MECE 6374: Fun Work #2

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Problem 1

Consider the ODE model

$$\ddot{y} + \dot{y} + y - \frac{1}{16}y^5 = 0 \tag{1.1}$$

(i) Write the model in a state space form $\dot{x} = f(x)$ and compute the equilibrium points.

(ii) Use linearization to determine the local stability properties of each equilibrium point.

Solution

i) Let $x_1 = y$, $x_2 = \dot{y}$. Then

$$\dot{x_1} = x_2 \tag{1.2}$$

$$\dot{x_2} + x_2 + x_1 - \frac{1}{16}x_1^5 = 0$$

$$\dot{x_2} = \frac{1}{16}x_1^5 - x_1 - x_2 \tag{1.3}$$

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{16}x_1^5 - x_1 - x_2 \end{bmatrix}$$

The equilibrium points $(\bar{x}_1, \bar{x}_2)$ are found by setting $\dot{x_1} = \dot{x_2} = 0$.

$$\dot{x_1} = x_2 = 0 \implies \bar{x}_2 = 0 \tag{1.4}$$

$$\dot{x_2} = \frac{1}{16}x_1^5 - x_1 - x_2 = 0 \tag{1.5}$$

Plugging in [1.4] into [1.5] gives us

$$\frac{1}{16}x_1^5 - x_1 = 0$$

$$x_1(\frac{1}{16}x_1^4 - 1) = 0$$

$$\bar{x}_1 = 0, \pm 2, \pm 2j \tag{1.6}$$

$$\bar{x}_2 = 0 \tag{1.7}$$

$$\boxed{\text{Equilibrium Points} : (0,0), (0,2), (0,-2)}$$

ii) $\dot{x_1} = x_2$ is already linearized so we only need to linearize $\dot{x_2} = f_2(x) = \frac{1}{16}x_1^5 - x_1 - x_2$.
The formula for linearization is

$$f_{2,lin}(x_1, x_2) = \cancelto{0}{f_2(\bar{x}_1, \bar{x}_2)} + \frac{\partial f_2}{\partial x_1}\bigg|_{(\bar{x}_1,\bar{x}_2)} (x_1 - \bar{x}_1) + \frac{\partial f_2}{\partial x_2}\bigg|_{(\bar{x}_1,\bar{x}_2)} (x_2 - \bar{x}_2) + \cancelto{0}{H.O.T.} \tag{1.8}$$

$$f_{2,lin}(x_1, x_2) = (\frac{5}{16}x_1^4 - 1)\bigg|_{(x_1,x_2)} (x_1 - \bar{x}_1) - x_2$$

<u>**Equilibrium Point: (0, 0)**</u>

$$\begin{aligned}
f_{1,lin}(x_1, x_2) &= x_2 \\
f_{2,lin}(x_1, x_2) &= (\frac{5}{16}x_1^4 - 1)\bigg|_{(0,0)} x_1 - x_2 \\
f_{2,lin}(x_1, x_2) &= -x_1 - x_2 \\
\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} &= \begin{bmatrix} x_2 \\ -x_1 - x_2 \end{bmatrix} \\
\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned}$$

The eigenvalues of the matrix $\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ are $\lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}j$
Because both of the eignevalues have negative real parts, we know that **this equilibrium point is locally stable.**

<u>**Equilibrium Point: (0, 2)**</u>

$$\begin{aligned}
f_{1,lin}(x_1, x_2) &= x_2 \\
f_{2,lin}(x_1, x_2) &= (\frac{5}{16}x_1^4 - 1)\bigg|_{(0,2)} (x_1 - 2) - x_2 \\
f_{2,lin}(x_1, x_2) &= 4(x_1 - 2) - x_2
\end{aligned}$$

Now we will define $\tilde{x}_1, \tilde{x}_2$, s.t.

$$\begin{aligned}
\tilde{x}_1 &= x_1 - 2 \\
\tilde{x}_2 &= x_2 \\
\implies \dot{\tilde{x}}_1 &= \dot{x}_1 \\
\implies \dot{\tilde{x}}_2 &= \dot{x}_2
\end{aligned}$$

