MECE 6388: HW #3

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2.2-7 Cubic performance index

Let

$$x_{k+1} = ax_k + bu_k,$$

where x_k and u_k are scalars, and

$$J = \frac{1}{3}s_N x_N^3 + \frac{1}{3}\sum_{k=0}^{N-1}(qx_k^3 + ru_k^3)$$

(a) Write state and costate equations and stationarity condition.

(b) When can we solve for u_k? Under this condition, eliminate u_k from the equation.

(c) Solve the open-loop control problem (i.e., x_N fixed, s_N=0, q=0).

Solution

To solve the costate and stationarity conditions, we must first define the Hamiltonian function, $H(x_k, u_k)$ We start by defining the Hamiltonian function, H^k,

$$H^k = L^k + \lambda_{k+1}^T f^k$$

where

$$L^k = \frac{1}{3}(qx_k^3 + ru_k^3)$$
$$f^k = ax_k + bu_k$$

Therefore

$$H^k = \frac{1}{3}(qx_k^3 + ru_k^3) + \lambda_{k+1}(ax_k + bu_k)$$

The state, costate, and stationarity condition are given by the following equations

$$x_{k+1} = ax_k + bu_k \tag{1.1}$$
$$\lambda_k = \frac{\partial H^k}{\partial x_k} = qx_k^2 + a\lambda_{k+1}$$
$$\lambda_k = qx_k^2 + a\lambda_{k+1} \tag{1.2}$$
$$0 = \frac{\partial H^k}{\partial u_k} = ru_k^2 + b\lambda_{k+1}$$
$$u_k^2 = -\frac{b}{r}\lambda_{k+1}$$
$$u_k = \sqrt{-\frac{b}{r}\lambda_{k+1}} \tag{1.3}$$

b) Given the constraints in our equation for u_k, we can see that the terms inside the square root must be positive. We know that $r > 0$ so that gives us the constraint

$$-b\lambda_{k+1} \geq 0$$

Assuming that this condition is true, we can plug (1.3) into (1.1) and (1.2) to give us the state and costate equations without u_k

$$x_{k+1} = ax_k + b(\sqrt{-\frac{b}{r}\lambda_{k+1}})$$

$$\boxed{\begin{aligned} x_{k+1} &= ax_k + (\sqrt{-\frac{b^3}{r}\lambda_{k+1}}) \\ \lambda_k &= qx_k^2 + a\lambda_{k+1} \end{aligned}}$$

c) To solve the open loop control problem, we let $s_N = q = 0, r = 1$.

Now we can write our state and costate equations solved in part b) as

$$\begin{aligned} \lambda_k &= a\lambda_{k+1} \\ \lambda_k &= a^{N-k}\lambda_N \qquad (1.4) \\ x_{k+1} &= ax_k + (\sqrt{-b^3\lambda_{k+1}}) \\ x_{k+1} &= ax_k + \sqrt{-b^3a^{N-k-1}\lambda_N} \qquad (1.5) \end{aligned}$$

Plugging in for k=0,1,2,3 gives us

$$\begin{aligned} x_1 &= ax_0 + \sqrt{-b^3a^{N-1}\lambda_N} \\ x_2 &= a^2x_0 + \sqrt{-b^3a^{N+1}\lambda_N} + \sqrt{-b^3a^{N-2}\lambda_N} \\ x_3 &= a^3x_0 + \sqrt{-b^3a^{N+3}\lambda_N} + \sqrt{-b^3a^{N}\lambda_N} + \sqrt{-b^3a^{N-3}\lambda_N} \\ x_k &= a^kx_0 + \sqrt{-b^3\lambda_N}\sum_{i=0}^{k-1}(\sqrt{a^{N+2k-3-3i}}) \\ x_k &= a^kx_0 + \sqrt{-b^3\lambda_N a^{N+2k-3}}\sum_{i=0}^{k-1}(a^{-\frac{3}{2}i}) \qquad (1.6) \end{aligned}$$

Using the formula for the sum of a geometric series, we can rewrite (1.6) as

$$x_k = a^kx_0 + \sqrt{-b^3\lambda_N a^{N+2k-3}}(\frac{1-a^{-\frac{3}{2}k}}{1-a^{-\frac{3}{2}}}) \qquad (1.7)$$

Since the final state is fixed ($x_N = r_N$) we can write

$$\begin{aligned} x_N = r_N &= a^Nx_0 + \sqrt{-b^3\lambda_N a^{3N-3}}(\frac{1-a^{-\frac{3}{2}N}}{1-a^{-\frac{3}{2}}}) \\ \frac{(r_N - a^Nx_0)(1-a^{-\frac{3}{2}})}{1-a^{-\frac{3}{2}N}} &= \sqrt{-b^3\lambda_N a^{3N-3}} \end{aligned}$$

$$\boxed{\lambda_N = -\frac{(r_N - a^Nx_0)^2(1-a^{-\frac{3}{2}})^2}{a^{3N-3}b^3(1-a^{-\frac{3}{2}N})^2}} \qquad (1.8)$$

We can use equation (1.4) to solve for λ_k now that we have λ_N.

$$\lambda_k = a^{N-k}\lambda_N$$

Once we have λ_k, we can solve for u_k using equation (1.3).

$$u_k = \sqrt{-\frac{b}{r}\lambda_{k+1}}$$

Finally, once we have u_k, we can solve for x_k forward in time using equation (1.1)

$$x_{k+1} = ax_k + bu_k$$

2.3-1 Digital control of harmonic oscillator

A harmonic oscillator is described by

$$\dot{x}_1 = x_2 \tag{2.1}$$

$$\dot{x}_2 = -\omega_n^2 x_1 + u \tag{2.2}$$

(a) Discretize the plant using a sampling period of T.

