MECE 6374: Fun Work #4

Eric Eldridge (1561585)

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Problem 1

Consider the following scalar nonlinear system

$$\dot{x} = -x^3$$

We are interested to examine the stability of the origin $\tilde{x} = 0$.

(a) Can you determine stability using the linearization of the nonlinear system?

(b) Consider the Lyapunov function

$$V(x) = x^4$$

Use Lyapunov's Direct method to determine the stability (global or local) of the origin $\tilde{x} = 0$.

Solution

(a) If we let $\dot{x} = f(x) = -x^3$, the equilibrium point $\bar{x}$ occurs when $f(\bar{x}) = 0$

$$-x^3 = 0 \implies \bar{x} = 0$$

The formula for linearization of $\dot{x}$ is

$$\begin{aligned}
f_{lin}(x) &= \cancelto{0}{f(x)} + \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} (x - \bar{x}) + \cancelto{0}{H.O.T.} \\
f_{lin}(x) &= \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} (x - \bar{x}) \\
f_{lin}(x) &= -3x^2\Big|_{x=0} x \\
f_{lin}(x) &= 0
\end{aligned}$$

The system $\dot{x} = 0x$ has the eigenvalue 0 which tells us that the linearized system is marginally stable which does not tell us anything about the behavior of the non-linear system near the equilibrium point 0.

(b) Lyapunov's 2nd method (Energy Method) states that if $V(0) = 0$ and $V(x) \geq 0$ for $x \neq 0$, then the function V(x) is said to have "energy-like" properties. This holds true for $V(x) = x^4$.

Further, Lyapnuov's 2nd method states that if V(x) decreases and $V(x) = 0$, then the system is stable.

$$\begin{aligned}
&V(x) = x^4 \\
&\dot{V}(x) = 4x^3\dot{x} \\
&\dot{V}(x) = 4x^3(-x^3) = -4x^6
\end{aligned}$$

We can see that $\dot{V}(x) < 0$ for all x. Therefore V(x) is global negative definite. V(x) is globally positive definite and radially unbound and $\dot{V}$ is globally negative definite which proves that **the system is globally asymptotically stable**.

Problem 2

Consider the nonlinear system

$$\dot{x}_1 = (x_1 - x_2)(x_1^2 + x_2^2 - 1)$$
$$\dot{x}_2 = (x_1 + x_2)(x_1^2 + x_2^2 - 1)$$

(a) Find all equilibrium points

(b) Use linearization and Lyapunov methods to show that (0, 0) is an asymptotically stable equilibrium.

(c) Is (0,0) globally stable?

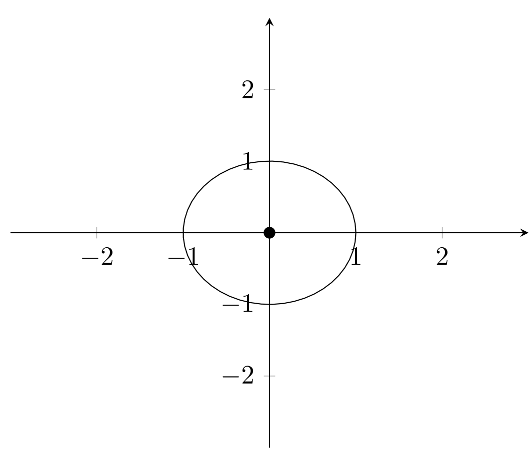
Solution

(a) The equilibrium points $(\bar{x}_1, \bar{x}_2)$ occur when $\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{aligned} (x_1 - x_2)(x_1^2 + x_2^2 - 1) &= 0 \\ \implies x_1 = x_2 \text{ or } x_1^2 + x_2^2 &= 1 \\ (x_1 + x_2)(x_1^2 + x_2^2 - 1) &= 0 \\ x_1 = x_2 \implies (x_2 + x_2)(x_2^2 + x_2^2 - 1) &= 0 \\ (2x_2)(2x_2^2 - 1) &= 0 \\ \implies x_2 = 0 \text{ or } x_2 &= \pm\frac{\sqrt{2}}{2} \\ x_1^2 + x_2^2 = 1 \implies (x_1 - x_2)(0) &= 0 \\ \implies x_1, x_2 \text{ can be anything so long as } x_1^2 + x_2^2 &= 1 \text{ is satisfied.} \end{aligned}$$

The points $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ both belong to the set of points of points described by $x_1^2 + x_2^2 = 1$. Therefore, the equilibrium points are:

$$\boxed{\text{Equilibrium Points} : (0,0), (x_1, x_2) \text{ s.t. } x_1^2 + x_2^2 = 1}$$



