MECE 6374: Fun Work #3

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Problem 1

Consider the rotational equations of a rigid body about its principal axes.

$$\begin{aligned}
\dot{\omega_1} &= 10\omega_2\omega_3 \\
\dot{\omega_2} &= -5\omega_3\omega_1 \\
\dot{\omega_3} &= 2\omega_1\omega_2
\end{aligned}$$

(a) Find the equilibrium points of this system.

(b) Linearize the system about these equilibrium points. Can you determine the stability of the these equilibrium points from the linearized systems?

Solution

(a) The equilibrium points $(\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3)$ occur when $\dot{\omega} = \begin{bmatrix} \dot{\omega_1} \\ \dot{\omega_2} \\ \dot{\omega_3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{aligned}
10\omega_2\omega_3 &= 0 \\
\implies \omega_2 &= 0, \text{ or } \omega_3 = 0 \\
-5\omega_3\omega_1 &= 0 \\
\implies \omega_1 &= 0, \text{ or } \omega_3 = 0 \\
2\omega_1\omega_2 &= 0 \\
\implies \omega_1 &= 0, \text{ or } \omega_2 = 0
\end{aligned}$$

This tells us that so long as any two of ω_1, ω_2, ω_3 are set to 0, there is an equilibrium point.

$$\boxed{\begin{aligned} \text{Equilibrium Points} : (\alpha, 0, 0), (0, \beta, 0), (0, 0, \gamma) \\ \text{where } \alpha,\ \beta,\ \gamma \in \mathbb{R} \end{aligned}}$$

(b) If we define functions f_1, f_2, f_3, s.t. $\dot{\omega} = \begin{bmatrix}\dot{\omega}_1\\ \dot{\omega}_2\\ \dot{\omega}_3\end{bmatrix} = \begin{bmatrix}f_1(\omega_1,\omega_2,\omega_3)\\ f_2(\omega_1,\omega_2,\omega_3)\\ f_3(\omega_1,\omega_2,\omega_3)\end{bmatrix}$

The formula for linearization of $\dot{\omega}$ is

$$f_{i,lin}(\omega_1,\omega_2,\omega_3) = \cancelto{0}{f_i(\bar{\omega}_1,\bar{\omega}_2,\bar{\omega}_3)} + \left.\frac{\partial f_i}{\partial \omega_1}\right|_{(\bar{\omega}_1,\bar{\omega}_2,\bar{\omega}_3)}(\omega_1-\bar{\omega}_1) + \left.\frac{\partial f_i}{\partial \omega_2}\right|_{(\bar{\omega}_1,\bar{\omega}_2,\bar{\omega}_3)}(\omega_2-\bar{\omega}_2) + \cdots$$
$$\cdots + \left.\frac{\partial f_i}{\partial \omega_3}\right|_{(\bar{\omega}_1,\bar{\omega}_2,\bar{\omega}_3)}(\omega_3-\bar{\omega}_3) + \cancelto{0}{H.O.T.}$$

$$f_{i,lin}(\omega_1,\omega_2,\omega_3) = \left.\frac{\partial f_i}{\partial \omega_1}\right|_{(\bar{\omega}_1,\bar{\omega}_2,\bar{\omega}_3)}(\omega_1-\bar{\omega}_1) + \left.\frac{\partial f_i}{\partial \omega_2}\right|_{(\bar{\omega}_1,\bar{\omega}_2,\bar{\omega}_3)}(\omega_2-\bar{\omega}_2) + \left.\frac{\partial f_i}{\partial \omega_3}\right|_{(\bar{\omega}_1,\bar{\omega}_2,\bar{\omega}_3)}(\omega_3-\bar{\omega}_3)$$

$$\begin{aligned}
f_{1,lin}(\omega_1,\omega_2,\omega_3) &= \left.10\omega_3\right|_{(\bar{\omega}_1,\bar{\omega}_2,\bar{\omega}_3)}(\omega_2-\bar{\omega}_2) + \left.10\omega_2\right|_{(\bar{\omega}_1,\bar{\omega}_2,\bar{\omega}_3)}(\omega_3-\bar{\omega}_3)\\
f_{2,lin}(\omega_1,\omega_2,\omega_3) &= \left.-5\omega_3\right|_{(\bar{\omega}_1,\bar{\omega}_2,\bar{\omega}_3)}(\omega_1-\bar{\omega}_1) - \left.5\omega_1\right|_{(\bar{\omega}_1,\bar{\omega}_2,\bar{\omega}_3)}(\omega_3-\bar{\omega}_3)\\
f_{3,lin}(\omega_1,\omega_2,\omega_3) &= \left.2\omega_2\right|_{(\bar{\omega}_1,\bar{\omega}_2,\bar{\omega}_3)}(\omega_1-\bar{\omega}_1) + \left.2\omega_1\right|_{(\bar{\omega}_1,\bar{\omega}_2,\bar{\omega}_3)}(\omega_2-\bar{\omega}_2)
\end{aligned}$$

Equilibrium Point: $(\alpha, 0, 0)$

$$\begin{aligned}
f_{1,lin}(\omega_1,\omega_2,\omega_3) &= \dot{\omega}_{1,lin} = 0\\
f_{2,lin}(\omega_1,\omega_2,\omega_3) &= \dot{\omega}_{2,lin} = -5\alpha\omega_3\\
f_{3,lin}(\omega_1,\omega_2,\omega_3) &= \dot{\omega}_{3,lin} = 2\alpha\omega_2
\end{aligned}$$
$$\begin{bmatrix}\dot{\omega}_{1,lin}\\ \dot{\omega}_{2,lin}\\ \dot{\omega}_{3,lin}\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -5\alpha\\ 0 & 2\alpha & 0\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}$$

The eigenvalues of the matrix $\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -5\alpha\\ 0 & 2\alpha & 0\end{bmatrix}$ are $\lambda_{1,2,3} = 0, \pm\sqrt{10}\alpha j$ These eigenvalues tell us that the linearized system is marginally stable which does not tell us anything about the behavior of the non-linear system near the equilibrium point $(\alpha, 0, 0)$.