(b) With the discretized plant, associate a performance index of

$$J = \frac{1}{2}[s_1(x_N^1)^2 + s_2(x_N^2)^2] + \frac{1}{2}\sum_{k=0}^{N-1}[q_1(x_k^1)^2 + q_2(x_k^2)^2 + ru_k^2]$$

where the state is $x_k = \begin{bmatrix} x_k^1 & x_k^2 \end{bmatrix}^T$. Write scalar equations for a digital optimal controller.

(c) Write a MATLAB subroutine to simulate the plant dynamics and use the time response program lsim.m to obtain zero-input state trajectories.

(d) Write a MATLAB subroutine to compute and store the optimal control gains and to update the control u_k given the current state x_k. Write a MATLAB driver program to obtain time response plots for the optimal controller.

Solution

a) To discretize the plant, we use the following equations.

$$A^s = e^{AT} \tag{2.3}$$

$$B^s = \int_0^T e^{A\tau} B d\tau \tag{2.4}$$

where A, B are found by putting the plant into the state space form $\dot{x} = Ax + Bu$. Equations (2.1) and (2.2) can be rewritten as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

which tells us that $A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We can solve for the eigenvalues of A, $\lambda_{1,2} = \pm\omega_n j$.

To solve equation (2.3), we will invoke the Cayley-Hamilton Theorem. The Cayley-Hamilton Theorem states that the matrix exponential e^{AT} can be solved by

$$e^{AT} = \sum_{k=0}^{n-1} \alpha_k A^k \tag{2.5}$$

$$e^{\lambda_i T} = \sum_{k=0}^{n-1} \alpha_k \lambda^k \tag{2.6}$$

where the α_i's are determined from the set of equations given by the eigenvalues of A.

Using (2.5), we can write

$$e^{AT} = \alpha_0 I + \alpha_1 A$$

To solve for α_0, α_1, we use equation (2.6)

$$e^{-\omega_n T j} = \alpha_0 - j\omega_n\alpha_1$$

$$cos(-\omega_n T) + jsin(-\omega_n T) = \alpha_0 - j\omega_n\alpha_1 \tag{2.7}$$

$$e^{\omega_n T j} = \alpha_0 + \omega_n\alpha_1 j$$

$$cos(\omega_n T) + jsin(\omega_n T) = \alpha_0 + j\omega_n\alpha_1 \tag{2.8}$$

(2.7)+(2.8) gives us

$$
\begin{aligned}
2cos(\omega_n T) &= 2\alpha_0 \\
\alpha_0 &= cos(\omega_n T) \qquad (2.9)
\end{aligned}
$$

Plugging in (2.9) to (2.7)

$$
\begin{aligned}
cos(-\omega_n T) + jsin(-\omega_n T) &= cos(\omega_n T) - j\omega_n \alpha_1 \\
sin(-\omega_n T) &= -\omega_n \alpha_1 \\
\alpha_1 &= \frac{sin(\omega_n T)}{\omega_n} \qquad (2.10)
\end{aligned}
$$

Now that we have α_0, α_1, we can find e^{AT} from (2.5)

$$
\begin{aligned}
e^{AT} &= (cos(\omega_n T))I + (\frac{sin(\omega_n T)}{\omega_n})A \\
e^{AT} &= \begin{bmatrix} cos(\omega_n T) & \frac{sin(\omega_n T)}{\omega_n} \\ -\omega_n sin(\omega_n T) & cos(\omega_n T) \end{bmatrix} \qquad (2.11)
\end{aligned}
$$

Now that we have A^s, we can plug (2.11) into (2.4) to solve for B^s

$$
\begin{aligned}
B^s &= \int_0^T e^{A\tau} B d\tau \\
B^s &= \int_0^T \begin{bmatrix} cos(\omega_n \tau) & \frac{sin(\omega_n \tau)}{\omega_n} \\ -\omega_n sin(\omega_n \tau) & cos(\omega_n \tau) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau \\
B^s &= \int_0^T \begin{bmatrix} \frac{sin(\omega_n \tau)}{\omega_n} \\ cos(\omega_n \tau) \end{bmatrix} d\tau \\
B^s &= \begin{bmatrix} \frac{-cos(\omega_n \tau)}{\omega_n^2}\Big|_0^T \\ \frac{sin(\omega_n \tau)}{\omega_n}\Big|_0^T \end{bmatrix} \\
B^s &= \begin{bmatrix} \frac{1}{\omega_n^2}(1 - cos(\omega_n T)) \\ \frac{sin(\omega_n T)}{\omega_n} \end{bmatrix} \qquad (2.12)
\end{aligned}
$$