(b) The formula for linearization of $\dot{x}$ is

$$
\begin{aligned}
f_{1,lin}(x) &= \cancelto{0}{f_1(\bar{x})} + \left.\frac{\partial f_1}{\partial x_1}\right|_{x=\bar{x}}(x_1-\bar{x}_1) \quad + \left.\frac{\partial f_1}{\partial x_2}\right|_{x=\bar{x}}(x_2-\bar{x}_2) + \cancelto{0}{H.O.T.} \\
f_{2,lin}(x) &= \cancelto{0}{f_2(\bar{x})} + \left.\frac{\partial f_2}{\partial x_1}\right|_{x=\bar{x}}(x_1-\bar{x}_1) \quad + \left.\frac{\partial f_2}{\partial x_2}\right|_{x=\bar{x}}(x_2-\bar{x}_2) + \cancelto{0}{H.O.T.} \\
f_{1,lin}(x) &= \left.\frac{\partial f_1}{\partial x_1}\right|_{x=\bar{x}}(x_1-\bar{x}_1) \quad + \left.\frac{\partial f_1}{\partial x_2}\right|_{x=\bar{x}}(x_2-\bar{x}_2) \\
f_{2,lin}(x) &= \left.\frac{\partial f_2}{\partial x_1}\right|_{x=\bar{x}}(x_1-\bar{x}_1) \quad + \left.\frac{\partial f_2}{\partial x_2}\right|_{x=\bar{x}}(x_2-\bar{x}_1) \\
f_{1,lin}(x) &= \left.((x_1^2+x_2^2-1)+(x_1-x_2)(2x_1))\right|_{(0,0)} x_1 + \left.(-(x_1^2+x_2^2-1)+(x_1-x_2)(2x_2))\right|_{(0,0)} x_2 \\
f_{2,lin}(x) &= \left.((x_1^2+x_2^2-1)+(x_1-x_2)(2x_1))\right|_{(0,0)} x_1 + \left.((x_1^2+x_2^2-1)+(x_1-x_2)(2x_2))\right|_{(0,0)} x_2 \\
f_{1,lin}(x) &= -x_1+x_2 \\
f_{2,lin}(x) &= -x_1-x_2 \\
\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\end{bmatrix} &= \begin{bmatrix}-x_1+x_2\\ -x_1-x_2\end{bmatrix} \\
\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\end{bmatrix} &= \begin{bmatrix}-1 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}
\end{aligned}
$$

The eigenvalues of the matrix $\begin{bmatrix}-1 & 1\\ -1 & -1\end{bmatrix}$ are $\lambda_{1,2} = -1 \pm 1j$

Because both of the eignevalues have negative real parts, we know that **this equilibrium point is asymptotically stable.**

We can also use Lyapunov's 2nd Method to solve this. If we choose the energy function $V(x) = x_1^2 + x_2^2$, we can see that V is global positive definite and radially unbounded. We can solve for $\dot{V}$:

$$
\begin{aligned}
\dot{V}(x) &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\
\dot{V}(x) &= 2x_1(x_1-x_2)(x_1^2+x_2^2-1) + 2x_2(x_1+x_2)(x_1^2+x_2^2-1) \\
\dot{V}(x) &= 2(x_1^2+x_2^2)(x_1^2+x_2^2-1)
\end{aligned}
$$

We can see that $\dot{V}$(x) is locally negative definite. Therefore (0,0) is **locally asymptotically stable**.

(c) As shown using Lyapunov's 2nd method in the latter half of part (b), we can see that the equilibrium point is <u>NOT</u> globally stable

Problem 3

Consider the following state-space system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{2x_1}{(1+x_1^2)^2}$$

(a) Can you determine stability of the equilibrium point $\bar{x} = 0$ using the Lyapunov function below?

$$V(x) = \frac{x_1^2}{1+x_1^2} + \frac{1}{2}x_2^2$$

(b) Can you determine global stability?

(c) Confirm the answers above by plotting the phase plane portrait of the system.

