MECE 6374: Fun Work #5

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Problem 1

Consider the following system

$$\dot{x}_1 = -x_1$$
$$\dot{x}_2 = -2x_2 + x_1 x_2^2$$

Use the variable gradient method (VGM) to construct a Lyapunov function for the system and determine the stability of the equilibrium point at the origin.

Solution

With VGM, we first assume $\nabla V = \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix} = \begin{bmatrix} \alpha_{11}x_1 + \alpha_{12}x_2 \\ \alpha_{21}x_1 + \alpha_{22}x_2 \end{bmatrix}$

Next step is find α_{ij} using the curl condition

$$\frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1}$$
$$\implies \alpha_{12} = \alpha 21$$

Next step is to find $\dot{V}$ and restrict the coefficients α_{ij} s.t. $\dot{V}$ is at least negative semi-definite.

$$\dot{V} = \frac{dV}{dt} = \frac{\partial V}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial V}{\partial x_2} \cdot \frac{dx_2}{dt}$$
$$\dot{V} = (\alpha_{11}x_1 + \alpha_{12}x_2)(-x_1) + (\alpha_{12}x_1 + \alpha_{22}x_2)(-2x_2 + x_1x_2^2)$$
$$\dot{V} = -\alpha_{11}x_1^2 - \alpha_{12}x_1x_2 - 2\alpha_{12}x_1x_2 - 2\alpha_{22}x_2^2 + \alpha_{22}x_1x_2^3$$
$$\dot{V} = -\alpha_{11}x_1^2 - 2\alpha_{22}x_2^2 - 3\alpha_{12}x_1x_2 + \alpha_{22}x_1x_2^3$$
$$\dot{V} = -\alpha_{11}x_1^2 - 2\alpha_{22}x_2^2 + x_1x_2(\alpha_{22}x_2^2 - 3\alpha_{12})$$

Let $\alpha_{11} = \alpha_{22} = 1, \alpha_{12} = \alpha_{21} = 0$. Then

$$\dot{V} = -x_1^2 - 2x_2^2 + x_1x_2^3$$

We can see that $\dot{V}$ **is locally negative definite.**

Now that we know the properties of $\dot{V}$, we will construct V(x) so that we can examine what properties the Lyapunov function has.

$$
\begin{aligned}
V(x) &= \int_0^{x_1} g_1(x)dx_1 + \int_0^{x_2} g_2(x)dx_2 \\
V(x) &= \int_0^{x_1} x_1dx_1 + \int_0^{x_2} x_2dx_2 \\
V(x) &= \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2
\end{aligned}
$$

It's clear that **V(x) is globally positive definite and radially unbounded.**

Combined with what we know about $\dot{V}$(x), we can see that **the equilibrium point (0,0) is locally, asymptotically stable**

Problem 2

Consider the following two nonlinear systems:

$$\dot{x}_1 = -x_1^3 + x_2$$
$$\dot{x}_2 = -x_1^3 - x_2^3$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1^3$$

(i) Can you determine the stability of the equilibrium point at (0, 0) using linearization?

(ii) Can you determine stability of the equilibrium point at (0, 0) using a quadratic type Lyapunov function?

(iii) Can you determine the stability of the equilibrium point at (0, 0) using a Lyapunov function $V(x_1, x_2) = x_1^4 + 2x_2^2$

Solution

(i) The first system can be linearized as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$$

The second system can also be linearized as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$$

Both systems have repeated eignevalues of 0. Therefore, we cannot determine the stability of the equilibrium points using linearization.

(ii) System 1: A quadratic type Lyapunov function takes the form

$$V(x) = \alpha x_1^2 + \beta x_2^2$$

We can see that this function is gpd and radially unbounded. Now we want to examine the properties of $\dot{V}$.

$$\dot{V}(x) = 2\alpha x_1 \dot{x}_1 + 2\beta x_2 \dot{x}_2$$
$$\dot{V}(x) = 2\alpha x_1(-x_1^3 + x_2) + 2\beta x_2(-x_1^3 - x_2^3)$$
$$\dot{V}(x) = -2\alpha x_1^4 + 2\alpha x_1 x_2 - 2\beta x_2^4 - 2\beta x_1^3 x_2$$

