MECE 6388: HW #1

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1.1-1

Find the critical points $u*$ (classify them) and the value of $L(u*)$ in Example 1.1-1 if

(a) $Q = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$, $S^T = \begin{bmatrix} 0 & 1 \end{bmatrix}$

(b) $Q = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$, $S^T = \begin{bmatrix} 0 & 1 \end{bmatrix}$

Sketch the contours of L and find the gradient L_u.

Solution

From Example 1.1-1, L(u) is defined as

$$L(u) = \frac{1}{2}u^T Qu + S^T u \tag{1.1}$$

The critical point is given by setting $L_u = 0$,

$$\begin{aligned} L_u &= Qu + S = 0 \\ u* &= -Q^{-1}S \end{aligned} \tag{1.2}$$

We plug (1.2) into (1.1)
For part a)

$$\begin{aligned} u* &= -\begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ u* &= -\begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

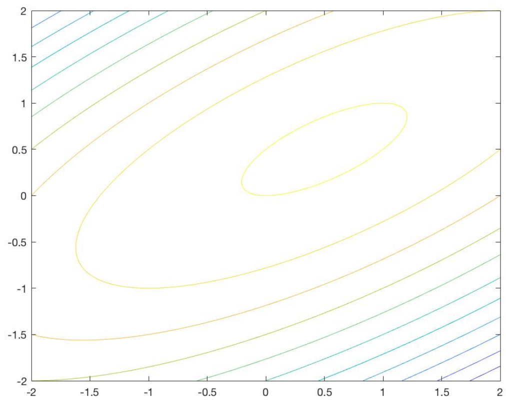
$$\boxed{u* = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

For part b)

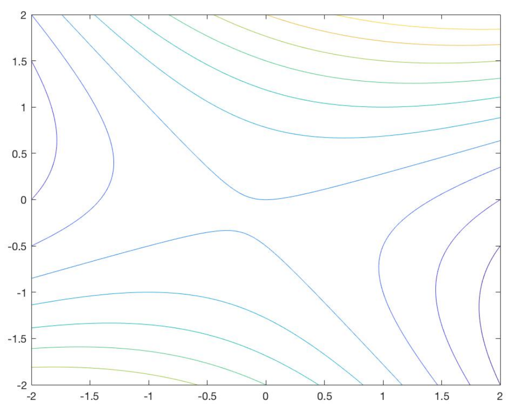
$$\begin{aligned} u* &= -\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ u* &= -\frac{1}{3}\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

$$\boxed{u* = -\frac{1}{3}\begin{bmatrix}1\\1\end{bmatrix}}$$

Contour map for part (a)



Contour map for part (b)



1.1-2

Find the minimum value of

$$L(x_1, x_2) = x_1^2 - x_1 x_2 + x_2^2 + 3x_1 \tag{2.1}$$

Find the curvature matrix at the minimum. Sketch the contours, showing the gradient at several points.

Solution

$$\frac{\partial L}{\partial x_1} = 2x_1 - x_2 + 3 \tag{2.2}$$
$$\frac{\partial L}{\partial x_2} = 2x_2 - x_1 \tag{2.3}$$

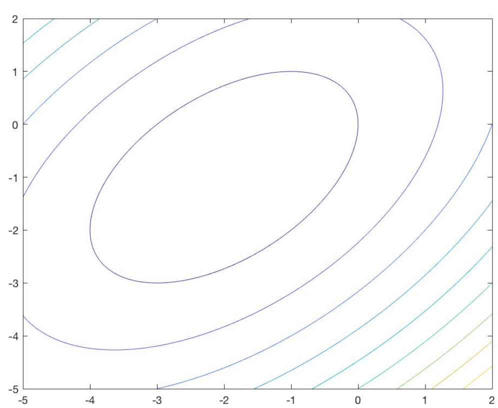
Setting $\frac{\partial L}{\partial x_1} and \frac{\partial L}{\partial x_2} = 0$ gives us the following

$$\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \tag{2.4}$$

To find if this critical point is a minimum or a maximum, we need to look at the curvature matrix.

$$L_{xx} = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 x_2} \\ \frac{\partial^2 L}{\partial x_1 x_2} & \frac{\partial^2 L}{\partial x_2^2} \end{bmatrix}$$
$$L_{xx} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \tag{2.5}$$

We can see from observation that L_{xx} is positive definite which means that the critcal point x* is a local minimum.