Solution

(a) The formula for linearization of $\dot{x}$ is

$$\begin{aligned}
f_{1,lin}(x) &= \cancelto{0}{f_1(\bar{x})} + \left.\frac{\partial f_1}{\partial x_1}\right|_{x=\bar{x}}(x_1-\bar{x}_1) \quad + \left.\frac{\partial f_1}{\partial x_2}\right|_{x=\bar{x}}(x_2-\bar{x}_2) + \cancelto{0}{H.O.T.} \\
f_{2,lin}(x) &= \cancelto{0}{f_2(\bar{x})} + \left.\frac{\partial f_2}{\partial x_1}\right|_{x=\bar{x}}(x_1-\bar{x}_1) \quad + \left.\frac{\partial f_2}{\partial x_2}\right|_{x=\bar{x}}(x_2-\bar{x}_2) + \cancelto{0}{H.O.T.} \\
f_{1,lin}(x) &= \left.\frac{\partial f_1}{\partial x_1}\right|_{x=\bar{x}}(x_1-\bar{x}_1) \quad + \left.\frac{\partial f_1}{\partial x_2}\right|_{x=\bar{x}}(x_2-\bar{x}_2) \\
f_{2,lin}(x) &= \left.\frac{\partial f_2}{\partial x_1}\right|_{x=\bar{x}}(x_1-\bar{x}_1) \quad + \left.\frac{\partial f_2}{\partial x_2}\right|_{x=\bar{x}}(x_2-\bar{x}_1) \\
f_{1,lin}(x) &= x_2 \\
f_{2,lin}(x) &= -\left.\frac{2(1+x_1^2)^2 - (2x_1)(2(1+x_1^2)(2x_1))}{(1+x_1^2)^4}\right|_{(0,0)} x_1 \\
f_{1,lin}(x) &= x_2 \\
f_{2,lin}(x) &= -2x_1 \\
\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\end{bmatrix} &= \begin{bmatrix}-x_1+x_2\\ -x_1-x_2\end{bmatrix} \\
\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\end{bmatrix} &= \begin{bmatrix}0 & 1\\ -2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}
\end{aligned}$$

The eigenvalues of the matrix $\begin{bmatrix}0 & 1\\ -2 & 0\end{bmatrix}$ are $\lambda_{1,2} = \pm\sqrt{2}j$

Because both of the eignevalues lie along the jω axis, we know that the linearized system is marginally stable. This tells us nothing about the non-linear system though. Instead we will attempt to solve using Lyapnuov's 2nd method.

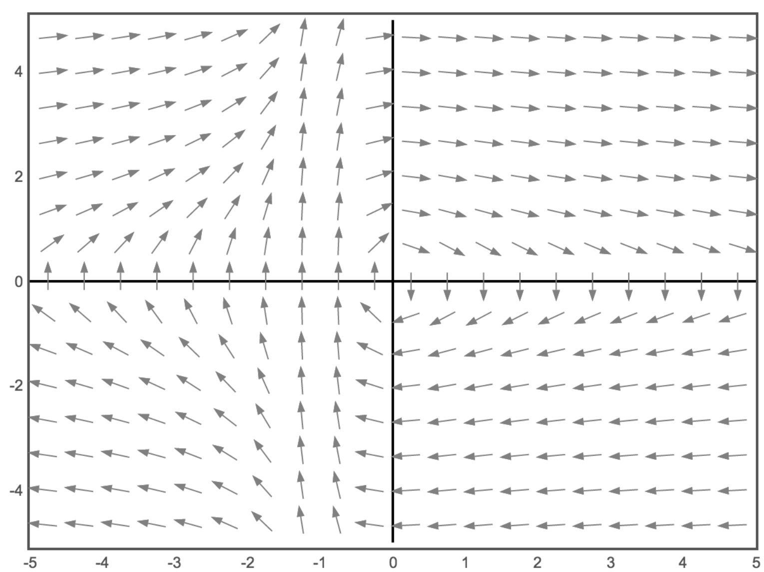
We see that $V(x) = \frac{x_1^2}{1+x_1^2} + \frac{1}{2}x_2^2$ is globally positive definite. However, the energy function is not radially unbounded. As $x_1 -> \infty$ and x_2 remains small, V(x) does not go to infinity. We will now look at the properties of $\dot{V}(x)$.

$$
\begin{aligned}
V(x) &= \frac{x_1^2}{1+x_1^2} + \frac{1}{2}x_2^2 \\
\dot{V}(x) &= \frac{2x_1\dot{x}_1(1+x_1^2) - 2x_1\dot{x}_1(x_1^2)}{(1+x_1^2)^2} + x_2\dot{x}_2 \\
\dot{V}(x) &= \frac{2x_1\dot{x}_1}{(1+x_1^2)^2} + x_2\dot{x}_2 \\
\dot{V}(x) &= \frac{2x_1x_2}{(1+x_1^2)^2} - \frac{2x_1x_2}{(1+x_1^2)^2} \\
\dot{V}(x) &= 0
\end{aligned}
$$

$\dot{V}(x)$ is globally negative semi-definite. Since V(x) is globally positive definite and unbounded, this leads us to the conclusion that **the system is stable at the equilibrium point (0,0)**.

