \dot{V}}{\partial x}|_{(0,0)} &= \begin{bmatrix} 2\alpha x_2 - 6\beta x_1^2 x_2 \\ 2\alpha x_1 - 2\beta x_1^3 \end{bmatrix}|_{(0,0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\frac{\partial^2 \dot{V}}{\partial x^2} &= \begin{bmatrix} -12\beta x_1 x_2 & 2\alpha - 6\beta x_1^2 \\ 2\alpha - 6\beta x_1^2 & 0 \end{bmatrix} \\
\frac{\partial^2 \dot{V}}{\partial x^2}|_{(0,0)} &= \begin{bmatrix} -12\beta x_1 x_2 & 2\alpha - 6\beta x_1^2 \\ 2\alpha - 6\beta x_1^2 & 0 \end{bmatrix}|_{(0,0)} = \begin{bmatrix} 0 & 2\alpha \\ 2\alpha & 0 \end{bmatrix}
\end{aligned}$$

We see that $\frac{\partial \dot{V}}{\partial x} \leq 0$ and $\frac{\partial^2 \dot{V}}{\partial x^2} \leq 0$. Therefore, we can conclude that $\dot{V}$ is locally negative semi-definite.

This implies that **the second system is locally stable at the equilibrium point (0,0)**.

(iii) If we choose $V(x) = x_1^4 + 2x_2^2$, then we know that the Lyapunov function is both globally negative definite and radially unbounded. Next we want to examine the properties of $\dot{V}(x)$.

<u>System 1:</u>

$$
\begin{aligned}
V(x) &= x_1^4 + 2x_2^2 \\
\dot{V}(x) &= 4x_1^3\dot{x}_1 + 4x_2\dot{x_2} \\
\dot{V}(x) &= 4x_1^3(-x_1^3 + x_2) + 4x_2(-x_1^3 - x_2^3) \\
\dot{V}(x) &= -4x_1^6 + \cancelto{0}{4x_1^3x_2} - 4x_2^4 - \cancelto{0}{4x_1^3x_2} \\
\dot{V}(x) &= -4x_1^6 - 4x_2^4
\end{aligned}
$$

It is clear here that $\dot{V}(x)$ is globally negative definite. Therefore, we know that **the first system is globally asymptotically stable.**

<u>System 2:</u>

$$
\begin{aligned}
V(x) &= x_1^4 + 2x_2^2 \\
\dot{V}(x) &= 4x_1^3\dot{x}_1 + 4x_2\dot{x_2} \\
\dot{V}(x) &= 4x_1^3(x_2) + 4x_2(-x_1^3) \\
\dot{V}(x) &= 0
\end{aligned}
$$

It is clear here that $\dot{V}(x)$ is globally negative semi-definite. Therefore, we know that **the second system is globally stable.**

Problem 3

Consider a structural system

$$M\ddot{q} + C\dot{q} + Kq = 0$$

Where q $\in \mathbb{R}^n$ is the vector of generalized coordinates and M, C, and K are positive definite mass, damping and stiffness matrices respectively.

(i) Write the system in state-space form.

(ii) Use the total energy of the system

$$V = \frac{1}{2}\dot{q}^T M\dot{q} + \frac{1}{2}q^T kq$$

as a Lyapunov function candidate to examine the stability of the origin (q,$\dot{q}$)=(0,0).

Solution

To write the system in state-space form, we will first define the state vector as

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

Next, we solve the system for $\ddot{q}$

$$\begin{aligned} M\ddot{q} + C\dot{q} + Kq &= 0 \\ \ddot{q} &= M^{-1}(-C\dot{q} - Kq) \\ \ddot{q} &= M^{-1}(-Cx_2 - Kx_1) \end{aligned}$$

We can then write the state space from as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= M^{-1}(-Cx_2 - Kx_1) \\ \dot{x} &= \begin{bmatrix} 0_n & I_n \\ -M^{-1}K & -M^{-1}C \end{bmatrix} x \end{aligned}$$

(ii)

If we choose $V = \frac{1}{2}\dot{q}^T M\dot{q} + \frac{1}{2}q^T kq$ to be the Lyapunov function, we can rewrite the function as $V = \frac{1}{2}x_2^T Mx_2 + \frac{1}{2}{}^{x}_{1}Tkx_1$ we can tell that the function V is gpd and radially unbounded. Next we want to examine the properties of $\dot{V}$.