$$
\boxed{x_{k+1} = \begin{bmatrix} cos(\omega_n T) & \frac{sin(\omega_n T)}{\omega_n} \\ -\omega_n sin(\omega_n T) & cos(\omega_n T) \end{bmatrix} x_k + \begin{bmatrix} \frac{1}{\omega_n^2}(1 - cos(\omega_n T)) \\ \frac{sin(\omega_n T)}{\omega_n} \end{bmatrix} u_k}
$$

b) Since we know that S_k is symmetric for all k, let

$$
S_k = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}
$$

Then the feedback gain is updated by

$$
\begin{aligned}
\delta &= B^T S_{k+1} B + r \\
\delta &= r + \frac{s_1 T^4}{4} + s_2 T^3 + s_3 T^2 \\
K_k &= B^T S_{k+1} A \frac{1}{\delta}
\end{aligned}
$$

We can write

$$
\begin{aligned}
k_1 &= (\frac{s_1 T^2}{2} + s_2 T)\frac{1}{\delta} \\
k_2 &+ (\frac{s_1 T^3}{2} + \frac{3 s_2 T^2}{2} + s_3 T)\frac{1}{\delta}
\end{aligned}
$$

We can write the closed-loop plant matrix

$$A_k^{cl} = A - BK_k$$
$$A_k^{cl} = \begin{bmatrix} a_{11}^{cl} & a_{12}^{cl} \\ a_{21}^{cl} & a_{22}^{cl} \end{bmatrix} = \begin{bmatrix} 1 - \frac{k_1 T^a}{2} & T - \frac{k_2 T^2}{2} \\ -k_1 T & 1 - k_2 T \end{bmatrix}$$

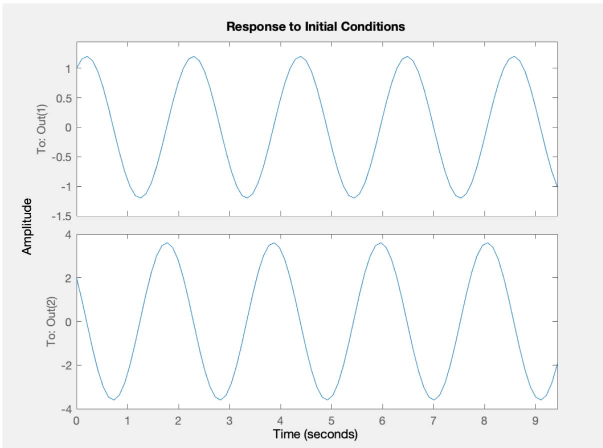
The updated cost kernel is

$$S_k = A^T S_{k+1} A_k^{cl} + Q$$

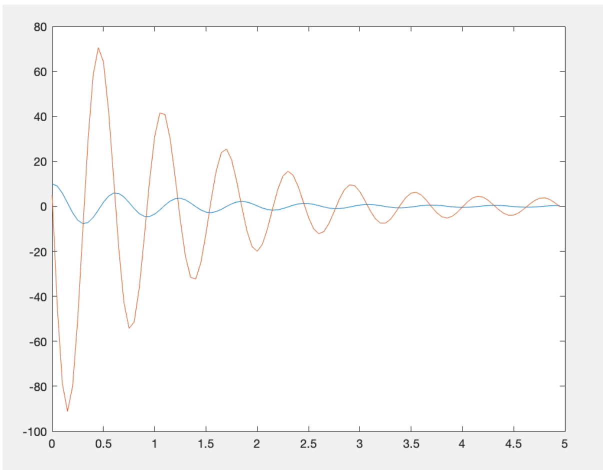
which yields the scalar updates:

$$s_1 = s_1 a_{11}^{cl} + s_2 a_{22}^{cl} + q_d$$
$$s_2 = s_1 a_{12}^{cl} + s_2 a_{22}^{cl}$$
$$s_3 = (s_1 T + s_2) a_{12}^{cl} + (s_2 T + s_3) a_{22}^{cl} + q_v$$

c) The unforced system (assuming $w_n = 3$, T=0.001) shown below is marginally stable.