Plugging in gives us

$$\begin{aligned}
\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} &= \begin{bmatrix} \tilde{x}_2 \\ 4\tilde{x}_1 - \tilde{x}_2 \end{bmatrix} \\
\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}
\end{aligned}$$

The eigenvalues of the matrix $\begin{bmatrix} 0 & 1 \\ 4 & -1 \end{bmatrix}$ are $\lambda_{1,2} = -2.56,\ 1.56$

Because one of the eignevalues has negative real part and one of the eigenvalues has positive real part, we know that **this equilibrium point is a saddle point.**

Equilibrium Point: (0, -2)

$$\begin{aligned}
f_{1,lin}(x_1, x_2) &= x_2 \\
f_{2,lin}(x_1, x_2) &= (\frac{5}{16}x_1^4 - 1)\Big|_{(0,-2)}(x_1 + 2) - x_2 \\
f_{2,lin}(x_1, x_2) &= 4(x_1 + 2) - x_2
\end{aligned}$$

Now we will define $\tilde{x}_1, \tilde{x}_2$, s.t.

$$\begin{aligned}
\tilde{x}_1 &= x_1 + 2 \\
\tilde{x}_2 &= x_2 \\
\implies \dot{\tilde{x}}_1 &= \dot{x}_1 \\
\implies \dot{\tilde{x}}_2 &= \dot{x}_2
\end{aligned}$$

Plugging in gives us

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} \tilde{x}_2 \\ 4\tilde{x}_1 - \tilde{x}_2 \end{bmatrix}$$
$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

The eigenvalues of the matrix $\begin{bmatrix} 0 & 1 \\ 4 & -1 \end{bmatrix}$ are $\lambda_{1,2} = -2.56,\ 1.56$

Because one of the eignevalues has negative real part and one of the eigenvalues has positive real part, we know that **this equilibrium point is a saddle point.**

Problem 2

Consider the following system

$$\dot{x_1} = x_1 - x_2^2 \tag{2.1}$$
$$\dot{x_2} = 6x_2 + x_1^2 - 7x_2^2 \tag{2.2}$$

a) Find all equilibrium points
b) Use linearization to determine the local stability and type of each equilibrium point and sketch the approximate phase portrait near each point.

Solution

a) The equilibrium points are found by setting $\dot{x_1} = \dot{x_2} = 0$.

$$\dot{x_1} = x_1 - x_2^2 = 0 \implies x_1 = x_2^2 \tag{2.3}$$
$$\dot{x_2} = 6x_2 + x_1^2 - 7x_2^2 = 0 \tag{2.4}$$

Plugging in [2.3] into [2.4] gives us

$$\begin{aligned} 6x_2 + (x_2^2)^2 - 7x_2^2 &= 0 \\ x_2^4 - 7x_2^2 + 6x_2 &= 0 \\ x_2(x_2^3 - 7x_2 + 6) &= 0 \\ x_2(x_2 - 2)(x_2 - 1)(x_2 + 3) &= 0 \end{aligned}$$

$$\bar{x}_2 = -3, 0, 1, 2$$
$$\bar{x}_1 = 9, 0, 1, 4$$

$$\boxed{\text{Equilibrium Points : } (9, -3), (0, 0), (1, 1), (4, 2)} \tag{2.5}$$

b) As stated in [1.8], the formula for linearization is

$$f_{i,lin}(x_1, x_2) = \left.\frac{\partial f_i}{\partial x_1}\right|_{(\bar{x}_1, \bar{x}_2)} (x_1 - \bar{x}_1) + \left.\frac{\partial f_i}{\partial x_2}\right|_{(\bar{x}_1, \bar{x}_2)} (x_2 - \bar{x}_2)$$

$$\boxed{\begin{aligned} f_{1,lin}(x_1, x_2) &= (x_1 - \bar{x}_1) - 2x_2\Big|_{(\bar{x}_1, \bar{x}_2)} (x_2 - \bar{x}_2) \\ f_{2,lin}(x_1, x_2) &= 2x_1\Big|_{(\bar{x}_1, \bar{x}_2)} (x_1 - \bar{x}_1) + (6 - 14x_2)\Big|_{(\bar{x}_1, \bar{x}_2)} (x_2 - \bar{x}_2) \end{aligned}}$$

Equilibrium Point: (9, -3)

$$f_{1,lin}(x_1, x_2) = (x_1 - 9) + 6(x_2 + 3)$$
$$f_{2,lin}(x_1, x_2) = 18(x_1 - 9) + 48(x_2 + 3)$$

Now we will define $\tilde{x}_1, \tilde{x}_2$, s.t.