To find the properties of $\dot{V}(x)$, we must look at $\frac{\partial \dot{V}}{\partial x}$ and $\frac{\partial^2 \dot{V}}{\partial x^2}$

$$\frac{\partial \dot{V}}{\partial x} = \begin{bmatrix} -8\alpha x_1^3 + 2\alpha x_2 - 6\beta x_1^2 x_2 \\ 2\alpha x_1 - 8\beta x_2^3 - 2\beta x_1^3 \end{bmatrix}$$

$$\frac{\partial \dot{V}}{\partial x}|_{(0,0)} = \begin{bmatrix} -8\alpha x_1^3 + 2\alpha x_2 - 6\beta x_1^2 x_2 \\ 2\alpha x_1 - 8\beta x_2^3 - 2\beta x_1^3 \end{bmatrix}|_{(0,0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{\partial^2 \dot{V}}{\partial x^2} = \begin{bmatrix} -24\alpha x_1^2 - 12\beta x_1 x_2 & 2\alpha - 6\beta x_1^2 \\ 2\alpha - 6\beta x_1^2 & -24\beta x_2^2 \end{bmatrix}$$

$$\frac{\partial^2 \dot{V}}{\partial x^2}|_{(0,0)} = \begin{bmatrix} -24\alpha x_1^2 - 12\beta x_1 x_2 & 2\alpha - 6\beta x_1^2 \\ 2\alpha - 6\beta x_1^2 & -24\beta x_2^2 \end{bmatrix}|_{(0,0)} = \begin{bmatrix} 0 & 2\alpha \\ 2\alpha & 0 \end{bmatrix}$$

We see that $\frac{\partial \dot{V}}{\partial x} \leq 0$ and $\frac{\partial^2 \dot{V}}{\partial x^2} \leq 0$. Therefore, we can conclude that $\dot{V}$ is locally negative semi-definite.

This implies that **the first system is locally stable at the equilibrium point (0,0)**.

System 2: A quadratic type Lyapunov function takes the form

$$V(x) = \alpha x_1^2 + \beta x_2^2$$

We can see that this function is globally positive definite and radially unbounded. Now we want to examine the properties of $\dot{V}$.

$$\begin{aligned}
\dot{V}(x) &= 2\alpha x_1 \dot{x}_1 + 2\beta x_2 \dot{x}_2 \\
\dot{V}(x) &= 2\alpha x_1(x_2) + 2\beta x_2(-x_1^3) \\
\dot{V}(x) &= 2\alpha x_1 x_2 - 2\beta x_1^3 x_2
\end{aligned}$$

To find the properties of $\dot{V}(x)$, we must look at $\frac{\partial \dot{V}}{\partial x}$ and $\frac{\partial^2 \dot{V}}{\partial x^2}$

$$\begin{aligned}
\frac{\partial \dot{V}}{\partial x} &= \begin{bmatrix} 2\alpha x_2 - 6\beta x_1^2 x_2 \\ 2\alpha x_1 - 2\beta x_1^3 \end{bmatrix} \\
\frac{\partial \dot{V}}{\partial x}|_{(0,0)} &= \begin{bmatrix} 2\alpha x_2 - 6\beta x_1^2 x_2 \\ 2\alpha x_1 - 2\beta x_1^3 \end{bmatrix}|_{(0,0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\frac{\partial^2 \dot{V}}{\partial x^2} &= \begin{bmatrix} -12\beta x_1 x_2 & 2\alpha - 6\beta x_1^2 \\ 2\alpha - 6\beta x_1^2 & 0 \end{bmatrix} \\
\frac{\partial^2 \dot{V}}{\partial x^2}|_{(0,0)} &= \begin{bmatrix} -12\beta x_1 x_2 & 2\alpha - 6\beta x_1^2 \\ 2\alpha - 6\beta x_1^2 & 0 \end{bmatrix}|_{(0,0)} = \begin{bmatrix} 0 & 2\alpha \\ 2\alpha & 0 \end{bmatrix}
\end{aligned}$$

We see that $\frac{\partial \dot{V}}{\partial x} \leq 0$ and $\frac{\partial^2 \dot{V}}{\partial x^2} \leq 0$. Therefore, we can conclude that $\dot{V}$ is locally negative semi-definite.