(b) Global stability cannot be determined because the Lyapnuov function we used in part (a) is not radially unbounded.

(c) The phase plane is shown below:



Problem 4

Consider the following system

$$\dot{x}_1 = -x_2 + x_1(x_1^2 + x_2^2 - 1)$$
$$\dot{x}_2 = x_1 + x_2(x_1^2 + x_2^2 - 1)$$

Determine the stability of the origin (0,0) using the following Lyapunov function

$$V(x) = x_1^2 + x_2^2$$

Classify the stability in terms of local/global and asymptotic properties

Solution

V(x) is globally positive definite and radially unbounded.
To determine the properties of $\dot{V}(x)$, we derive the function.

$$\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$

Plugging in $\dot{x}_1$ and $\dot{x}_2$

$$\dot{V}(x) = 2x_1(-x_2 + x_1(x_1^2 + x_2^2 - 1)) + 2x_2(x_1 + x_2(x_1^2 + x_2^2 - 1))$$
$$\dot{V}(x) = 2x_1(\cancelto{0}{-x_2} + x_1(x_1^2 + x_2^2 - 1)) + 2x_2(\cancelto{0}{x_1} + x_2(x_1^2 + x_2^2 - 1))$$
$$\dot{V}(x) = 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1)$$

We can see that $\dot{V}$(x) is locally negative definite for small x_1, x_2. Therefore, the system is **locally asymptotically stable** near the equilibrium point (0,0).

Problem 5

Examine the stability (local or global, asymptotic or not) of the origin (0,0) of the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -2x_1 - x_2^3$$

(a) Using a linearization approach

(b) Using a Lyapunov function candidate of the form $V(x_1, x_2) = ax_1^2 + bx_2^2$.

Solution

(a) The formula for linearization of $\dot{x}$ about the point (0,0) is

$$
\begin{aligned}
f_{1,lin}(x) &= \cancelto{0}{f_1(x)} + \left.\frac{\partial f_1}{\partial x_1}\right|_{(0,0)} (x_1) \quad + \left.\frac{\partial f_1}{\partial x_2}\right|_{(0,0)} (x_2) + \cancelto{0}{H.O.T.} \\
f_{2,lin}(x) &= \cancelto{0}{f_2(x)} + \left.\frac{\partial f_2}{\partial x_1}\right|_{(0,0)} (x_1) \quad + \left.\frac{\partial f_2}{\partial x_2}\right|_{(0,0)} (x_2) + \cancelto{0}{H.O.T.} \\
f_{1,lin}(x) &= \left.\frac{\partial f_1}{\partial x_1}\right|_{(0,0)} (x_1) \quad + \left.\frac{\partial f_1}{\partial x_2}\right|_{(0,0)} (x_2) \\
f_{2,lin}(x) &= \left.\frac{\partial f_2}{\partial x_1}\right|_{(0,0)} (x_1) \quad + \left.\frac{\partial f_2}{\partial x_2}\right|_{(0,0)} (x_2) \\
f_{1,lin}(x) &= x_2 \\
f_{2,lin}(x) &= -2x_1 - \left.3x_2^2\right|_{(0,0)} x_2 \\
f_{1,lin}(x) &= x_2 \\
f_{2,lin}(x) &= -2x_1 \\
\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} &= \begin{bmatrix} x_2 \\ -2x_1 \end{bmatrix} \\
\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned}
$$

The eigenvalues of the matrix $\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$ are $\lambda_{1,2} = \pm\sqrt{2}j$

Because both of the eignevalues have negative real parts, we know that **this equilibrium point is marginally stable.**

(b) Derive V(x) to find

$$\dot{V}(x) = 2ax_1\dot{x}_1 + 2bx_2\dot{x}_2$$

Plugging in $\dot{x}_1$ and $\dot{x}_2$

$$\dot{V}(x) = 2ax_1x_2 + 2bx_2(-2x_1 - x_2^3)$$
$$\dot{V}(x) = 2ax_1x_2 - 4bx_1x_2 - 2bx_2^4$$

We can see that if $a = 2b$ and $b > 0$, the Lyapunov function $V(x_1, x_2) = ax_1^2 + bx_2^2$ is globally positive definite and radially unbounded. Further, we can see that $\dot{V}$(x) is globally negative semi-definite. This, tells us that at the equilibrium point (0,0), the system will be **globally stable**.