1.2-2 Shortest distance between 2 points

Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two given points.
Find the third point $P_3 = (x_3, y_3)$ such that $d_1 + d_2$ is minimized with the constraint $d_1 = d_2$, where d_1 is the distance from P_3 to P_1 and d_2 is the distance from P_3 to P_2.

Solution

We define d_1, d_2 as

$$d_1 = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2} \tag{3.1}$$
$$d_2 = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2} \tag{3.2}$$

We want to minimize $L(x_3, y_3) = d_1 + d_2$ subject to the constraint $f(x_3, y_3) = 0$. Since d_1 and d_2 are positive we can instead minimize $L(x_3, y_3) = d_1^2 + d_2^2$ so that

$$L(x_3, y_3) = (x_3 - x_1)^2 + (y_3 - y_1)^2 + (x_3 - x_2)^2 + (y_3 - y_2)^2 \tag{3.3}$$
$$f(x_3, y_3) = (x_3 - x_1)^2 + (y_3 - y_1)^2 - [(x_3 - x_2)^2 + (y_3 - y_2)^2] \tag{3.4}$$

First we define the Hamiltonian, H(x,u,λ) as

$$\begin{split} H(x_3, y_3, \lambda) = (x_3 - x_1)^2 + (y_3 - y_1)^2 + (x_3 - x_2)^2 + (y_3 - y_2)^2 + \cdots \\ \cdots + \lambda((x_3 - x_1)^2 + (y_3 - y_1)^2 - [(x_3 - x_2)^2 - (y_3 - y_2)^2]) \end{split} \tag{3.5}$$

To find the critical point using the Hamiltonian, we must meet the following three conditions:

$$\frac{\partial H}{\partial \lambda} = f(x_3, y_3) = 0$$
$$(x_3 - x_1)^2 + (y_3 - y_1)^2 - (x_3 - x_2)^2 - (y_3 - y_2)^2 = 0$$
$$2x_3(x_2 - x_1) + 2y_3(y_2 - y_1) = x_2^2 + y_2^2 - x_1^2 - y_1^2 \tag{3.6}$$
$$\frac{\partial H}{\partial x_3} = L_{x_3} + \lambda f_{x_3} = 0 \tag{3.7}$$
$$\frac{\partial H}{\partial y_3} = L_{y_3} + \lambda f_{y_3} = 0 \tag{3.8}$$

We have $f(x_3, y_3)$ so we need to find $\frac{\partial H}{\partial x_3}$ and $\frac{\partial H}{\partial y_3}$

$$\frac{\partial H}{\partial x_3} = 2(x_3 - x_1) + 2(x_3 - x_2) + 2\lambda(x_3 - x_1) - 2\lambda(x_3 - x_2) = 0$$
$$\frac{\partial H}{\partial x_3} = 4x_3 - 2x_1 - 2x_2 - 2\lambda(x_1 - x_2) = 0$$
$$x_3 = \frac{1}{2}[\lambda(x_1 - x_2) + x_1 + x_2] \tag{3.9}$$
$$\frac{\partial H}{\partial y_3} = 2(y_3 - y_1) + 2(y_3 - y_2) + 2\lambda(y_3 - y_1) - 2\lambda(y_3 - y_2) = 0$$
$$\frac{\partial H}{\partial y_3} = 4y_3 - 2y_1 - 2y_2 - 2\lambda(y_1 - y_2) = 0$$
$$y_3 = \frac{1}{2}[\lambda(y_1 - y_2) + y_1 + y_2] \tag{3.10}$$

We have 3 independent equations and 3 unknowns which means this problem is solvable:

$$
\begin{aligned}
&2x_3(x_2 - x_1) + 2y_3(y_2 - y_1) = x_2^2 + y_2^2 - x_1^2 - y_1^2 \\
&x_3 = \frac{1}{2}[\lambda(x_1 - x_2) + x_1 + x_2] \\
&y_3 = \frac{1}{2}[\lambda(y_1 - y_2) + y_1 + y_2]
\end{aligned}
$$

Solving (3.9) and (3.10) for λ and setting them equal to one another gives us

$$
\frac{2x_3 - x_1 - x_2}{x_1 - x_2} = \frac{2y_3 - y_1 - y_2}{y_1 - y_2} \tag{3.11}
$$

$$
2x_3 - x_1 - x_2 = \frac{x_1 - x_2}{y_1 - y_2}(2y_3 - y_1 - y_2)
$$

$$
x_3 = \frac{(x_1 - x_2)(2y_3 - y_2 - y_1)}{2(y_1 - y_2)} + x_1 + x_2 \tag{3.12}
$$