d) The response plots for the optimal controller is shown by



The MATLAB code is given below

```
clear
clc
close all

wn=10;
t0=0;
tf=5;
N=100;

T0=(tf-t0)/N;
A_d=[cos(wn*T0), sin(wn*T0)/wn; -wn*sin(wn*T0), cos(wn*T0)];
B_d=[1/(wn^2)*(1-cos(wn*T0)); sin(wn*T0)/wn];
s1=1;
s2=0;
s3=1;
Q=eye(2);
r=1;
x0=[10;5];


delta_array=zeros(1,N);
Sk_array=cell(1,N);
xk_array=cell(1,N);
uk_array=zeros(1,N);
xk_array{1}=x0;
Kk_array=cell(1,N);
Sk_array{N}=[s1 s2;s2 s3];
xk1_array=zeros(1,N);
xk2_array=zeros(1,N);

for k=1:(N-1)
    delta_array(1,N-k)=r+((s1/4)*T0^4)+(s2*T0^3)+s3*T0^2;
    k1=((s1*T0^2)/2+s2*T0)/delta_array(1,N-k);
    k2=((s1*T0^3)/2+(3*s2*T0^2)/2+s3*T0)/delta_array(1,N-k);
    Kk_array{N-k}=[k1 k2];
    a_cl11=1-(k1*T0^2)/2;
    a_cl12=T0-(k2*T0^2)/2;
    a_cl21=-k1*T0;
    a_cl22=1-k2*T0;
    s1=s1*a_cl11+s2*a_cl21+Q(1,1);
    s2=s1*a_cl12+s2*a_cl22;
    s3=(s1*T0+s2)*a_cl12+(s2*T0+s3)*a_cl22+Q(2,2);
    Sk_array{N-k}=[s1 s2;s2 s3];
end
for k=1:(N-1)
    uk_array(1,k)=-Kk_array{k}*xk_array{k};
    xk_array{k+1}=A_d*xk_array{k}+B_d*uk_array(1,k);
end
for k=1:N
    xk_array{k}(1)
    xk1_array(1,k)=xk_array{k}(1);
    xk2_array(1,k)=xk_array{k}(2);
end
xk_array{N}
uk_array(1,N-1)

T=0:T0:4.955;
U=uk_array;

plot(T,xk1_array)
hold on
plot(T,xk2_array)

```

2.3-2 Digital control of an unstable system

Repeat the previous problem for

$$\dot{x}_1 = x_2 \tag{3.1}$$
$$\dot{x}_2 = a^2 x_1 + bu \tag{3.2}$$

Solution

a) To discretize the plant, we use the following equations.

$$A^s = e^{AT} \tag{3.3}$$
$$B^s = \int_0^T e^{A\tau} B d\tau \tag{3.4}$$

where A, B are found by putting the plant into the state space form $\dot{x} = Ax + Bu$. Equations (2.1) and (2.2) can be rewritten as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ a^2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ b \end{bmatrix}$$

which tells us that $A = \begin{bmatrix} 0 & 1 \\ a^2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ b \end{bmatrix}$. We can solve for the eigenvalues of A, $\lambda_{1,2} = \pm a$.
To solve equation (3.3), we will invoke the Cayley-Hamilton Theorem. The Cayley-Hamilton Theorem states that the matrix exponential e^{AT} can be solved by

$$e^{AT} = \sum_{k=0}^{n-1} \alpha_k A^k \tag{3.5}$$
$$e^{\lambda_i T} = \sum_{k=0}^{n-1} \alpha_k \lambda^k \tag{3.6}$$

where the α_i's are determined from the set of equations given by the eigenvalues of A.
Using (3.5), we can write

$$e^{AT} = \alpha_0 I + \alpha_1 A$$

To solve for α_0, α_1, we use equation (3.6)

$$e^{-aT} = \alpha_0 - a\alpha_1 \tag{3.7}$$
$$e^{aT} = \alpha_0 + a\alpha_1 \tag{3.8}$$

(3.7)+(3.8) gives us

$$(e^{-aT} + e^{aT}) = 2\alpha_0$$
$$\alpha_0 = \frac{1}{2}(e^{-aT} + e^{aT})$$
$$\alpha_0 = cosh(aT) \tag{3.9}$$

Plugging in (3.9) to (3.7)

$$e^{-aT} = \frac{1}{2}(e^{-aT} + e^{aT}) - a\alpha_1$$
$$a\alpha_1 = \frac{1}{2}(e^{aT} - e^{-aT})$$
$$\alpha_1 = \frac{1}{a} sinh(aT) \tag{3.10}$$

Now that we have α_0, α_1, we can find e^{AT} from (2.5)

$$e^{AT} = (cosh(aT))I + \frac{1}{a}sinh(aT)A$$
$$e^{AT} = \begin{bmatrix} cosh(aT) & \frac{1}{a}sinh(aT) \\ asinh(aT) & cosh(aT) \end{bmatrix}$$

Now that we have A^s, we can use equation (2.4) to solve for B^s

$$B^s = \int_0^T e^{A\tau} B d\tau$$
$$B^s = \int_0^T \begin{bmatrix} cosh(a\tau) & \frac{1}{a}sinh(a\tau) \\ asinh(a\tau) & cosh(a\tau) \end{bmatrix} \begin{bmatrix} 0 \\ b \end{bmatrix} d\tau$$
$$B^s = \int_0^T \begin{bmatrix} \frac{b}{a}sinh(a\tau) \\ bcosh(a\tau) \end{bmatrix} d\tau$$
$$B^s = \begin{bmatrix} \frac{b}{a^2}cosh(a\tau)|_0^T \\ \frac{b}{a}sinh(a\tau)|_0^T \end{bmatrix}$$
$$B^s = \begin{bmatrix} \frac{b}{a^2}(cosh(aT) - 1) \\ \frac{b}{a}sinh(aT) \end{bmatrix}$$

$$\boxed{x_{k+1} = \begin{bmatrix} cosh(aT) & \frac{1}{a}sinh(aT) \\ asinh(aT) & cosh(aT) \end{bmatrix} x_k + \begin{bmatrix} \frac{b}{a^2}(cosh(aT) - 1) \\ \frac{b}{a}sinh(aT) \end{bmatrix} u_k}$$

b) Since we know that S_k is symmetric for all k, let

$$S_k = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$$

Then the feedback gain is updated by

$$\delta = B^T S_{k+1} B + r$$
$$\delta = r + \frac{s_1 T^4}{4} + s_2 T^3 + s_3 T^2$$
$$K_k = B^T S_{k+1} A \frac{1}{\delta}$$