This implies that **the second system is locally stable at the equilibrium point (0,0)**.

(iii) If we choose $V(x) = x_1^4 + 2x_2^2$, then we know that the Lyapunov function is both globally negative definite and radially unbounded. Next we want to examine the properties of $\dot{V}(x)$.

System 1:

$$
\begin{aligned}
V(x) &= x_1^4 + 2x_2^2 \\
\dot{V}(x) &= 4x_1^3\dot{x}_1 + 4x_2\dot{x}_2 \\
\dot{V}(x) &= 4x_1^3(-x_1^3 + x_2) + 4x_2(-x_1^3 - x_2^3) \\
\dot{V}(x) &= -4x_1^6 + \cancelto{0}{4x_1^3x_2} - 4x_2^4 - \cancelto{0}{4x_1^3x_2} \\
\dot{V}(x) &= -4x_1^6 - 4x_2^4
\end{aligned}
$$

It is clear here that $\dot{V}(x)$ is globally negative definite. Therefore, we know that **the first system is globally asymptotically stable.**

System 2:

$$
\begin{aligned}
V(x) &= x_1^4 + 2x_2^2 \\
\dot{V}(x) &= 4x_1^3\dot{x}_1 + 4x_2\dot{x}_2 \\
\dot{V}(x) &= 4x_1^3(x_2) + 4x_2(-x_1^3) \\
\dot{V}(x) &= 0
\end{aligned}
$$

It is clear here that $\dot{V}(x)$ is globally negative semi-definite. Therefore, we know that **the second system is globally stable.**

Problem 3

Consider a structural system

$$M\ddot{q} + C\dot{q} + Kq = 0$$

Where q $\in \mathbb{R}^n$ is the vector of generalized coordinates and M, C, and K are positive definite mass, damping and stiffness matrices respectively.

(i) Write the system in state-space form.

(ii) Use the total energy of the system

$$V = \frac{1}{2}\dot{q}^T M\dot{q} + \frac{1}{2}q^T kq$$

as a Lyapunov function candidate to examine the stability of the origin (q,$\dot{q}$)=(0,0).

Solution

To write the system in state-space form, we will first define the state vector as

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

Next, we solve the system for $\ddot{q}$

$$\begin{aligned} M\ddot{q} + C\dot{q} + Kq &= 0 \\ \ddot{q} &= M^{-1}(-C\dot{q} - Kq) \\ \ddot{q} &= M^{-1}(-Cx_2 - Kx_1) \end{aligned}$$

We can then write the state space from as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= M^{-1}(-Cx_2 - Kx_1) \\ \dot{x} &= \begin{bmatrix} 0_n & I_n \\ -M^{-1}K & -M^{-1}C \end{bmatrix} x \end{aligned}$$

(ii)

If we choose $V = \frac{1}{2}\dot{q}^T M\dot{q} + \frac{1}{2}q^T kq$ to be the Lyapunov function, we can rewrite the function as $V = \frac{1}{2}x_2^T Mx_2 + \frac{1}{2}{}_1^{x}Tkx_1$ we can tell that the function V is gpd and radially unbounded. Next we want to examine the properties of $\dot{V}$.

$$
\begin{aligned}
V &= \frac{1}{2}\dot{q}^T M\dot{q} + \frac{1}{2}q^T kq \\
\dot{V} &= \frac{1}{2}\ddot{q}^T M\dot{q} + \frac{1}{2}\dot{q}^T M\ddot{q} + \frac{1}{2}\dot{q}^T Kq + \frac{1}{2}q^T K\dot{q} \\
\dot{V} &= \frac{1}{2}\dot{x_2}^T Mx_2 + \frac{1}{2}x_2^T M\dot{x_2} + \frac{1}{2}x_2^T Kx_1 + \frac{1}{2}x_1^T Kx_2 \\
\dot{V} &= \frac{1}{2}(M^{-1}(-Cx_2 - Kx_1))^T Mx_2 + \frac{1}{2}x_2^T M(M^{-1}(-Cx_2 - Kx_1)) + \frac{1}{2}x_2^T Kx_1 + \frac{1}{2}x_1^T Kx_2 \\
\dot{V} &= \frac{1}{2}(-M^{-1}Cx_2 - \cancelto{0}{M^{-1}Kx_1})^T Mx_2 + \frac{1}{2}x_2^T(-Cx_2 - \cancelto{0}{Kx_1}) + \cancelto{0}{\frac{1}{2}x_2^T Kx_1} + \cancelto{0}{\frac{1}{2}x_1^T Kx_2} \\
\dot{V} &= -\frac{1}{2}x_2^T C^T \cancelto{I}{M^{-1}M}x_2 - \frac{1}{2}x_2^T Cx_2 \\
\dot{V} &= -x_2^T C^T x_2
\end{aligned}
$$

