MECE 6388: HW #4

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3.3-3 Optimal Control of Newton's System

Let

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u$$

have performance index

$$J = \frac{1}{2}x^T(T)x(T) + \frac{1}{2}\int_0^T (x^T x + ru^2)dt,$$

where x = $\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$.

(a) Find the Riccati equation. Write it as three scalar differential equations. Find the feedback gain in terms of the scalar components of S(t).

(b) Write subroutines to find and simulate the optimal control using MATLAB.

(c) Find analytic expressions for the steady-state Riccati solution and gain.

Solution

a) We can rewrite the state equation as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

We can see from the state equation and the performance index that A= $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, B= $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, Q= $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The Ricatti equation for the given system and parameters is described as

$$-\dot{S} = A^T S + SA - SBR^{-1}B^T S + Q$$

We can define S as $\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$.

Plugging these values into the Ricatti equation gives

$$-\begin{bmatrix} \dot{s_1} & \dot{s_2} \\ \dot{s_2} & \dot{s_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} + \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} * \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} * \frac{1}{r} * \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ s_1 & s_2 \end{bmatrix} + \begin{bmatrix} 0 & s_1 \\ 0 & s_2 \end{bmatrix} - \frac{1}{r} \begin{bmatrix} s_2 \\ s_3 \end{bmatrix} \begin{bmatrix} s_2 & s_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & s_1 \\ s_1 & 2s_2 + 1 \end{bmatrix} - \frac{1}{r} \begin{bmatrix} s_2^2 & s_2 s_3 \\ s_2 s_3 & s_3^2 \end{bmatrix}$$
$$-\begin{bmatrix} \dot{s_1} & \dot{s_2} \\ \dot{s_2} & \dot{s_3} \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{r}s_2^2 & s_1 - \frac{1}{r}s_2 s_3 \\ s_1 - \frac{1}{r}s_2 s_3 & 2s_2 - \frac{1}{r}s_3^2 + 1 \end{bmatrix}$$

$$\boxed{\begin{aligned}\dot{s}_1 &= \frac{1}{r}s_2^2 - 1\\ \dot{s}_2 &= \frac{1}{r}s_2 s_3 - s_1\\ \dot{s}_3 &= \frac{1}{r}s_3^2 - 2s_2 - 1\end{aligned}}$$

The feedback gain can be written as

$$K = R^{-1}B^T S$$
$$K = \frac{1}{r}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}s_1 & s_2\\ s_2 & s_3\end{bmatrix}$$

$$\boxed{K = \frac{1}{r}\begin{bmatrix}s_2 & s_3\end{bmatrix}}$$

c) The steady-state Ricatti equation is given by the [], but with $\dot{S} = 0$. Therefore

$$0 = A^T S + SA - SBR^{-1}B^T S + Q$$

Plugging in for A, S, B, r, Q

$$0_{2x2} = \begin{bmatrix}1 - \frac{1}{r}s_2^2 & s_1 - \frac{1}{r}s_2 s_3\\ s_1 - \frac{1}{r}s_2 s_3 & 2s_2 - \frac{1}{r}s_3^2 + 1\end{bmatrix}$$

This can be written as a series of 3 scalar differential equations with 3 unknown variables (s_1, s_2, s_3)

$$0 = \frac{1}{r}s_2^2 - 1$$
$$0 = \frac{1}{r}s_2 s_3 - s_1$$
$$0 = \frac{1}{r}s_3^2 - 2s_2 - 1$$

Eqn () can be solved for s_2

$$0 = \frac{1}{r}s_2^2 - 1$$
$$s_2 = \pm\sqrt{r}$$
$$s_2 = \sqrt{r}$$

We use only the positive value of s_2 because S should be positive definite.
Eqn () can be solved for s_3.

$$0 = \frac{1}{r}s_3^2 - 2s_2 - 1$$
$$\frac{1}{r}s_3^2 = 2s_2 + 1$$
$$s_3 = \sqrt{r(2\sqrt{r} + 1)}$$
$$s_3 = \sqrt{2r^{\frac{3}{2}} + r}$$

Finally, we can use Eqn () to solve for s_1.

$$
\begin{aligned}
0 &= \frac{1}{r}s_2 s_3 - s_1 \\
s_1 &= \frac{1}{r}s_2 s_3 \\
s_1 &= \frac{1}{r}\sqrt{r}\sqrt{2r^{\frac{3}{2}} + r} \\
s_1 &= \sqrt{2r^{\frac{1}{2}} + 1}
\end{aligned}
$$

Gathering s_1, s_2, and s_3 terms

$$
\boxed{
\begin{aligned}
s_1 &= \sqrt{2r^{\frac{1}{2}} + 1} \\
s_2 &= \sqrt{r} \\
s_3 &= \sqrt{2r^{\frac{3}{2}} + r}
\end{aligned}
}
$$

3.3-4 Uncontrolled Newton's System

Consider the system of Problem 3.3-3. Solve the Lyapunov equation (3.3-9) to find the cost kernel S(t) if u = 0. Sketch the scalar components of S (t).