We can write

$$k_1 = (\frac{s_1 T^2}{2} + s_2 T)\frac{1}{\delta}$$
$$k_2 + (\frac{s_1 T^3}{2} + \frac{3 s_2 T^2}{2} + s_3 T)\frac{1}{\delta}$$

We can write the closed-loop plant matrix

$$A_k^{cl} = A - BK_k$$
$$A_k^{cl} = \begin{bmatrix} a_{11}^{cl} & a_{12}^{cl} \\ a_{21}^{cl} & a_{22}^{cl} \end{bmatrix} = \begin{bmatrix} 1 - \frac{k_1 T^a}{2} & T - \frac{k_2 T^2}{2} \\ -k_1 T & 1 - k_2 T \end{bmatrix}$$

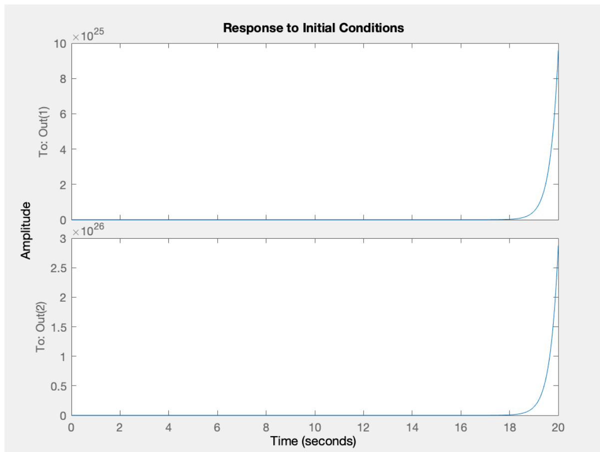
The updated cost kernel is

$$S_k = A^T S_{k+1} A_k^{cl} + Q$$

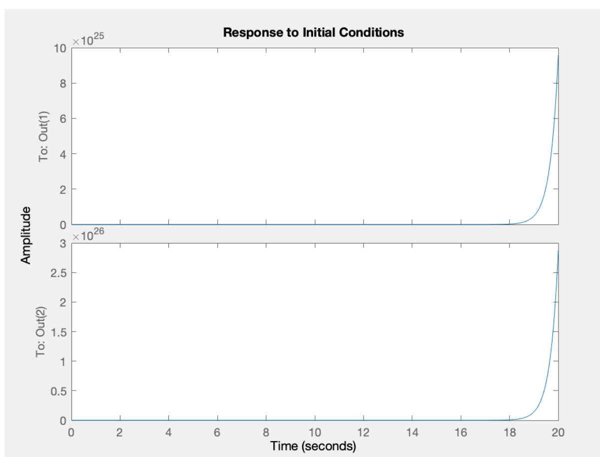
which yields the scalar updates:

$$s_1 = s_1 a_{11}^{cl} + s_2 a_{22}^{cl} + q_d$$
$$s_2 = s_1 a_{12}^{cl} + s_2 a_{22}^{cl}$$
$$s_3 = (s_1 T + s_2) a_{12}^{cl} + (s_2 T + s_3) a_{22}^{cl} + q_v$$

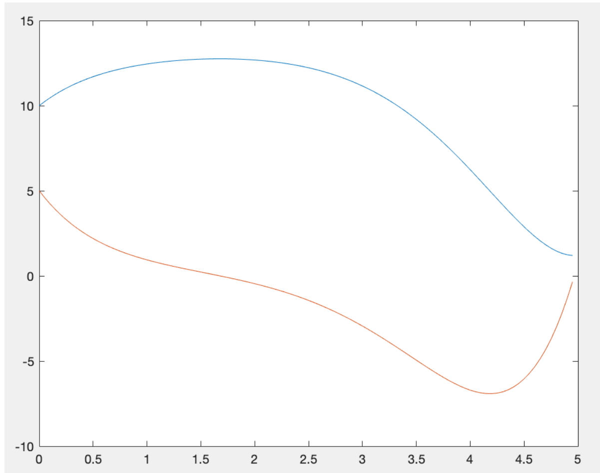
The unforced system (assuming a=3, b=2,T=0.001) shown below is clearly unstable.



c) The unforced system (assuming $w_n = 3$, T=0.001) shown below is marginally stable.