We can see that $\dot{V}$ is globally negative semi-definite. Therefore, the system is <u>Globally Stable</u> at the equilibrium point (0,0).

To test if we can further define our system as globally asymptotically stable, we turn to LaSalle's Theorem. LaSalle's Theorem states that if a system is locally/globally stable AND the set $\dot{V}(\vec{x}) = \vec{0}$ contains no trajectories other than $\vec{x} = \vec{0}$, then we can say that the system is locally/globally asymptotically stable.

$$\dot{V} = -x_2^T C^T x_2$$

Therefore, if $\dot{V} = 0$, then $x_2 = 0 \ \forall \ t \implies \dot{x}_2 = 0 \ \forall \ t$

$$
\begin{aligned}
\dot{x}_2 &= M^{-1}(-C\cancelto{0}{x_2} - Kx_1) \\
\dot{x}_2 &= -M^{-1}(Kx_1) \\
\dot{x}_2 &= -M^{-1}(Kx_1) = 0 \\
\implies x_1 &= 0
\end{aligned}
$$

Therefore we have shown that $\dot{V}(\vec{x}) = \vec{0}$ only when $\vec{x} = \vec{0}$. This satisfies Lasalle's Theorem and we have show that **our system is Globally Asymptotically Stable at the equilibrium point (0,0)**.

Problem 4

For the system

$$\dot{x}_1 = -x_1(1 - x_1^2 - x_2^2)$$
$$\dot{x}_2 = -x_2(1 - x_1^2 - x_2^2)$$

show that the origin (0,0) is Locally Asymptotically Stable. Estimate a region of attraction.

Solution

A quadratic type Lyapunov function takes the form

$$V(x) = \alpha x_1^2 + \beta x_2^2$$

We can see that this function is globally positive definite and radially unbounded for $\alpha, \beta > 0$. Now we want to examine the properties of $\dot{V}$.

$$\begin{aligned}
\dot{V}(x) &= 2\alpha x_1 \dot{x}_1 + 2\beta x_2 \dot{x}_2 \\
\dot{V}(x) &= 2\alpha x_1(-x_1(1 - x_1^2 - x_2^2)) + 2\beta x_2(-x_2(1 - x_1^2 - x_2^2)) \\
\dot{V}(x) &= -2\alpha x_1^2 - 2\alpha x_1^4 - 2\alpha x_1^2 x_2^2 - 2\beta x_2^2 - 2\beta x_1^2 x_2^2 - 2\beta x_2^4 \\
\dot{V}(x) &= -2[(\alpha x_1^2)(1 - x_1^2 - x_2^2) + (\beta x_2^2)(1 - x_1^2 - x_2^2)] \\
\dot{V}(x) &= -2(\alpha x_1^2 + \beta x_2^2)(1 - x_1^2 - x_2^2)
\end{aligned}$$

We can see that for $(1 - x_1^2 - x_2^2) > 0$, that $\dot{V}(x)$ is negative, therefore $\dot{V}(x)$ is locally negative definite. This implies that **our system is Locally Asymptotically Stable at the equilibrium point (0,0).**

Because $\dot{V}(x)$ is negative when $(1 - x_1^2 - x_2^2) > 0$, the region of attraction can be estimated to be when $(x_1^2 + x_2^2) < 1$ which is the area inside of the unit circle in the phase plane.