Solution

From problem 3.3-3 and the constraint u=0, we have A= $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, B= $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, Q= $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $S(T) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $r = 0$,

Again, we can define S as the symmetric matrix $\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$.

Plugging these values into the steady-state Ricatti equation gives

$$\begin{aligned}
-\begin{bmatrix} \dot{s}_1 & \dot{s}_2 \\ \dot{s}_2 & \dot{s}_3 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} + \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} * \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ s_1 & s_2 \end{bmatrix} + \begin{bmatrix} 0 & s_1 \\ 0 & s_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & s_1 \\ s_1 & 2s_2 + 1 \end{bmatrix}
\end{aligned}$$

This can be written as a series of 3 scalar equation

$$\begin{aligned}
\dot{s}_1 &= -1 \\
\dot{s}_2 &= -s_1 \\
\dot{s}_3 &= -2s_2 - 1
\end{aligned}$$

Equation () can be solved for s_1

$$\begin{aligned}
\dot{s}_1 &= -1 \\
s_1(t) &= \int (-1dt) \\
s_1(t) &= -t + C_1 \\
s_1(T) &= -T + C_1 = 1 \\
\implies C_1 &= 1 + T \\
s_1(t) &= -t + 1 + T
\end{aligned}$$

Equation () can be solved for s_2

$$\begin{aligned}
\dot{s}_2 &= -s_1 \\
\dot{s}_2 &= t - (1 + T) \\
s_2(t) &= \int (t - (1 + T))dt \\
s_2(t) &= \frac{1}{2}t^2 - (1 + T)t + C_2 \\
s_2(T) &= \frac{1}{2}T^2 - (1 + T)T + C_2 = 0 \\
\implies C_2 &= \frac{1}{2}T^2 + T \\
s_2(t) &= \frac{1}{2}t^2 - (1 + T)t + \frac{1}{2}T^2 + T
\end{aligned}$$

Equation () can be solved for s_3

$$
\begin{aligned}
\dot{s}_3 &= -2s_2 - 1 \\
\dot{s}_3 &= -t^2 + 2(1+T)t - T^2 - 2T - 1 \\
s_3(t) &= -\frac{1}{3}t^3 + (1+T)t^2 - (1+2T+T^2)t + C_3 \\
s_3(T) &= -\frac{1}{3}T^3 + (1+T)T^2 - (1+2T+T^2)T + C_3 = 1 \\
\implies C_3 &= \frac{1}{3}T^3 + T^2 + T - 1 \\
s_3(t) &= -\frac{1}{3}t^3 + (1+T)t^2 - (1+2T+T^2)t + \frac{1}{3}T^3 + T^2 + T - 1
\end{aligned}
$$

Gathering s_1, s_2, s_3 terms

$$
\boxed{
\begin{aligned}
&s_1(t) = -t + 1 + T \\
&s_2(t) = \frac{1}{2}t^2 - (1+T)t + \frac{1}{2}T^2 + T \\
&s_3(t) = -\frac{1}{3}t^3 + (1+T)t^2 - (1+2T+T^2)t + \frac{1}{3}T^3 + T^2 + T - 1
\end{aligned}
}
$$

3.3-5 Uncontrolled Harmonic Oscillator

Repeat problem 3.3-4 for the system in Example 3.3-5. Let S(T) = I, Q = I , $w_n^2 = 1$, $\delta = 0.5$.

Solution

From example 3.3-5 and the constraint u=0, we have
A= $\begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\delta\omega_n \end{bmatrix}$, B= $\begin{bmatrix} 0 \\ b \end{bmatrix}$, Q= $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, S(T)= $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, r=0 (no input),
Again, we can define S as the symmetric matrix $\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$.
Plugging these values into the steady-state Ricatti equation gives

$$\begin{aligned} -\begin{bmatrix} \dot{s}_1 & \dot{s}_2 \\ \dot{s}_2 & \dot{s}_3 \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} * \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} + \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} * \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -s_2 & -s_3 \\ s_1 - s_2 & s_2 - s_3 \end{bmatrix} + \begin{bmatrix} -s_2 & s_1 - s_2 \\ -s_3 & s_2 - s_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 - s_2 & s_1 - s_2 - s_3 \\ s_1 - s_2 - s_3 & 2s_2 - 2s_3 + 1 \end{bmatrix} \end{aligned}$$

This can be written as a series of 3 scalar equations

$$\boxed{\begin{aligned} \dot{s}_1 &= s_2 - 1 \\ \dot{s}_2 &= s_2 + s_3 - s_1 \\ \dot{s}_3 &= 2s_3 - 2s_2 - 1 \end{aligned}}$$