d) The response plots for the optimal controller is shown by



The MATLAB code is given below

```
clear
clc
close all

a=1;
b=1;
t0=0;
tf=5;
N=100;

T0=(tf-t0)/N;
A_d=[cosh(a*T0), (1/a)*sinh(a*T0); a*sinh(a*T0), cosh(a*T0)];
B_d=[b/(a^2)*(cosh(a*T0)-1); (b/a)*sinh(a*T0)];
s1=100;
s2=0;
s3=100;
Q=eye(2);
r=1;
x0=[10;5];


delta_array=zeros(1,N);
Sk_array=cell(1,N);
xk_array=cell(1,N);
uk_array=zeros(1,N);
xk_array{1}=x0;
Kk_array=cell(1,N);
Sk_array{N}=[s1 s2;s2 s3];
xk1_array=zeros(1,N);
xk2_array=zeros(1,N);

Acl=eye(2);
for k=1:(N-1)
    delta_array(1,N-k)=r+((s1/4)*T0^4)+(s2*T0^3)+s3*T0^2;
    k1=((s1*T0^2)/2+s2*T0)/delta_array(1,N-k);
    k2=((s1*T0^3)/2+(3*s2*T0^2)/2+s3*T0)/delta_array(1,N-k);
    Kk_array{N-k}=[k1 k2];
    a_cl11=1-(k1*T0^2)/2;
    a_cl12=T0-(k2*T0^2)/2;
    a_cl21=-k1*T0;
    a_cl22=1-k2*T0;
    s1=s1*a_cl11+s2*a_cl21+Q(1,1);
    s2=s1*a_cl12+s2*a_cl22;
    s3=(s1*T0+s2)*a_cl12+(s2*T0+s3)*a_cl22+Q(2,2);
    Sk_array{N-k}=[s1 s2;s2 s3];
end
for k=1:(N-1)
    uk_array(1,k)=-Kk_array{k}*xk_array{k};
    xk_array{k+1}=A_d*xk_array{k}+B_d*uk_array(1,k);
end
for k=1:N
    xk_array{k}(1)
    xk1_array(1,k)=xk_array{k}(1);
    xk2_array(1,k)=xk_array{k}(2);
end
xk_array{N}
uk_array(1,N-1)

T=0:T0:4.955;
U=uk_array;

plot(T,xk2_array)
```

2.4-1 Steady-state behavior

In this problem we consider a rather unrealistic discrete system because it is simple enough to allow an analytic treatment. Thus, let the plant

$$x_{k+1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k$$

have the performance index of

$$J_0 = \frac{1}{2} s_N^T x_N + \frac{1}{2} \sum_{k=0}^{N-1} (x_k^T \begin{bmatrix} q_1 & q_2 \\ q_2 & q_1 \end{bmatrix} x_k + r u_k^2)$$

(a) Find the optimal steady-state (i.e., N $\to \infty$) Riccati solution S_∞^* and show that it is positive definite. Find the optimal steady-state gain K_∞^* and determine when it is nonzero.

(b) Find the optimal steady-state closed-loop plant and demonstrate its stability.

(c) Now the suboptimal constant feedback

$$u_k = -K_\infty^* x_k$$

is applied to the plant. Find scalar updates for the components of the sub-optimal cost kernel S_k. Find the suboptimal steady-state cost kernel S_∞ and demonstrate that $S_\infty = S_\infty^*$.

Solution

To solve this problem we can analytically solve equation (2.4-12) from the book

$$S = A^T [S - SB(B^T SB + R)^{-1} B^T S] A + Q$$

Plugging in A,B,R=r,Q, assuming $S = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix}$

$$S = A^T [S - SB(\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + r)^{-1} B^T S] A + Q$$

$$S = A^T [S - \frac{1}{s_4 + r} \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix}] A + Q$$

$$S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} [\begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} - \frac{1}{s_4 + r} \begin{bmatrix} s_2 s_3 & s_2 s_4 \\ s_3 s_4 & s_4^2 \end{bmatrix}] \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} q_1 & q_2 \\ q_2 & q_1 \end{bmatrix}$$

$$\begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & s_1 - \frac{s_2 s_3}{r + s_4} \end{bmatrix} + \begin{bmatrix} q_1 & q_2 \\ q_2 & q_1 \end{bmatrix}$$

$$\begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 \\ q_2 & q_1 + s_1 - \frac{s_2 s_3}{r + s_4} \end{bmatrix}$$

This allows us to solve for s_i in terms of q_1, q_2, and r

$$s_1 = q_1$$
$$s_2 = q_2$$
$$s_3 = q_2$$
$$s_4 = q_1 + s_1 - \frac{s_2 s_3}{r + s_4}$$
$$s_4^2 + r s_4 = 2 q_1 r + 2 q_1 s_4 - s_2 s_3$$
$$s_4^2 + r s_4 = 2 q_1 r + 2 q_1 s_4 - s_2 s_3$$
$$s_4^2 + (r - 2q_1) s_4 + (s_2 s_3 - 2 q_1 r) = 0$$
$$s_4 = \frac{(2q_1 - r) \pm \sqrt{(r - 2_q 1)^2 - 4(s_2 s_3 - 2 q_1 r)}}{2}$$