Now that we have x_3 in terms of y_3, we plug (3.12) into (3.6)

$$
\begin{aligned}
[x_1 + x_2 + \frac{(x_1 - x_2)(2y_3 - y_1 - y_2)}{(y_1 - y_2)}](x_2 - x_1) &= 2y_3(y_1 - y_2) + x_2^2 + y_2^2 - x_1^2 - y_1^2 \\
\cancel{-x_1^2} + \cancel{x_2^2} - \frac{(x_1 - x_2)(2y_3 - y_1 - y_2)}{(y_1 - y_2)} &= 2y_3(y_1 - y_2) + \cancel{x_2^2} + y_2^2 - \cancel{x_1^2} - y_1^2 \\
-(x_1 - x_2)^2(2y_3 - y_1 - y_2) &= 2y_3(y_1 - y_2)^2 + (y_2^2 - y_1^2)(y_1 - y_2) \\
2y_3[(x_1 - x_2)^2 + (y_1 - y_2)^2] &= (x_1 - x_2)^2(y_1 + y_2) + (y_2^2 - y_1^2)(y_1 - y_2)
\end{aligned}
$$

$$
\boxed{y_3 = \frac{(x_1 - x_2)^2(y_1 + y_2) + (y_2^2 - y_1^2)(y_1 - y_2)}{2(x_1 - x_2)^2 + (y_1 - y_2)^2}}
$$

Plugging y_3 into (3.12) gives us

$$
\boxed{x_3 = \frac{(x_1 - x_2)([\frac{(x_1 - x_2)^2(y_1 + y_2) + (y_2^2 - y_1^2)(y_2 - y_1)}{(x_1 - x_2)^2 + (y_1 - y_2)^2} - y_1 - y_2] - y_2 - y_1)}{2(y_1 - y_2)} + x_1 + x_2}
$$

1.2-5 Rectangles with maximum area, minimum perimeter

(a) Find the rectangle of maximum area with perimeter p. That is, maximize

$$L(x, y) = xy \tag{4.1}$$

subject to

$$f(x, y) = 2x + 2y - p = 0 \tag{4.2}$$

(b) Find the rectangle of minimum perimeter with area a2. That is, minimize

$$L(x, y) = 2x + 2y \tag{4.3}$$

subject to

$$f(x, y) = xy - a^2 = 0 \tag{4.4}$$

(c) In each case, sketch the contours of L(x, y) and the constraint. Optimization problems related like these two are said to be dual.

Solution

(a) The Hamiltonian for this problem is

$$\begin{aligned} H(x, y, \lambda) &= L(x, y) + \lambda f(x, y) \\ H(x, y, \lambda) &= xy + \lambda(2x + 2y - p) \end{aligned} \tag{4.5}$$

To find the critical point using the Hamiltonian, we must meet the following three conditions:

$$\frac{\partial H}{\partial \lambda} = f(x, y) = 0 \tag{4.6}$$

$$\frac{\partial H}{\partial x} = L_x + \lambda f_x = 0 \tag{4.7}$$

$$\frac{\partial H}{\partial y} = L_y + \lambda f_y = 0 \tag{4.8}$$

$$\frac{\partial H}{\partial \lambda} = 2x + 2y - p = 0 \tag{4.9}$$

$$\frac{\partial H}{\partial x} = y + 2\lambda = 0 \tag{4.10}$$

$$\frac{\partial H}{\partial y} = x + 2\lambda = 0 \tag{4.11}$$

Solving equations (4.9), (4.10), (4.11) yields $x^* = y^* = \frac{p}{4}$ and $\lambda = -\frac{p}{8}$ Plugging x,y into L(x,y), we get

$$\boxed{L^*(x, y) = xy|_{(x^*, y^*)} = \frac{1}{16}p^2}$$

(b) The Hamiltonian for this problem is

$$H(x, y, \lambda) = L(x, y) + \lambda f(x, y)$$
$$H(x, y, \lambda) = 2x + 2y + \lambda(xy - a^2) \tag{4.12}$$

To find the critical point using the Hamiltonian, we must meet the following three conditions:

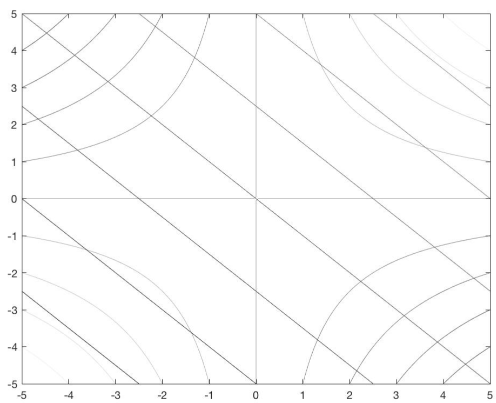
$$\frac{\partial H}{\partial \lambda} = f(x, y) = 0 \tag{4.13}$$
$$\frac{\partial H}{\partial x} = L_x + \lambda f_x = 0 \tag{4.14}$$
$$\frac{\partial H}{\partial y} = L_y + \lambda f_y = 0 \tag{4.15}$$

$$\frac{\partial H}{\partial \lambda} = xy - a^2 = 0 \tag{4.16}$$
$$\frac{\partial H}{\partial x} = 2 + \lambda y = 0 \tag{4.17}$$
$$\frac{\partial H}{\partial y} = 2 + \lambda x = 0 \tag{4.18}$$

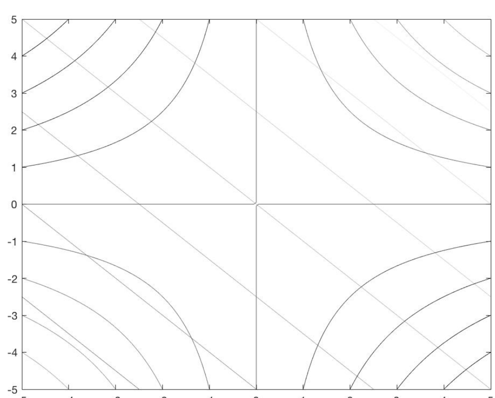
Solving equations (4.16), (4.17), (4.18) yields $x^* = y^* = a$ and $\lambda = -\frac{2}{a}$ Plugging x,y into L(x,y), we get

$$\boxed{L^*(x, y) = xy|_{(x^*, y^*)} = 4a}$$

(c) Contour map for part (a)



Contour map for part (b)



1.2-6 Linear Quadratic Case

Minimize

$$L(x,u) = \frac{1}{2}x^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \frac{1}{2}u^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} u \tag{5.1}$$

if

$$x = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} u \tag{5.2}$$

Solution

The constraint above can be written as

$$f(x,u) = Ix - \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} u - \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 0 \tag{5.3}$$

The Hamiltonian for this problem is

$$\begin{aligned} H(x,u,\lambda) &= L(x,u) + \lambda f(x,u) \\ H(x,u,\lambda) &= \frac{1}{2}x^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \frac{1}{2}u^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} u + \lambda^T (Ix - \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} u - \begin{bmatrix} 1 \\ 3 \end{bmatrix}) \end{aligned} \tag{5.4}$$

To find the critical point using the Hamiltonian, we must meet the following three conditions:

$$\frac{\partial H}{\partial \lambda} = 0 \tag{5.5}$$

$$\frac{\partial H}{\partial x} = L_x + \lambda f_x = 0 \tag{5.6}$$

$$\frac{\partial H}{\partial u} = L_u + \lambda f_u = 0 \tag{5.7}$$

$$\frac{\partial H}{\partial \lambda} = Ix - \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} u - \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 0 \tag{5.8}$$

$$\frac{\partial H}{\partial x} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \lambda = 0 \tag{5.9}$$

$$\frac{\partial H}{\partial u} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} u - \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \lambda = 0 \tag{5.10}$$

Solving equations (5.8), (5.9), (5.10) yield the following

$$\lambda = -\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} u - \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \lambda = 0$$

$$u = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \lambda = \begin{bmatrix} -1 & -4 \\ 0 & 4 \end{bmatrix} x$$

$$x = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} u$$

$$x = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -4 \\ 0 & 4 \end{bmatrix} x = \begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} x$$

$$(I + \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix})x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$x = \begin{bmatrix} 3 & 0 \\ 1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.333 \\ 0.533 \end{bmatrix} \tag{5.11}$$

$$u = \begin{bmatrix} -1 & -4 \\ 0 & 4 \end{bmatrix} x = \begin{bmatrix} -2.47 \\ 2.13 \end{bmatrix} \tag{5.12}$$

Now that we have solved for x^* and u^*, we can plug (5.11) and (5.12) into (5.1) to get our minimum point

$$L(x, u) = \frac{1}{2} x^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \frac{1}{2} u^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} u$$

$$L^*(x, u) = \frac{1}{2} \begin{bmatrix} 0.33 & 0.53 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0.33 \\ 0.53 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -2.47 & 2.13 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2.47 \\ 2.13 \end{bmatrix}$$

$$\boxed{L^*(x, u) = 3.44}$$