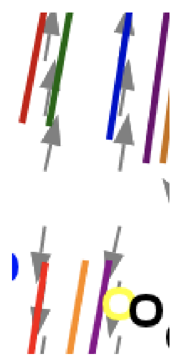
$$\tilde{x}_1 = x_1 - 9 \implies \dot{\tilde{x}}_1 = \dot{x}_1$$
$$\tilde{x}_2 = x_2 + 3 \implies \dot{\tilde{x}}_2 = \dot{x}_2$$

Plugging in gives us

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} \tilde{x}_1 + 6\tilde{x}_2 \\ 18\tilde{x}_1 + 48\tilde{x}_2 \end{bmatrix}$$
$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 18 & 48 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

The eigenvalues of the matrix $\begin{bmatrix} 1 & 6 \\ 18 & 48 \end{bmatrix}$ are $\lambda_{1,2} = -1.20, 50.82$

Because one of the eignevalues has negative real part and one of the eigenvalues has positive real part, we know that **this equilibrium point is a saddle point.**

The phase portrait near the point looks like the following: 

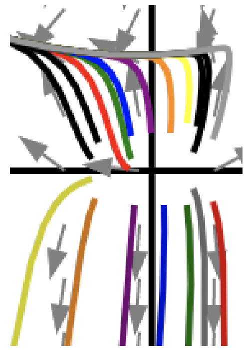
Equilibrium Point: (0, 0)

$$f_{1,lin}(x_1, x_2) = x_1$$
$$f_{2,lin}(x_1, x_2) = 6x_2$$

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 6x_2 \end{bmatrix}$$
$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

The eigenvalues of the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$ are $\lambda_{1,2} = 1, 6$

Because both of the eignevalues have positive real parts we know that **this equilibrium point is an unstable node.**

The phase portrait near the point looks like the following: 

Equilibrium Point: (1, 1)

$$f_{1,lin}(x_1, x_2) = (x_1 - _1) - 2(x_2 - 1)$$
$$f_{2,lin}(x_1, x_2) = 2(x_1 - 1) - 8(x_2 - 1)$$

Now we will define $\tilde{x}_1, \tilde{x}_2$, s.t.

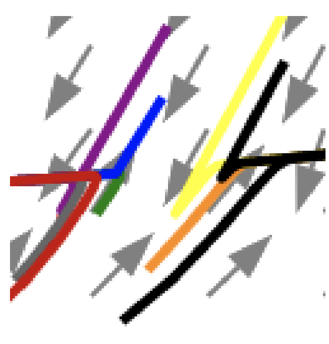
$$\tilde{x}_1 = x_1 - 1 \implies \dot{\tilde{x}}_1 = \dot{x}_1$$
$$\tilde{x}_2 = x_2 - 1 \implies \dot{\tilde{x}}_2 = \dot{x}_2$$

Plugging in gives us

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} \tilde{x}_1 - 2\tilde{x}_2 \\ 2\tilde{x}_1 - 8\tilde{x}_2 \end{bmatrix}$$
$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & -8 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

The eigenvalues of the matrix $\begin{bmatrix} 1 & -2 \\ 2 & -8 \end{bmatrix}$ are $\lambda_{1,2} = -7.53, 0.53$

Because one of the eignevalues has negative real part and one of the eigenvalues has positive real part, we know that **this equilibrium point is a saddle point.**

The phase portrait near the point looks like the following: 

Equilibrium Point: (4, 2)

$$f_{1,lin}(x_1, x_2) = (x_1 - 4) - 4(x_2 - 2)$$
$$f_{2,lin}(x_1, x_2) = 8(x_1 - 4) - 22(x_2 - 2)$$