Equilibrium Point: $(\mathbf{0}, \beta, \mathbf{0})$

$$
\begin{aligned}
f_{1,lin}(\omega_1, \omega_2, \omega_3) &= \dot{\omega}_{1,lin} = 10\beta\omega_3 \\
f_{2,lin}(\omega_1, \omega_2, \omega_3) &= 0 \\
f_{3,lin}(\omega_1, \omega_2, \omega_3) &= \dot{\omega}_{3,lin} = 2\beta\omega_1 \\
\begin{bmatrix} \dot{\omega}_{1,lin} \\ \dot{\omega}_{2,lin} \\ \dot{\omega}_{3,lin} \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 10\beta \\ 0 & 0 & 0 \\ 2\beta & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}
\end{aligned}
$$

The eigenvalues of the matrix $\begin{bmatrix} 0 & 0 & 10\beta \\ 0 & 0 & 0 \\ 2\beta & 0 & 0 \end{bmatrix}$ are $\lambda_{1,2,3} = 0, \pm\sqrt{20}\beta$ These eigenvalues tell us that, no matter the value of β, the linearized system is unstable which tells us that the non-linear system near the equilibrium point $(0, \beta, 0)$ is unstable.

Equilibrium Point: $(\mathbf{0}, \mathbf{0}, \gamma)$

$$
\begin{aligned}
f_{1,lin}(\omega_1, \omega_2, \omega_3) &= \dot{\omega}_{1,lin} = 10\gamma\omega_2 \\
f_{2,lin}(\omega_1, \omega_2, \omega_3) &= \dot{\omega}_{3,lin} = -5\gamma\omega_1 \\
f_{3,lin}(\omega_1, \omega_2, \omega_3) &= 0 \\
\begin{bmatrix} \dot{\omega}_{1,lin} \\ \dot{\omega}_{2,lin} \\ \dot{\omega}_{3,lin} \end{bmatrix} &= \begin{bmatrix} 0 & 10\gamma & 0 \\ -5\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}
\end{aligned}
$$

The eigenvalues of the matrix $\begin{bmatrix} 0 & 10\gamma & 0 \\ -5\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are $\lambda_{1,2,3} = 0, \pm 5\sqrt{2}\gamma$ These eigenvalues tell us that the linearized system is marginally stable which does not tell us anything about the behavior of the non-linear system near the equilibrium point $(\alpha, 0, 0)$.

Problem 2

Consider the following Loventz attractor system.

$$
\begin{aligned}
\dot{x}_1 &= -\sigma x_1 + \sigma x_2 \\
\dot{x}_2 &= \rho x_1 - x_2 - x_1 x_3 \\
\dot{x}_3 &= -\beta x_3 + x_1 x_2
\end{aligned}
$$

where $\sigma = 10$, $\beta = \frac{8}{3}$, and ρ is a parameter. Compute the equilibrium points of the system. How do these equilibrium points change as ρ varies from 0 to ∞?

Solution

The equilibrium points $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ occur when $\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$
\begin{aligned}
-\sigma \bar{x}_1 + \sigma \bar{x}_2 &= 0 \\
\implies \bar{x}_1 &= \bar{x}_2 \\
\rho \bar{x}_1 - \bar{x}_2 - \bar{x}_1 \bar{x}_3 &= 0 \\
\rho \bar{x}_2 - \bar{x}_2 - \bar{x}_2 \bar{x}_3 &= 0 \\
\bar{x}_2(\rho - 1 - \bar{x}_3) &= 0 \\
\implies \bar{x}_2 &= 0, \text{ or } \bar{x}_3 = \rho - 1 \\
-\beta \bar{x}_3 + \bar{x}_1 \bar{x}_2 &= 0 \\
-\beta \bar{x}_3 + \bar{x}_2^2 &= 0 \\
\bar{x}_2 = 0 \implies \bar{x}_1, \bar{x}_3 &= 0, \\
\bar{x}_3 = \rho - 1 \implies -\beta(\rho - 1) + \bar{x}_2^2 &= 0 \implies \bar{x}_2 = \bar{x}_1 = \pm\sqrt{\beta(\rho - 1)}
\end{aligned}
$$

$$
\boxed{\begin{aligned} \text{Equilibrium Points : } (0,0,0), (\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho - 1), \\ (-\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, \rho - 1) \end{aligned}}
$$

For $\rho < 1$, the only equilibrium point is at the origin, $(0, 0, 0)$.

For $\rho = 1$, there are equilibrium points at $(0, 0, 0), (0, 0, -1)$

For $\rho > 1$, there are equilibrium points at $(0, 0, 0), (\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho - 1)$, $(-\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, \rho - 1)$