Now that we have s_1, s_2, s_3, s_4, we have

$$\boxed{S_\infty = \begin{bmatrix} q_1 & q_2 \\ q_2 & s_4 \end{bmatrix}}$$

We can see that S_∞ is symmetric and is positive definite if $q_1 > 0$, $q_1 s_4 - q_2^2 > 0$. These are both true so we have shown it is symmetric and positive definite.
To find K_∞ we use the following equation (2.4-13) from the book

$$K_\infty = (B^T S_\infty B + R)^{-1} B^T S_\infty A$$
$$K_\infty = \frac{1}{r + s_4} \begin{bmatrix} 0 & 1 \end{bmatrix} S_\infty \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\boxed{K_\infty = \frac{1}{r + s_4} \begin{bmatrix} 0 & q_2 \end{bmatrix}}$$

b) We now look at evaluating the closed loop system

$$A_{cl} = A - BK_\infty$$
$$A_{cl} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & q_2 \end{bmatrix}$$
$$A_{cl} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{q_2}{r+s_4} \end{bmatrix}$$

This gives us eigevnalues of

$$\boxed{\lambda_{1,2} = 0, -\frac{q_2}{r + s_4}}$$

Because q_2, r, s_4 are all positive, we see that both eigenvalues are stable and the closed-loop plant is stable.

2.4-2 Analytic Riccati solution

Let A=$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$, B=$\begin{bmatrix}0\\1\end{bmatrix}$, S_N=I, Q=I

(a) Let r=0.1. Find the Hamiltonian matrix H and its eigenvalues and eigenvectors. Find the analytic expression for Riccati solution S_k. Find the steady-state solution S_∞ using (2.4-42). FInd the optimal steady-state gain K_∞ using (2.4-63) and also using Ackermann's formula.

(b) Let r=1. Find the Hamiltonian matrix and its eigenstructure. Find the steady-state solution S_∞ and gain K_∞. (Hint: See the discussion following (2.4-63).)

Solution

The Hamiltonian matrix H is given by the following equation

$$H = \begin{bmatrix} A^{-1} & A^{-1}BR^{-1}B^T \\ QA^{-1} & A^T + QA^{-1}BR^{-1}B^T \end{bmatrix}$$
$$H = \begin{bmatrix} 1 & -1 & 0 & -10 \\ 0 & 1 & 0 & 10 \\ 1 & -1 & 1 & -10 \\ 0 & 1 & 1 & 11 \end{bmatrix}$$

We can use MATLAB to determine the eigenvalues and eigenvectors of the matrix H as

$$\lambda_{1,2,3,4} = 10.7802, 2.7654, 0.3616, 0.0928$$
$$v_{1,2,3,4} = \begin{bmatrix} -0.1013 \\ -0.9907 \\ 0.0104 \\ 0.0899 \end{bmatrix}, \begin{bmatrix} -0.4740 \\ -0.8369 \\ 0.2685 \\ 0.0534 \end{bmatrix}, \begin{bmatrix} -0.5081 \\ 0.3243 \\ -0.7959 \\ 0.0573 \end{bmatrix}, \begin{bmatrix} -0.5113 \\ 0.4639 \\ -0.5636 \\ 0.4537 \end{bmatrix}$$

The analytic expression for the Ricatti solution S_k is given by

$$S_k = A^T[S_{k+1} - S_{k+1}B(B^T S_{k+1}B + R)^{-1}B^T S_{k+1})]A + Q$$

To find the steady-state solution (S_∞), we are instructed to use equation (2.4-42)

$$S_\infty = W_{21}W_{11}^{-1}$$

Therefore to find S_∞ we must define and solve W.
First, let us define M to be a diagonal matrix containing n eigenvalues outside the unit circle. Then we can define D as

$$D = \begin{bmatrix} M & 0 \\ 0 & M^{-1} \end{bmatrix}$$
$$D = \begin{bmatrix} 10.7802 & 0 & 0 & 0 \\ 0 & 2.7654 & 0 & 0 \\ 0 & 0 & 0.0928 & 0 \\ 0 & 0 & 0 & 0.3616 \end{bmatrix}$$

W is defined as a nonsingular matrix whose columns are the eigenvectors of H such that

$$W^{-1}HW = D$$

Matching our eigenvectors that we found earlier to their eigenvalues in the D matrix, we can find W to be

$$W = \begin{bmatrix} -0.5113 & -0.5081 & -0.1013 & 0.4740 \\ 0.4639 & 0.3243 & -0.9907 & 0.8369 \\ -0.5636 & -0.7959 & 0.0104 & -0.2685 \\ 0.4537 & 0.0573 & 0.0899 & -0.0534 \end{bmatrix}$$

Now all that's left to solve for S_∞ is to partition W to solve for W_{11}, W_{12}, W_{21}, W_{22}.