$$
\begin{aligned}
V &= \frac{1}{2}\dot{q}^T M\dot{q} + \frac{1}{2}q^T kq \\
\dot{V} &= \frac{1}{2}\ddot{q}^T M\dot{q} + \frac{1}{2}\dot{q}^T M\ddot{q} + \frac{1}{2}\dot{q}^T Kq + \frac{1}{2}q^T K\dot{q} \\
\dot{V} &= \frac{1}{2}\dot{x_2}^T Mx_2 + \frac{1}{2}x_2^T M\dot{x_2} + \frac{1}{2}x_2^T Kx_1 + \frac{1}{2}x_1^T Kx_2 \\
\dot{V} &= \frac{1}{2}(M^{-1}(-Cx_2 - Kx_1))^T Mx_2 + \frac{1}{2}x_2^T M(M^{-1}(-Cx_2 - Kx_1)) + \frac{1}{2}x_2^T Kx_1 + \frac{1}{2}x_1^T Kx_2 \\
\dot{V} &= \frac{1}{2}(-M^{-1}Cx_2 - \cancelto{0}{M^{-1}Kx_1})^T Mx_2 + \frac{1}{2}x_2^T(-Cx_2 - \cancelto{0}{Kx_1}) + \cancelto{0}{\frac{1}{2}x_2^T Kx_1} + \cancelto{0}{\frac{1}{2}x_1^T Kx_2} \\
\dot{V} &= -\frac{1}{2}x_2^T C^T \cancelto{I}{M^{-1}M}x_2 - \frac{1}{2}x_2^T Cx_2 \\
\dot{V} &= -x_2^T C^T x_2
\end{aligned}
$$

We can see that $\dot{V}$ is globally negative semi-definite. Therefore, the system is Globally Stable at the equilibrium point (0,0).

To test if we can further define our system as globally asymptotically stable, we turn to LaSalle's Theorem. LaSalle's Theorem states that if a system is locally/globally stable AND the set $\dot{V}(\vec{x}) = \vec{0}$ contains no trajectories other than $\vec{x} = \vec{0}$, then we can say that the system is locally/globally asymptotically stable.

$$\dot{V} = -x_2^T C^T x_2$$

Therefore, if $\dot{V} = 0$, then $x_2 = 0 \; \forall \; t \implies \dot{x}_2 = 0 \; \forall \; t$

$$
\begin{aligned}
\dot{x}_2 &= M^{-1}(-C\cancelto{0}{x_2} - Kx_1) \\
\dot{x}_2 &= -M^{-1}(Kx_1) \\
\dot{x}_2 &= -M^{-1}(Kx_1) = 0 \\
\implies x_1 &= 0
\end{aligned}
$$

Therefore we have shown that $\dot{V}(\vec{x}) = \vec{0}$ only when $\vec{x} = \vec{0}$. This satisfies Lasalle's Theorem and we have show that **our system is Globally Asymptotically Stable at the equilibrium point (0,0)**.

Problem 4

For the system

$$\dot{x}_1 = -x_1(1 - x_1^2 - x_2^2)$$
$$\dot{x}_2 = -x_2(1 - x_1^2 - x_2^2)$$

show that the origin (0,0) is Locally Asymptotically Stable. Estimate a region of attraction.

Solution

A quadratic type Lyapunov function takes the form

$$V(x) = \alpha x_1^2 + \beta x_2^2$$

We can see that this function is globally positive definite and radially unbounded for $\alpha, \beta > 0$. Now we want to examine the properties of $\dot{V}$.

$$\begin{aligned}
\dot{V}(x) &= 2\alpha x_1 \dot{x}_1 + 2\beta x_2 \dot{x}_2 \\
\dot{V}(x) &= 2\alpha x_1(-x_1(1 - x_1^2 - x_2^2)) + 2\beta x_2(-x_2(1 - x_1^2 - x_2^2)) \\
\dot{V}(x) &= -2\alpha x_1^2 - 2\alpha x_1^4 - 2\alpha x_1^2 x_2^2 - 2\beta x_2^2 - 2\beta x_1^2 x_2^2 - 2\beta x_2^4 \\
\dot{V}(x) &= -2[(\alpha x_1^2)(1 - x_1^2 - x_2^2) + (\beta x_2^2)(1 - x_1^2 - x_2^2)] \\
\dot{V}(x) &= -2(\alpha x_1^2 + \beta x_2^2)(1 - x_1^2 - x_2^2)
\end{aligned}$$

We can see that for $(1 - x_1^2 - x_2^2) > 0$, that $\dot{V}(x)$ is negative, therefore $\dot{V}(x)$ is locally negative definite. This implies that **our system is Locally Asymptotically Stable at the equilibrium point (0,0).**

Because $\dot{V}(x)$ is negative when $(1 - x_1^2 - x_2^2) > 0$, the region of attraction can be estimated to be when $(x_1^2 + x_2^2) < 1$ which is the area inside of the unit circle in the phase plane.