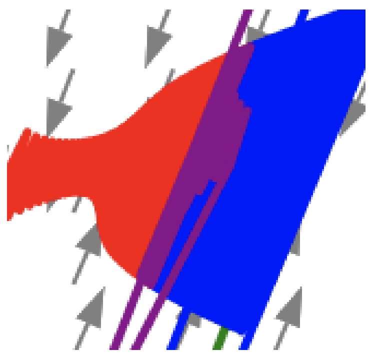
Now we will define $\tilde{x}_1, \tilde{x}_2$, s.t.

$$\tilde{x}_1 = x_1 - 2 \implies \dot{\tilde{x}}_1 = \dot{x}_1$$
$$\tilde{x}_2 = x_2 - 4 \implies \dot{\tilde{x}}_2 = \dot{x}_2$$

Plugging in gives us

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} \tilde{x}_1 - 4\tilde{x}_2 \\ 8\tilde{x}_1 - 22\tilde{x}_2 \end{bmatrix}$$
$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 8 & -22 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

The eigenvalues of the matrix $\begin{bmatrix} 1 & -4 \\ 8 & -22 \end{bmatrix}$ are $\lambda_{1,2} = -20.51, -0.49$ Because both of the eignevalues has negative real parts, we know that **this equilibrium point is a stable node.**

The phase portrait near the point looks like the following: 

Problem 3

Consider the following mechanical system with position dependent damping and stiffness

$$\ddot{q} + c(q)\dot{q} + k(q) = 0 \tag{3.1}$$

Show that if $c(q) > 0$ for all q, the system has no limit cycles.

Solution

First, let's put the system in the state-space form. Let $x_1 = q$, $x_2 = \dot{q}$, then

$$\dot{x_1} = f_1(x_1, x_2) = x_2 \tag{3.2}$$

$$\dot{x_2} = f_2(x_1, x_2) = -c(q)x_2 - k(x_1) \tag{3.2}$$

Bendixson's Criteria states that if $(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2})$ does not change sign in a region R, then no limit cycles exist.

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -c(q) \tag{3.3}$$

Because we know that $c(q) > 0$ for all q, we know that Bendixson's Criteria is satisfied for all $(x_1, x_2) \in \mathbb{R}$. Therefore, there are no limit cycles for this system.

Problem 4

For the system below:

$$\dot{x_1} = 4x_1x_2^2 \tag{4.1}$$
$$\dot{x_2} = 4x_1^2x_2 \tag{4.2}$$

(a) Find all the equilibrium points. Are they isolated?

(b) Show that the system has no limit cycles

Solution

a) The equilibrium points are found by setting $\dot{x_1} = \dot{x_2} = 0$.

$$\dot{x_1} = 4x_1x_2^2 = 0 \implies x_1 = 0 \text{ or } x_2 = 0 \tag{4.3}$$
$$\dot{x_2} = 4x_1^2x_2 = 0 \implies x_1 = 0 \text{ or } x_2 = 0 \tag{4.4}$$

$$\boxed{\text{Equilibrium Points : } (0, a), (b, 0)}$$

where $a, b \in \mathbb{R}$

b) As stated in problem 3, Bendixson's Criteria states that if $(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2})$ does not change sign in a region R, then no limit cycles exist.

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 4x_2^2 + 4x_1^2 > 0 \tag{4.5}$$

Because x_1^2 and x_2^2 are both always greater than zero, we know that Bendixson's Criteria is satisfied for all $(x_1, x_2) \in \mathbb{R}$. Therefore, there are no limit cycles for this system.

Problem 5

Consider the nonlinear system

$$\dot{x_1} = x_2 \tag{5.1}$$

$$\dot{x_2} = ax_1 + bx_2 - x_1^2 x_2 - x_1^3 \tag{5.2}$$

where a, b are constants. Find condition on a and b s.t. the system has no limit cycles in the phase plane.

Solution

As stated in problem 3, Bendixson's Criteria states that if $(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2})$ does not change sign in a region R, then no limit cycles exist.

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 1 + b - x_1^2 \tag{5.3}$$

We can see that Bendixson's Criteria does not depend on the value for a and that for the system to have no limit cycles in the phase plane b must be defined to be less than -1.

$$\boxed{a \in \mathbb{R}, b < -1}$$