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$
$$W_{11} = \begin{bmatrix} -0.5113 & -0.5081 \\ 0.4639 & 0.3243 \end{bmatrix}$$
$$W_{12} = \begin{bmatrix} -0.1013 & 0.4740 \\ -0.9907 & 0.8369 \end{bmatrix}$$
$$W_{21} = \begin{bmatrix} -0.5636 & -0.7959 \\ 0.4537 & 0.0573 \end{bmatrix}$$
$$W_{22} = \begin{bmatrix} 0.0104 & -0.0534 \\ 0.0899 & -0.0534 \end{bmatrix}$$

We now have everything we need to solve S_∞ so we plug W_{11} and W_{21} into equation ()

$$S_\infty = W_{21} W_{11}^{-1}$$
$$S_\infty = \begin{bmatrix} -0.5636 & -0.7959 \\ 0.4537 & 0.0573 \end{bmatrix} \begin{bmatrix} -0.5113 & -0.5081 \\ 0.4639 & 0.3243 \end{bmatrix}^{-1}$$

$$\boxed{S_\infty = \begin{bmatrix} 2.6687 & 1.7266 \\ 1.7266 & 2.8812 \end{bmatrix}}$$

To find K_∞, we plug into our equation

$$K_\infty = (B^T S_\infty B + R)^{-1} B^T S_\infty A$$

$$\boxed{K_\infty = \begin{bmatrix} 0.5792 & 1.5456 \end{bmatrix}}$$

Ackermann's formula can be used to solve the Algebraic Ricatti Equation (ARE).
The optimal closed-loop poles are the stable eigenvalues of H^{-1}. Using MATLAB, we can calculate these to be

$$\lambda_{1,2,3,4} = 10.7802, 2.7654, 0.3616, 0.0928$$

Therefore the desired closed loop characteristic polynomial is

$$\Delta^{cl}(z) = z^2 - 0.4544z + 0.0335$$

The reachability matrix is

$$U_2 = \begin{bmatrix} B & AB \end{bmatrix}$$
$$U_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Finally, we define e_n as the last column of the nxn identity matrix. For our n=2 case

$$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

According to Ackermann's formula

$$K_\infty = e_n^T U_n^{-1} \Delta^d(A)$$
$$K_\infty = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{-1} (A^2 - 0.4544A + 0.0335)$$

$$\boxed{K_\infty = \begin{bmatrix} 0.5792 & 1.5456 \end{bmatrix}}$$

As we can see, these two methods produce the same result as we would expect.

b) The Hamiltonian matrix H is given by the following equation

$$H = \begin{bmatrix} A^{-1} & A^{-1}BR^{-1}B^T \\ QA^{-1} & A^T + QA^{-1}BR^{-1}B^T \end{bmatrix}$$
$$H = \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

To find the eigenstructure, we first find the eigenvalues and eigenvectors of H^{-1}

$$\lambda_{1,2,3,4} = 2.1220 + 1.0538i, 2.1220 - 1.0538i, 0.3780 + 0.1877i, 0.3780 - 0.1877i$$
$$v_{1,2,3,4} = \begin{bmatrix} -0.3330 + 0.3128i \\ -0.7032 + 0.0000i \\ 0.0186 - 0.2962i \\ 0.4374 + 0.1320i \end{bmatrix}, \begin{bmatrix} -0.3330 - 0.3128i \\ -0.7032 + 0.0000i \\ 0.0186 + 0.2962i \\ 0.4374 - 0.1320i \end{bmatrix}, \begin{bmatrix} 0.4374 - 0.1320i \\ -0.2472 + 0.1642i \\ 0.7032 + 0.0000i \\ -0.1044 + 0.4448i \end{bmatrix}, \begin{bmatrix} 0.4374 + 0.1320i \\ -0.2472 - 0.1642i \\ 0.7032 + 0.0000i \\ -0.1044 - 0.4448i \end{bmatrix}$$

Now we create the diagonal matrix M, using the stable eigenvalues of H^{-1}

$$M = \begin{bmatrix} 0.3616 & 0 \\ 0 & 0.0928 \end{bmatrix}$$

and the associate eigenvectors

$$\begin{bmatrix} X \\ \Lambda \end{bmatrix} = \begin{bmatrix} 0.4374 - 0.1320i & 0.4374 + 0.1320i \\ -0.2472 + 0.1642i & -0.2472 - 0.1642i \\ 0.7032 + 0.0000i & 0.7032 + 0.0000i \\ -0.1044 + 0.4448i & -0.1044 - 0.4448i \end{bmatrix}$$
$$X = \begin{bmatrix} 0.4374 - 0.1320i & 0.4374 + 0.1320i \\ -0.2472 + 0.1642i & -0.2472 - 0.1642i \end{bmatrix}$$
$$\Lambda = \begin{bmatrix} 0.7032 + 0.0000i & 0.7032 + 0.0000i \\ -0.1044 + 0.4448i & -0.1044 - 0.4448i \end{bmatrix}$$

Now that we have Λ and X, we can plug in to find S_∞.

$$S_\infty = \Lambda X^{-1}$$

$$\boxed{S_\infty = \begin{bmatrix} 2.9471 & 2.3692 \\ 2.3692 & 4.6131 \end{bmatrix}}$$

We can also solve for K_∞ using the following equation

$$K_\infty = \frac{1}{r} B^T \Lambda M X^{-1}$$

$$\boxed{K_\infty = \begin{bmatrix} 0.4221 & 1.2439 \end{bmatrix}}$$